Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 25 (2008) 1209–1220
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Structurally stable perturbations of polynomials in the Riemann sphere

J. Iglesias [a,*], A. Portela [a], A. Rovella [b]

[a] *Universidad de La República, Facultad de Ingenieria, IMERL, Julio Herrera y Reissig 565, Uruguay*
[b] *Universidad de La República, Facultad de Ciencias, Centro de Matemática, Iguá 4225, C.P. 11400, Montevideo, Uruguay*

Received 28 March 2007; accepted 12 February 2008

Available online 21 March 2008

Abstract

The perturbations of complex polynomials of one variable are considered in a wider class than the holomorphic one. It is proved that under certain conditions on a polynomial p of the plane, the C^r conjugacy class of a map f in a C^1 neighborhood of p depends only on the geometric structure of the critical set of f. This provides the first class of examples of structurally stable maps with critical points and nontrivial nonwandering set in dimension greater than one.

Résumé

Nous considérons les perturbations des polynômes complexes en une variable dans une classe plus vaste que la classe holomorphe. Si f est une application appartenant à un voisinage C^1 d'un polynôme p du plan, nous prouvons, sous certaines conditions sur p, que la classe de conjugaison C^r de f ne dépend que de la structure géométrique du lieu critique de f. Ceci fournit la première classe d'exemples, en dimension supérieure à une, d'applications structurellement stables ayant des points critiques et un ensemble nonerrant nontrivial.

MSC: 37C20; 37C75; 58K05

Keywords: Structural stability; Criticat set; Complex polynomials

1. Introduction

Given a manifold without boundary M, denote by $C^r_W(M)$ the set of C^r endomorphisms of M, considered with the strong (or Whitney) topology. The set of critical points of $f \in C^r_W(M)$ is denoted by S_f. Two maps f and g are *topologically equivalent* if there exists a homeomorphism h such that $hf = gh$. The problem of determining the classes of topological equivalence is central in the theory of dynamical systems. In particular, a great effort has been made to classify those maps that are topologically equivalent to its neighbors. If C is a topological space of self

* Corresponding author.
E-mail addresses: jorgei@fing.edu.uy (J. Iglesias), aldo@fing.edu.uy (A. Portela), leva@cmat.edu.uy (A. Rovella).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2008.02.001

mappings, then f is C- structurally stable if there exists a neighborhood of f such that every g in that neighborhood is topologically equivalent to f. Obviously, the concept depends on the space and topology under consideration.

The examples of structurally stable maps on manifolds without boundary that are already known are the following:

(1) A C^1 diffeomorphism of a compact manifold is C^1 structurally stable if and only if it satisfies Axiom A and the strong transversality condition. This theorem is the result of the work of many authors, from the sixties to the nineties. The "only if" part is due to C. Robinson [10] and the other direction was obtained by R. Mañé [6], fifteen years later.
It is still not known if there exist C^r structurally stable diffeomorphisms that are not C^1 structurally stable.
(2) Any C^r expanding map of a compact manifold is C^r structurally stable. This was proved by M. Shub [11] in the sixties.
(3) In the case of one dimensional maps of the circle there are some possible combinations giving conditions for structural stability.
The same occurs for rational maps of the Riemann sphere. This case will be specially considered in the sequel. For example a polynomial map of degree d is stable in the d dimensional space of parameters corresponding to its coefficients, if p is hyperbolic and satisfies the *no critical relations* property: $p^n(S'_p) \cap p^m(S'_p) = \emptyset$ for every $0 \leqslant n < m$, where S'_p is the set of finite critical points of p. It is not known, however, if the converse of this assertion is true.

Therefore there are no examples of noninvertible nonexpanding structurally stable maps with or without critical points, in dimensions greater than one. In the attempt to construct the simplest possible examples, we consider $C^1_W(\mathbb{C})$ neighborhoods of polynomials and look for $C^r_W(\mathbb{C})$ stable maps there. The theorem of Mañé, Sad and Sullivan of stability of rational mappings [7], implies the statement (3) above and also that within the family of degree d polynomials, the stable ones are dense.

It will be clear later that no polynomial can be $C^r_W(\mathbb{C})$ structurally stable, because the critical points of holomorphic maps are nongeneric in those spaces of smooth maps. Indeed, let f and g be topologically equivalent (also called conjugate) and h the *conjugacy* between them, i.e. the homeomorphism such that $hf = gh$; then h carries generic critical points of f to critical points of g and critical values of f to critical values of g. Therefore, some geometric conditions must be imposed on the critical sets of maps f and g in order to obtain the existence of a conjugacy between them. The concept that will be used is the following:

Definition 1. Two maps f and g are *geometrically equivalent* if there exist orientation preserving C^1 diffeomorphisms of M, φ and ψ, such that $\varphi f = g\psi$.

If for some positive α, the map ψ is α close to the identity in C^0 topology, then the maps f and g are said α-geometrically equivalent.

This concept, introduced by R. Thom, is now a central concept in global analysis. It is a concept of geometric nature: it implies, for example, that the set of (generic) critical points and critical values of f and g are diffeomorphic and that the degree of the maps are the same. However, it has no dynamical meaning: for example, two quadratic polynomials of the sphere are always geometrically equivalent. The concept of geometric equivalence has no significance relative to future iterates of the map: the fact that two maps f and g are equivalent in this sense does not imply that their iterates f^2 and g^2 are also equivalent. It is clear, on the other hand, that if two maps are topologically equivalent, then the homeomorphism realizing the conjugacy carries information about the local behavior of the maps; therefore, under generic conditions, topological equivalence implies geometric equivalence. The aim now is to establish conditions implying the converse statement.

Note that if a polynomial p satisfies the no critical relations property (item (3) above) then no (finite) critical point of p is periodic or preperiodic.

The main result in this work is the following; after its proof, in the last section, some other more general statements will be discussed.

Theorem 1. *Let p be a polynomial that satisfies the no critical relations property. The following conditions are equivalent*:

(1) *The Julia set of* p *is connected and hyperbolic.*
(2) *There exists a neighborhood* $\mathcal{U}$ *of* p *in* $C^1_W(\mathbb{C})$, *and* $\alpha > 0$ *such that, if two maps belonging to* $\mathcal{U}$ *are* α*-geometrically equivalent, then they are topologically equivalent.*

The implication (1) $\Rightarrow$ (2) is the most difficult part of the statement. It contains the proof that, under certain conditions on the polynomial p, it suffices to prove that the sets of critical points and values of two maps C^1 close to p have the same geometry, to obtain that the maps are equivalent from the dynamical point of view. The reason why α-geometric equivalence is needed is explained in an example in Section 3: it may happen that the diffeomorphism ψ is identifying components of S_f and S_g that are not close to each other.

The dynamical structure of a polynomial p satisfying the hypothesis (1) of the theorem is well known. Recall that the Julia set is connected if and only if every critical point (other than ∞) has bounded orbit. The hyperbolicity of p is equivalent to the fact that every critical point is attracted to a periodic attractor or superattractor, and the hypothesis of no critical relations implies that there are no finite superattractors. Within this context the polynomial is stable under small perturbations of its coefficients. The proof of this fact is based on the construction of conjugacies in the Fatou components of p, that come from the holomorphic local conjugacies at the periodic points (see the theorems of Schröder and Böttcher in the references [12,8]). Then these conjugacies are glued together via the application of the λ lemma [7]. When the perturbation is taken in the C^1 Whitney topology, then nonholomorphic maps arise, including some with wild critical sets. All the above techniques rely on the conformal structure of the maps in question and therefore cannot be in general applied in this wider context. To deal with the structure of the nonwandering set one has a basic result, a theorem by F. Przytycki [9], that implies that under the hypothesis (1), the polynomial p is C^1 Ω-stable. This means that for a small C^1 perturbation f of p the restrictions of f and p to respective nonwandering sets are topologically equivalent. This theorem is used in Section 2 to prove that the complement of the nonwandering set of f is the union of the basins of the periodic attractors of f. This is a fundamental step in the proof. In particular, every component of the complement of the nonwandering set of f is periodic or preperiodic. This extends Sullivan's theorem of nonexistence of wandering Fatou components, to Whitney C^1 perturbations of hyperbolic polynomials. It justifies, moreover, the denomination of Fatou component of f for a component of the complement of the nonwandering set of f, and also the concept of *analytic continuation* for Fatou components.

Some work is needed to prove that geometrically equivalent maps f and g are conjugate when restricted to corresponding Fatou components. The proof of this and that these conjugacies extend to the whole plane deserve Section 3. As a consequence of this part of the theorem the first examples of C^3-structurally stable maps having critical points are shown:

Corollary 1. *Let* p *be a hyperbolic polynomial map having connected Julia set. In each neighborhood* $\mathcal{U}$ *of* p *in* $C^\infty_W(\mathbb{C})$ *there exists some* f *that is* C^3 *structurally stable.*

The proof also implies the existence of C^3 structurally stable maps in $C^3(S^2)$ with uniform topology, see final section. It will become clear in subsequent sections that no polynomial can be C^1 approximated by a C^2 structurally stable map. See Remark 1 in Section 4.

For the proof of the converse ((2) $\Rightarrow$ (1)): to prove hyperbolicity it will be shown that if a critical point of p belongs to the Julia set of p, then there exists a C^1 perturbation of p that is geometrically but not topologically equivalent to it. Less evident is the fact that the Julia set of p must be connected in order to obtain the properties stated in part (2). See Section 4. See the remarks at the end of the article concerning some questions about the problem of stability.

2. Whitney perturbations of *p*

In this section a polynomial p satisfying the hypothesis (1) of Theorem 1 is fixed and f is a small C^1 Whitney perturbation of p. The objective is to show that the picture of the dynamics of f is the same as that of p. The following properties are satisfied by a polynomial p verifying the hypothesis (1) of Theorem 1:

(1) The point ∞ is an attractor. The basin of ∞, $B_\infty(p)$, is connected and simply connected.
(2) Its boundary, $\partial B_\infty(p)$, is a curve (not necessarily a Jordan curve), and is equal to $\Omega'(p)$, the set of nonwandering points of p that are not periodic attractors. (Clearly $\Omega'(p)$ is the Julia set of p, also denoted J_p.)

(3) Every component of the complement of the closure of $B_\infty(p)$ is simply connected and its boundary is a Jordan curve.
(4) The components of the Fatou set of p, are the periodic components and their preimages.

See for example [12] or [8].

Theorem 2. *There exists a neighborhood $\mathcal{U}$ of p in $C^1_W(\mathbb{C})$, such that each $f \in \mathcal{U}$ satisfies conditions* 1 *to* 4 *above.*

The remaining of this section is devoted to the proof of this theorem. The first result is trivial and one of the reasons why Whitney topology is considered. See for example reference [4], where the properties of Whitney topology are clearly exposed. If f were a C^r perturbation of p in the topology $C^r(S^2)$, then the intersection of the critical set of f with a neighborhood of ∞ may possibly become a nonconnected set with $d-1$ components, where d is the degree of p, and the analytic continuation of the fixed point at ∞ may not be critical anymore.

Lemma 1. *For every f in a neighborhood of p in $C^0_W(\mathbb{C})$, the point at ∞ is an attractor.*

This means that under this hypothesis, f is a proper map of $\mathbb{C}$ and there exists a disc D with the property that $f(D)$ contains the closure of D and such that the future orbit of any point outside D diverges.

Now consider a C^1_W perturbation f of p. The hypothesis on p imply that the Julia set of p is hyperbolic and hence expanding, in the sense that $|p'(z)| > 1$ for every $z \in J_p$ where the norm is considered with respect to a hyperbolic metric in an open set containing J_p. This implies that p is C^1-Ω stable by the theorem of Przytycki. For f close to p define $\Omega'(f) = \Omega(f) \setminus \{\text{periodic attractors}\}$. Obviously periodic attractors of p are carried by the conjugacy h to attracting periodic points of f, so that h must carry J_p onto $\Omega'(f)$.

Lemma 2. *If f is C^1_W close to p, then $\Omega'(f) = \partial B_\infty(f)$.*

Proof. To prove that $\partial B_\infty(f) \subset \Omega'(f)$, observe first that there exists a neighborhood U of $\Omega'(p)$ and a neighborhood $\mathcal{U}$ of p such that $f^{-1}(U) \subset U$ and $\bigcap_{n\geqslant 0} f^{-n}(U) = \Omega'(f)$ for every $f \in \mathcal{U}$. This holds because p is a hyperbolic polynomial and by C^1 Ω-stability. If $x \notin \Omega'(f)$, then there exists an $n = n_x \geqslant 0$ such that $f^n(x) \notin U$, then x belongs to the basin of an attractor and cannot belong to ∂B_∞.

To prove the other inclusion take a point $z \in \Omega'(f)$ and V a neighborhood of z. It is known that the restriction of p to J_p is locally eventually onto; by conjugation, this also holds for the restriction of f to $\Omega'(f)$. Using this and the other inclusion, already proved, there exist $n > 0$ and $x \in V \cap \Omega'(f)$ such that $f^n(x)$ belongs to the boundary of $B_\infty(f)$. Let $U \subset V$ be a neighborhood of z such that $U \cap \Omega'(f) = V \cap \Omega'(f)$ and U does not intersect the set of critical points of f^n. Then $x \in U$ and f^n is open in U, so $f^n(U) \cap B_\infty(f) \neq \emptyset$ and hence U, and also V, intersect $B_\infty(f)$. □

Note that it was not used that the basin of ∞ is simply connected.

Proof of Theorem 2. The first assertion of (1) follows from Lemma 1. As the perturbation f is a proper map of the plane, it follows that the restriction of $f : \mathbb{C} \setminus f^{-1}(S_f) \to \mathbb{C} \setminus f(S_f)$ to a component of its domain is a covering map (see [5], Proposition 2). Then the basin of ∞ must be connected. Simple connectivity is now a consequence of the fact that the boundary of $B_\infty(f)$ is connected (by Lemma 2 and the theorem of Przytycki). Also (2) is an immediate consequence of the above arguments.

Let V be a component of the complement of the closure of $B_\infty(f)$. It is clear that the boundary of V is contained in the boundary of $B_\infty(f)$, from which it follows that V is simply connected. Moreover, the boundary of V is a Jordan curve, because the contrary assumption implies that the complement of the unbounded component of the boundary of V contains points of the boundary of V and this contradicts the fact that the boundary of V is contained in the boundary of $B_\infty(f)$. This proves (3). To prove the remaining statement it is sufficient to show that every point in the complement of the closure of $B_\infty(f)$ is attracted to a periodic attractor. For this an argument similar to that of the proof of Lemma 2 works: indeed, if U is a small neighborhood of the boundary of $B_\infty(p)$, then the complement of

U is a compact set contained in the union of the basins of the periodic attractors of p, and the conclusion follows because this condition is open in the topology under consideration. $\square$

Given f in a small C^1 neighborhood $\mathcal{U}$ of p, one can define $\mathcal{C}_f$ as the set of Fatou components of f, which are the components of the complement of $\Omega(f)$. The result above implies that given any $g \in \mathcal{U}$ there exists a natural map $a : \mathcal{C}_f \to \mathcal{C}_g$ such that $a(f(C)) = g(a(C))$ for each $C \in \mathcal{C}_f$. The bijection is defined first assigning immediate basins of attractors of f to elements of $\mathcal{C}_g$ corresponding by analytic continuation.

3. Construction of conjugacies

In this section, the C^1 Whitney perturbations of a polynomial p satisfying conditions (1) of the theorem will be considered. By the theorem of Böttcher, any complex polynomial is holomorphically conjugate to $z \to z^d$ locally at ∞, where d is the degree of the polynomial; moreover, under the hypothesis of part (1) of the theorem (as the Julia set is connected, ∞ is the unique critical point of p in B_∞), the conjugacy extends to the whole basin of ∞.

The same proof of Böttcher theorem yields a conjugacy between f and p in a neighborhood of ∞. This local conjugacy (obviously not holomorphic) dynamically extends to a conjugacy that is close to the identity in the whole basin:

Lemma 3. *Given any $\epsilon > 0$ there exists a C^1 neighborhood $\mathcal{U}$ of p such that for every $f \in \mathcal{U}$ there exists a map $h : B_\infty(f) \to B_\infty(p)$ such that $hf = ph$ in $B_\infty(f)$ and $|h(z) - z| < \epsilon$.*

Proof. Let U be a neighborhood of J_p and $\mathcal{U}_0$ a C^1 neighborhood of p such that for every f in $\mathcal{U}_0$, $f^{-1}(U) \subset U$ and f is λ-expanding in U, where $\lambda > 1$.

Let $H : \{|z| > 1\} \to B_\infty(p)$, be Böttchers conjugacy between p and z^d. By the same argument given in [8] to prove Böttcher theorem, the following assertion holds:

Given $\epsilon > 0$ and $k > 0$ there exists a neighborhood $\mathcal{U}_1 \subset \mathcal{U}_0$ such that, for every $f \in \mathcal{U}_1$, there exists a homeomorphism $h : B_\infty(f) \to B_\infty(p)$ such that $hf = ph$ and $|h(z) - z| < \epsilon$ for every z such that $|H^{-1}(z)| > 1 + k$.

The number k is now chosen so that there is a fundamental domain D for the restriction of f to $B_\infty(f)$ contained in $U \cap h^{-1}(H(\{z\colon |z| > 1 + k\}))$. It remains to prove that h is ϵ-close to the identity in the whole $B_\infty(f)$. Given any $z \in U$ such that $f(z) \in D$, let w be such that $h(z) = w$ so that $p(w) = h(f(z))$. Let also w' be close to z such that $p(w') = f(z)$, and note that

$$|w - z| = |w - w'| + |z - w'| \leqslant \lambda^{-1}\epsilon + \delta,$$

where it was used that h is ϵ-close to the identity in D and where δ is the distance between local inverses of f and p. Now diminish the neighborhood $\mathcal{U}_1$ in such a way that $\delta \leqslant \epsilon(1 - \lambda^{-1})$ to obtain that $|h(z) - z| < \epsilon$ also holds for z in the preimage of D. Then continue for every $f^{-n}(D)$ by induction. $\square$

3.1. Conjugacy in bounded domains

Let $\{c_1, \ldots, c_r\}$ be the set of finite critical points of p. These points are all contained in the basins of the bounded attractors. For every i let V_i be a small neighborhood of c_i, such that $V_i \cap V_j = \emptyset$. Let α be a positive number less than the distance between any two different V_i. Then there exists a C^1_W neighborhood $\mathcal{U}$ of p, such that for every $f \in \mathcal{U}$, the critical set S_f is contained in $V = \bigcup V_i$, and so every critical point of f belongs to the basin of a periodic attractor of f. Assume that f and g are α-geometrically equivalent maps C^1 close to p. This means that there exist diffeomorphisms of the plane φ and ψ such that $\varphi f = g\psi$, moreover, the choice of α assures that the map ψ must carry $S_f \cap V_i$ to $S_g \cap V_i$. Begin with a fixed attracting point of p and consider its smooth continuation x_f for $f \in \mathcal{U}$. The basin of x_f is denoted by B_f and the immediate basin by U_f. Note that Theorem 2 implies that U_f is simply connected. The objective throughout this section is to prove that there exists a homeomorphism h realizing the equivalence of $f|_{U_f}$ and $g|_{U_g}$. This map will be produced as an extension of the restrictions of φ to a neighborhood of the set of critical values and of ψ to a neighborhood of the set of critical points of f in U_f.

Lemma 4. *If f and g are geometrically equivalent maps C^1 close to p, then their restrictions to U_f and U_g are topologically equivalent.*

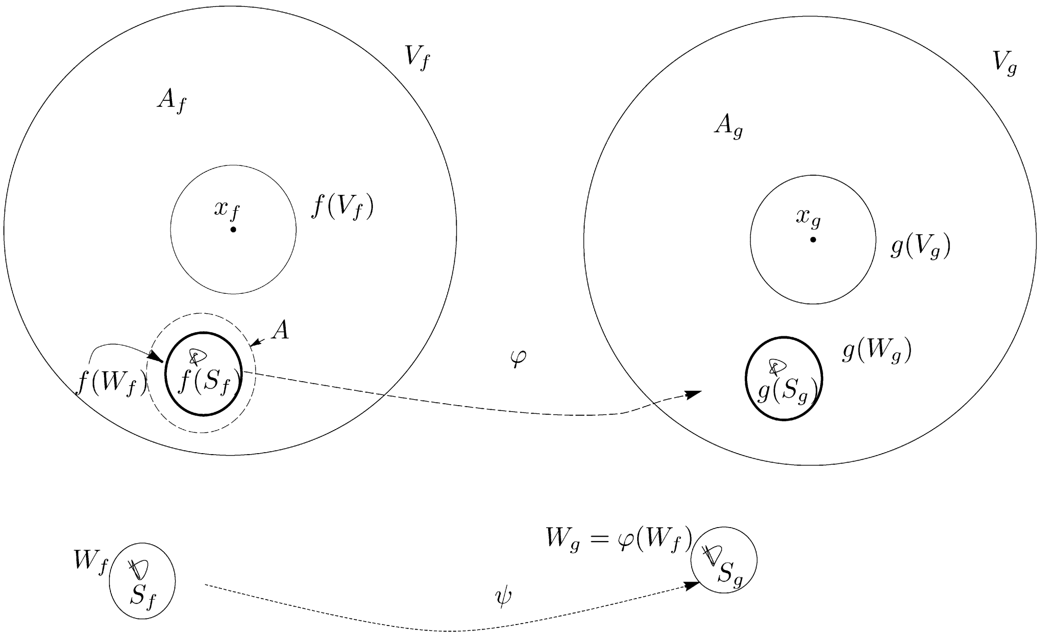

Fig. 1.

Proof. It will be assumed first that p has only one critical point c in B_p. Let V_f be a neighborhood of x_f, such that $f|_{V_f}$ is a diffeomorphism and the annulus $A_f = V_f \setminus f(V_f)$ is a fundamental domain. It is also possible to choose V_f and a topological disc W_f, containing S_f, such that $\overline{f(W_f)}$ is also a topological disc contained in the interior of A_f (see Fig. 1). For the map g define corresponding V_g, A_g and W_g. Moreover, W_g is chosen so that $\varphi(f(W_f)) = g(\psi(W_f)) = g(W_g)$.

Identifying the boundaries of the annulus A_f (resp. A_g) via f (resp. g) one obtains tori $A_f|f$ and $A_g|g$. There exists an orientation preserving homeomorphism h (that can be chosen C^0 close to the identity because W_f and W_g are arbitrary small):

$$h : A_f|f \to A_g|g, \tag{1}$$

realizing a conjugacy between the maps induced by f and g to the given domains, and such that the restriction of h to $f(W_f)$ is equal to φ (recall that φ is orientation preserving). Moreover, one can dynamically extend h to the whole V_f. It is claimed now that there exists (a unique) extension of h to $U_f \setminus W'_f$, where $W'_f = \bigcup_{m \geqslant 0} f^{-m}(f(W_f))$. First extend h to the preimage of V_f. Observe that $f : f^{-1}(V_f \setminus f(W_f)) \to V_f \setminus f(W_f)$ is a covering map of degree d', from which it follows that $hf : f^{-1}(V_f \setminus f(W_f)) \to V_g \setminus g(W_g)$ is a degree d' covering map. Also the restriction of g to $g^{-1}(V_g \setminus g(W_g))$ is a degree d' covering map onto $V_g \setminus g(W_g)$. To show that there exists a lift $\tilde{h}$ of hf, one can consider induced maps in homotopy groups. The domains of g and hf are open connected sets in the plane with the same connectivity, and the proximity of the maps implies that the action on relative generators of the induced maps are equal. This implies that there exists a unique lift $\tilde{h}$ of hf such that $g\tilde{h} = hf$ and $\tilde{h}(x_f) = x_g$. The uniqueness of $\tilde{h}$ implies that it extends h.

$$\begin{tikzcd}
 & g^{-1}(V_g \setminus g(W_g)) \arrow[d, "g"] \\
f^{-1}(V_f \setminus f(W_f)) \arrow[r, "hf"'] \arrow[ur, dotted, "\tilde{h}"] & V_g \setminus g(W_g)
\end{tikzcd}$$

The same argument shows how to extend h to the whole $U_f \setminus W'_f$. Finally one must extend h to U_f.

To define h in W_f and its preimages, other details must be taken into account, relative to the fact that the restrictions of h and ψ to the boundary of W_f may be equal or not. In the first case, h can be extended to W_f as equal to ψ and then to the remaining part of U_f dynamically. But in the other case h and ψ differ in the boundary of W_f, so the definition of h started in formula (1) must be changed. Note that the set of points of ∂W_f where h and ψ are equal is open and closed in ∂W_f, so it suffices to find a way of make them coincide at just one point. Note that the definition of h in V_f is somehow arbitrary; Let A be a small annulus contained in V_f and whose interior boundary is equal to the boundary of $f(W_f)$. Let D be a Dehn twist supported on A, define $h_i = h \circ D^i$ and $\tilde{h}_i$ as the lift of h_i. This implies that if r is a point in the boundary of W_f, then $\tilde{h}_i(r)$ takes all the possible values of $g^{-1}(h(f(r)))$. So one can choose i such that $\tilde{h}_i(r) = \psi(r)$. This finishes the proof in the case that there exists just one critical point in B_p.

If there are more than one critical point in the basin of x_f then the arguments are similar, so we explain the differences and omit the details. Let $c_1, \dots, c_r$ be the critical points of p in that basin. Let f, x_f, V_f and A_f be as above; let W_f^j $j = 1, \dots, r$ be a small disc containing the components of S_f close to c_j. For every $1 \leqslant j \leqslant r$ there exists an $n_j \geqslant 1$ such that $f^{n_j}(W_f^j) \subset A_f$. Define h in V_f in such a way that its restriction to $f^{n_j}(W_f^j)$ is equal to

$$g^{n_j-1}\varphi f^{-(n_j-1)}|_{f(W_f^j)}.$$

As above, h is extended to V_f and the same argument shows how to define it in

$$U_f \setminus \bigcup_{n \geqslant 0} f^{-n}\left(\bigcup_{j=1}^{r} f^{n_j}(W_f^j)\right).$$

For each j, let A'_j be a small annulus whose interior boundary is equal to boundary of $f^{n_j}(W_f^j)$ and let D_j by a Dehn twist supported on A'_j. To make $\tilde{h}$ coincide with ψ one just needs to compose h with adequate iterates of the maps D_j. □

This previous result concerned with fixed domains. Suppose now that the polynomial p has an attracting cycle $\alpha_p = \{x_p^1, \dots, x_p^n\}$. For every f close to p in C^1 topology, denote by $U_f^1, \dots, U_f^n$ the components of the immediate basin of the attractor $\alpha_f = \{x_f^1, \dots, x_f^n\}$, C^1 continuation of α_p. Define also $U_f = \bigcup U_f^j$. The following is an easy generalization of the previous Lemma 4, and its proof is omitted.

Lemma 5. *If f and g are geometrically equivalent maps C^1 close to p, then they are also topologically equivalent when restricted to the grand orbits of U_f and U_g.*

Using that every component of the complement of the set $\Omega'(f)$ is preperiodic and the previous results, it follows that:

Corollary 2. *Given $\epsilon > 0$ there exists a C^1 neighborhood $\mathcal{U}$ of p such that, if f and g are geometrically equivalent maps in $\mathcal{U}$, then there exists $h : \mathbb{R}^2 \setminus \Omega'(f) \to \mathbb{R}^2 \setminus \Omega'(g)$, homeomorphism that conjugates f and g and such that $|h(z) - z| < \epsilon$.*

Proof. It remains to show the uniform proximity to the identity in the bounded domains, but this is similar to the unbounded case (Lemma 3); the main fact that makes the above arguments work is the following: given any neighborhood U of $\partial B_\infty(p)$ and given a fundamental domain D_i in each periodic component of the Fatou set of f, there exists a positive integer N such that $f^{-N}(\bigcup D_i \cup F) \subset U$, where F denotes the union of the nonperiodic components of the Fatou set of f. □

Example. The reason why the α-geometric equivalence is needed is explained in the following example. Assume that p is a polynomial with two attracting fixed points x_1 and x_2 with immediate basins B_1 and B_2. Assume that B_1 contains a critical point c_1 and B_2 contains two critical points c_2 and c_3. Let f and g be C^1 perturbations of p such that the following holds:

1. $S_f \cap B_1$ is homeomorphic to a circle and $S_f \cap B_2$ is equal to $\{c_2, c_3\}$.
2. $S_g \cap B_1 = \{c_1\}$ and S_g has two components in B_2, one of them is a point and the other a circle.

The maps f and g may be chosen as geometrically equivalent, but cannot be topologically equivalent. Note that necessarily the image under ψ of $S_f \cap B_1$ is contained in B_2.

3.2. Extension to the boundary of $B_\infty(f)$

It is already known that there exists a conjugacy h between the restrictions of f and g to the Fatou components of f. It was also explicit that the conjugacy h can be chosen as close to the identity as wished, by diminishing the neighborhood $\mathcal{U}$ of p (see Lemma 3 and Corollary 2). On the other hand, the theorem of Przytycki provides a conjugacy h_p of these maps in the boundaries of the respective domains. It remains to prove that h can be continuously extended to the closure of $B_\infty(f)$, and that in the boundary is equal to the conjugacy of Przytycki. We first show that h extends continuously to the boundary and that it is close to the identity.

Let ϵ_0 be a constant of expansivity of the restriction of p to its Julia set, that is, for every $z \neq w$ in J_p there exists $N > 0$ such that $|p^N(z) - p^N(w)| > \epsilon_0$.

Corollary 3. *There exists a C^1 neighborhood $\mathcal{U}$ of p such that, for geometrically equivalent maps f and g in $\mathcal{U}$, it holds that the conjugacy h of Corollary* 2 *extends to the boundary of $B_\infty(f)$ and is a homeomorphism.*

Proof. Choose $\mathcal{U}$ such that the distance between the identity and h is less than $\epsilon_0/2$. Let $x \in \partial B_\infty(f)$ and $x_n \to x$, where $x_n \notin \partial B_\infty(f)$. We claim that $h(x_n)$ converges. Otherwise, one can choose cluster points $z \neq y$ of $h(x_n)$. By the choice of ϵ_0 there exists $N > 0$ such that $|g^N(y) - g^N(z)| > \epsilon_0$. Then $hf^N(x_n)$ accumulates at $g^N(y)$ and $g^N(z)$, but as $f^N(x_n)$ converges to $f^N(x)$, a contradiction appears because h is $\epsilon_0/2$ close to the identity. Define h in the boundary as the limit of $h(x_n)$. The claim implies that h is continuous and surjective. Finally h is injective because two points z and w with the same image would verify that $|f^n(z) - f^n(w)|$ eventually becomes greater than ϵ_0, while $h(f^n(z)) = h(f^n(w))$ for every $n > 0$. □

4. Proof of Theorem 1

Proof of (1) ⇒ (2). This has been already done in the previous section. Corollary 3, gives the map h, defined in the whole plane, realizing the conjugacy between f and g. As h is close to the identity, then its restriction to the nonwandering set must coincide with the conjugacy of Przytycki.

Proof of (2) ⇒ (1). The hypothesis give a C^1_W neighborhood $\mathcal{U}$ of p such that geometric and topological equivalence are the same in $\mathcal{U}$. Maps of class C^3 are dense in $\mathcal{U}$ and their critical points have a generic structure. The proof of the following lemma can be found in [5].

Lemma 6. *Let c be a simple critical point of p, that is, $p'(c) = 0 \neq p''(0)$. There exist a neighborhood U of c, a C^3 neighborhood $\mathcal{U}_0$ of p and an open and dense subset $\mathcal{G}$ of $\mathcal{U}_0$ such that, for every $f \in \mathcal{G}$, the intersection $S_f \cap U$ is diffeomorphic to a circle.*

Moreover, there exists $f \in \mathcal{G}$ such that the restriction of f to $S_f \cap U$ is injective and $S_f \cap U$ contains exactly three cusp type points.

Remark 1. Here we use some elementary facts about singularities of differentiable mappings in dimension two, a classical reference is the book by Golubitsky and Guillemin [3].

We do not know if there exists a neighborhood $\mathcal{U}_0$ of p such that the restriction of every map $f \in \mathcal{G} \cap \mathcal{U}_0$ to $S_f \cap U$ is injective. It is known, however, that there exists at least one cusp type point in the boundary of the unbounded component of the complement of $S_f \cap U$.

The classification of critical points for generic maps is very easy in dimension two. Indeed, if c is a critical point of a generic map f, then the kernel of Df_c has dimension one. The critical point c is a *fold point* if the kernel of Df_c is not equal to the tangent space of S_f at c and is a *cusp point* otherwise. Moreover, normal forms are known for both kind of critical points:

The normal form of a fold point is $(x, y) \to (x^2, y)$.

The normal form of a cusp point is $(x, y) \to (x^3 - xy, y)$.

It can also be shown also that fold points are C^2 persistent and cusp points are C^3 persistent. As well as maps having critical points cannot be C^1 structurally stable, it can be concluded now that maps with cusp type points cannot be C^2 structurally stable, because a conjugacy between two maps must carry cusp critical points to critical points of the same type.

As any generic perturbation of a polynomial has a cusp type point, it follows, as asserted in the introduction, that in a small neighborhood of a polynomial no map can be C^2 structurally stable.

It follows also that if $f \in \mathcal{G}$ and the restriction of f to S_f is injective, then the same holds in a C^3 neighborhood of f.

Suppose that every critical point of p is simple, and let $S_p = \{c_i\colon 1 \leqslant i \leqslant d-1\}$; for each i, let U_i be a small neighborhood of c_i, and $\mathcal{G}_i$ the generic set associated with c_i as in Lemma 6.

Recall that p satisfies the noncritical relations property, so the degree of p is d, and the number of finite critical values of p is $d-1$.

Define $\mathcal{G}' \subset \mathcal{U}$ as the set of maps f such that $f|_{S_f}$ is injective and f belongs to every $\mathcal{G}_i$. It follows that S_f has $d-1$ connected components, each one of them homeomorphic to the circle and such that the restriction of f to S_f is injective. The proof that $\mathcal{G}'$ is nonempty is left to the last corollary. This, together with the following proposition, will provide the examples of structurally stable maps.

Proposition 1. *If $f \in \mathcal{G}'$, then f is C^3-geometrically stable.*

Proof. Let g be a C^3 perturbation of f, let $\{C_1(g), \dots, C_{d-1}(g)\}$ be the components of the set of critical values of g.

Let φ be a diffeomorphism of the plane close to the identity that carries $C_i(f)$ onto $C_i(g)$ preserving cusps. For each i choose a curve α_i joining the image of a cusp point $z_i \in C_i(f)$ with infinity. This can be done without any intersection, that is, the curves α_i are simple, disjoint and the intersection of α_i with $\bigcup C_i(f)$ is the set $\{z_i\}$. Let $\beta_i = \varphi(\alpha_i)$ and define $H(f)$ as the complement of the union of $\tilde{S}_f := f^{-1}(f(S_f))$ with $\bigcup_i f^{-1}(\alpha_i)$ and $H(g)$ as the union of the unbounded components of the complement of the union of $\tilde{S}_g$ with $\bigcup_i g^{-1}(\beta_i)$. See Fig. 2 below with $d = 2$.

Each component of $H(f)$ corresponds to a unique component of $H(g)$ by proximity. Moreover, these components of $H(f)$ are simply connected, and the restriction of f to each of them is a diffeomorphism onto its image. Therefore, for each component $H_j(f)$ of $H(f)$ there exists a unique diffeomorphism ψ_j that satisfies $\varphi f = g\psi_j$, and whose image is the corresponding component of $H(g)$. These diffeomorphisms can be extended to a unique diffeomorphism ψ of the plane such that $\varphi f = g\psi$. □

Proof of the connectedness of the Julia set of p. Observe that there exists a neighborhood of ∞ foliated by curves homeomorphic to circles that are invariant under f. This foliation $\mathcal{F}_f$ is invariant and must be preserved by conjugacies. If J_p is not connected, then there exists a critical point $c = c_1$ of p contained in $B_\infty(p)$; c is the critical point of p closest to ∞ (i.e. the circle of the foliation that contains c is the boundary of an open neighborhood of ∞ that does not contain any other finite critical point). Assume first that c is a simple critical point of p. By the proof of Proposition 1 two maps f and g in $\mathcal{G}_1$ that are equal outside the neighborhood U_1 of c, are geometrically equivalent. To arrive to a contradiction it suffices to find f and g as above that are not topologically equivalent.

Let A be a p-invariant neighborhood of ∞ that contains $p(c)$, does not intersect U_1 and whose boundary is a circle of the foliation $\mathcal{F}_p$. If f is a perturbation of p with support U_1 ($f = p$ outside U_1) then the foliations $\mathcal{F}_f$ and $\mathcal{F}_p$ coincide in A. Perturb p in U_1 such that the perturbation f belongs to $\mathcal{G}_1$ and such that there exist two cusp points that belong to the component of S_f contained in U_1 whose images belong to the same leaf of the foliation $\mathcal{F}_f$. This is possible but is not generic; a new perturbation g supported in U_1 and belonging to $\mathcal{G}_1$ can be found such that the image of the three cusps belong to different leaves of the foliation.

To treat the case of c not simple, assume that the order of c is k. Given a neighborhood U_0 of c there exists a C^∞ perturbation q of p such that:

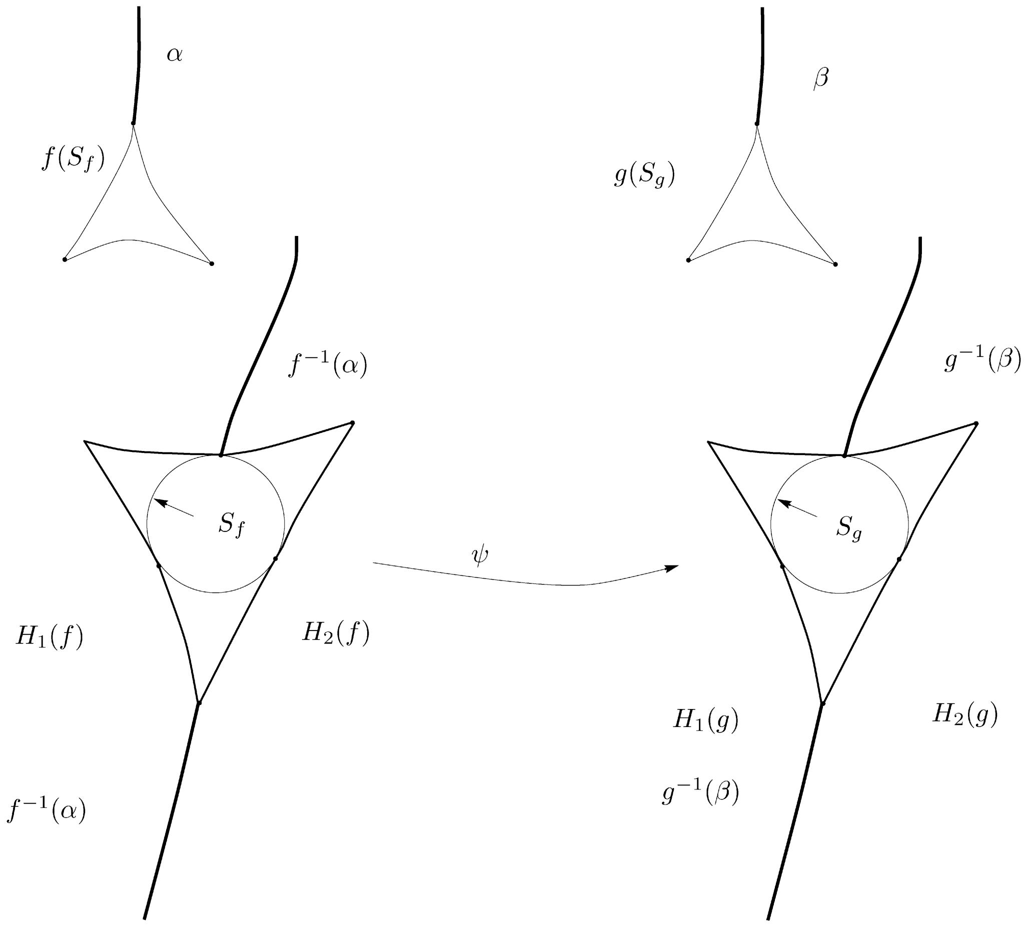

Fig. 2.

- $q = p$ outside U_0.
- There exists an arbitrary small neighborhood $U_0' \subset U_0$ of c such that q is holomorphic in U_0'.
- q has k critical points in U_0 all contained in U_0'.

Once this q was obtained, one can proceed as above. □

Proof of the hyperbolicity of *p*. The first step is to prove that the Julia set cannot have critical points if some type of C^1 stability is required. The proof is very simple, which contrasts with the fact that the problem is open when only holomorphic perturbations are allowed.

Proposition 2. *If p has a critical point in its Julia set, then in every C^1 neighborhood of p there exists an f that is geometrically but not topologically equivalent to p.*

Proof. Let $\mathcal{U}$ be a C^1 neighborhood of p and c be a critical point of p in J_p. This implies that there exist expanding periodic points accumulating at c. An argument based in J. Franks lemma [2] will imply the existence of a map f in a C^1 neighborhood of p such that f and p have the same sets of critical points but f has a new attracting periodic orbit. Indeed, if ε is such that $f \in \mathcal{U}$ if the C^1 distance between p an f is less than ε, then take a periodic orbit of p contained in J_p and containing a point z close to c in such a way that $|p'(z)| < \varepsilon$. Let $K = |(p^n)'(z)|$, where n is the period of the orbit of z. Note that there exists a neighborhood of the orbit of z such that the restriction of p to this neighborhood is a diffeomorphism onto its image. Under these conditions, Franks' lemma asserts that there exists a map $f \in \mathcal{U}$ such that:

- The orbit of z under f is the same as that of p.
- For every $0 < j < n$, the differential of f at $f^j(z)$ is equal to that of p at the same point. Moreover, f is also conformal at z, and $|f'(z)| < |p'(z)|/K$.
- The support of the perturbation is an arbitrary small neighborhood of the orbit of z not intersecting the critical set of p or the set of periodic attractors of p.
- The perturbation f is a diffeomorphism onto its image when restricted to the support of the perturbation.

The first three items imply that f has a new periodic attractor (the orbit of z) and so it is not topologically equivalent to p. It is geometrically equivalent to p because the support of the perturbation is disjoint with the set of critical points of p. □

To conclude the proof of the hyperbolicity of p, one has to show that every critical point is attracted to a periodic attractor. First of all note that every periodic point of p must be hyperbolic: under the contrary assumption one can perturb in a neighborhood of the nonhyperbolic orbit to obtain a map that is geometrically but not topologically equivalent to p. This implies that the Fatou set of p does not contain Leau components neither Siegel discs. Herman rings are forbidden since the Julia set of p is connected. Finally, as the set of critical points do not intersect the Julia set and there are no superattractors, the conclusion is immediate from the classification theorem of Sullivan, see [8] or [12]. □

Proof of Corollary 1. First perturb p to a polynomial p_0 having no critical relations. It suffices to show that in every C^∞ neighborhood of p_0 there exists a map $f \in \mathcal{G}'$, because by Proposition 1 this map will be geometrically equivalent to every map g in a C^3 neighborhood of it, and then (1) ⇒ (2) of Theorem 1 implies the topological equivalence between f and g. It is very easy to give an example that is generic in the sense of Lemma 6 and such that the restriction of f to S_f is injective. It suffices to do it locally, and as the critical points of p_0 are nondegenerate, it suffices to give just an example of a perturbation f of $p(z) = z^2$ such that $f \in \mathcal{G}'$. An explicit example is: $(x, y) \to (x^2 - y^2 + \lambda y, 2xy)$, $\lambda \neq 0$. So to construct an example of a C^3 structurally stable map, just take $p(z) = z^2 + \epsilon$ (ϵ small so that J_p is connected and hyperbolic) and then perturb in a neighborhood of 0 so that the new map f has the representation above in that neighborhood. □

Example. We exhibit a map f that is C^3 geometrically stable and C^3 Ω-stable but cannot be C^3 approximated by a C^3 structurally stable map. Begin with $p(z) = z^2 - 3$ and perturb it, as in Corollary 1, to $f(x, y) = (x^2 - y^2 + \lambda y - 3, 2xy)$. It was shown that f is C^3 geometrically stable and C^3 Ω-stable. If g is C^3 close to f, the critical set S_g is a simple closed curve with three cusp type points contained in the basin of ∞. There exists a leave γ of the foliation by circles that contains the image of a cusp c and another critical value $f(z)$ (we can assume that z is a critical point of fold type). As the set $\bigcup_{n>0} f^{-n}(f^n(c))$ is dense in γ, we can find a small perturbation g_1 of g supported in a neighborhood of c, such that for some n $g_1^n(c) = g_1^n(z)$. Then perturb g to a map g_2 such that $g_2^n(c_{g_2}) \cap g_2^n(S_{g_2} \setminus c_{g_2}) = \emptyset$, where c_{g_2} is the set of cusps of g_2. Then g_1 and g_2 cannot be topologically equivalent.

4.1. Further considerations

Throughout this discussion, M is a manifold of dimension at least two and $I^r(M)$ denotes the space of maps p having a strong C^r neighborhood where geometric equivalence implies topological equivalence. So we have proved that a polynomial without critical relations belongs to $I^1(S^2 \setminus \infty)$ iff its Julia set is hyperbolic and connected. However, the arguments used imply also other results.

Theorem 3. *If R is a hyperbolic rational map without critical relations* (*hence without superattractors*), *then $R \in I^1(S^2)$. It follows that any rational map can be C^∞ approximated by $C^3(S^2)$ structurally stable maps.*

The first assertion follows directly from the arguments of the proof that $(1) \to (2)$. To prove the second one first perturb to a rational map that has no critical relations.

Another corollary of the arguments of the proof of Theorem 1 give C^1 structurally stable maps:

Corollary 4. *The map $z \to z^d$ is C^1 structurally stable in $C^1_W(\mathbb{C} \setminus 0)$.*

About stability, some of the results here attained are now briefly commented. The case of compact M will be now considered. Let $E^r(M)$ be the set of nonexpanding noninvertible endomorphisms, and $St^r(M)$ the set of C^r structurally stable maps.

As far as we know, there exist no examples in $St^1(M) \cap E^1(M)$ if M has dimension at least two. Note that $S_f = \emptyset$ is a necessary condition for a map f to be C^1 stable. The theorem of N. Aoki, K. Moriyasu and N. Sumi in [1] implies that a map in $St^1(M)$ must satisfy Axiom A and, as is the case for diffeomorphisms, also the strong transversality condition. However, these conditions are not sufficient for stability, as was shown by an example of F. Przytycki in [9]. It seems difficult to find examples of structurally stable maps having saddle type basic pieces: indeed, unstable manifolds of a basic piece may have self intersections and can also visit different basic pieces. On the other hand, the arguments in this article seem to be extendable to prove stability in other situations, where the maps have only expanding or attracting basic pieces. We conjecture that if a noninvertible Axiom A map has no saddle type basic pieces, then there must be critical points in the immediate basin of any attractor. It would follow that a C^1 structurally stable map must have saddle type basic pieces.

Acknowledgement

We would like to thank Jan Kiwi for many helpful suggestions. Also the referee has pointed out some improvements to the work. The first and second authors were partially supported by PEDECIBA Uruguay.

References

[1] N. Aoki, K. Moriyasu, N. Sumi, C^1 maps having hyperbolic periodic points, Fund. Math. 169 (2001) 1–49.
[2] J. Franks, Necessary conditions for stability of diffeomorphisms, Trans. Amer. Math. Soc. 158 (1971) 301–308.
[3] M. Golubitsky, V. Guillemin, Stable Mappings and Their Singularities, Graduate Texts in Mathematics, vol. 14, Springer, New York, 1973.
[4] M. Hirsch, Differential Topology, Springer-Verlag, New York, 1976.
[5] J. Iglesias, A. Portela, Real Perturbations of complex polynomials, Bull. Braz. Math. Soc. (N.S.) 38 (1) (2007) 129–155.
[6] R. Mañé, A proof of the C^1 stability conjecture, Publ. Math. Inst. Hautes Études Sci. 66 (1987) 161–210 (in English).
[7] R. Mañé, P. Sad, D. Sullivan, On the dynamics of rational maps, Ann. Sci. École Norm. Sup. (4) 16 (2) (1983) 193–217.
[8] J. Milnor, Dynamics in One Complex Variable, Annals of Mathematics Studies, vol. 160, Princeton Univ. Press, 2006.
[9] F. Przytycki, On Ω-stability and structural stability of endomorphisms satisfying Axiom A, Studia Math. 60 (1976) 61–77.
[10] C. Robinson, Structural stability of C^1 diffeomorphisms, J. Differential Equations 22 (1976) 28–73.
[11] M. Shub, Endomorphisms of compact differentiable manifolds, Amer. J. Math. 91 (1969) 175–199.
[12] N. Steinmetz, Rational Iteration, de Gruyter Studies in Mathematics, vol. 16, 1993.