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Nodal domains and spectral minimal partitions

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Abstract

We consider two-dimensional Schrödinger operators in bounded domains. We analyze relations between the nodal domains of eigenfunctions, spectral minimal partitions and spectral properties of the corresponding operator. The main results concern the existence and regularity of the minimal partitions and the characterization of the minimal partitions associated with nodal sets as the nodal domains of Courant-sharp eigenfunctions.

Keywords: Optimal partitions; Eigenvalues; Nodal domains; Spectral minimal partitions

1. Introduction and main results

We consider mainly two-dimensional Laplace operators in bounded domains. We would like to analyze the relations between the nodal domains of the eigenfunctions of the Dirichlet Laplacians and the partitions by k open sets D_i which are optimal in the sense that the maximum over the D_i's of the ground state energy of the Dirichlet realization of the Laplacian in D_i is minimal.

1.1. Definitions and notations

Let us consider a Schrödinger operator

$$H = -\Delta + V \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ with Dirichlet boundary condition.

In the whole article (except in Section 3) , we will consider that Ω satisfies the following condition of smoothness:

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Assumption 1.1. Ω has compact and piecewise $\mathcal{C}^{1,+}$ boundary, i.e. piecewise $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. Moreover Ω satisfies the interior cone property.

This allows a finite number of corners (and cracks) of opening $\alpha\pi$ (defined in Section 2).

The other general assumption is that

Assumption 1.2. The potential V belongs to $L^\infty(\Omega)$.

Under these assumptions (which will not be recalled at each statement), H is selfadjoint if viewed as the Friedrichs extension of the quadratic form associated to H with form domain $W_0^{1,2}(\Omega)$ and form core $\mathcal{C}_0^\infty(\Omega)$. We denote H by $H(\Omega)$. We are interested in the eigenvalue problem for $H(\Omega)$ and note that under our assumptions $H(\Omega)$ has compact resolvent and its spectrum, which will be denoted by $\sigma(H(\Omega))$ is discrete and consists of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ with finite multiplicities which tend to infinity, so that

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \tag{1.2}$$

and such that the associated eigenfunctions u_k can be chosen to form an orthonormal basis for $L^2(\Omega)$.

Without loss of generality we can assume that the u_k are real valued and by elliptic regularity (see also Proposition 2.8 in Section 2) we have:

$$u_k \in \mathcal{C}^{1,\alpha}(\Omega) \cap \mathcal{C}_0^0(\overline{\Omega}), \tag{1.3}$$

for any $\alpha < 1$.

We know that u_1 can be chosen to be strictly positive in Ω, but the other eigenfunctions u_k ($k \geqslant 2$) must have zerosets. We define for any function $u \in \mathcal{C}_0^0(\overline{\Omega})$

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1.4}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of such a function will be called $\mu(u)$.

We now introduce the notions of partition and minimal partition.

Definition 1.3. Let $1 \leqslant k \in \mathbb{N}$. We will call *partition* (or k-partition if we want to indicate the cardinality of the partition) of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of mutually disjoint subsets of Ω:

$$D_i \cap D_j = \emptyset, \quad \forall i \neq j \quad \text{and} \quad \bigcup_{i=1}^k D_i \subset \Omega. \tag{1.5}$$

We call it *open* if the D_i are open sets of Ω, *connected* if the D_i are connected.

We denote by $\mathfrak{O}_k$ the set of open connected k-partitions of Ω.

We now introduce the notion of spectral minimal partition sequence.

Definition 1.4. Let $H = H(\Omega)$ as above. For $\mathcal{D}$ in $\mathfrak{O}_k$, we introduce

$$\Lambda(\mathcal{D}) = \max_i \lambda(D_i), \tag{1.6}$$

where $\lambda(D_i)$ is the ground state energy of $H(D_j)$.

Remark 1.5. When D is not sufficiently regular, we define $\lambda(D)$ differently. See Definition 3.1.

Definition 1.6. For any integer $k \geqslant 1$, we define

$$\mathfrak{L}_k = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda(\mathcal{D}). \tag{1.7}$$

We call the sequence $\{\mathfrak{L}_k\}_{k \geqslant 1}$ the *spectral minimal partition sequence* of $H(\Omega)$.

For given k, we call a k-partition $\mathcal{D} \in \mathfrak{O}_k$ minimal, if $\mathfrak{L}_k = \Lambda(\mathcal{D})$.

Remark 1.7. If $k=2$, it is rather well known (see for example [18] or [12]) that $\mathfrak{L}_2$ is the second eigenvalue and the associated minimal 2-partition is the nodal partition associated to the second eigenfunction.

We now introduce the notion of strong partition.

Definition 1.8. A partition $\mathcal{D}=\{D_i\}_{i=1}^k$ of Ω in $\mathfrak{O}_k$ is called *strong* if

$$\mathrm{Int}\left(\overline{\bigcup_i D_i}\right)\setminus\partial\Omega=\Omega. \tag{1.8}$$

Attached to a partition, we can naturally associate a closed set in $\overline{\Omega}$ defined by

$$N(\mathcal{D})=\overline{\bigcup_i(\partial D_i\cap\Omega)}. \tag{1.9}$$

This leads us to introduce the definition of a regular closed set. This definition is modeled on some (but not all) of the properties of the nodal set of an eigenfunction of a Schrödinger operator (see [23] and Section 2).

Definition 1.9. A closed set $N\subset\overline{\Omega}$ is regular (and write $N\in\mathcal{M}(\Omega)$) if N meets the following requirements:

(i) There are finitely many distinct $x_i\in\Omega\cap N$ and associated positive integers ν_i with $\nu_i\geqslant 2$ such that, in a sufficiently small neighborhood of each of the x_i, N is the union of $\nu_i(x_i)$ $\mathcal{C}^{1,+}$ curves (non self-crossing) with one end at x_i (and each pair defining at x_i a positive angle in $(0,2\pi)$) and such that in the complement of these points in Ω, N is locally diffeomorphic to a $\mathcal{C}^{1,1-}$ (i.e. $\mathcal{C}^{1,\alpha}$ for any $\alpha\in(0,1)$) curve.
(ii) $\partial\Omega\cap N$ consists of a (possibly empty) finite set of points z_i, such that, at each z_i, ρ_i $\mathcal{C}^{1,+}$ half-lines belonging to N (with $\rho_i\geqslant 1$) hit the boundary.
(iii) Moreover the half curves meet with equal angle at each critical point of $N\cap\Omega$ and also at each point of $N\cap\partial\Omega$ together with the boundary.

Complementarily, we introduce the notion of regular partition.

Definition 1.10. A strong partition $\mathcal{D}$ is regular (and we write in this case $\mathcal{D}\in\mathcal{R}(\Omega)$) if there exists a regular closed set N such that $\mathcal{D}=\mathcal{D}(N)$, where $\mathcal{D}(N)$ is the family of the connected components of $\Omega\setminus N$ belongs (by definition) to $\mathcal{R}(\Omega)$.

In Fig. 1, we give the example of a regular partition which cannot correspond to nodal domains because the associated graph[1] is not bipartite.

1.2. Main results

Although some of the statements could be obtained under weaker assumptions we assume below that Ω is bounded and connected.

It has been proved[2] by Conti–Terracini and Verzini [12] that

Theorem 1.11. *For any k, there exists a minimal regular strong k-partition.*

The first aim of this paper is to show the

Theorem 1.12. *Any minimal k-partition has a connected, regular and strong representative.*

[1] See the next subsection for definitions.
[2] But these papers treat only smoother boundaries than assumed in the whole article. So we will prove here a slight generalization.

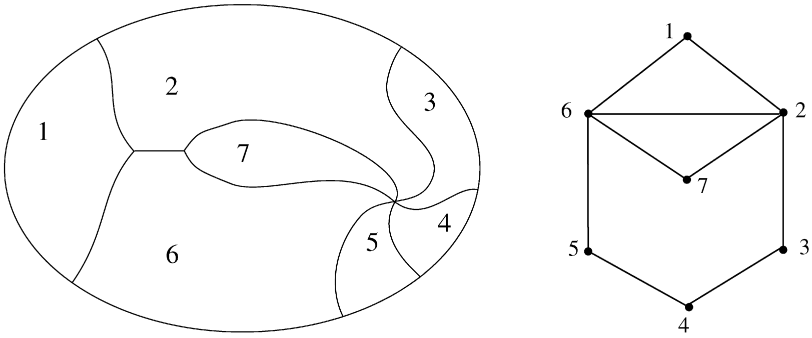

Fig. 1. An example of regular partition and associated graph.

Here we need to explain what we mean by representative (which involves implicitly the notion of capacity). This involves indeed a notion of equivalence classes. Two k-partitions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are equivalent if there is a labeling such that for any ground state u_i associated with D_i, there is a ground state $\tilde{u}_i$ associated with $\widetilde{D}_i$ such that $u_i = \tilde{u}_i$ in $W_0^{1,2}(\Omega)$, and conversely.

Once this notion is introduced it is natural to look for the existence of a regular representative and uniqueness will always be inside this class.

In general, there is no reason for a minimal partition to be unique (and here we speak of uniqueness of equivalence classes). This can for example occur in presence of symmetries. However, we will show that a uniqueness property always holds for subpartitions of a given minimal partition. More precisely, we have

Theorem 1.13. *Let $\mathcal{D}$ be a minimal k-partition relative to $\mathfrak{L}_k(\Omega)$. Let $\mathcal{D}' \subset \mathcal{D}$ be a subpartition of $\mathcal{D}$ into $1 \leqslant k' < k$ elements and assume that*

$$\Omega' := \operatorname{Int}\left(\bigcup_{D_i \in \mathcal{D}'} \overline{D_i}\right),$$

is connected. Then $\mathfrak{L}_k(\Omega) = \mathfrak{L}_{k'}(\Omega')$ and the k'-minimal partition of Ω' is unique.

A natural question is whether a minimal partition is the partition induced by an eigenfunction (in this case, we will more shortly speak of nodal partition). Theorem 1.14 gives a simple criterion for a partition to be associated to a nodal set. For this we need some additional definitions.

We say that D_i, D_j are *neighbors* and we write $D_i \sim D_j$, if the set $D_{i,j} := \operatorname{Int}(\overline{D_i \cup D_j}) \setminus \partial\Omega$ is connected. We then construct for each $\mathcal{D}$ a graph $G(\mathcal{D})$ by associating to each D_i a vertex and to each pair of neighbors (D_i, D_j) an edge. This is an undirected graph without multiple edges or loops. Following [14], we will say that the graph is *bipartite* if it can be colored by two colors (two neighbors having different colors). We recall that the graph associated to a collection of nodal domains of an eigenfunction is always bipartite. In this case, we say that the partition is *admissible*. We have now the following converse theorem:

Theorem 1.14. *If the graph of the minimal partition of Ω is bipartite, this is a partition associated to the nodal set of an eigenfunction of $H(\Omega)$ corresponding to $\mathfrak{L}_k(\Omega)$.*

This theorem was already obtained in [18] by adding a strong a priori regularity assumption on the partition and the assumption that Ω is simply connected. Any subpartition of cardinality two (D_i, D_j) corresponds indeed to a nodal partition of some eigenfunction associated to the second eigenvalue of $H(D_{i,j})$. This implies the Pair Compatibility Condition (see in Appendix B) and Theorem B.1 can be applied.

The proof given here is more general (but more difficult) and is actually a byproduct of the proof of Theorem 1.12, which will directly give an eigenfunction whose nodal domains form the partition.

A natural question is now to determine how general is the situation described in the previous theorem. The surprise is that this will only occur in the so-called Courant-sharp situation. Before stating precisely our second main result we need to introduce some further statements and notations. The Courant Nodal Theorem says:

Theorem 1.15. *Let $k \geqslant 1$, $\lambda_k = \lambda_k(\Omega)$ the k-th eigenvalue of $H(\Omega)$ and u any real associated eigenfunction. Then the number of nodal domains $\mu(u)$ of u satisfies $\mu(u) \leqslant k$.*

When the number of nodal domains $\mu(u)$ satisfies

$$\mu(u) = k,$$

we will say, as in [4], that u is *Courant-sharp*.

Definition 1.16. For any integer $k \geqslant 1$, we denote by $L_k(\Omega)$ (or simply L_k) the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains.

In general, we will show in Corollary 5.6, that

$$\lambda_k(\Omega) \leqslant \mathfrak{L}_k(\Omega) \leqslant L_k(\Omega). \tag{1.10}$$

The last goal consists in giving the full picture of cases of equality:

Theorem 1.17. *If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\mathfrak{L}_k(\Omega) = \lambda_k(\Omega)$, then*

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega).$$

In addition, one can find in the eigenspace associated to λ_k an eigenfunction u such that $\mu(u) = k$.

In other words, the only case when the k nodal domains of an eigenfunction of $H(\Omega)$ form a minimal partition is the case when this eigenfunction is Courant-sharp.

1.3. Organization of the paper

The paper is organized as follows. We first start in Section 2 by recalling and extending (up to the boundary) results on the local properties of the nodal set of an eigenfunction. Section 3 is devoted to the analysis of the geometrical properties of minimal partitions in $\mathbb{R}^N$. Section 4 gives stronger results but limited to the two-dimensional case, which is our main subject. This gives in particular the proof of our first Main Theorem 1.12. Sections 5 and 6 are devoted to additional properties of the minimal partitions. We discuss different notions related to the spectrum and revisit Pleijel's theorem and its proof. Section 7 gives the proof of the second Main Theorem 1.17 permitting to show that when a minimal k-partition is a nodal family then the corresponding eigenvalue is the k-th one. In Section 8, we complete the proofs and the statements concerning subpartitions. In Sections 9 and 10 we analyze in great detail the various spectra of specific $H(\Omega)$ in connection with minimal partitions. This leads in particular to nice conjectures and open problems. Finally, we develop in two appendices useful results which will complete some proofs or help the reader.

2. Preliminaries: Hölder regularity of nodal sets

It is a well-known property of nodal sets of eigenfunctions to be the union of curves ending either at interior singular points or at the boundary. This section is devoted to the analysis of the regularity of the nodal curves in the Hölder spaces $\mathcal{C}^{1,\varepsilon}$, for some $\varepsilon > 0$. A word of caution must be entered at this point: with regularity we mean *global regularity* of the nodal branch up to the singularities or the boundary. This is not a completely obvious issue (basically because of the lack of regularity of our solutions and, possibly, of the boundary of the domain) and will require a reconsideration of the well-known asymptotic estimates about critical points of eigenfunctions. To start with, we recall the classical local regularity result by Hartman and Wintner ([17], Corollary 1), stating that interior critical points of non-zero solutions to our class of equations are isolated and have finite (local) multiplicity m. In addition the solution satisfies, for some $c \neq 0$, the asymptotic formula

$$u(r,\theta) = cr^{m+1}\cos\big((m+1)(\theta+\theta_0)\big) + o(r^{m+1}), \quad r = |z - z_0|. \tag{2.1}$$

Here we identify $\mathbb{R}^2$ with $\mathbb{C}$ and use either z, or (x, y), or (r, θ) for a point of $\mathbb{R}^2$, with the standard notations:

$$z = r\exp i\theta, \quad z = x + iy, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

We shall need a refined version of it which is stated below:

Theorem 2.1. *Let Ω be open and $V \in L^\infty(\Omega)$. Assume $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ solves*

$$-\Delta u + V(x,y)u = 0,$$

in the distributional sense.

Let $z_0 = (x_0, y_0) \in \Omega$ be such that $u(x_0, y_0) = 0$ and $\nabla u(x_0, y_0) = 0$; then, in a neighborhood of z_0,

(a) *There are an integer n, a complex-valued function ξ of class $\mathcal{C}^{0,+}$ such that $\xi(z_0) \neq 0$ and*

$$u_x + iu_y = r^n e^{-in\theta}\xi(x,y), \quad r = |z - z_0|. \tag{2.2}$$

(b) *There is a function $\tilde{\xi}$ of class $\mathcal{C}^{0,+}$ such that $\tilde{\xi}(x_0, y_0) = 0$ and*

$$u(x,y) = \frac{r^{n+1}}{n+1}\big(\Re\big(\xi(x_0,y_0)\big)\cos(n+1)\theta + \Im\big(\xi(x_0,y_0)\big)\sin(n+1)\theta + \tilde{\xi}(x,y)\big). \tag{2.3}$$

(c) *There exists a positive radius R such that $u^{-1}(\{0\}) \cap B(z_0, R)$ is composed by $2n$ $\mathcal{C}^{1,+}$-simple arcs which all end in z_0 and whose tangent lines at z_0 divide the disc into $2n$ angles of equal amplitude.*

Proof. We follow the paper by Hartman and Wintner [17] and write $w = u_y + iu_x$ and set $z_0 = 0$. It is shown there that, if

$$u = \mathrm{o}\big(|z|^k\big), \tag{2.4}$$

for some integer $k \geqslant 0$, then the Cauchy formula is available:

$$2\pi i \frac{w(\zeta)}{\zeta^k} = \int_{|z|=R} \frac{w(z)}{z^k(z-\zeta)}\,dz - \int_{|z|<R} \frac{V(z)u(z)}{z^k(z-\zeta)}\,dx\,dy, \tag{2.5}$$

where $R > 0$ is fixed and the double integral over the disk is absolutely convergent. We now show that the left-hand side is Hölder continuous in ζ in a neighborhood of the origin. The line integral is smooth in ζ, since the integrand has no singularities on the circle. Concerning the second term, notice that we can find a constant K such that

$$\left|\int_{|z|<R} \frac{V(z)u(z)}{z^k}\left(\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2}\right)dx\,dy\right| \leqslant \int_{|z|<R} \frac{|V(z)u(z)|}{|z|^k}\left|\frac{|\zeta_1 - \zeta_2|}{|z-\zeta_1||z-\zeta_2|}\right|dx\,dy$$
$$\leqslant K|\zeta_1 - \zeta_2|\big|\log|\zeta_1 - \zeta_2|\big|.$$

Now we show that (2.4) cannot be verified for every integer. To this aim, we integrate Eq. (2.5) over the disk and, taking absolute values, we obtain:

$$2\pi \int_{|z|<R} \frac{|w(z)|}{|z|^k}\,dx\,dy \leqslant 2\pi R \int_{|z|=R} \frac{|w(z)|}{|z|^k}|dz| + 2\pi R \int_{|z|<R} \frac{|V(z)||u(z)|}{|z|^k}\,dx\,dy. \tag{2.6}$$

Following [17] and using the identity

$$u(r,\theta) = \int_0^r \big(u_x(\rho,\theta)\cos\theta + u_y(\rho,\theta)\sin\theta\big)\,d\rho, \tag{2.7}$$

we observe that:

$$\big|u(z)\big| \leqslant \int_0^1 \big|zw(tz)\big|\,dt,$$

implies

$$\int_{|z|<R} \frac{|V(z)||u(z)|}{|z|^k}\,dx\,dy \leqslant K \int_{|z|<R} \frac{|zw(z)|}{|z|^k}\,dx\,dy \leqslant KR \int_{|z|<R} \frac{|w(z)|}{|z|^k}\,dx\,dy.$$

Thus, for R sufficiently small, inequality (2.6) leads to

$$\int_{|z|<R} \frac{|w(z)|}{|z|^k}\,dx\,dy \leqslant 2R \int_{|z|=R} \frac{|w(z)|}{|z|^k}|dz|. \tag{2.8}$$

We have now fixed $R > 0$ such that (2.8) is satisfied. Let us assume that $w(z_0) \neq 0$ for some $|z_0| < R$. Then, for a constant K independent of k, there holds

$$\left|w(z_0)\right| \leqslant K\left(\frac{|z_0|}{R}\right)^k, \quad k = 1, 2, \ldots.$$

Let us take the limit $k \to +\infty$ in this inequality. Then the limit of the sequence $(|z_0|/R)^k$ does not vanish, in contradiction with $|z_0| < R$. This completes the proof of point (a) in the statement of the theorem. Point (b) follows from point (a) together with the identity (2.7).

To prove point (c) we choose a branch of the nodal set and we choose, as a regular parametrization the path $z(t) = r(t)e^{i\theta(t)}$, where the pair $(r(t), \theta(t))$ solves the following system of ordinary differential equations:

$$\begin{cases} \dot r = \frac{1}{r^{n+1}}(xu_y - yu_x), \\ \dot\theta = \frac{1}{r^{n+2}}(xu_x + yu_y). \end{cases}$$

Here $\dot r$ and $\dot\theta$ denote respectively the derivative of $r(t)$ and $\theta(t)$ with respect to t.

One can easily prove using points (a) and (b) that both functions $t \mapsto \dot r(t)$ and $t \mapsto r(t)\dot\theta(t)$ are Hölder continuous; therefore both r and θ are Hölder continuous functions. Hence they can be extended through the singularity. Since the parametrization is regular ($\dot z \neq 0$), the assertion follows from the equation

$$\dot z = \dot r(t)e^{i\theta(t)} + ir(t)\dot\theta(t)e^{i\theta(t)}. \qquad \Box$$

Remark 2.2. Theorem 2.1 extends, with the same argument, to the case when the potential V has a singularity at z_0, provided there exists $\beta < 1$ and K such that

$$\left|V(x, y)\right| \leqslant \frac{K}{|z - z_0|^\beta}.$$

This fact will be useful when we shall consider the case of domains with corners or cracks; indeed such singular potentials result as conformal factors associated with the complex exponentials.
Note that this singular situation was also analyzed, but for the interior problem, in [21] and [22] and that in this case the authors obtain a better regularity.

In order to examine the regularity up to the boundary of the nodal partition associated to an eigenfunction we now extend a known result by Alessandrini [2,3] (which treats the convex case) to our setting. The proof exploits the classical Kellog–Warschawski theorem on the boundary regularity of conformal mappings which states that any conformal map on a $\mathcal{C}^{1,\varepsilon}$ domain extends continuously on the boundary keeping the same regularity (see the book by Pommerenke [26], Theorem 3.6 in Chapter 3).

Theorem 2.3. *Let $\varepsilon > 0$ and Ω be an open set with $\mathcal{C}^{1,\varepsilon}$ boundary and $V \in L^\infty(\Omega)$. Assume $u \in W_0^{1,2}(\Omega)$ solves*

$$-\Delta u + V(x, y)u = 0,$$

in the distributional sense. Then the associated nodal partition is regular. More precisely if $\overline{u^{-1}(\{0\})}$ intersects $\partial\Omega$ at z_0, then there exist an integer m and $R > 0$ such that $\overline{u^{-1}(\{0\})} \cap B(z_0, R)$ is composed by m $\mathcal{C}^{1,\varepsilon}$-simple arcs which all end in z_0 and whose tangent lines at z_0 divide the tangent cone $\Gamma(z_0)$ into $m + 1$ angles of equal opening.

Proof. The result immediately follows from Theorem 2.1 in the case of the half-plane: indeed one can extend u by a reflection to the other half-plane and reduce to the case of the interior zeros. The general case reduces to that of the half-space through the Riemann mapping theorem. Indeed, by [26] (Theorem 3.6 in Chapter 3) the Hölder regularity

$\mathcal{C}^{1,\varepsilon}$ of $\partial\Omega$ implies the same regularity property for the extensions, up to boundary, of the Riemann map f and of its inverse. Since the composition of $\mathcal{C}^{1,\varepsilon}$ maps enjoys the same regularity property, the statement follows. □

Now we wish to extend Theorem 2.3 to the case of domains possessing corners or cracks. To be precise we start with the following

Definition 2.4. Let $\varepsilon \in (0,1]$. We say that $\partial\Omega$ has a $\mathcal{C}^{1,\varepsilon}$-corner of opening $\alpha\pi$ $(0 \leqslant \alpha \leqslant 2)$ at z_0 if, in a sufficiently small neighborhood, $\partial\Omega$ contains the union of two curves of class $\mathcal{C}^{1,\varepsilon}$ (non self-crossing) ending at z_0, and such that Ω lies in the curvilinear sector of angle opening $\alpha\pi$ spanned by the two arcs, which does not intersect other components of the boundary $\partial\Omega$.

Remark 2.5. Note that the boundary $\partial\Omega$ can have several corners of angle opening α_i at the same point z_0: of course the sum of all the angles does not exceed 2π. Moreover, it is worthwile noticing that we allow the presence of cracks (i.e. corners of angle opening 2π where the two curves coincide), exterior cusps (i.e. corners of angle opening 2π spanned by two distinct curves), as well as angles of any possible positive angles, (positiveness is required by the interior cone property). Finally, a corner can be a point of smoothness of the boundary, when its angle opening is π.

Our next goal is to prove the following result

Theorem 2.6. *Let $V \in L^\infty(\Omega)$ and $\varepsilon \in]0,1]$. Assume $u \in W_0^{1,2}(\Omega)$ solves $-\Delta u + V(x,y)u = 0$ in the distributional sense, in a neighborhood of some $z_0 \in \partial\Omega$, a $\mathcal{C}^{1,\varepsilon}$-corner of opening $\alpha\pi$ $(0 < \alpha \leqslant 2)$. If $\overline{u^{-1}(\{0\})}$ intersects $\partial\Omega$ at z_0, then there exist an integer m and $R > 0$ such that $\overline{u^{-1}(\{0\})} \cap B(z_0,R)$ is composed by m $\mathcal{C}^{1,\varepsilon'(\alpha)}$-simple arcs which all end at z_0 and whose tangent lines at z_0 divide the tangent cone $\Gamma(z_0)$ into $m+1$ angles of equal amplitude. In addition*

$$\varepsilon'(\alpha) = \begin{cases} \varepsilon \min(\alpha, 1/\alpha) & \text{if } 1/2 < \alpha \leqslant 2, \\ 2^n \varepsilon\alpha & \text{if } 1/2^{(n+1)} < \alpha \leqslant 1/2^n. \end{cases}$$

To prove the theorem we shall first straighten the corner and then apply Theorem 2.3. We shall need the following basic result.

Proposition 2.7. *Let $\varepsilon \in]0,1]$ and let C be a Hölder-continuous arc ending at the origin, without self-intersections. Let $w(\tau)$, $\tau \in [0,\bar\tau]$ be a regular parametrization of C such that $w(0) = 0$ and $w'(0) \neq 0$, and define the curve $\mathcal{C}^{1/\alpha}$ by the parametrization*

$$t \mapsto v(t) := \big(w(t^\alpha)\big)^{1/\alpha}.$$

Then, for any $\alpha > 0$

$$C \in \mathcal{C}^{1,\varepsilon} \quad \Longrightarrow \quad C^{1/\alpha} \in \mathcal{C}^{1,\varepsilon\min(1,\alpha)}.$$

Proof. We have

$$v'(t) = w'(t^\alpha)u(t^\alpha), \qquad u(\tau) := \big(\tau w(\tau)\big)^{-1+1/\alpha}.$$

Obviously v defines a regular parametrization of $C^{1/\alpha}$. At first we remark that u is Hölder continuous with exponent ε. Indeed

$$\left|\frac{w(\tau_1)}{\tau_1} - \frac{w(\tau_2)}{\tau_2}\right| = \left|\int_0^1 \big(w'(\tau_2 s) - w'(\tau_1 s)\big)\,ds\right| \leqslant K|\tau_1 - \tau_2|^\varepsilon.$$

Therefore the product $w'(\tau)u(\tau)$ is of the same class $\mathcal{C}^{0,\varepsilon}$ and the composition $v'(t) = w'(t^\alpha)u(t^\alpha)$ is in the Hölder space $\mathcal{C}^{0,\varepsilon\min(1,\alpha)}$. □

Proof. We are in position to prove Theorem 2.6. We consider separately the two cases:

First, we assume the case of openings satisfying the inequality $1/2 < \alpha \leqslant 2$. We straighten the corner as in Proposition 2.7, by the map $z \to z^{1/\alpha}$. Then, through this composition, the boundary $\partial\Omega^{1/\alpha}$ becomes smooth (of class $\mathcal{C}^{1,\varepsilon\min(1,\alpha)}$) while the potential V has to be multiplied by the conformal factor $2\alpha^2|z-z_0|^{2(\alpha-1)}$, which is singular whenever $\alpha < 1$. As already observed in Remark 2.2, this is not a problem if $\alpha > 1/2$. Thanks to Theorem 2.3, the nodal set of the composition $z \mapsto u(z^\alpha)$ is the union of arcs of class $\mathcal{C}^{1,\varepsilon\min(1,\alpha)}$. Now we take its inverse image through the map $z \to z^\alpha$ and, applying again Proposition 2.7 we obtain the desired value of $\varepsilon'(\alpha)$.

Next we turn to the case when the opening is too small, that is when $1/2^{(n+1)} < \alpha \leqslant 1/2^n$, for some $n \geqslant 1$. Using again Theorem 3.6 in Chapter 3 of [26], one can easily construct, locally in Ω, a conformal map of class $\mathcal{C}^{1,\varepsilon}$ up to the boundary such that the image of one of the two arcs is a straight segment. Next step is to reflect the domain about this line and extend the function on the reflected corner, in such a way to double the opening, which is now 2α. In this procedure, the second arc, being composed with a $\mathcal{C}^{1,\varepsilon}$ map, still remains in the same Hölder class. We iterate this reflection procedure n times, until $1/2 < 2^n\alpha \leqslant 1$, and we afterward proceed as in the proof of the case $\alpha > 1/2$. □

Using the same technique of straightening the angles by conformal maps, one can easily prove the following

Proposition 2.8. *Let $V \in L^\infty(\Omega)$ and $\varepsilon \in]0,1]$. Assume $u \in W_0^{1,2}$ solves $-\Delta u + V(x,y)u = 0$ in the distributional sense, in a neighborhood of some $z_0 \in \partial\Omega$, a $\mathcal{C}^{1,\varepsilon}$-corner of opening $\alpha\pi$ $(0 < \alpha \leqslant 2)$. Then, if $\alpha \leqslant 1$, $u \in \mathcal{C}^{1,\varepsilon}$, locally at z_0; otherwise, if $1 < \alpha \leqslant 2$, we only have $u \in \mathcal{C}^{0,1/\alpha}$.*

3. Optimal partitions in N dimensions

In the recent literature, an *optimal partition problem* is a minimization problem of the form

$$\min\{\mathcal{F}(D_1,\dots,D_k):\ D_i \in \mathcal{A}(D_i),\ D_i \cup D_j = \emptyset \text{ for } i \neq j\}$$

where k is a fixed integer, $\mathcal{A}(\Omega)$ is the given class of all *admissible domains* and $\mathcal{F}:\mathcal{A}(\Omega)^k \to [0,+\infty)$ is the *cost function*. When $k = 1$ it is called a shape optimization problem. Both optimal shape and partition problems may fail to admit a solution: a minimizer exists, in general, only for an associated relaxed problem (see, for instance, [8,9]). In order to recover compactness and obtain the existence of a minimizer for the original problem, two strategies have been proposed in the recent literature: the first one consists in imposing some capacitary constraint on the admissible domains and has been mainly developed in [32,6]. A second approach, introduced in [7], requires the cost function to be monotonic with respect to set inclusion and gives existence of an optimal partition in the class of *quasi-open* partitions (a set is termed quasi-open if it can be arbitrarily approximated, in capacity, by open sets). This section deals with the existence of a minimizer of the cost function defined in Definition 1.4 as the maximal first eigenvalue for the Dirichlet problem of the elements of the partition:

$$\Lambda(D_1,\dots,D_k) = \max_i \lambda_1(D_i),$$

and we require the elements of the partition to be open connected sets. It is worthwhile noticing that this is a much stronger admissibility assumption than the one in [7] and will require a more detailed analysis of the regularity of the interfaces, though the general existence theory developed in [7] could very well be applied in this case, giving the existence of a quasi-open minimal partition. In order to establish the regularity of any minimizing partition, we shall exploit the strategy already developed in [11,12]. We shall first prove the validity of some optimality conditions, expressed by the system of differential inequalities (I1)–(I2), generalizing the domain derivative condition of [28]. Some regularity results regarding the solutions of a different, though related, optimal partition problems are outlined in [10].

To begin with, let $\Omega \subset \mathbb{R}^N$ be a connected, open bounded domain. Note that at this stage we do not need any regularity of the boundary. However we will need it later (in Section 4), in the 2-dimension case, in order to describe the local structure of nodal lines at their intersection with the boundary.

Definition 3.1. For any measurable $D \subset \Omega$ and for $V \in L^\infty(\Omega)$, let $\lambda_1(D)$ denotes the first eigenvalue of the Dirichlet realization of the Schrödinger operator in the following generalized sense. We define

$$\lambda_1(D) = +\infty,$$

if $\{u \in W_0^{1,2}(\Omega),\ u \equiv 0 \text{ a.e. on } \Omega \setminus D\} = \{0\}$, and

$$\lambda_1(D) = \min\left\{ \frac{\int_\Omega (|\nabla u(x)|^2 + V(x)u(x)^2)\,dx}{\int_\Omega |u(x)|^2\,dx}\colon\ u \in W_0^{1,2}(\Omega) \setminus \{0\}, u \equiv 0 \text{ a.e. on } \Omega \setminus D \right\},$$

otherwise. We call groundstate any function ϕ achieving the above minimum.

We shall always assume that

$$\lambda_1(\Omega) > 0.$$

Remark 3.2. The presence of an L^∞ potential V does not create particular problems. We prefer, to simplify the notation, to explain all the proofs with the additional assumption that V is identically 0. In this case the positivity of $\lambda_1(\Omega)$ is effectively satisfied. In the general case, we can always assume this property by adding a constant to V.

We observe that the minimization problem always possesses a (possibly not unique) non-negative solution $\phi \geqslant 0$. We shall always make this choice. Next we consider the following class of minimal partition problems:

$$\mathfrak{L}_{k,p} := \inf_{\mathcal{B}_k} \left(\frac{1}{k} \sum_{i=1}^{k} \big(\lambda_1(D_i)\big)^p \right)^{1/p}, \tag{3.1}$$

$$\mathfrak{L}_k := \inf_{\mathcal{B}_k} \max_{i=1,\dots,k} \big(\lambda_1(D_i)\big) \tag{3.2}$$

where the minimization is taken over the class of partitions in k "disjoint" measurable subsets of Ω

$$\mathcal{B}_k := \left\{ \mathcal{D} = (D_1, \dots, D_k)\colon \bigcup_{i=1}^{k} D_i \subset \Omega,\ |D_i \cap D_j| = 0 \text{ if } i \neq j \right\},$$

where, for a Lebesgue-measurable set A, $|A|$ denotes the measure of A.

Remark 3.3. The values $\mathfrak{L}_k$ considered in this section can be viewed as a relaxation of those defined in the introduction. We have indeed replaced "open" by "measurable". We keep the same notation, for we shall prove as a part of our regularity theory that, in all the interesting cases, the two definitions coincide.

The main result of this section is the following

Theorem 3.4. *Let $\mathcal{D} = (\widetilde{D}_1, \dots, \widetilde{D}_k) \in \mathcal{B}_k$ be any minimal partition associated with $\mathfrak{L}_k$ and let $(\tilde{\phi}_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(\widetilde{D}_i))_i$. Then there exist $a_i \geqslant 0$, not all vanishing, such that the functions $\tilde{u}_i = a_i \tilde{\phi}_i$ verify in Ω the differential inequalities in the distributional sense*

(I1) $-\Delta \tilde{u}_i \leqslant \mathfrak{L}_k \tilde{u}_i$, $\forall i = 1, \dots, k$,
(I2) $-\Delta(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j) \geqslant \mathfrak{L}_k(\tilde{u}_i - \sum_{j \neq i} \tilde{u}_j)$, $\forall i = 1, \dots, k$.

Remark 3.5. Note that at this stage we do not know whether the $\widetilde{D}_i$'s are connected and consequently whether the $\tilde{\phi}_i$'s are unique. It will be shown in the next section that these properties are true in two dimensions.

The following results were proved in [12]:

Theorem 3.6. *Let $p \in [1, +\infty)$ and let $\mathcal{D} = (D_1, \dots, D_k) \in \mathcal{B}_k$ be a minimal partition associated with $\mathfrak{L}_{k,p}$ and let $(\phi_i)_i$ be any set of positive eigenfunctions normalized in L^2 corresponding to $(\lambda_1(D_i))_i$. Then there exist $a_i > 0$, such that the functions $u_i = a_i \phi_i$ satisfy in Ω the differential inequalities in the distribution sense*

(I1) $-\Delta u_i \leqslant \lambda_1(D_i) u_i$,
(I2) $-\Delta(u_i - \sum_{j \neq i} u_j) \geqslant \lambda_1(D_i) u_i - \sum_{j \neq i} \lambda_1(D_j) u_j$.

Remark 3.7. In particular, this implies that $U = (u_1, \dots, u_k)$ is in the class $\mathcal{S}^*$ as defined in [11]. Hence Theorem 8.3 in [11] ensures the Lipschitz continuity of the u_i's in the interior of Ω. Therefore we can choose a partition made of open representatives $D_i = \{u_i > 0\}$.

Moreover, taking the limit as $p \to +\infty$, the following result was shown in [12]:

Theorem 3.8. *There holds*

$$\lim_{p\to+\infty} \mathfrak{L}_{k,p} = \mathfrak{L}_k.$$

Moreover, there exists a minimizer of $\mathfrak{L}_k$ *such that* (I1)–(I2) *hold for suitable non-negative multiples* $u_i = a_i\phi_i$ *of an appropriate set of associated eigenfunctions.*

Let us start the proof of Theorem 3.4.

Let $(\widetilde{D}_1, \dots, \widetilde{D}_k) \in \mathcal{B}_k$ be a particular minimal partition associated with $\mathfrak{L}_k$ and let $(\tilde{\phi}_1, \dots, \tilde{\phi}_k)$ be any choice of associated eigenfunctions. The existence of such a minimal partition was proved, in a slightly less general framework in [12]. To recover the proof of the existence under the assumptions of our paper, the reader can follow the argument below just deleting the penalization term in the definition of $\mathcal{F}_{k,p}$. We wish to prove that (I1)–(I2) hold for a suitable set of multiples of the $\tilde{\phi}_j$'s. We consider, for a given

$$q \in \big(1, N/(N-2)\big), \tag{3.3}$$

(or $q \in (1, +\infty)$ when $N = 2$), the penalized Rayleigh quotient:

$$\mathcal{F}_{k,p}(u_1, \dots, u_k) = \left(\frac{1}{k}\sum_{i=1}^{k}\left(\frac{\int_\Omega |\nabla u_i(x)|^2\,dx}{\int_\Omega |u_i(x)|^2\,dx}\right)^p\right)^{1/p} + \sum_{i=1}^{k}\left(1 - \frac{\int_\Omega u_i(x)^q \tilde{\phi}_i(x)^q\,dx}{(\int_\Omega u_i(x)^{2q}\,dx \int_\Omega \tilde{\phi}_i(x)^{2q}\,dx)^{1/2}}\right).$$

We consider the minimization problem

$$\mathcal{M}_{k,p} = \inf\big\{\mathcal{F}_{k,p}(u_1, \dots, u_k):\ (u_1, \dots, u_k) \in \mathcal{U}\big\}, \tag{3.4}$$

where

$$\mathcal{U} = \big\{(u_1, \dots, u_k) \in \big(W_0^{1,2}(\Omega)\big)^k:\ u_i \cdot u_j = 0,\ \text{for } i \neq j,\ u_i \geqslant 0,\ u_i \not\equiv 0,\ \forall i = 1, \dots, k\big\}. \tag{3.5}$$

We note that the condition on q permits to have (weak and strong) continuity and differentiability in $W_0^{1,2}(\Omega)$ of the penalization term, which involves integrals of powers of u_i. This will be used later to apply the direct method of the Calculus of Variations and to differentiate $\mathcal{F}_{k,p}$ at the minimum.

It is also worthwhile noticing that $\mathcal{F}_{k,p}$ is invariant by multiplication:

$$\mathcal{F}_{k,p}(a_1u_1, \dots, a_ku_k) = \mathcal{F}_{k,p}(u_1, \dots, u_k), \quad \forall a_i \neq 0. \tag{3.6}$$

Recalling Definition 3.1 we have:

Proposition 3.9. *There holds, for every* $p \in [1, +\infty)$,

$$\frac{1}{k^{1/p}}\mathfrak{L}_k \leqslant \mathfrak{L}_{k,p} \leqslant \mathcal{M}_{k,p} \leqslant \mathfrak{L}_k.$$

Proof. It is an immediate consequence of Jensen and Hölder inequalities. □

Lemma 3.10. *For every* $p \in [1, +\infty)$*, the value* $\mathcal{M}_{k,p}$ *is achieved.*

Proof. Using the invariance by multiplication (3.6), we can choose a bounded minimizing sequence, having as weak limit the configuration $(u_1, \dots, u_k) \in \mathcal{U}$. Now the assertion simply follows from the weak lower semi-continuity of the norm and the compact embeddings of $W_0^{1,2}(\Omega)$ into $L^s(\Omega)$ for any $s \in [1, +\infty)$, whenever $N = 2$, and for any $s \in [1, 2N/(N-2))$ when $N \geqslant 3$. □

Lemma 3.11. *Let $\Lambda > 0$ and let $U = (u_1, \dots, u_k)$ be any minimizer of $\mathcal{M}_{k,p}$ normalized in such a way that*

$$\left(\int_\Omega |\nabla u_i|^2\,dx\right)^{p-1} = \left(\Lambda \int_\Omega |u_i|^2\,dx\right)^p, \quad \forall i = 1, \dots, k. \tag{3.7}$$

Define

$$f_i(u)(x) = \frac{-\gamma q}{2(\int_\Omega u(x)^{2q}\,dx \int_\Omega \tilde{\phi}_i(x)^{2q}\,dx)^{1/2}}\left[u(x)^{q-1}\tilde{\phi}_i(x)^q - \frac{\int_\Omega u(x)^q \tilde{\phi}_i(x)^q\,dx}{\int_\Omega u(x)^{2q}\,dx} u(x)^{2q-1}\right], \tag{3.8}$$

where

$$\gamma = \Lambda^{-p}\left(\frac{1}{k}\sum_{i=1}^k \left(\frac{\int_\Omega |\nabla u_i(x)|^2\,dx}{\int_\Omega |u_i(x)|^2\,dx}\right)^p\right)^{1-1/p}. \tag{3.9}$$

Then U satisfies the differential inequalities in the distribution sense

(I1) $-\Delta u_i \leqslant \lambda_1(D_i)u_i + f_i(u_i)$,
(I2) $-\Delta(u_i - \sum_{j\neq i} u_j) \geqslant \lambda_1(D_i)u_i + f_i(u_i) - \sum_{j\neq i}(\lambda_1(D_j)u_j + f_j(u_j))$.

Proof. For a fixed index i, let us introduce

$$\hat{u}_i = u_i - \sum_{j\neq i} u_j.$$

Let $\varphi \geqslant 0$, $\varphi \in W_0^{1,2}(\Omega)$, and, for $t > 0$ very small, let us define a new test function $V = (v_1, \dots, v_k)$, belonging to $(W_0^{1,2}(\Omega))^k$, as follows:

$$v_j = \begin{cases} (\hat{u}_i + t\varphi)^+, & \text{if } j = i,\\ (-u_j + t\varphi)^- = (\hat{u}_i + t\varphi)^- \chi_{\{u_j>0\}}, & \text{if } j \neq i. \end{cases}$$

We first remark that there is differentiability (with respect to t) of all the terms which do not involve derivatives. Indeed, since the map $u \to (u^+)^r$ is differentiable, we have, for any set of functions $\eta_j \in L^s(\Omega)$ and $r > 1$:

$$\int_\Omega \eta_j v_j^r\,dx = \begin{cases} \int_\Omega \eta_j u_j^r\,dx + rt\int_\Omega \eta_j u_j^{r-1}\varphi\,dx + o(t), & \text{if } j = i,\\ \int_\Omega \eta_j u_j^r\,dx - rt\int_\Omega \eta_j u_j^{r-1}\varphi\,dx + o(t), & \text{if } j \neq i. \end{cases}$$

By the Sobolev Embedding Theorem, this expansion holds with respect to the $W_0^{1,2}(\Omega)$-norm provided $s \in (1, +\infty]$ and $r \leqslant (1 - 1/s)(2N/(N-2))$. As a first application, letting

$$\alpha_j = \frac{1}{t}\left\{\int_\Omega |v_j|^2\,dx - \int_\Omega |u_j|^2\,dx\right\},$$

and $r = 2$, we have

$$\alpha_j = \begin{cases} 2\int_\Omega u_j\varphi\,dx + o(1), & \text{if } j = i,\\ -2\int_\Omega u_j\varphi\,dx + o(1), & \text{if } j \neq i. \end{cases}$$

Moreover, letting

$$\beta_j = \frac{1}{t}\left\{\left(1 - \frac{\int_\Omega v_j(x)^q\tilde{\phi}_j(x)^q\,dx}{(\int_\Omega v_j(x)^{2q}\,dx\int_\Omega \tilde{\phi}_j(x)^{2q}\,dx)^{1/2}}\right) - \left(1 - \frac{\int_\Omega u_j(x)^q\tilde{\phi}_j(x)^q\,dx}{(\int_\Omega u_j(x)^{2q}dx\int_\Omega \tilde{\phi}_j(x)^{2q}\,dx)^{1/2}}\right)\right\},$$

we find, recalling that $q \in (1, N/(N-2))$, by the usual differentiation rules

$$\beta_j = \begin{cases} \dfrac{-q}{(\int_\Omega u_j(x)^{2q}\,dx \int_\Omega \tilde\phi_j(x)^{2q}\,dx)^{1/2}} \Bigg[\displaystyle\int_\Omega u_j(x)^{q-1}\tilde\phi_j(x)^q \varphi(x)\,dx \\ \qquad - \dfrac{\int_\Omega u_j(x)^{2q-1}\varphi(x)\,dx \int_\Omega u_j(x)^q \tilde\phi_j(x)^q\,dx}{(\int_\Omega u_j(x)^{2q}\,dx)} \Bigg] + \mathrm{o}(1), \quad \text{if } j = i, \\ \dfrac{+q}{(\int_\Omega u_j(x)^{2q}\,dx \int_\Omega \tilde\phi_j(x)^{2q}\,dx)^{1/2}} \Bigg[\displaystyle\int_\Omega u_j(x)^{q-1}\tilde\phi_j(x)^q \varphi(x)\,dx \\ \qquad - \dfrac{\int_\Omega u_j(x)^{2q-1}\varphi(x)\,dx \int_\Omega u_j(x)^q \tilde\phi_j(x)^q\,dx}{(\int_\Omega u_j(x)^{2q}dx)} \Bigg] + \mathrm{o}(1), \quad \text{if } j \neq i. \end{cases}$$

On the other hand, differentiation with respect to t may fail when we consider the gradient integrals. Let us denote

$$\delta_j = \frac{1}{t}\left\{ \int_\Omega |\nabla v_j|^2\,dx - \int_\Omega |\nabla u_j|^2\,dx \right\}.$$

Although $t\delta_j \to 0$ as $t \to 0$, the δ_j's themselves can be unbounded in general, for they involve boundary integrals which are not necessarily finite for functions in $W_0^{1,2}(\Omega)$. On the other hand, from the definition

$$\int_\Omega |\nabla v_j|^2\,dx \leqslant \int_\Omega \left|\nabla(u_j - t\varphi)\right|^2 dx, \quad \text{if } j \neq i,$$

we can easily deduce that

$$\delta_j \leqslant -2\int_\Omega \nabla u_j \cdot \nabla\varphi\,dx + \mathrm{o}(1), \quad \text{if } j \neq i, \text{ and } \varphi \geqslant 0, \tag{3.10}$$

while, from

$$\begin{aligned} t\sum_j \delta_j &= \sum_j \left\{ \int_\Omega |\nabla v_j|^2\,dx - \int_\Omega |\nabla u_j|^2\,dx \right\} \\ &= \int_\Omega \left|\nabla(\hat u_i + t\varphi)\right|^2 dx - \int_{\{\hat u_i + t\varphi = 0\}} |\nabla \hat u_i + t\varphi|^2\,dx - \int_\Omega |\nabla \hat u_i|^2\,dx + \int_{\{\hat u_i = 0\}} |\nabla \hat u_i|^2\,dx \\ &= 2t\int_\Omega \nabla\hat u_i \cdot \nabla\varphi\,dx + t^2\int_\Omega |\nabla\varphi|^2\,dx, \end{aligned}$$

we easily conclude that

$$\sum_j \delta_j = 2\int_\Omega \nabla\hat u_i \cdot \nabla\varphi\,dx + \mathrm{o}(1). \tag{3.11}$$

Let us estimate, for a fixed index j, the difference:

$$\left(\frac{\int_\Omega |\nabla v_j(x)|^2\,dx}{\int_\Omega |v_j(x)|^2\,dx}\right)^p - \left(\frac{\int_\Omega |\nabla u_j(x)|^2\,dx}{\int_\Omega |u_j(x)|^2\,dx}\right)^p = pt\Lambda^p\big(\delta_j - \lambda_1(D_j)\alpha_j + \mathrm{o}(\delta_j)\big),$$

here we used the normalization condition (3.7), which implies

$$\int_\Omega |\nabla u_j|^2\,dx = \left(\frac{\lambda_1(D_j)}{\Lambda}\right)^p. \tag{3.12}$$

On the other hand, we have:

$$\left(1 - \frac{\int_\Omega v_j(x)^q\tilde\phi_j(x)^q\,dx}{(\int_\Omega v_j(x)^{2q}\,dx \int_\Omega \tilde\phi_j(x)^{2q}\,dx)^{1/2}}\right) - \left(1 - \frac{\int_\Omega u_j(x)^q\tilde\phi_j(x)^q\,dx}{(\int_\Omega u_j(x)^{2q}\,dx \int_\Omega \tilde\phi_j(x)^{2q}\,dx)^{1/2}}\right) = t\beta_j.$$

Now we prove inequality (I1). We select $j \neq i$ and we replace only the j'th component u_j by v_j. We obtain, as $t \to 0^+$,

$$0 \leqslant \frac{1}{t}\big(\mathcal{F}_{k,p}(u_1,\ldots,v_j,\ldots,u_k) - \mathcal{F}_{k,p}(u_1,\ldots,u_j,\ldots,u_k)\big)$$
$$= \frac{1}{\gamma}\big(\delta_j - \lambda_1(D_j)\alpha_j - \gamma\beta_j\big) + \mathrm{o}(\delta_j) + \mathrm{o}(1).$$

This inequality and the boundedness of the α_j's, the β_j's and γ gives a lower bound of the δ_j's. On the other hand (3.10) gives an upper bound of the δ_j's, which are consequently bounded as $t \to 0$. Hence we can deduce from (3.10) and the last inequality that

$$0 \leqslant -2\int_\Omega \big(\nabla u_j \cdot \nabla\varphi - \lambda_1(D_j)u_j\varphi - f_j(u_j)\varphi\big)\,dx.$$

Since this holds for every pair of indices $i \neq j$ (though here i does not appear) and every non-negative test function φ, inequality (I1) is proved.

To prove inequality (I2), we argue by contradiction and we assume the existence of $\varphi \geqslant 0$ such that

$$\int_\Omega \nabla\hat{u}_i \cdot \nabla\varphi\,dx < \int_\Omega \Big(\lambda_1(D_i)u_i(x) + f_i(u_i)(x) - \sum_{j\neq i}\lambda_1(D_j)u_j(x) + f_j(u_j)(x)\Big)\varphi(x)\,dx,$$

or, in other words,

$$\int_\Omega \nabla\hat{u}_i \cdot \nabla\varphi\,dx < \sum_i\Big(\lambda_1(D_j)\frac{\alpha_j}{2} + \gamma\frac{\beta_j}{2}\Big) + \mathrm{o}(1). \tag{3.13}$$

Now, by the minimization property of U, we have,

$$0 \leqslant \mathcal{F}_{k,p}(v_1,\ldots,v_k) - \mathcal{F}_{k,p}(u_1,\ldots,u_k) = \frac{t}{\gamma}\sum_j\big(\delta_j - \lambda_1(D_j)\alpha_j - \gamma\beta_j\big) + \mathrm{o}(t),$$

in contradiction with (3.13) and (3.11). □

Theorem 3.4 will easily follow from the next two results:

Lemma 3.12. *As $p \to +\infty$ any family of minimizers of* (3.4) *satisfying the normalization condition* (3.7) *with $\Lambda = \mathfrak{L}_k$ converges, up to a subsequence, to a multiple $(a_1\tilde{\phi}_1,\ldots,a_k\tilde{\phi}_k)$ ($a_i \geqslant 0$, not all vanishing) strongly in $(W_0^{1,2}(\Omega))^k$.*

Proof. From Proposition 3.9 we have for the minimizers $u_{i,p}$ and the corresponding $D_{i,p}$ (we now mention the reference to p which will then tend to $+\infty$),

$$\frac{1}{k}\sum_i\int_\Omega |\nabla u_{i,p}|^2\,dx = \frac{1}{k}\sum_i\Big(\frac{\lambda_1(D_{i,p})}{\Lambda}\Big)^p \leqslant \Big(\frac{\mathcal{M}_{k,p}}{\mathfrak{L}_k}\Big)^p \leqslant 1,$$

while

$$\frac{1}{k}\sum_i\int_\Omega |\nabla u_{i,p}|^2\,dx \geqslant \Big(\frac{\mathfrak{L}_{k,p}}{\mathfrak{L}_k}\Big)^p \geqslant \frac{1}{k}.$$

Hence the family is bounded in $(W_0^{1,2}(\Omega))^k$ and does not vanish. We extract a sequence $(u_{i,p_n})_{n\in\mathbb{N}}$ possessing a limit, in the weak $(W_0^{1,2}(\Omega))^k$-topology and in any $(L^r(\Omega))^k$, for subcritical r's. We denote by $(\tilde{u}_i)_{i=1,\ldots,k}$ this limit. We infer that the weak limit cannot be identically zero. We have indeed:

$$\lambda_1(D_{i,p_n}) = \frac{\int_\Omega |\nabla u_{i,p_n}|^2\,dx}{\int_\Omega |u_{i,p_n}|^2\,dx} \leqslant k^{1/p_n}\mathfrak{L}_k, \quad \forall i = 1,\ldots,k,$$

so that

$$\frac{1}{k}\sum_i \int_\Omega |u_{i,p_n}|^2\,dx \geqslant \frac{1}{k^{1/p_n}\mathfrak{L}_k}\sum_i \int_\Omega |\nabla u_{i,p_n}|^2\,dx.$$

We further remark that, if for some i the weak limit happens to be zero, then the strong limit vanishes too.

We claim that, for suitable non-negative a_i's, $\tilde{u}_i = a_i\tilde{\phi}_i$. This is obvious if $\tilde{u}_i \equiv 0$. If not, since by Proposition 3.9

$$\lim_{p\to+\infty}(\mathcal{M}_{k,p} - \mathfrak{L}_{k,p}) = 0,$$

we deduce that, whenever $\tilde{u}_i \not\equiv 0$,

$$1 - \frac{\int_\Omega \tilde{u}_i(x)^q \tilde{\phi}_i(x)^q\,dx}{(\int_\Omega \tilde{u}_i(x)^{2q}\,dx \int_\Omega \tilde{\phi}_i(x)^{2q}\,dx)^{1/2}} = 0,$$

and therefore that $\tilde{u}_i$ is a multiple of $\tilde{\phi}_i$. To pass from weak to strong convergence we first notice that each $u_{i,p_n} - \tilde{u}_i$ converges weakly and strongly in L^2 to zero. Now we recall that u_{i,p_n} satisfy inequalities (I1)–(I2) of Lemma 3.11. Let us multiply (I1) by $(u_{i,p_n} - \tilde{u}_i)_+$, the positive part of $u_{i,p_n} - \tilde{u}_i$, (I2) by $(u_{i,p_n} - \tilde{u}_i)_-$ and take the difference. We obtain

$$\int_\Omega \nabla u_{i,p_n}\cdot\nabla(u_{i,p_n} - \tilde{u}_i)\,dx \leqslant \int_\Omega \sum_{j\neq i} \nabla u_{j,p_n}\cdot\nabla(u_{i,p_n} - \tilde{u}_i)^-\,dx + \mathrm{o}(1).$$

Since $u_{i,p_n}(x)u_{j,p_n}(x)$ vanishes almost everywhere and $\tilde{u}_i \geqslant 0$, we infer

$$\begin{aligned}\int_\Omega \nabla u_{i,p_n}\cdot\nabla(u_{i,p_n} - \tilde{u}_i)\,dx &\leqslant \sum_{j\neq i}\int_\Omega \nabla u_{j,p_n}\cdot\nabla\tilde{u}_i\,dx + \mathrm{o}(1)\\ &= \sum_{j\neq i}\int_\Omega \nabla\tilde{u}_j\cdot\nabla\tilde{u}_i\,dx + \mathrm{o}(1) = \mathrm{o}(1).\end{aligned}$$

Thus we can deduce strong convergence from the weak. We have indeed

$$\left\|\nabla(u_{i,p_n} - \tilde{u}_i)\right\|^2 = \int_\Omega \nabla u_{i,p_n}\cdot\nabla(u_{i,p_n} - \tilde{u}_i)\,dx - \int_\Omega \nabla\tilde{u}_i\cdot\nabla(u_{i,p_n} - \tilde{u}_i)\,dx = \mathrm{o}(1). \qquad \square$$

Lemma 3.13. *Let $U_n = (u_{1,p_n},\ldots,u_{k,p_n})$ $(n\in\mathbb{N})$ as in the proof of Lemma* 3.12. *Then its limit, as $n\to+\infty$, $\widetilde{U} := (\tilde{u}_1,\ldots,\tilde{u}_k)$ verifies the inequalities in the statement of Theorem* 3.4.

Proof. First of all, we wish to pass to the limit in formulas (I1)–(I2) of Lemma 3.11, in the sense of distributions. From the previous lemma we deduce that $-\Delta u_{i,p_n} \to -\Delta\tilde{u}_i$ in $H^{-1}(\Omega)$. Hence one can pass to the limit in inequality (I1). Let us turn to (I2). We remark that also $f_i(u_{i,p_n})$ converge to $f_i(\tilde{u}_i)$, provided $\tilde{u}_i \not\equiv 0$. For such i's inequality (I2) passes to the limit, because so does its right hand. On the other hand, when the limit $\tilde{u}_i$ does vanish then (I2) holds because of (I1) and the fact that $-\Delta\tilde{u}_i = 0$. In order to end the proof, we have to prove convergence of the eigenvalues $\lambda_1(D_{i,p_n})$ to $\mathfrak{L}_k$ whenever $\tilde{u}_i$ does not vanish identically. At first we notice that the $\lambda_1(D_{i,p_n})$'s do converge, thanks to the strong convergence of the u_{i,p_n}'s to limits $\lambda_1(\widetilde{D}_i) \leqslant \mathfrak{L}_k$. Assuming $\lambda_1(\widetilde{D}_i) < \mathfrak{L}_k$ we deduce from (3.12), using again the strong convergence, that $\tilde{u}_i \equiv 0$. $\square$

Remark 3.14.

(a) Thanks to [11], Theorem 8.3 all the $u_{i,p}$'s and their limits $\tilde{u}_i$ are locally Lipschitz continuous in the interior of Ω and continuous up to the boundary, for (I1), if the boundary $\partial\Omega$ is Lipschitz, or has the interior cone property; moreover, they are globally Lipschitz up to the boundary, if $\partial\Omega$ is of class $\mathcal{C}^{1,+}$.

(b) Of course, since the eigenfunctions are normalized in L^2

$$a_i \stackrel{\text{def}}{=} \|\tilde{u}_i\|_2.$$

In general, it may happen that some of the a_i's vanish; we denote

$$\mathbf{k_0} = \big\{ i \in \{1, \dots, k\} \colon a_i = 0 \big\}. \tag{3.14}$$

(c) Going back to the proof of Lemma 3.12, we can extract a subsequence with the further property that the $u_{i,p_n}/\|u_{i,p_n}\|_2$ converge strongly in $L^2(\Omega)$ and weakly in $W_0^{1,2}(\Omega)$ to $\tilde{\phi}_i$, also for those indices i for which the component u_{i,p_n} normalized as in (3.12) strongly converges to 0.
(d) We also infer from (3.12) that

$$\lambda_1(\widetilde{D}_i) \leqslant \mathfrak{L}_k, \quad \forall i \in \{1, \dots, k\},$$

while

$$\lambda_1(\widetilde{D}_i) = \mathfrak{L}_k, \quad \forall i \notin \mathbf{k_0}.$$

(e) Obviously the system of differential inequalities (I1)–(I2) are fulfilled by the set $(\tilde{u}_i)$ with $i \notin \mathbf{k}_0$.
(f) For $i \notin \mathbf{k}_0$, the $\widetilde{D}_i$'s are open and possess finitely many connected components.

4. Further results in two dimensions

The aim of this section is to refine the analysis of the geometrical features of the minimal partition to obtain, in the two-dimensional case, that every open optimal partition is regular and strong. In order to achieve this goal we shall extensively make use of the optimality conditions expressed by the system of differential inequalities (I1)–(I2). As a final result we shall obtain that the nodal set associated with a minimal partition is a finite union of Hölder continuous closed arcs.

In this section, we recall from the introduction that we work under Assumptions 1.1 and 1.2. This implies that $\partial\Omega$ has finitely many connected components. For simplicity, we shall omit the potential V in the following discussion. All the arguments can be straightforwardly extended (sometimes at the price to replace $\mathcal{C}^2$ by $\mathcal{C}^{1,1-} := \bigcup_{\alpha<1} \mathcal{C}^{1,\alpha}$) in order to cover the case of a non-vanishing bounded potential. A special caution is only due in the proof of Theorem 4.6, where the needed extra argument is outlined.

4.1. *Case when* $\mathbf{k}_0 = \emptyset$

In this section we discard in a first step all the identically vanishing components. Hence, from now on, we will assume that

$$\mathbf{k}_0 = \emptyset.$$

Relabeling if necessary, and taking a smaller k, we assume that the components of

$$U = (u_1, \dots, u_k)$$

are non-negative, non-vanishing $W_0^{1,2}(\Omega)$-functions, such that $u_i(x)u_j(x) \equiv 0$ almost everywhere in Ω (for $i \neq j$), satisfying the two differential inequalities (I1)–(I2) of Theorem 3.4. As a consequence of Remark 3.14(a), they are continuous on the closure of Ω. Hence, by expanding the set of indices if necessary, we can always assume that the sets

$$D_i = \{u_i > 0\}$$

are open and connected.

Let us define the set of zeros of U as

$$\mathcal{Z} = \big\{ x \in \Omega \colon u_i(x) = 0, \ \forall i = 1, \dots, k \big\},$$

and define the *multiplicity* $m(x)$ *of* $x \in \Omega$ by

$$m(x) = \sharp \big\{ i \colon \operatorname{meas}\big(\{u_i > 0\} \cap B_r(x)\big) > 0, \ \forall r > 0 \big\}. \tag{4.1}$$

We shall denote by

$$\mathcal{Z}_h = \big\{ x \in \Omega \colon m(x) \geqslant h \big\}$$

the set of points of multiplicity greater than or equal to the integer h and by

$$\mathcal{Z}^h = \{x \in \Omega:\ m(x) = h\}.$$

We remark that, by definition, $\mathcal{Z}^0$ is open. Let us now consider $\mathcal{Z}^1$. This is the object of:

Proposition 4.1.

$$\mathcal{Z}^1 = \bigcup_{i=1}^{k} D_i.$$

Proof. If a ball $B_r(x_0)$ intersects only D_i we deduce from (I1)–(I2) that u_i is a non-negative solution to the differential equation $-\Delta u = \Lambda u$ on $B_r(x_0)$. By the strong maximum principle then u_i is strictly positive on $B_r(x_0)$ and therefore $B_r(x_0) \subset D_i$ and all the points of $B_r(x_0)$ have multiplicity equal to one. □

To continue the analysis of the topological properties related to points of multiplicity two and more, we shall consider the simplicial homology groups with coefficients in $\mathbb{Z}_2$, $H_n(X)$, for $n = 0, 1$. We recall that $\operatorname{rank}(H_0(X))$ is the number of connected components of X. For open subsets of an Euclidean space, as the fundamental group π_1 is already Abelian, there holds $\operatorname{rank}(\pi_1(X)) = \operatorname{rank}(H_1(X))$. Finally for planar bounded open subsets, $\operatorname{rank}(H_1(X)) + 1$ is the number of connected components of ∂X. A reference book for the algebraic topology concepts is Greenberg and Harper's book [16].

Proposition 4.2. *If $H_1(\Omega)$ is finite, so is $H_1(D_i)$ for every i.*

Proof. Let us consider a loop $\gamma \subset D_i$ which is homotopically trivial in Ω but not in D_i; hence denoting by Σ the inner region of γ, we have that $\Sigma \subset \Omega$ but $\Sigma \not\subset D_i$. Let $j \neq i$; then either D_j is contained in Σ or in its complement, for it is connected. To prove the proposition, we argue by contradiction and we assume that $H_1(D_i)$ is infinite. Then also $\operatorname{rank}(\pi_1(D_i))$ is infinite; thus we infer the existence of at least one loop $\gamma = \partial\Sigma$ such that $\Sigma \cap \bigcup_{j\neq i} D_j = \emptyset$ and $\Sigma \not\subset D_i$. Since all the u_j's ($j \neq i$) vanish identically in Σ, we deduce from (I1)–(I2) that $-\Delta u_i = \Lambda u_i$ in Σ and, by the strong maximum principle, that u_i is strictly positive there; thus $\Sigma \subset D_i$, a contradiction. □

Let us consider

$$\Gamma_{i,j} = \partial D_i \cap \partial D_j \cap \mathcal{Z}^2,$$
$$D_{i,j} = D_i \cup D_j \cup \Gamma_{i,j}.$$

Our next goal consists in showing that the $\Gamma_{i,j}$ consist of a finite number of (possibly open) arcs. This will require some topological considerations.

Proposition 4.3. *Let $x_0 \in \Omega$ such that $m(x_0) = 2$. Then $u_i - u_j$ is in $\mathcal{C}^{1,1-}$ in some neighborhood of x_0 and $\nabla(u_i - u_j)(x_0) \neq 0$. Furthermore $\mathcal{Z}^2$ is locally a $\mathcal{C}^{1,+}$-curve through x_0.*

Proof. Relabeling, we can always assume $x_0 \in \partial\{u_1 > 0\} \cap \partial\{u_2 > 0\}$; thus for all r small enough $B(x_0, r) \cap D_i = \emptyset$ for all $i > 2$. Then $u = u_1 - u_2$ satisfies the equation $-\Delta u = \Lambda u$ in $B(x_0, r)$ and is consequently $\mathcal{C}^{1,1-}$ near x_0. Therefore the zero set of u near x_0 (by a standard result on the zero set) is made up by a finite number of even regular curves starting from x_0. But there are actually only two arcs meeting at x_0. Indeed, if not, at least one of the D_i's should be disconnected. In this case the zero set is actually locally a regular line passing through x_0 and the Boundary Point Lemma (see [15]) gives the proposition. □

Proposition 4.4. *If $H_1(\Omega)$ is finite, so is $H_1(D_{i,j})$ for every $i \neq j$.*

Proof. This is an obvious statement if $\Gamma_{i,j} = \emptyset$.

Let us consider a loop $\gamma \subset D_{i,j}$ which is trivial in Ω but not in $D_{i,j}$; hence $\gamma = \partial\Sigma$ with $\Sigma \subset \Omega$ but $\Sigma \not\subset D_{i,j}$. Let $\ell \notin \{i, j\}$; then either D_ℓ is contained in Σ or in its complement. Thus, arguing by contradiction we infer the existence of a loop $\gamma = \partial\Sigma$ such that $\Sigma \cap \bigcup_{\ell\neq i,j} D_\ell = \emptyset$ and $\Sigma \not\subset D_{i,j}$. Since all the u_ℓ's ($\ell \neq i, j$) vanish identically in Σ, we deduce from (I1)–(I2) that $-\Delta(u_i - u_j) = \Lambda(u_i - u_j)$ in Σ and, from Proposition 4.3 and the unique continuation principle we infer that $\mathcal{Z} \cap \Sigma \subset \Gamma_{i,j}$; thus $\Sigma \subset D_{i,j}$, a contradiction. □

Lemma 4.5. *If $H_1(D_{i,j})$ is finite, then $\Gamma_{i,j}$ has finitely many connected components.*

Proof. To prove the statement we will take advantage of the Mayer–Vietoris theorem. The Mayer–Vietoris sequence is usually proven to be exact for a triad X, X_1 and X_2 where X_1 and X_2 are open subsets of the topological space X and X is the union of X_1 and X_2 (such a triplet is called and admissible triad). Here we would like to apply the Mayer–Vietoris sequence to the triad $X = D_{i,j}$, $X_1 = D_i \cup \Gamma_{i,j}$ and $X_2 = D_j \cup \Gamma_{i,j}$, but these latter two are not open in $D_{i,j}$. However, the Mayer–Vietoris sequence is still available because, thanks to Proposition 4.3, the $\Gamma_{i,j}$'s are regular embedded one-dimensional submanifolds in $D_{i,j}$. Hence each $D_\ell \cup \Gamma_{i,j}$ is a Euclidean neighborhood retract in $D_{i,j}$ and therefore has the same homology as the corresponding D_ℓ ($\ell = i, j$). Thus, following [16], the triplet $D_{i,j}$, $D_i \cup \Gamma_{i,j}$, $D_j \cup \Gamma_{i,j}$ is a proper excision triad and thus the Mayer–Vietoris sequence is exact:

$$H_1(D_{i,j}) \stackrel{\partial_*}{\to} H_0(\Gamma_{i,j}) \stackrel{i_*\oplus -j_*}{\to} H_0(D_i \cup \Gamma_{i,j}) \oplus H_0(D_j \cup \Gamma_{i,j}).$$

The assertion then follows as a consequence of Propositions 4.2 and 4.4 taking into account the connectedness of D_ℓ ($\ell = i, j$). Indeed, since both $H_1(D_{i,j})$ and $H_0(D_i \cup \Gamma_{i,j}) \oplus H_0(D_j \cup \Gamma_{i,j}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are finite, we have that both the range of ∂_* and the coker of $i_* \oplus -j_*$ are finite and thus so is $H_0(\Gamma_{i,j})$. □

The following results follow from [11]:

Theorem 4.6. *We have for all $i = 1, \dots, k$:*

(a) $u_i \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.
(b) *The function $\sum_{i=1}^k |\nabla u_i|$ admits a continuous representative in Ω.*
(c) *If $x_0 \in \mathcal{Z}_3$, then*

$$\lim_{x\to x_0} \sum_{i=1}^k \left|\nabla u_i(x)\right| = 0.$$

(d) *If $x_0 \in \mathcal{Z}_3$, then*

$$\lim_{x\to x_0} \frac{u_i(x)}{|x - x_0|} = 0.$$

Proof. Part (a) is indeed stated as Theorem 8.3 of [11]. To prove part (b), we first observe that locally in $\bigcup_{i=0}^2 \mathcal{Z}^i$ (which is open in Ω) the u_i's satisfy the differential equation $-\Delta(u_i - u_j) = \Lambda(u_i - u_j)$: hence they are regular (of class $\mathcal{C}^1(\bigcup_{i=0}^2 \mathcal{Z}^i)$). The continuity up to the $\mathcal{Z}_3$ (which is indeed the boundary of $\bigcup_{i=0}^2 \mathcal{Z}^i$) is then a consequence of the vanishing of the limit stated in part (c), which is indeed Theorem 9.3 in [11]. The proof of Theorem 9.3 was originally performed in the absence of the L^∞ potential, but all the arguments can be promptly adapted to cover also this case. A special care is needed when, at the beginning, the function $\sum_{i=1}^k |\nabla u_i|^2$ is shown to be a subsolution to a linear differential equation. Subsequently the mean value property is applied. This is not exactly true in the present situation, for the linear problem is now perturbed by a term of the form $\nabla w \cdot \nabla((\Lambda + V(x))w)$, where w is an auxiliary function (which can be taken, by the way, the same as defined in (4.3)). After integrating by parts one easily sees that the contribution of this term is negligible. This observation permits us to follow, from then on, exactly the same arguments of the quoted paper. □

Lemma 4.7. *Let $x_0 \in \mathcal{Z}_3$. Then there exists a sequence $\{x_n\} \subset \Omega$ such that $m(x_n) = 2$ and $x_n \to x_0$ as $n \to +\infty$.*

Proof. Assume not, then there would be an element y_0 of $\mathcal{Z}_3$ with a positive distance d from $\mathcal{Z}^2$. Let $r < d/2$; then the ball $B(y_0, r)$ intersects at least three of the D_i's. Therefore there exist $i \in \{1, \dots, k\}$, $x \in D_i$ and $z_0 \in \mathcal{Z}_3$ such that

$\rho = d(x, z_0) = d(x, \mathcal{Z}_3) < d(x, \mathcal{Z}^2)$. Then the ball $B(x, \rho)$ is tangent from the interior of D_i to $\mathcal{Z}_3$ in z_0. Furthermore u_i solves an elliptic PDE (in the sense of [15]) and it is positive on D_i. Thus we infer from the Boundary Point Lemma (see Lemma 3.4 and formula (3.11) in [15]) that

$$\liminf_{h\to 0} \frac{u_i(z_0 + h\nu)}{h} > 0, \tag{4.2}$$

where ν denotes the inner normal to $\partial B(x, \rho)$ at z_0, in contrast with Theorem 4.6(c). □

A simple but important consequence of this discussion is the following result:

Proposition 4.8. *If $H_1(\Omega)$ is finite, then $\overline{\mathcal{Z}_3}$, the closure of $\mathcal{Z}_3$ in $\overline{\Omega}$, has a finite number of connected components.*

Proof. Indeed, we have $\overline{\mathcal{Z}_3} \subset \bigcup_{i,j} (\overline{\Gamma_{i,j}} \setminus \Gamma_{i,j})$, and each $\Gamma_{i,j}$ is the union of finitely many arcs, each of them being homeomorphic to the real line (non-compact case) or to a circle. In the first case they have connected and closed α and ω-limits.

We recall that the α and ω-limits of a parametrized arc $\Gamma(t)$ are the sets of the limit points as the parameter tends to $-\infty$ and $+\infty$ respectively: one easily sees that bounded arcs have compact and connected α and ω-limits. Therefore each $\overline{\Gamma_{i,j}} \setminus \Gamma_{i,j}$ contributes with finitely many connected components by the previous proposition. □

In addition we have:

Proposition 4.9. *If $H_1(\Omega)$ is finite, then $\overline{\mathcal{Z}_3 \cup \mathcal{Z}^0}$ has a finite number of connected components.*

Proof. Indeed, by Theorem 4.6 and Lemma 4.7, the boundary $\partial(\overline{\mathcal{Z}_3 \cup \mathcal{Z}^0})$ is the same as $\partial\overline{\mathcal{Z}_3}$ and this last one has finitely many connected components. □

Our next goal is the following

Theorem 4.10. *An isolated connected component of $\overline{\mathcal{Z}_3 \cup \mathcal{Z}^0}$ consists of a single point.*

This theorem implies straightforwardly, having in mind that $\mathcal{Z}^0$ is open, that

Corollary 4.11. *If $H_1(\Omega)$ is finite, then $\mathcal{Z}^0 = \emptyset$.*

Proof of Theorem 4.10. To prove the theorem we focus on a connected component Y_0 of $\overline{\mathcal{Z}_3 \cup \mathcal{Z}^0}$ and we show that it is reduced to a single point. First we consider an open connected neighborhood $\mathcal{N}$ of Y_0 in $\mathbb{R}^2$ having a regular boundary and such that $\overline{\mathcal{N}} \cap \overline{\mathcal{Z}_3 \cup \mathcal{Z}^0} = Y_0$. Since Y_0 is connected, we can choose $\mathcal{N}$ in such a way that its boundary has exactly one or two connected components (each diffeomorphic to S^1), depending on whether Y_0 disconnects $\mathbb{R}^2$ or not. This can be achieved by taking one connected component of a regular sublevel of a non-negative $\mathcal{C}^\infty$ function having Y_0 as null set. Furthermore we can take the measure of $\mathcal{N} \setminus Y_0$ small enough that none of the D_i's is completely enclosed in $\mathcal{N}$. We may assume that $\partial\mathcal{N}$ intersects transversally the $\Gamma_{i,j}$'s and $\partial\Omega$. Thus, after possibly cutting off some portions of $\mathcal{N}$, we can assume that each oriented arc in $\Gamma_{i,j}$ intersects $\partial\mathcal{N}$ exactly once. Then each $D_i \cap \mathcal{N}$ has a finite number of connected components. Moreover, we can manage to have these intersections simply connected. Indeed, assuming not, and arguing as in the proof of Proposition 4.2, at least one of the other D_j's should be entirely contained in $D_i \cap \mathcal{N} \setminus Y_0$. That we have excluded by taking the measure of $\mathcal{N} \setminus Y_0$ small enough. We can label the connected components of $D_i \cap \mathcal{N}$ and $(\mathbb{R}^2 \setminus \Omega) \cap \mathcal{N}$ clockwise from 1 to h, according to their intersection with $\partial\mathcal{N}$. By taking a double covering of $\mathcal{N}$, still denoted by $\mathcal{N}$, if necessary, we may assume that h is an even integer. Indeed, assuming h to be odd, we choose some point $y_0 \in Y_0$ and perform a double covering branched at y_0 (in complex notation $f(z) = (z - y_0)^2$). With some abuse of notation, we use the same symbols for D_i, $\mathcal{N}$ and Y_0 and their pre-images, and we introduce $\tilde{u}_i(z) = u_i(f(z))$. Note that

$$\Delta\tilde{u}_i(z) = 8|z - y_0|^2 \Delta u_i\big(f(z)\big).$$

If the original h was even we set $\tilde{u}_i = u_i$ and we define, in both cases, the auxiliary function

$$w(x) = \begin{cases} \sigma(x)\tilde{u}_i(x) & \text{if } x \in D_i, \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

where σ is a sign assignment compatible with the partition of $\mathcal{N}$. The function w satisfies the linear equation:

$$-\Delta w = \Lambda a(x) w \quad \text{in } \Omega \cap (\mathcal{N} \setminus Y_0), \tag{4.4}$$

$$w = 0 \quad \text{on } (\partial\Omega \cap \mathcal{N}) \cup Y_0, \tag{4.5}$$

where $a \equiv 1$ if no double covering has been performed, otherwise $a(x) = 8|x - y_0|^2$ is the conformal factor. Of course w is regular outside the singular component Y_0. Also, w vanishes identically on the open set $N \cap \mathcal{Z}^0$; hence, thanks to Theorem 4.6(c), w is in $\mathcal{C}^1(\Omega \cap \mathcal{N})$.

We claim that w actually solves (4.4) in the whole of $\Omega \cap \mathcal{N}$. This is not a trivial fact, for there is no a priori bound on the Hausdorff measure of the boundary ∂Y_0 of the singular set. In order to overcome this problem we make the following construction:

Proposition 4.12. *There exists a family of neighborhoods $\mathcal{N}_\delta \subset \mathcal{N}$, decreasing to Y_0 and bounded by a finite number of regular arcs, with the property that*

$$\lim_{\delta \to 0} \int_{\partial(\mathcal{N}_\delta \cap \Omega)} |\nabla w| \, ds = 0, \tag{4.6}$$

where ds denotes the measure on the union of these regular arcs.

Postponing the proof of the proposition, we end the proof of Theorem 4.10. Testing (4.4) with any function $\varphi \in C_0^\infty(\mathcal{N} \cap \Omega)$, it follows, by decomposing the integration on $\mathcal{N} \setminus \mathcal{N}_\delta$ and on $\mathcal{N}_\delta$ and then making an integration by parts, that for the second integral:

$$\begin{aligned} \left| \int_{\mathcal{N} \cap \Omega} \big(\nabla w \cdot \nabla\varphi - \Lambda a(x) u\varphi\big)\, dx \right| &= \left| \int_{\mathcal{N}_\delta \cap \Omega} \big(\nabla w \nabla\varphi - \Lambda a(x) u\varphi\big)\, dx - \int_{\partial(\mathcal{N}_\delta \cap \Omega)} \varphi \nabla w \cdot \nu \, ds \right| \\ &\leqslant C\Bigg(\sup_{\mathcal{N}_\delta \cap \Omega \cap \operatorname{supp}(\varphi)} \big(|w| + |\nabla w|\big) + \int_{\partial(\mathcal{N}_\delta \cap \Omega)} |\nabla w| \, ds \Bigg). \end{aligned}$$

We have now to show that the right-hand side is $\mathrm{o}(1)$. Proposition 4.12 ensures that, as $\delta \to 0$,

$$\int_{\partial(\mathcal{N}_\delta \cap \Omega)} |\nabla w| \, ds = \mathrm{o}(1).$$

So it remains to show that, as $\delta \to 0$,

$$\sup_{\mathcal{N}_\delta \cap \Omega \cap \operatorname{supp}(\varphi)} \big(|w| + |\nabla w|\big) = \mathrm{o}(1).$$

This can be done by Theorem 4.6. Indeed, both w and ∇w are uniformly continuous on $\operatorname{supp}(\varphi)$ which is compactly contained in Ω, and the distance of $\mathcal{N}_\delta \cap \operatorname{supp}(\varphi)$ to Y_0 tends to zero. Finally, we recall that w and its gradient vanish identically on Y_0. Hence w solves (4.4) on the whole of $\mathcal{N} \cap \Omega$. By a classical local regularity result by Hartman and Wintner ([17], Corollary 1), we know that interior critical points of solutions to such class of equations are isolated and have finite (local) multiplicity m, and satisfy (2.1).

Hence we are left with the case when Y_0 is contained in a connected component of the boundary. We notice that, in this case, we do not need the double covering, for a sign assignment compatible with the partition of $\mathcal{N}$ always exists, since $\partial\Omega$ disconnects $\partial\mathcal{N}$. Now we need an extension of Theorem 2.1, suitable to cover the case of domains possessing a finite number of $\mathcal{C}^{1,+}$-corners: namely Theorems 2.3 and 2.6 in Section 2.

In particular, these results guarantee finiteness of critical points also at the boundary. □

Proof of Proposition 4.12. First of all, let us consider a triplet of adjacent domains separated by the two arcs $\Gamma_{i,j}$ and $\Gamma_{j,k}$. We claim that the distance of the arcs, relative to D_j, must vanish. In other words, we claim that

$$\inf\left\{\int_0^1 \left|\dot\gamma(s)\right| ds\colon\ \gamma(0)\in\Gamma_{i,j},\ \gamma(s)\in D_j,\ \forall s\in(0,1),\ \gamma(1)\in\Gamma_{j,k}\right\}=0.$$

Indeed, if not, one could find in D_j a ball, tangent to the boundary $\partial D_j\cap Y_0$ at, say, x_0 with positive distance from both $\Gamma_{i,j}$ and $\Gamma_{j,k}$. By the Boundary Point Lemma, at x_0 the gradient of w cannot vanish (in the weak sense of (4.2)), in contradiction with the fact that $x_0\in Y_0$.

As a second remark, by integrating the equation $-\Delta u_i=\Lambda u_i$ over the set $\{u_i>\varepsilon\}$ and using the Divergence Theorem, we obtain,

$$\lim_{\varepsilon\to 0^+}\int_{\partial\{u_i>\varepsilon\}}|\nabla u_i|\,ds\leqslant C\int_\Omega u_i\,dx<+\infty.$$

The Divergence Theorem is applicable because the level sets (according to Hartman and Wintner's regularity result) of the eigenfunctions of $-\Delta+V(x)$ are compact and piecewise $\mathcal{C}^1$ when the potential V is bounded. Since the components of $\partial\{u_i>\varepsilon\}$ converge to $\Gamma_{i,j}$ as $\varepsilon\to 0$, in $\mathcal{C}^1$ as parametrized curves, we obtain that

$$\int_{\Gamma_{i,j}}|\nabla w|\,ds<+\infty,\quad \forall i,j.$$

If the arc $\Gamma_{i,j}$ meets Y_0, then we can choose an orientation for its parametrization $\gamma(t)$ in a way that

$$\lim_{t\to+\infty}\operatorname{dist}\bigl(\gamma(t),Y_0\bigr)=0,$$

and we have:

$$\int_0^{+\infty}\left|\nabla w\bigl(\gamma(t)\bigr)\right|\left|\gamma'(t)\right|dt<+\infty;$$

$$\gamma(t)\notin Y_0,\quad \forall t\geqslant 0;$$

$$\lim_{t\to+\infty}\operatorname{dist}\bigl(\gamma(t),Y_0\bigr)=0.$$

Therefore we conclude

$$\lim_{\delta\to 0}\int_{\Gamma_{i,j}\cap B_\delta(Y_0)}|\nabla w|\,ds=0,\quad \forall i,j,$$

where $B_\delta(Y_0)=\{x\in\mathbb{R}^2,\ d(x,Y_0)<\delta\}$.

Now we illustrate the construction of the boundary of the neighborhood $\mathcal{N}_\delta$. We start by taking the component of $\Gamma_{i,j}\cap B_\delta(Y_0)$ ending at Y_0; then we can jump from this arc to the next $\Gamma_{j,k}$, still remaining in $B_\delta(Y_0)\cap D_j$, following an arc of arbitrarily short length. We proceed in this way passing from one arc to the next until we complete the loop. □

The next result straightforwardly follows from Theorems 2.3 and 2.6 applied to the auxiliary function w defined in (4.3); it completes our analysis of the asymptotic expansion of the nodal set at multiple intersection points.

Theorem 4.13. *Let $x_0\in\Omega$ be an isolated point in $\overline{\mathcal{Z}}_3$ with $m(x_0)=h\geqslant 3$. Then there exist an integer $n\geqslant h$ (the local multiplicity), $c\in\mathbb{R}\setminus 0$, $\theta_0\in(-\pi,\pi]$ such that*

$$\sum_{i=1}^h u_i(r,\theta)=cr^{n/2}\left|\cos\left(\frac{n}{2}(\theta+\theta_0)\right)\right|+\mathrm{o}(r^{n/2}),$$

as $r \to 0$*, where* (r, θ) *denotes a system of polar coordinates around* x_0 *and* n *is the* local multiplicity *of* x_0.

Moreover the nodal set in a neighborhood of x_0 *is the union of* n *closed arcs of class* $\mathcal{C}^{1,+}$ *meeting at* x_0 *and spanning angles of opening* $2\pi/n$.

If the boundary $\partial\Omega$ *presents a* $\mathcal{C}^{1,+}$*-corner of amplitude* $\alpha\pi$ *at* x_0 *and the nodal set hits* x_0 *from inside the corner, then there exist an integer* n *and* $R > 0$ *such that the component of* $\overline{u^{-1}(\{0\})} \cap B(x_0, R)$ *lying inside the corner is composed by* n $\mathcal{C}^{1,+}$*-simple arcs which all end in* x_0 *and whose tangent lines at* x_0 *divide the sector into* $n+1$ *angles of equal opening angle* $\pi\alpha/(n+1)$.

4.2. General case

Let us come back to the general case. We no more assume a priori that $\mathbf{k}_0 = \emptyset$, nor the connectedness of the D_i's. Then we obtain:

Theorem 4.14. *If* $N = 2$ *and* Ω *is a connected open set satisfying Assumption* 1.1*, then the assertion of Theorem* 3.4 *holds with all* a_i*'s strictly positive.*

Moreover any minimizing partition $\mathcal{D}$ *admits an open regular connected representative.*

Proof. Assume that some of the a_i vanish or, in other words, that $\mathbf{k}_0 \neq \emptyset$. Let us denote

$$\mathcal{W}_0 = \Omega \setminus \bigcup_{i \notin \mathbf{k}_0} D_i$$

the nodal set. Then the 2-dimensional measure of $\mathcal{W}_0$ is positive, since it contains the supports of $\tilde{\phi}_i$ for all $i \in \mathbf{k}_0$.

Now, as already remarked, inequalities (I1) and (I2) are still available when we discard all the vanishing components and we take $U = (u_i)_{i \notin \mathbf{k}_0}$. Hence Theorem 4.13 applies and, as a direct consequence, we find that the zero set $\mathcal{Z}$, being the union of a finite number of closed $\mathcal{C}^{1,+}$ curves has vanishing measure in $\mathbb{R}^2$, a contradiction. This also implies that the partition associated with the D_i is strong and that the zero set is indeed the nodal set $N(\mathcal{D})$ as defined in (1.9).

Now assume by contradiction that some elements of the partition are not connected. We can anyway choose a set of first eigenfunctions having each a connected support, but we clearly have an open, non-empty set of multiplicity zero points. This contradicts Corollary 4.11. □

Remark 4.15. Note that Theorem 4.14 completes the proof of Theorem 1.12.

Proof of Theorem 1.14. But we also get the proof of Theorem 1.14 in the following way. If the graph associated to $\widetilde{D}$ is bipartite we can find $\epsilon_i = \pm 1$ satisfying $\epsilon_i \epsilon_j = -1$ if $D_i \sim D_j$ and such that $u := \sum_i \epsilon_i a_i \tilde{\varphi}_i$ is in $W_0^{1,2}(\Omega)$ and satisfies

$$(-\Delta + V)u = \mathfrak{L}_k(\Omega)u \tag{4.7}$$

in $\Omega \setminus \mathcal{Z}_3$. But we have proven that $\mathcal{Z}_3$ consists of isolated points (which cannot be the support of a distribution in $W^{-1,2}(\Omega)$), the dual of $W_0^{1,2}(\Omega)$. Hence (4.7) is satisfied in Ω and u is actually an eigenfunction of $H(\Omega)$ corresponding to $\mathfrak{L}_k(\Omega)$. We refer to Appendix B for a complementary discussion.

5. More on nodal sets and partitions

We continue by discussing more deeply the links between the various spectral sequences.

The first important property is given by:

Proposition 5.1. *Let* $H(\Omega)$ *be defined as above. Then*

$$\mathfrak{L}_k(\Omega) < \mathfrak{L}_{k+1}(\Omega) \quad \textit{for } k \geqslant 1. \tag{5.1}$$

Proof. We take indeed a minimal $(k+1)$-partition of Ω. We have proved that this partition is regular. If we take any subpartition by k elements of the previous partitions. This cannot be a minimal k-partition (it has not the "strong partition" property). So the inequality in (5.1) is strict. $\square$

The second property concerns the *domain monotonicity*.
It is indeed immediate to verify:

Proposition 5.2. *If $\Omega \subset \widetilde{\Omega}$, then*

$$\mathfrak{L}_k(\widetilde{\Omega}) \leqslant \mathfrak{L}_k(\Omega), \quad \forall k \geqslant 1.$$

We observe indeed that each partition of Ω is a partition of $\widetilde{\Omega}$.

Remark 5.3. The analysis of the equality in the proposition will involve the capacity of $\widetilde{\Omega} \setminus \Omega$. See [4] and references therein.

We now come back to a definition which was briefly mentioned in the introduction. Having in mind Definition 1.16, we denote, for any integer $k \geqslant 1$, by $L_k(\Omega)$ the smallest eigenvalue whose eigenspace contains an eigenfunction with k nodal domains. We take $L_k = +\infty$, if there are no eigenfunctions with k nodal domains. We call this sequence the spectral nodal sequence.

Proposition 5.4. *Let λ be an eigenvalue corresponding to an eigenfunction with k nodal domains. Then*

$$\mathfrak{L}_k \leqslant \lambda. \tag{5.2}$$

Proof of Proposition 5.4. If u is an eigenfunction associated with λ and with k nodal domains, then, taking as $\mathcal{B}_0$ the collection of these nodal domains, we obtain:

$$\inf_{\mathcal{B}\in\mathfrak{O}_k} \Lambda(\mathcal{B}) \leqslant \Lambda(\mathcal{B}_0) \leqslant \lambda. \qquad \square \tag{5.3}$$

Proposition 5.5. *Let $\lambda = \lambda_k$ be an eigenvalue $H(\Omega)$. Then*

$$\lambda_k \leqslant \inf_{\mathcal{B}\in\mathfrak{O}_k} \Lambda(\mathcal{B}). \tag{5.4}$$

Proof of Proposition 5.5. The basic idea (which is already present in Courant's theorem) is simply the following. We can assume (using Proposition 5.1)

$$\lambda_{k-1} < \lambda_k.$$

Attached to a minimal $\mathcal{B}_k$ (hence regular), we have a k-dimensional space in $W_0^{1,2}(\Omega)$ generated by the ground states of the D_i $(i = 1, \dots, k)$. We can find in this space a non-trivial element which is orthogonal to the eigenspace corresponding to the eigenvalues which are $\leqslant \lambda_{k-1}$, whose energy is $\mathfrak{L}_k$, hence by the Minimax Principle $\lambda_k \leqslant \mathfrak{L}_k$.

Suppose now that we have the equality $\lambda_k = \mathfrak{L}_k$. Again by the proof of the Minimax Principle, this non-trivial element should be an eigenfunction which is consequently Courant-sharp and we have consequently $\lambda_k = L_k = \mathfrak{L}_k$. $\square$

The following corollary is just a rephrasing of Propositions 5.5 and 5.4.

Corollary 5.6. *We have*

$$L_k \geqslant \mathfrak{L}_k \geqslant \lambda_k, \quad \forall k \geqslant 1. \tag{5.5}$$

In particular, if $L_k = \lambda_k$ (also called the Courant-sharp case in [4]) the nodal domain of a corresponding eigenfunction gives a minimal partition.

Remarks 5.7.

(i) For the one-dimensional case the standard Sturm–Liouville theory leads easily to the following

$$L_k = \mathfrak{L}_k = \lambda_k, \quad \forall k \geqslant 1. \tag{5.6}$$

(ii) It is easy to show, that for a given H

$$\mathfrak{L}_1 = L_1 = \lambda_1, \tag{5.7}$$

(by the property of the ground state) and that

$$\mathfrak{L}_2 = L_2 = \lambda_2, \tag{5.8}$$

by the orthogonality of u_2 to the ground state combined with Courant's nodal theorem. (See also [12], Corollary 4.1 (Case $V = 0$, but the extension to $V \in L^\infty$ is not a problem).)

(iii) The sequence L_k is not necessarily monotone: see for example (9.6).

One also observes that, using (5.5) and the property that $\lambda_k \to +\infty$,

$$\lim_{k\to+\infty} \mathfrak{L}_k = +\infty. \tag{5.9}$$

6. Playing around Pleijel's argument

It is a well-known result of Pleijel ([25]) that the one-dimensional result that the k-th eigenfunction of a Sturm–Liouville operator on an interval has only k nodal domains cannot be extended to higher dimension. The k-th Courant-sharp eigenfunctions (i.e. k-th eigenfunctions with k nodal domains) can only be found for a finite number of k's.

We will show in this section, that the arguments behind the proof of this theorem give also many informations on the spectral minimal partition sequence in comparison with the spectral sequence and the nodal sequence.

Let us look at a universal lower bound for $\mathfrak{L}_k(\Omega)$. We actually obtain:

Proposition 6.1. *Considering the Dirichlet Laplacian, we have*

$$\mathfrak{L}_k(\Omega) \geqslant k\frac{\pi j^2}{|\Omega|},$$

where $|\Omega|$ denotes the area of Ω and j is the smallest positive 0 *of the Bessel function J_0:*

$$j \sim 2.4048\ldots. \tag{6.1}$$

Proof. For any D_j of a partition, we have by the Faber–Krahn inequality (see [5])

$$|D_j|\lambda(D_j) \geqslant \pi j^2.$$

The Faber–Krahn inequality gives indeed:

$$\lambda(D) \geqslant \frac{\lambda(B_{1/\sqrt{\pi}})}{|D|}, \tag{6.2}$$

for any open set D.

The lowest eigenvalue for the disk of radius 1 is known to be:

$$\lambda(B_1) = j^2, \quad \text{with } \pi j^2 \sim 18.1695. \tag{6.3}$$

Summing up over j, we obtain

$$\pi j^2 k \leqslant \sum_j |D_j|\lambda(D_j) \leqslant |\Omega| \max \lambda(D_j). \tag{6.4}$$

Taking the infimum over the partition leads to the result. □

Remark 6.2. Using Corollary 5.6, this implies

$$L_k(\Omega) \geqslant k\frac{\pi j^2}{|\Omega|}.$$

We conclude this section with a classical result of Pleijel:

Theorem 6.3. *The set of the k's such that one can find, for a k-th eigenvalue of $H(\Omega)$, an eigenfunction with k nodal domains is finite.*

This holds in larger generality for bounded potentials and also for higher dimensions.

Let us describe for completeness how Pleijel's theorem is proved. The Weyl asymptotics (see for example [27]) says that

$$\lambda_n \sim \frac{4\pi n}{|\Omega|}, \tag{6.5}$$

as $n \to +\infty$.

If u_n is an eigenfunction associated to λ_n with n nodal domains, we obtain immediately a contradiction for n large between (6.5) and (6.4) (applied with the family of nodal domains of u_n), having in mind the value of j given in (6.1). So u_n cannot have n nodal domains! More precisely, if there exists a smallest $n(k)$ such that $\lambda_{n(k)} = L_k$, we obtain asymptotically

$$\liminf_{k\to+\infty} \frac{n(k)}{k} \geqslant \frac{j^2}{4} > 1. \tag{6.6}$$

A more difficult question is to determine whether L_k is always finite.

Proposition 6.4. *In the case of the Laplacian and if Ω is regular, we have*[3]

$$\limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leqslant \lambda_1(Hx_1)/|\Omega|, \tag{6.7}$$

where Hx_1 is the regular hexagon of area 1.

For the proof, we just use a (non-strong) partition of Ω by equal hexagons of area at most $\frac{|\Omega|}{k}$.

Remark 6.5. Adding a potential V does not create any difficulty and the previous discussions can be easily adapted to go from $H_0(\Omega)$ to $H(\Omega)$. Concerning the values $L_k(\Omega, V)$ we obtain immediately,

$$L_k(\Omega, V) \geqslant k\frac{\pi j^2}{|\Omega|} - \sup|V|. \tag{6.8}$$

On the other hand, using the minimax, there are no problem to show that

$$\mathfrak{L}_k(\Omega, V) \geqslant k\frac{\pi j^2}{|\Omega|} - \sup|V|. \tag{6.9}$$

This implies in particular that

$$\liminf_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega, V)}{k} \geqslant \frac{\lambda_1(B_{1/\sqrt{\pi}})}{|\Omega|} > \frac{4\pi}{|\Omega|}, \tag{6.10}$$

is satisfied in full generality.

[3] Thanks to M. Van den Berg for discussions. In particular, he conjectures the existence of the limit $\lim_{k\to+\infty} \frac{\mathfrak{L}_k}{k}$ and that the limit is actually $\lambda_1(Hx_1)$.

7. Minimal partitions and Courant-sharp

The main object of this section is the proof of Theorem 1.17.

7.1. Preliminaries

The proof in this section uses essentially [18,19] (or easy extensions[4] of it). The minimal partitions which are involved in the proof are indeed regular. Although not very important here, it seems useful to mention this for possible extensions to higher dimensions where we do not have the fine results established in Section 4.

7.2. Definition of an exhausting family $N(u,\alpha)$

Let u be an eigenfunction of $H(\Omega)$ with k nodal domains and consider $N(u) \in \mathcal{M}(\Omega)$. First we consider the finite sets of points

$$C^*(N) := \mathcal{Z}_3 \cup \big(N(u) \cap \partial\Omega\big). \tag{7.1}$$

From each of these points an arc emanates which ends either in the point itself (irregular loop) or ends in another point in $C^*(N)$. We call the collection of these arcs $\mathcal{A}_*$. Then we consider those components of $N(u)$ whose intersection with $C^*(N)$ is empty. They have to be pairwise disjoint embedded circles (without selfintersections) and we call the collection of these circles $\mathcal{A}_{**}$. Let us introduce

$$\mathcal{A} = \mathcal{A}_* \cup \mathcal{A}_{**}.$$

Note that each arc (or loop) $A \in \mathcal{A}$ is rectifiable (because $N(u_m)$ is regular by Theorem 2.3) and we can associate to A naturally a middle point x_A in the natural way (x_A is chosen arbitrarily if $A \in \mathcal{A}_{**}$). We have a natural arc length parametrization starting from the point x_A, but we prefer to parametrize A as a parametrized curve $[-1,+1] \ni t \mapsto L(A,t)$ such that

$$L(A,0) = x_A, \quad L(A,-1) = y_A^-, \quad L(A,1) = y_A^+,$$

where y_{A^-} and y_{A^+} are the end points in the case of an arc, and where $y_{A^-} = y_{A^+}$ is the irregular point in the case of an irregular loop (i.e. in $\mathcal{A}_*$) and the opposite point in the case of a regular loop (i.e. in $\mathcal{A}_{**}$).

For each $\alpha \in (0,1)$, we can consider the set

$$N(u,\alpha) = \big\{N(u) \setminus L\big(A,(-1+\alpha, 1-\alpha)\big)\big\} \tag{7.2}$$

and complete the definition by

$$N(u,0) = \emptyset \quad \text{and} \quad N(u,1) = N(u). \tag{7.3}$$

Note that by construction for every $0 < \alpha$, $N(u,\alpha)$ contains all the critical points and $N(u) \cap \partial\Omega$; this will be important below.

7.3. Proof of Theorem 1.17

We assume for contradiction that for some k, $\mathfrak{L}_k = L_k$, but that $\lambda_k < \lambda_m = L_k$ for some $m > k$.

Taking the smallest m with this property, we can in addition assume that

$$\lambda_{m-1} < \lambda_m. \tag{7.4}$$

Let u_m be a normalized eigenfunction such that u_m has $\mu(u_m) = k$ nodal domains D_i $(i = 1, 2, \ldots, k)$. Then we associate with the exhausting family $N(u_m,\alpha)$ the decreasing family of open sets:

$$\Omega(\alpha) = \Omega \setminus N(u_m,\alpha). \tag{7.5}$$

[4] Except that these papers assume that Ω is simply connected.

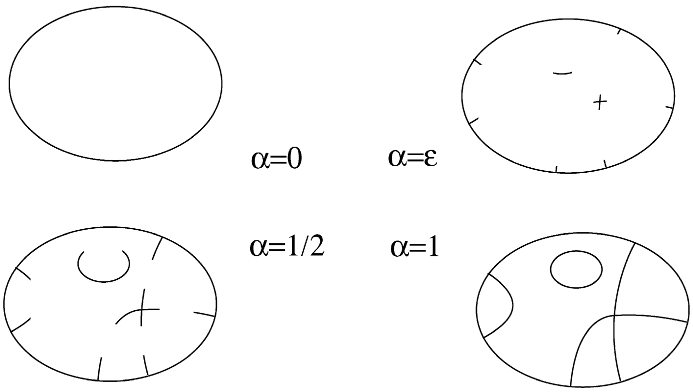

Fig. 2. An example of exhausting family.

We want to consider the spectrum of

$$H(\alpha) := H\big(\Omega(\alpha)\big). \tag{7.6}$$

(We suppress the dependence on Ω and V.)

Then, $H(0)$ is our initial operator $H(\Omega)$ and

$$H(1) = \bigoplus_{i=1}^{k} H(D_i). \tag{7.7}$$

Hence $H(1)$ has as lowest eigenvalue λ_m with multiplicity k. By construction $\sigma(H(0)) = \sigma(H)$. Furthermore $\lambda_1(H(1))$ has multiplicity k and

$$\lambda_{k+1}\big(H(1)\big) > \lambda_k\big(H(1)\big) = \lambda_m\big(H(0)\big).$$

Lemma 7.1. *For any ℓ, $\lambda_\ell(H(\alpha))$ is monotonically increasing with α.*

Proof. We just note that the form domains $\mathcal{Q}(\alpha)$ of the quadratic forms $q(\alpha)$ associated to $H(\alpha)$, $0 \leqslant \alpha \leqslant 1$, satisfy $\mathcal{Q}(\alpha) \subset \mathcal{Q}(\alpha')$ for $\alpha' \leqslant \alpha$. □

Lemma 7.2. *For any ℓ, $\lambda_\ell(H(\alpha))$ depends continuously upon α.*

Proof. Although there is a lot of literature[5] on the subject (see [24,29,30,34–36,31]), it is difficult to give a reference corresponding to this crack situation. This statement is proved in Dauge and Helffer [13] (at least for the case of one crack). These authors treat the case when the boundary condition on the crack is Neumann. The Dirichlet case does not create new problems (the important point being the monotonicity which is evident in the case of Dirichlet). Note that we have strong resolvent convergence and that the left continuity and the right continuity should be treated separately. □

We continue with

Lemma 7.3. *For each $\alpha \in [0, 1]$, $\lambda_m \in \sigma(H(\alpha))$.*

Proof. By construction of $N(u_m, \alpha)$, the restriction of u_m to $\Omega(\alpha)$ is indeed an eigenfunction of $H(\alpha)$. □

We recall, see the two first lines of this subsection, that we are inside a proof by contradiction and consider first the following case.

[5] We thank P. Stollmann and M. Dauge for useful discussions.

Case (a): λ_m is simple.

Lemma 7.4. *There is a minimal $\alpha_1 \in (0,1)$ such that*

$$\lambda_1(\alpha_1) < \lambda_2(\alpha_1) \leqslant \cdots \leqslant \lambda_{m-1}(\alpha_1) = \lambda_m(\alpha_1), \tag{7.8}$$

with $\lambda_m(\alpha_1) = \lambda_m$.

Proof. By assumption $m > k$ and $H(0)$ has m eigenvalues smaller or equal to λ_m whereas $H(1)$ has just k eigenvalues smaller or equal to λ_m, hence less. By continuity with respect to α at least one eigenvalue has to become larger than λ_m. □

Next we consider this eigenvalue of $H(\alpha_1)$. The restriction of u_m to $\Omega(\alpha_1)$ gives a first eigenfunction but there exists a second real valued normalized eigenfunction v of $H(\alpha_1)$ such that v is orthogonal to u_m in $L^2(\Omega(\alpha_1))$:

$$\langle u_m | v\rangle = 0, \tag{7.9}$$

where $\langle\cdot|\cdot\rangle$ denotes the L^2-scalar product.

We now play inside the two-dimensional eigenspace spanned by u_m and v.

Lemma 7.5. *There exists $\beta_0 > 0$, such that $\forall \beta \in (-\beta_0, +\beta_0)$ the function $w_\beta = u_m + \beta v$ has exactly k nodal domains.*

Furthermore, the family of the nodal domains of w_β gives, for $\beta \neq 0$, a minimal bipartite partition of Ω, which is distinct of the partition associated to $N(u_m)$.

Proof. We recall that this construction is done for $\alpha = \alpha_1 \in (0,1)$. For each $A \in \mathcal{A}$, let

$$I_A := L\big(A, (-1+\alpha_1, 1-\alpha_1)\big),$$

and let $\mathcal{V}_A \subset \Omega$ be an open neighborhood of $\overline{I_A}$, whose regular boundary crosses $N(u_m)$ twice (transversally) and such that each component of the open set $\Omega(\alpha_1) \setminus \bigcup_{A\in\mathcal{A}} \overline{\mathcal{V}_A}$ (which is contained in $\Omega(1)$), is contained in a unique nodal domain of u_m.

We assume that we have colored these nodal domains by $+$ or $-$, and this permits us to write, for each $A \in \mathcal{A}$, the decomposition

$$\partial\mathcal{V}_A \cap \Omega(\alpha_1) = b_A^+ \cup b_A^-,$$

where $b_A^\pm$ is contained in a positive or negative nodal domain of u_m.

The first claim is now that there exists β_0 such that if we add βv, with $|\beta| \leqslant \beta_0$, the number of nodal domains of $u_m + \beta v$ can only increase. Using Hopf's boundary point lemma for $u = u_m$, [15], we have $|\nabla u_m(x)| > 0$ for $x \in (N(u_m) \setminus N(u_m, \alpha_1)) \cap \Omega$ and using the property that v vanishes at the boundary of $\Omega(\alpha)$, we obtain the existence of β_0 such that $u_m + \beta v$ is strictly positive on each b_A^+ and strictly negative on b_A^-. It is then clear that associated to each positive D_i, there is at least one nodal domain of $u_m + \beta v$, with non-trivial intersection with D_i and contained in $D_i \cup (\cup_A \mathcal{V}_A)$. All these nodal domains are necessarily disjoint and this proves the first claim.

Let us now show that we cannot increase the number of nodal domains. If it was the case, this would give an upper bound for $\mathfrak{L}_{k+1}$ and using (5.1), we would obtain, using the strict monotonicity of the sequence $\mathfrak{L}_\ell$ (see (5.1)) with respect to ℓ, $\lambda_m = \mathfrak{L}_k < \mathfrak{L}_{k+1} \leqslant \lambda_m$, hence a contradiction.

So w_β has also exactly k nodal domains corresponding also to a minimal k-partition $\mathcal{D}' \in \mathfrak{O}_k$ of Ω. But $\mathcal{D}' \neq \mathcal{D}$ since both functions, u_m and v (and hence w_β for $\beta \neq 0$) are linearly independent. In addition, one can verify that $G(\mathcal{D}')$ is bipartite. □

Now we can complete the proof of Theorem 1.17 for *case* (a). Indeed by Theorem 1.14, w_β (more precisely the natural extension of w_β to Ω) is an eigenfunction of $H = H(0)$ and therefore this would imply that λ_m has multiplicity at least two, contradicting our assumption that λ_m is simple.

Remark 7.6. Note that for these proofs we only need weak versions of our results because we work only with strong regular partitions (satisfying in addition the equal angle meeting property). So the techniques of [18] (as recalled in Appendix B) are also relevant.

In the non-simply connected case, if one wants to apply [18], one should also verify a global compatibility condition for each homotopy class of Ω. Because w_β is an eigenfunction in $\Omega(\alpha_1)$, this is a consequence of the property that any path in Ω is homotopic to a path in $\Omega(\alpha_1)$. It is then easy to verify this additional cycle-compatibility condition introduced in [18], because w_β is an eigenfunction of $H(\alpha_1)$.

Case (b): λ_m has multiplicity greater than one.

Assume that λ_m has multiplicity $\ell > 1$, so that the $m+\ell$ first eigenvalues of $H = H(0)$ satisfy:

$$\lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \lambda_{m-1} < \lambda_m = \cdots = \lambda_{m+\ell-1} < \lambda_{m+\ell}. \tag{7.10}$$

Of course, all the previous constructions can be done but arriving at the last line of case (a), we loose the contradiction. The idea is that we have to choose our w more carefully.

λ_m being an eigenvalue of $H(\alpha)$ for any α, we can associate to λ_m the eigenspace $U(\alpha,\lambda_m)$ of $H(\alpha)$. Inside $U(\alpha,\lambda_m)$, we then introduce the subspace $\underline{U}(\alpha,\lambda_m)$ consisting of functions which are restrictions to $\Omega(\alpha)$ of eigenfunctions of $H(\Omega)$. Of course, $\underline{U}(\alpha,\lambda_m)$ contains u_m but could be larger.

We then need to show the following lemma.

Lemma 7.7. *For $\alpha=\alpha_1$, the inclusion of $\underline{U}(\alpha_1,\lambda_m)$ in $U(\alpha_1,\lambda_m)$ is strict.*

Proof. For any $\alpha<\alpha_1$, we can choose a normalized v_α in $U(\alpha,\lambda_{m-1}(\alpha))$. Then it is clear, observing that $\lambda_{m-1}(\alpha) < \lambda_m$, that

- v_α is orthogonal in $L^2(\Omega(\alpha))$ to $\underline{U}(\alpha_1,\lambda_m)$ which is (more precisely, can be identified to) a subspace of $U(\alpha,\lambda_m)$ for any $\alpha<\alpha_1$.
- v_α is bounded independently of α in $W^{1,2}(\Omega(\alpha_1))$.

Then we can by compactness, find a sequence $w_{\alpha(n)}$ such that $\alpha(n)$ tends to α_1 as $n\to+\infty$ and $v_{\alpha(n)}$ converges weakly to some v_{α_1} in $W^{1,2}(\Omega(\alpha_1))$ and strongly in $W^{1,s}(\Omega(\alpha_1))$ for $s<1$ by compactness.

Now, it is clear that

- v_{α_1} is orthogonal to $\underline{U}(\alpha_1,\lambda_m)$.
- $\|v_{\alpha_1}\| = 1$.
- $(-\Delta+V)v_{\alpha_1} = \lambda_m v_{\alpha_1}$ in $\Omega(\alpha_1)$.

With a small additional work, one can show that $v_{\alpha_1}\in W_0^{1,2}(\Omega(\alpha_1))$. So v_{α_1} is effectively in the form domain for the Dirichlet problem in $\Omega(\alpha_1)$ and in $U(\alpha_1,\lambda_m)\cap \underline{U}(\alpha_1,\lambda_m)^\perp$. □

End of the proof of case (b).

The argument is then as in case (a), but, using Lemma 7.7, we can choose a non-trivial v in $U(\alpha_1,\lambda_m)\setminus\underline{U}(\alpha_1,\lambda_m)$. But on one hand w_β cannot belong to $\underline{U}(\alpha_1,\lambda_m)$ (because v does not). On the other hand, we have obtained some $\beta\neq 0$ such that w_β extends as an eigenfunction of $H(\Omega)$ hence by definition in $\underline{U}(\alpha_1,\lambda_m)$ and this gives the contradiction.

Theorem 1.17 has an immediate consequence.

Corollary 7.8. *Let Ω satisfy Assumption* 1.1 *and assume that $V\in L^\infty(\Omega)$. Then*

$$\left|\{k \mid \mathfrak{L}_k = L_k\}\right| < \infty. \tag{7.11}$$

Proof. This is an immediate consequence of Theorem 1.17 and of Pleijel's Theorem 6.3. □

Remark 7.9. It is now easier to analyze the situation for the disk and for rectangles (at least in the irrational case), since we have just to check for which eigenvalues one can find associated Courant-sharp eigenfunctions. This will be done in Sections 9 and 10.

8. Further properties of subpartitions

All the statements of this section illustrate the rigidity of the structure of the subpartitions. This can be very efficient for disproving that a partition is minimal. In particular we will prove Theorem 1.13.

The following proposition is useful:

Proposition 8.1. *Under Assumptions* 1.1 *and* 1.2, *let* $\mathcal{D} = (D_i)_{i\in\{1,\dots,k\}}$ *be a minimal k-partition for* $\mathfrak{L}_k(\Omega)$. *Then, for any subset* $I \in \{1,\dots,k\}$, *the associated subpartition* $\mathcal{D}^I = (D_i)_{i\in I}$ *satisfies*

$$\mathfrak{L}_k = \Lambda(\mathcal{D}^I) = \mathfrak{L}_{|I|}(\Omega^I), \tag{8.1}$$

where

$$\Omega^I := \mathrm{Int}\Big(\overline{\bigcup_{i\in I} D_i}\Big).$$

Proof. We prove this proposition by contradiction. If it was not the case, we would construct (starting of a minimal $|I|$-partition of Ω^I) a new minimal partition $\widetilde{D}$ of Ω, for which the $\lambda(\widetilde{D}_i)$'s are not equal in contradiction with what we proved in Section 4 (see also (d) in Remark 3.7 together with the fact that $k_0 = \emptyset$ in our case). □

As a consequence of a more general theorem in [4], we have (with very weak assumptions on Ω) the analogous of Proposition 8.1

Proposition 8.2. *Suppose u is Courant-sharp. Denote the associated nodal domains by* $\{D_i\}_1^k$. *Let L be a subset of* $\{1,2,\dots,k\}$ *with* $\#L = \ell < k$ *and let* $\Omega^L = \mathrm{Int}(\overline{\bigcup_{i\in L} D_i}) \setminus \partial\Omega$. *Then*

$$\lambda_\ell(\Omega^L) = \lambda_k \tag{8.2}$$

where $\lambda_j(\Omega^L)$ *are the eigenvalues of* $H(\Omega^L)$.

Moreover, if Ω^L *is connected,* $u|_{\Omega^L}$ *is Courant-sharp and* $\lambda_\ell(\Omega^L)$ *is simple.*

Proposition 8.3. *Under the assumptions of the introduction on* Ω *and* V, *let* $\mathcal{D} = (D_1,\dots,D_k)$ *be a minimizing partition associated to* $\mathfrak{L}_k$. *Let* $\mathcal{D}' \subset \mathcal{D}$ *be any subpartition into* $1 \leqslant k' \leqslant k$ *elements which is bipartite relatively to*

$$\Omega' := \mathrm{Int}\Big(\bigcup_{D_i\in\mathcal{D}'} \overline{D_i}\Big).$$

Then

(a) $\mathfrak{L}_k = \lambda_{k'}(\Omega')$;
(b) *If* $k' < k$, *and* Ω' *is connected then* $\lambda_{k'}(\Omega')$ *is simple.*

Proof. The point (a) is shown like Theorem 1.14 at the end of Section 4. The proof of (b) is an immediate consequence of Proposition 8.2, if we add the assumption that there exists a bipartite subpartition $\mathcal{D}''$ such that $\mathcal{D}' \subset \mathcal{D}'' \subset \mathcal{D}$ with $k' < k'' \leqslant k$.

The general proof is a little more tricky. Given the minimal partition $\mathcal{D}$ of Ω, and the subpartition $\mathcal{D}'$ (with associated bipartite graph), there is a subfamily $\mathcal{A}_0$ of the set $\mathcal{A}$ of arcs, which are supposed to be closed (see the discussion in Section 7) with the following properties

- $\widetilde{\Omega} := \Omega \setminus \mathcal{A}_0$ has for the same partition a bipartite graph relatively to $\widetilde{\Omega}$;
- $\mathcal{A}_0$ does not contain arcs belonging to the intersection of the boundaries of two neighbors of $\mathcal{D}'$ in Ω'.

But we are in a Courant-sharp situation, so

$$\mathfrak{L}_k(\Omega) = \mathfrak{L}_k(\widetilde{\Omega}) = \lambda_k(\widetilde{\Omega}), \tag{8.3}$$

and we can apply the Courant-sharp theorem relatively to $\widetilde{\Omega}$ and the partition $\mathcal{D}$ (which is now the nodal family associated to an eigenfunction on $\widetilde{\Omega}$). □

Remark 8.4. In the preceding proof, $\widetilde{\Omega}$ is not unique, but it is interesting to emphasize that what we have shown is that, once a minimal k-partition of Ω is given, then all the possible $\widetilde{\Omega}$'s should share the property (8.3).

In order to complete the proof of Theorem 1.13, it remains to establish our uniqueness result at the level of the subpartitions of a minimal partition.

Proposition 8.5 *(Uniqueness). Let $\mathcal{D}$ be a minimal k-partition relative to $\mathfrak{L}_k(\Omega)$. Let $\mathcal{D}' \subset \mathcal{D}$ be any subpartition of $\mathcal{D}$ into $1 \leqslant k' < k$ elements and let*

$$\Omega' = \mathrm{Int}\left(\bigcup_{D_i \in \mathcal{D}'} \overline{D_i}\right),$$

be connected. Then $\mathfrak{L}_{k'}(\Omega')$ is uniquely achieved.

We know already from Proposition 8.1 that $\mathfrak{L}_{k'}(\Omega') = \mathfrak{L}_k(\Omega)$. The proof of uniqueness is by contradiction. Let I and J two subsets of $\{1, \dots, k\}$ such that $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, k\}$.

If we have indeed two minimal subpartitions of Ω_I for some I of cardinality strictly less than k, we can complete them by the open sets D_j $(j \in J)$. Now take a pair (D_i, D_j), with $i \sim j$ $(i \in I$ and $j \in J)$. This should exist if Ω is connected and Ω_I is connected. There is necessarily another $\widetilde{D}_{\tilde{i}}$ of the other partition which meets D_i and is a neighbor of D_j. But the eigenfunction u_{ij} of D_{ij} and $\tilde{u}_{ij}$ of $\widetilde{D}_{ij}$ should be proportional on D_j to u_j. It is then clear that by unique continuation D_i should coincide with $\widetilde{D}_i$ and u_i should be proportional to $\tilde{u}_i$. Possibly iterating the argument, we arrive to a contradiction when the two partitions are different.

9. Example 1: The case of the disk

In this section, we analyze the case of the disk. Although the spectrum is explicitly computable, we are mainly interested in the ordering of the eigenvalues corresponding to different angular momenta. In particular this will give a first example where the inequalities in (5.5) can be strict.

Consider the Dirichlet realization H_0 in the unit disk $B_1 \subset \mathbb{R}^2$. We have in polar coordinates:

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

and the Dirichlet boundary conditions require that any eigenfunction u satisfies $u(r,\theta) = 0$ for $r = 1$. We analyze for any $\ell \in \mathbb{N}$ the eigenvalues $\lambda_{\ell,j}$ of

$$\left(-\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{\ell^2}{r^2}\right) f_{\ell,j} = \lambda_{\ell,j} f_{\ell,j}, \quad \text{in } (0,1).$$

We observe that the operator is self adjoint for the scalar product in $L^2((0,1), r\,dr)$.

The corresponding eigenfunctions of the eigenvalue problem take the form

$$u(r,\theta) = f_{\ell,j}(r)(a\cos\ell\theta + b\sin\ell\theta), \quad \text{with } a^2 + b^2 > 0, \tag{9.1}$$

where the $f_{\ell,j}(r)$ are suitable Bessel functions satisfying for $\ell = 0$, $f'_{0,j}(0) = 0$ and $f_{0,j}(1) = 0$ and for $\ell > 0$, $f_{\ell,j}(0) = f_{\ell,j}(1) = 0$. For the corresponding $\lambda_{\ell,j}$'s, we find (see in Appendix A) the following ordering.

$$\begin{aligned} &\lambda_1 = \lambda_{0,1} < \lambda_2 = \lambda_3 = \lambda_{1,1} < \lambda_4 = \lambda_5 = \lambda_{2,1} < \lambda_6 = \lambda_{0,2} < \lambda_7 = \lambda_8 = \lambda_{3,1} < \cdots \\ &\quad \cdots < \lambda_9 = \lambda_{10} = \lambda_{1,2} < \lambda_{11} = \lambda_{12} = \lambda_{4,1} < \cdots \\ &\quad \cdots < \lambda_{13} = \lambda_{14} = \lambda_{2,2} < \lambda_{15} = \lambda_{0,3} < \cdots. \end{aligned} \tag{9.2}$$

We recall that the zeros $j_{\ell,k}$ of the Bessel functions are related to the eigenvalues by the relation

$$\lambda_{\ell,k} = (j_{\ell,k})^2. \tag{9.3}$$

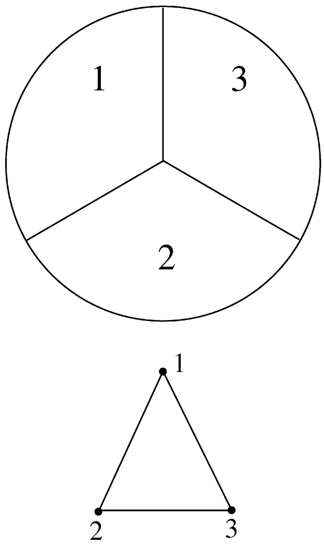

Fig. 3. The candidate for the 3-minimal partition of the disk and associated graph.

We hence have from (9.1)

$$\begin{aligned}
&\mu(u_1) = 1,\\
&\mu(u) = 2, \text{ for any eigenfunction } u \text{ associated to } \lambda_2 = \lambda_3,\\
&\mu(u) = 4, \text{ for any eigenfunction } u \text{ associated to } \lambda_4 = \lambda_5,\\
&\mu(u_6) = 2,\\
&\mu(u) = 6, \text{ for any eigenfunction } u \text{ associated to } \lambda_7 = \lambda_8,\\
&\mu(u) = 4, \text{ for any eigenfunction } u \text{ associated to } \lambda_9 = \lambda_{10},\\
&\mu(u) = 8, \text{ for any eigenfunction } u \text{ associated to } \lambda_{11} = \lambda_{12},\\
&\mu(u) = 8, \text{ for any eigenfunction } u \text{ associated to } \lambda_{13} = \lambda_{14},\\
&\mu(u_{15}) = 3.
\end{aligned} \tag{9.4}$$

Hence

$$L_1 = \lambda_1, \quad L_2 = \lambda_2, \quad L_3 = \lambda_{15}, \quad L_4 = \lambda_4 \tag{9.5}$$

and this implies

$$L_3 > \lambda_6 > L_4 > \lambda_3. \tag{9.6}$$

In addition, let us show that

$$L_3 > \mathfrak{L}_3. \tag{9.7}$$

This can be seen as follows. We can split B_1 in three sectors with opening angle $2\pi/3$. Call such a sector $S_{1/3}$ then the corresponding eigenvalue $\lambda(S_{\frac{1}{3}})$ (whose approximate eigenvalue can be recovered from (A.5) with $\ell = \frac{1}{2}$, $j = 1$ and from (9.3)) satisfies:

$$\mathfrak{L}_3 \leqslant \lambda(S_{\frac{1}{3}}).$$

We then observe that by monotonicity that:

$$\lambda(S_{\frac{1}{3}}) < \lambda(S_{\frac{1}{4}})$$

and we can recognize that $\lambda(S_{\frac{1}{4}}) = \lambda_4$ observing that $\lambda_4 = \lambda_{\ell,j}$ with $\ell = 2$ and $j = 1$ (see (9.1)). The proof of (9.7) is then a consequence of (9.6).

Remark 9.1. An open problem is to prove or disprove the equality $\mathfrak{L}_3 = \lambda(S_{\frac{1}{3}})$.

A possibility for trying to find other configurations could be to look at the second eigenvalue of a half disk problem $\{(x_1^2 + x_2^2) < 1\} \cap \{x_2 > 0\}$, where we take Dirichlet on the circular part and a part of the straight basis and Neumann on the other part. More precisely, we take Neumann on $x_2 = 0$, $|x_1| < t$, where t is a free parameter. The nodal domains of the second eigenfunction (completed by symmetry) could give an alternative candidate for such a minimal partition.

In the case of the disk, we have

Proposition 9.2. *Except the cases $k = 1$, 2 and 4, minimal partitions never correspond to nodal domains.*

Proof. According to Theorem 1.17, it is enough to investigate when the k-th eigenvalue corresponds to Courant-sharp eigenfunctions. □

One can in addition use the twisting trick (as done in [20]) for eliminating all the eigenvalues $\lambda_{\ell,m}$, for which $m \geqslant 2$ and $\ell > 0$. This trick goes roughly as follows. When $\ell > 0$, we can divide the disk as the union of a smaller disk and of its complementary, each of these sets being the union of at least two nodal domains. Then by small rotation of the small disk, we obtain a new partition which has the same energy. If the initial one was minimal, the new one should be also minimal, but it is easy to show that the new one has not the "equal angle meeting" property of a regular partition. This gives the contradiction.

So we have finally to analyze the eigenvalues $\lambda_{0,k}$ and the family $\lambda_{\ell,1}$.

For the first family, we observe that $\lambda_{0,k}$ can neither be the k-th eigenvalue as soon as $k \geqslant 2$.

For the second family, which occurs only for $k = 2\ell$ even, inspection of the tables leads to the condition $k \leqslant 4$, we observe indeed that $\lambda_{0,2} < \lambda_{3,1}$.

We also observe that when k is odd, we obtain that necessarily $L_k = \lambda_{0,k}$.

Remark 9.3. It could be interesting to determine when $\mathfrak{L}_k$ is an eigenvalue of the Laplacian on the double covering. Again, one can show (as done in [20]) that this cannot be the case for k large.

10. Example 2: the case of the rectangle

Note that for the case of a rectangle, the spectrum and the properties of the eigenfunctions are analyzed as toy models in [25], Section 4. This was also used for testing general conjectures in [4].

For a rectangle of sizes a and b, the spectrum is given by $\pi^2(m^2/a^2 + n^2/b^2)$ $((m,n) \in (\mathbb{N}^*)^2)$.

The first remark is that all the eigenvalues are simple if $\frac{a^2}{b^2}$ is irrational. Except for specific remarks for the square, we now assume

$$(a/b)^2 \text{ is irrational.}$$

So we can associate to each eigenvalue $\lambda_{m,n}$, an (essentially) unique eigenfunction $u_{m,n}$ such that $\mu(u_{m,n}) = nm$. Given $k \in \mathbb{N}^*$, the lowest eigenvalue corresponding to k nodal domains is given by

$$L_k = \pi^2 \inf_{mn=k} (m^2/a^2 + n^2/b^2).$$

The behavior of L_k can depend dramatically of the arithmetical properties of k but what is important for us is that in any case we have

$$L_k \geqslant 2\pi^2 k/(ab). \tag{10.1}$$

This immediately implies

$$\liminf_{k \to +\infty} \frac{L_k}{k} \geqslant \frac{2\pi^2}{ab}. \tag{10.2}$$

We note that the right-hand side can also be written in the form $\lambda([0,1]^2)/|\Omega|$.

Remark 10.1. Note that these estimates are much better than the estimates obtained by Faber–Krahn inequality (see in Section 6).

As we have seem in (6.7) and using[6] the comparison between the lowest eigenvalue of the hexagon and of the square,

$$\lambda(Hx_1) \sim 18.59013 < \lambda\big([0,1]^2\big) = 2\pi^2 \sim 19.7392, \tag{10.3}$$

this will imply

$$\liminf_{k\to+\infty} \frac{L_k}{k} > \limsup_{k\to+\infty} \frac{\mathfrak{L}_k}{k}. \tag{10.4}$$

This implies that $L_k > \mathfrak{L}_k$ for k large.

Remark 10.2. In the case when $(\frac{a}{b})^2$ is rational we could have problems in the case of multiplicities. We have then to control the nodal sets of the eigenfunctions corresponding to the degenerate eigenvalues which are $\leqslant \mathfrak{L}_k$.

We now describe all the possible situations.

Lemma 10.3. *In the irrational case, $\lambda_{m,n}$ cannot lead to a Courant-sharp situation if* $\inf(m,n) \geqslant 3$.

Proof. Applying Proposition 8.2, it is sufficient to analyze the case when $m = n = 3$. It is then enough to show that $\lambda_{3,3}$ cannot be the ninth eigenvalue.

Because the eigenvalues corresponding to $\max(m,n) \leqslant 3$ are obviously below $\lambda_{3,3}$, let us assume by contradiction that $\lambda_{3,3} \leqslant \lambda_{1,4}$ and that $\lambda_{3,3} \leqslant \lambda_{4,1}$.

This reads

$$\frac{9}{a^2} + \frac{9}{b^2} \leqslant \frac{1}{a^2} + \frac{16}{b^2},$$

and

$$\frac{9}{a^2} + \frac{9}{b^2} \leqslant \frac{1}{a^2} + \frac{16}{b^2}.$$

So, we obtain

$$\frac{8}{7} \leqslant \frac{a^2}{b^2} \leqslant \frac{7}{8},$$

hence a contradiction. □

The next step is given in

Lemma 10.4. *In the irrational case, $\lambda_{m,n}$ cannot lead to a Courant-sharp situation if $m = 2$ and $n \geqslant 4$ or if $m \geqslant 4$ and $n = 2$.*

Again it is enough to look at the case $(m = 2, n = 4)$, and to show that it cannot be the eighth eigenvalue. Using the same idea as in the previous lemma, we assume by contradiction that $\lambda_{2,4} \leqslant \lambda_{1,5}$ and that $\lambda_{2,4} \leqslant \lambda_{3,1}$.

This reads

$$\frac{4}{a^2} + \frac{16}{b^2} \leqslant \frac{1}{a^2} + \frac{25}{b^2},$$

[6] We thank V. Bonnaillie-Noël and G. Vial for giving us a precise numerical approximation of $\lambda(Hx_1)$. According to A. El Soufi, it seems unknown that the hexagon gives the minimal eigenvalue between all the polygons (of same area) permitting to realize a perfect partition of the plane. We just compare here the square and the hexagon.

and

$$\frac{4}{a^2}+\frac{16}{b^2}\leqslant\frac{9}{a^2}+\frac{1}{b^2}.$$

So, we obtain

$$\frac{1}{3}\leqslant\frac{a^2}{b^2}\leqslant\frac{1}{3}.$$

But this gives $\frac{a^2}{b^2}=\frac{1}{3}$, which is excluded by the assumption that $\frac{a^2}{b^2}$ is irrational.

Let us now analyze *the Courant-sharp property for the remaining cases.*

In the case $m=2$, $n=3$, an eigenfunction corresponding to $\lambda_{m,n}$ is Courant-sharp if

$$\frac{4}{a^2}+\frac{9}{b^2}\leqslant\frac{9}{a^2}+\frac{1}{b^2},$$

and

$$\frac{4}{a^2}+\frac{9}{b^2}\leqslant\frac{1}{a^2}+\frac{16}{b^2}.$$

So, we obtain

$$\frac{8}{5}\leqslant\frac{a^2}{b^2}\leqslant\frac{5}{3}.$$

The case $m=3$, $n=2$, is obtained by exchanging the role of a and b. So, we obtain

$$\frac{8}{5}\leqslant\frac{b^2}{a^2}\leqslant\frac{5}{3}.$$

In the case $m=2$, $n=2$, we obtain similarly

$$\frac{3}{5}\leqslant\frac{a^2}{b^2}\leqslant\frac{5}{3}.$$

For the case $m=1$, $n=k$, we obtain

$$\frac{1}{a^2}+\frac{k^2}{b^2}\leqslant\frac{4}{a^2}+\frac{1}{b^2}.$$

So, we obtain simply

$$\frac{k^2-1}{3}<\frac{a^2}{b^2}.$$

Finally, *the case* $m=k$, $n=1$ leads to the condition

$$\frac{k^2-1}{3}<\frac{b^2}{a^2}.$$

A candidate for the 3*-minimal partition on the square*.

In the case of the square $(-\frac{1}{2},\frac{1}{2})\times(-\frac{1}{2},\frac{1}{2})$, an argument similar to the case of the disk shows that $\mathfrak{L}_3$ (which should be smaller than $\mathfrak{L}_4$) is strictly less than L_3. We observe indeed that λ_4 is Courant-sharp, so $\mathfrak{L}_4=\lambda_4$, and there is no eigenfunction corresponding to $\lambda_2=\lambda_3$ with three nodal domains (by Courant's theorem). Assuming that there is a minimal partition which is symmetric with $\{y=0\}$, and intersecting the partition with the half-square $(-\frac{1}{2},\frac{1}{2})\times(0,\frac{1}{2})$, one is reduced to analyze a family of Dirichlet–Neumann problems. Numerical computations[7] performed by V. Bonnaillie-Noël (in January 2006) and G. Vial lead to a natural candidate (see Fig. 4) for a symmetric minimal partition.

The complete structure is recovered from the half-square by symmetry with respect to the horizontal axis. We observe numerically that the three lines of $N(\mathcal{D})$ meet at the center $(0,0)$ of the square. As expected by the theory they meet at $(0,0)$ with equal angle $\frac{2\pi}{3}$ and start from the boundary orthogonally.

[7] See http://www.bretagne.ens-cachan.fr/math/simulations/MinimalPartitions/.

Fig. 4. The candidate for the minimal 3-partition of the square (upper part).

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Appendix A. Zeros of Bessel functions

Let $j_{\ell,k}$ the k-th zero of the Bessel function corresponding to the integer $\ell \in \mathbb{N}$. The reference is the book by G.N. Watson [33]. The most important statement for us is that $j_{\ell,k} = j_{\ell',k'}$ imply if ℓ and ℓ' are positive integers, that $\ell = \ell'$ and $k = k'$. We refer to the subsection 15.28 (pp. 484–485) in [33]. Note that the proof of this result is based on deep results by Siegel about algebraic numbers.

Here is a list of approximate values after the handbook of [1], p. 409, we keep only the values which are less than approximately 13.

	$\ell =$ 0	1	2	3	4	5	6	7	8
$k = 1$	2.40	3.83	5.14	6.38	7.59	8.77	9.93	11.08	12.22
2	5.52	7.02	8.42	9.76	11.06	12.34	.		
3	8.65	10.17	11.62	13.02	..				
4	11.79	13.32	.						

(A.1)

This leads to the following ordering of the zeros:

$$\begin{aligned} &j_{0,1} < j_{1,1} < j_{2,1} < j_{0,2} < j_{3,1} < j_{1,2} < j_{4,1} < j_{2,2} < j_{0,3} < \cdots \\ &\quad \cdots < j_{5,1} < j_{3,2} < j_{6,1} < j_{1,3} < j_{7,1} < j_{2,3} < j_{0,4} < j_{8,1}. \end{aligned} \tag{A.2}$$

Note that, using Sturm–Liouville theory (see for example the proof in [33]), the following inequalities are always true:

$$j_{\ell,k} < j_{\ell+1,k} < j_{\ell,k+1}, \quad \forall \ell \in \mathbb{R}^+,\ \forall k \in \mathbb{N}^*. \tag{A.3}$$

As a corollary, we obtain

$$j_{\ell,k} \leqslant j_{0,k+\ell}, \tag{A.4}$$

with strict inequality for $\ell > 0$.

It is also useful, for the analysis of the problem for the double covering of the disk, to have the half integer results (see in [1], p. 467).

$$\begin{array}{cccccccc} & \ell = \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \frac{11}{2} & \frac{13}{2} \\ k=1 & \pi & 4.49 & 5.76 & 6.99 & 8.18 & 9.36 & 10.51 \\ 2 & 2\pi & 7.73 & 9.09 & 10.42 & 11.7 & & \\ 3 & 3\pi & 10.9 & 12.52 & 13.69 & .. & & \\ 4 & 4\pi & 14.06 & & & & & \end{array} \tag{A.5}$$

Appendix B. Alternative approach in the regular case

Although not needed in this article, we recall some of the results of statements of [18,19]. The main theorem is the following

Theorem B.1. *Suppose that Ω is regular and simply connected and that, for some regular closed set N satisfying in addition the "equal angle meeting condition". Suppose that, for some $\lambda \in \mathbb{R}$, the associated family $\mathcal{D}(N) = \{D_1, \ldots, D_\mu\}$ is admissible and satisfies a Pair Compatibility Condition, which means that λ is an eigenvalue of $H(D_{i,j})$ for which D_i and D_j are the two nodal domains of some corresponding eigenfunction. Then there is an eigenfunction u of $H(\Omega)$ with corresponding eigenvalue λ such that the family of nodal domains of u is $\mathcal{D}(N)$.*

Remarks B.2.

- If Ω is not simply connected then the result does not hold in general. One should add a non-holonomy condition (see [19]). In the case of minimal partitions, we have seen that this condition is reduced to the bipartite condition.
- Note that the Pair Compatibility Condition is weaker than to assume that λ is the second eigenvalue of $H(D_{i,j})$ for each pair of neighbors (D_i, D_j).

As an application of this theorem, the authors obtain:

Corollary B.3. *Let Ω be simply connected, $k \in \mathbb{N}$ $(k \geqslant 2)$ and let $\mathcal{D}^{\min} = (D_i)_{i=1,\ldots,k}$ be a minimal admissible strong regular*[8] *partition. Then there is an eigenfunction u of $H(\Omega)$ associated with*

$$\lambda = \max_i \bigl(\lambda(D_i)\bigr),$$

such that $\mathcal{D}^{\min}$ is the family of the k nodal domains of u.

Proof. Let us apply Theorem B.1. We take as $\lambda = \max_i(\lambda(D_i))$.

The first point is that all the $\lambda(D_i)$ should be equal. If not, one could by deformation of the D_i's in a neighborhood of regular points of their boundary find a new partition $\widetilde{\mathcal{D}}$, which would decrease $\max_i(\lambda(D_i))$.

The second point is to observe that considering two neighbors D_i and D_j, then λ should be the second eigenvalue of $H(D_{i,j})$. If it was not the case for some pair (i, j), the two nodal domains of the second eigenfunction of $H(D_{i,j})$ will give two new open sets D'_i and D'_j with $\lambda(D'_i) = \lambda(D'_j)$, in contradiction with the assumption of minimality and the first point of the proof.

Hence the Pair Compatibility Condition is satisfied. □

Remark B.4. As mentioned in the introduction, the case where $k = 2$ corresponds to a rather well known characterization of the second eigenvalue of $H(\Omega)$. The admissibility condition is of course automatically satisfied in this case.

[8] The notion of regularity was actually stronger there, but the Pair Compatibility Condition gives actually some regularity assumption on the boundaries.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Applied Math. Series, vol. 55, National Bureau of Standards, 1964.
[2] G. Alessandrini, Critical points of solutions of elliptic equations in two variables, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 14 (2) (1988) 229–256.
[3] G. Alessandrini, Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains, Comment. Math. Helv. 69 (1) (1994) 142–154.
[4] A. Ancona, B. Helffer, T. Hoffmann-Ostenhof, Nodal domain theorems à la Courant, Documenta Math. 9 (2004) 283–299.
[5] C. Bandle, Isoperimetric Inequalities and Applications, Monographs and Studies in Mathematics, vol. 7, Pitman, 1980.
[6] P. Bucur, J.P. Zolesio, N-dimensional shape optimization under capacitary constraints, J. Differential Equations 123 (1995) 504–522.
[7] P. Bucur, G. Buttazzo, A. Henrot, Existence results for some optimal partition problems, in: Monographs and Studies in Mathematics, vol. 8, 1998, pp. 571–579.
[8] G. Buttazzo, G. Dal Maso, Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions, Appl. Math. Optim. 23 (1991) 17–49.
[9] G. Buttazzo, G. Dal Maso, An existence result for a class of shape optimization problems, Arch. Rat. Mech. Anal. 122 (1993) 183–195.
[10] L.A. Cafferelli, Fang Hua Lin, An optimal partition problem for eigenvalues, J. Sci. Comput. 31 (2007) 5–14.
[11] M. Conti, S. Terracini, G. Verzini, A variational problem for the spatial segregation of reaction–diffusion systems, Indiana Univ. Math. J. 54 (3) (2005) 779–815.
[12] M. Conti, S. Terracini, G. Verzini, On a class of optimal partition problems related to the Fucik spectrum and to the monotonicity formula, Calc. Var. 22 (2005) 45–72.
[13] M. Dauge, B. Helffer, Eigenvalue variation II. Multidimensional problems, J. Differential Equations 104 (2) (1993) 263–297.
[14] R. Diestel, Graph Theory, second ed., Graduate Texts in Mathematics, vol. 173, Springer, 2000.
[15] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1983.
[16] M.J. Greenberg, J.R. Harper, Algebraic Topology: A First Course, Mathematics Lecture Note Series, vol. 58, Benjamin/Cummings Publishing Co., Inc., Reading, MA, 1981. Advanced Book Program.
[17] P. Hartman, A. Wintner, On the local behavior of solutions of non-parabolic partial differential equations, Amer. J. Math. 75 (1953) 449–476.
[18] B. Helffer, T. Hoffmann-Ostenhof, Converse spectral problems for nodal domains, Extended version, Preprint, Sept. 2005, mp_arc 05343.
[19] B. Helffer, T. Hoffmann-Ostenhof, Converse spectral problems for nodal domains. Preprint, April 2006, Moscow Math. J., in press.
[20] B. Helffer, T. Hoffmann-Ostenhof, On nodal patterns and spectral optimal partitions, Unpublished Notes, December 2005.
[21] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, Local properties of solutions of Schrödinger equations, Comm. Partial Differential Equations 17 (1992) 491–522.
[22] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, N. Nadirashvili, Interior Hölder estimates for solutions of Schrödinger equations and the regularity of nodal sets, Comm. Partial Differential Equations 20 (1995) 1241–1243.
[23] T. Hoffmann-Ostenhof, P. Michor, N. Nadirashvili, Bounds on the multiplicity of eigenvalues for fixed membranes, GAFA 9 (1999) 1169–1188.
[24] T. Kato, Perturbation Theory for Linear Operators, second ed., Springer, 1977.
[25] A. Pleijel, Remarks on Courant's nodal theorem, Comm. Pure Appl. Math. 9 (1956) 543–550.
[26] C. Pommerenke, Boundary Behavior of Conformal Maps, Grundlehren der Mathematischen Wissenshaften, vol. 295, Springer, 1992.
[27] M. Reed, B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, 1978.
[28] N. Shimakura, La pemiére valuer propre du Laplacien pour le problème de Dirichlet, J. Math. Pures Appl. 62 (1983) 129–152.
[29] B. Simon, Lower semi-continuity of positive quadratic forms, Proc. Roy. Soc. Edinburgh 29 (1977) 267–273.
[30] B. Simon, A canonical decomposition for quadratic forms with applications to monotone convergence theorems, J. Funct. Anal. 28 (1978) 377–385.
[31] P. Stollmann, A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains, Math. Z. 219 (1995) 275–287.
[32] V. Sverak, On optimal shape design, J. Math. Pures Appl. 72 (1993) 537–551.
[33] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1966.
[34] J. Weidmann, Continuity of the eigenvalues of self-adjoint operators with respect to the strong operator topology, Integral Equations Operator Theory 3 (1980) 138–142.
[35] J. Weidmann, Monotone continuity of the spectral resolution and the eigenvalues, Proc. R. Soc. Edinburgh Sect. A 85 (1980) 131–136.
[36] J. Weidmann, Stetige Abhängigkeit der Eigenwerte und Eigenfunktionen elliptischer Differentialoperatoren vom Gebiet, Math. Scand. 54 (1984) 51–69.

Further reading

[1] V. Bonnaillie-Noël, B. Helffer, G. Vial, Numerical simulations for nodal domains and spectral minimal partitions, Preprint, 2007, available on http://hal.archives-ouvertes.fr/hal-00150455.
[2] M. Conti, S. Terracini, G. Verzini, An optimal partition problem related to nonlinear eigenvalues, J. Funct. Anal. 198 (2003) 160–196.
[3] B. Kawohl, Remarks on some old and current eigenvalue problems, in: A. Alvino, E. Fabes, G. Talenti (Eds.), Partial Differential Equations of Elliptic Type, in: Symposia Mathematica, vol. XXXV, Cambridge University Press, 1994.