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Existence and uniqueness for a nonlinear parabolic/Hamilton–Jacobi coupled system describing the dynamics of dislocation densities

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Abstract

We study a mathematical model describing the dynamics of dislocation densities in crystals. This model is expressed as a 1D system of a parabolic equation and a first order Hamilton–Jacobi equation that are coupled together. We examine an associated Dirichlet boundary value problem. We prove the existence and uniqueness of a viscosity solution among those assuming a lower-bound on their gradient for all time including the initial time. Moreover, we show the existence of a viscosity solution when we have no such restriction on the initial data. We also state a result of existence and uniqueness of entropy solution for the initial value problem of the system obtained by spatial derivation. The uniqueness of this entropy solution holds in the class of bounded-from-below solutions. In order to prove our results on the bounded domain, we use an "extension and restriction" method, and we exploit a relation between scalar conservation laws and Hamilton–Jacobi equations, mainly to get our gradient estimates.

Résumé

Nous étudions un modèle mathématique décrivant la dynamique de densités de dislocations dans les cristaux. Ce modèle s'écrit comme un système 1D couplant une équation parabolique et une équation de Hamilton–Jacobi du premier ordre. Nous examinons un problème de Dirichlet associé. On montre l'existence et l'unicité d'une solution de viscosité dans la classe des fonctions ayant un gradient minoré pour tout temps ainsi qu'au temps initial. De plus, on montre l'existence d'une solution de viscosité sans cette condition sur la donnée initiale. On présente également un résultat d'existence et d'unicité d'une solution entropique pour le problème d'évolution obtenu par dérivation spatiale. L'unicité de cette solution entropique a lieu dans la classe des solutions minorées. Pour montrer nos résultats sur le domaine borné, on utilise une méthode de « prolongement et restriction », et on profite essentiellement d'une relation entre les lois de conservation scalaire et les équations de Hamilton–Jacobi, pour obtenir des contrôles du gradient.

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1. Introduction

1.1. Physical motivation

A dislocation is a defect, or irregularity within a crystal structure that can be observed by electron microscopy. The theory was originally developed by Vito Volterra in 1905. Dislocations are a non-stationary phenomena and their motion is the main explanation of the plastic deformation in metallic crystals (see [24,15] for a recent and mathematical presentation).

Geometrically, each dislocation is characterized by a physical quantity called the Burgers vector, which is responsible for its orientation and magnitude. Dislocations are classified as being positive or negative due to the orientation of its Burgers vector, and they can move in certain crystallographic directions.

Starting from the motion of individual dislocations, a continuum description can be derived by adopting a formulation of dislocation dynamics in terms of appropriately defined dislocation densities, namely the density of positive and negative dislocations. In this paper we are interested in the model described by Groma, Czikor and Zaiser [14], that sheds light on the evolution of the dynamics of the "two type" densities of a system of straight parallel dislocations, taking into consideration the influence of the short range dislocation–dislocation interactions. The model was originally presented in $\mathbb{R}^2 \times (0, T)$ as follows:

$$\begin{cases} \dfrac{\partial \theta^+}{\partial t} + \mathbf{b} \cdot \dfrac{\partial}{\partial \mathbf{r}} \left[\theta^+ \left\{ (\tau_{\text{sc}} + \tau_{\text{eff}}) - AD \dfrac{\mathbf{b}}{\theta^+ + \theta^-} \cdot \dfrac{\partial}{\partial \mathbf{r}} (\theta^+ - \theta^-) \right\} \right] = 0, \\ \dfrac{\partial \theta^-}{\partial t} - \mathbf{b} \cdot \dfrac{\partial}{\partial \mathbf{r}} \left[\theta^- \left\{ (\tau_{\text{sc}} + \tau_{\text{eff}}) - AD \dfrac{\mathbf{b}}{\theta^+ + \theta^-} \cdot \dfrac{\partial}{\partial \mathbf{r}} (\theta^+ - \theta^-) \right\} \right] = 0. \end{cases} \tag{1.1}$$

Where $T > 0$, $\mathbf{r} = (x, y)$ represents the spatial variable, $\mathbf{b}$ is the Burgers vector, $\theta^+(\mathbf{r}, t)$ and $\theta^-(\mathbf{r}, t)$ denote the densities of the positive and negative dislocations respectively. The quantity A is defined by the formula $A = \mu/[2\pi(1-\nu)]$, where μ is the shear modulus and ν is the Poisson ratio. D is a non-dimensional constant. Stress fields are represented through the self-consistent stress $\tau_{\text{sc}}(\mathbf{r}, t)$, and the effective stress $\tau_{\text{eff}}(\mathbf{r}, t)$. $\frac{\partial}{\partial \mathbf{r}}$ denotes the gradient with respect to the coordinate vector $\mathbf{r}$. An earlier investigation of the continuum description of the dynamics of dislocation densities has been done in [13]. However, a major drawback of these investigations is that the short range dislocation–dislocation correlations have been neglected and dislocation–dislocation interactions were described only by the long-range term which is the self-consistent stress field. Moreover, for the model described in [13], we refer the reader to [8,9] for a one-dimensional mathematical and numerical study, and to [3] for a two-dimensional existence result.

In our work, we are interested in a particular setting of (1.1) where we make the following assumptions:

(a1) the quantities in Eqs. (1.1) are independent of y,
(a2) $\mathbf{b} = (1, 0)$, and the constants A and D are set to be 1,
(a3) the effective stress is assumed to be zero.

Remark 1.1. (a1) gives that the self-consistent stress τ_{sc} is null; this is a consequence of the definition of τ_{sc} (see [14]).

Assumptions (a1)–(a3) permit rewriting the original model as a **1D** problem in $\mathbb{R} \times (0, T)$:

$$\begin{cases} \theta_t^+(x,t) - \left(\theta^+(x,t) \left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)} \right) \right)_x = 0, \\ \theta_t^-(x,t) + \left(\theta^-(x,t) \left(\dfrac{\theta_x^+(x,t) - \theta_x^-(x,t)}{\theta^+(x,t) + \theta^-(x,t)} \right) \right)_x = 0. \end{cases} \tag{1.2}$$

Let, unless otherwise stated, I denotes the open and bounded interval,

$$I = (0, 1),$$

of the real line. We examine an associated Dirichlet problem that we are going to give an idea of its physical derivation in the forthcoming arguments of this subsection.

To illustrate some physical motivations of the boundary value problem that we are going to study, we consider a constrained channel deforming in simple shear (see [14]). A channel of width 1 in the x-direction and infinite extension in the y-direction is bounded by walls that are impenetrable for dislocations. The motion of the positive and negative dislocations corresponds to the x-direction. This is a simplified version of a system studied by Van der Giessen and coworkers [5], where the simplifications stem from the fact that:

- only a single slip system is assumed to be active, such that reactions between dislocations of different type need not be considered;
- the boundary conditions reduce to "no flux" conditions for the dislocation fluxes at the boundary walls.

The mathematical formulation of this model, as expressed in [14], is the system (1.2) posed on

$$I_T = I \times (0, T).$$

We consider an integrated form of (1.2) and we let:

$$\rho_x^{\pm} = \theta^{\pm}, \quad \theta = \theta^+ + \theta^-, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-, \tag{1.3}$$

in order to obtain, for special values of the constants of integration, the following system of PDEs in terms of ρ and κ:

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{in } I_T, \\ \kappa(x,0) = \kappa^0(x) & \text{in } I, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \rho_t = \rho_{xx} & \text{in } I_T, \\ \rho(x,0) = \rho^0(x) & \text{in } I, \end{cases} \tag{1.5}$$

where $T > 0$ is a fixed constant. Enough regularity on the initial data will be given in order to impose the physically relevant condition:

$$\kappa_x^0 \geqslant |\rho_x^0|. \tag{1.6}$$

This condition is natural: it indicates nothing but the positivity of the dislocation densities $\theta^{\pm}(x,0)$ at the initial time (see (1.3)).

In order to formulate heuristically the boundary conditions at the walls located at $x = 0$ and $x = 1$, we note that the dislocation fluxes at the walls must be zero, which requires that

$$\overbrace{\partial_x(\theta^+ - \theta^-)}^{\Phi} = 0, \quad \text{at } x = 0 \text{ and } x = 1. \tag{1.7}$$

Rewriting system (1.2) in a special integrated form in terms of ρ, κ and Φ, we get

$$\kappa_t = (\rho_x/\kappa_x)\Phi \quad \text{and} \quad \rho_t = \Phi. \tag{1.8}$$

Using (1.7) and (1.8), we can formally deduce that ρ and κ are constants along the boundary walls. Therefore, this paper focuses attention on the study of the following coupled Dirichlet boundary value problems:

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x, & \text{in } I_T, \\ \kappa(x,0) = \kappa^0(x), & \text{in } I, \\ \kappa(0,t) = \kappa(0,0) \quad \text{and} \quad \kappa(1,t) = \kappa(1,0), & \forall t \in [0,T], \end{cases} \tag{1.9}$$

and

$$\begin{cases} \rho_t = \rho_{xx}, & \text{in } I_T, \\ \rho(x,0) = \rho^0(x), & \text{in } I, \\ \rho(0,t) = \rho(1,t) = 0, & \forall t \in [0,T]. \end{cases} \tag{1.10}$$

Besides (1.6), there is a second natural assumption concerning ρ^0 and κ^0 that has to do with the balance of the physical model that starts with the same number of positive and negative dislocations. In other words, if n^+ and n^- are the total number of positive and negative dislocations respectively at $t = 0$ then:

$$\rho^0(1) - \rho^0(0) = \int_0^1 \rho_x^0(x)\,dx = \int_0^1 \big(\theta^+(x,0) - \theta^-(x,0)\big)\,dx = n^+ - n^- = 0, \tag{1.11}$$

this shows that $\rho^0(1)=\rho^0(0)$ and this is what appears in (1.10).

1.2. Main results

In this paper, we show the existence and uniqueness of a viscosity solution κ of (1.9) in the class of all Lipschitz continuous viscosity solutions having special "bounded from below" spatial gradients. However, we show the existence of a Lipschitz continuous viscosity solution of (1.9) when this restriction is relaxed.

Let $\mathfrak{I}$ be a certain interval of $\mathbb{R}$. Denote $\mathrm{Lip}(\mathfrak{I})$ by:

$$\mathrm{Lip}(\mathfrak{I})=\{f:\mathfrak{I}\mapsto\mathbb{R};\ f \text{ is a Lipschitz continuous function}\}.$$

We prove the following theorems:

Theorem 1.2 *(Existence and uniqueness of a viscosity solution). Let $T>0$ and $\epsilon>0$ be two constants. Take*

$$\kappa^0\in\mathrm{Lip}(I),$$

and $\rho^0\in C_0^\infty(I)$ satisfying:

$$\kappa_x^0\geqslant G_\epsilon(\rho_x^0)\quad \text{a.e. in } I, \tag{1.12}$$

where

$$G_\epsilon(x)=\sqrt{x^2+\epsilon^2}. \tag{1.13}$$

Given the solution ρ of the heat equation (1.10)*, there exists a viscosity solution $\kappa\in\mathrm{Lip}(\bar{I}_T)$ of* (1.9)*, unique among those satisfying:*

$$\kappa_x\geqslant G_\epsilon(\rho_x)\quad \text{a.e. in } \bar{I}_T. \tag{1.14}$$

Theorem 1.3 *(Existence of a viscosity solution, case $\epsilon=0$). Let $T>0$, $\kappa^0\in\mathrm{Lip}(I)$ and $\rho^0\in C_0^\infty(I)$. If the condition* (1.6) *is satisfied a.e. in I, and if ρ is the solution of* (1.10)*, then there exists a viscosity solution $\kappa\in\mathrm{Lip}(\bar{I}_T)$ of* (1.9) *satisfying:*

$$\kappa_x\geqslant|\rho_x|,\quad \text{a.e. in } \bar{I}_T. \tag{1.15}$$

Remark 1.4. In the limit case where $\epsilon=0$, we remark that having (1.15) was intuitively expected due to the positivity of the dislocation densities θ^+ and θ^-. This reflects in some way the well-posedness of the model (1.2) of the dynamics of dislocation densities. We also remark that our result of existence of a solution of (1.9) under (1.15) still holds if we start with $\kappa_x^0=\rho_x^0=0$ on some sub-intervals of I. In other words, we can imagine that we start with the probability of the formation of no dislocation zones.

A relation between scalar conservation laws and Hamilton–Jacobi equations will be exploited to get almost all our gradient controls of κ. This relation, that will be made precise later, will also lead to a result of existence and uniqueness of a bounded entropy solution of the following equation:

$$\begin{cases}\theta_t=\left(\dfrac{\rho_x\rho_{xx}}{\theta}\right)_x & \text{in } Q_T=\mathbb{R}\times(0,T),\\ \theta(x,0)=\theta^0(x) & \text{in } \mathbb{R},\end{cases} \tag{1.16}$$

where ρ is the solution of the initial value problem:

$$\begin{cases}\rho_t=\rho_{xx} & \text{in } Q_T,\\ \rho(x,0)=\rho^0(x) & \text{in } \mathbb{R}.\end{cases} \tag{1.17}$$

Eq. (1.16) is deduced formally by taking a spatial derivation of (1.4). The uniqueness of this entropy solution is always restricted to the class of bounded entropy solutions with a special lower-bound.

Remark 1.5. The above result of existence and uniqueness of entropy solution for the initial value problem (1.16) is shown in the case $I = \mathbb{R}$. This also leads to a result of existence and uniqueness for the original problem (1.2). We will just present the statements of these results without going into the proofs. However, we refer the interested reader to [17, Theorem 1.3, Corollary 1.4] for the details. We also present, at the end of Section 4 of this paper, some viscosity results in the case $I = \mathbb{R}$.

Entropy results (case of the real space $\mathbb{R}$).

Theorem 1.6 *(Existence and uniqueness of an entropy solution). Let $T > 0$. Take $\theta^0 \in L^\infty(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$ such that,*

$$\theta^0 \geqslant \sqrt{(\rho_x^0)^2 + \epsilon^2} \quad \text{a.e. in } \mathbb{R},$$

for some constant $\epsilon > 0$. Then, given ρ the solution of (1.17)*, there exists an entropy solution $\theta \in L^\infty(\bar{Q}_T)$ of* (1.16)*, unique among the entropy solutions satisfying*:

$$\theta \geqslant \sqrt{\rho_x^2 + \epsilon^2} \quad \text{a.e. in } \bar{Q}_T.$$

Corollary 1.7 *(Existence and uniqueness for problem (1.2)). Let $T > 0$ and $\epsilon > 0$. Let θ_0^+ and θ_0^- be two given functions representing the initial positive and negative dislocation densities respectively. If the following conditions are satisfied*:

(1) $\theta_0^+ - \theta_0^- \in C_0^\infty(\mathbb{R})$,
(2) $\theta_0^+, \theta_0^- \in L^\infty(\mathbb{R})$,

together with,

$$\theta_0^+ + \theta_0^- \geqslant \sqrt{(\theta_0^+ - \theta_0^-)^2 + \epsilon^2} \quad \text{a.e. in } \mathbb{R},$$

then there exists a solution $(\theta^+, \theta^-) \in (L^\infty(Q_T))^2$ to the system (1.2)*, in the sense of Theorem* 1.6*, unique among those satisfying*:

$$\theta^+ + \theta^- \geqslant \sqrt{(\theta^+ - \theta^-)^2 + \epsilon^2} \quad \text{a.e. in } \bar{Q}_T.$$

1.3. Organization of the paper

The paper is organized as follows. In Section 2, we start by stating the definition of viscosity and entropy solutions with some of their properties. In Section 3, we introduce the essential tools used in the proof of the main results. Section 4 is devoted to the proof of Theorems 1.2 and 1.3. We also present, at the end of this section, some further results in the case $I = \mathbb{R}$, namely Theorems 4.2 and 4.3. Finally, Appendix A is an appendix containing a sketch of the proof to the classical comparison principle of scalar conservation laws adapted to our case with low regularity.

2. Definitions and preliminaries

The reader will notice throughout this section that all the results are valid in the case of working on the real space $\mathbb{R}$, and not on a bounded interval I as it is expected. Indeed, this is due to the fact that the technique of the proof that we use depends mainly on extending the problem to the whole space where we can exploit all the forthcoming results of this section. This will be made more clear in the following sections.

Recall that $Q_T = \mathbb{R} \times (0, T)$. We will deal with two types of equations:

1. Hamilton–Jacobi equation:

$$\begin{cases} u_t + F(x, t, u_x) = 0 & \text{in } Q_T, \\ u(x, 0) = u^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{2.1}$$

2. Scalar conservation laws:

$$\begin{cases} v_t + (F(x,t,v))_x = 0 & \text{in } Q_T, \\ v(x,0) = v^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{2.2}$$

where

$$F : \mathbb{R} \times [0,T] \times \mathbb{R} \to \mathbb{R},$$
$$(x,t,u) \mapsto F(x,t,u)$$

is called the Hamiltonian for the Hamilton–Jacobi equations and the flux function for the scalar conservation laws. This function is always assumed to be continuous, while additional and specific regularity will be given when needed.

Remark 2.1. The major part of this work concerns a Hamiltonian/flux function of a special form, namely:

$$F(x,t,u) = g(x,t) f(u), \tag{2.3}$$

where such forms often arise in problems of physical interest including traffic flow [26] and two-phase flow in porous media [12].

2.1. Viscosity solution: definition and properties

Definition 2.2 *(Viscosity solution: non-stationary case).*

(1) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity sub-solution of

$$u_t + F(x,t,u_x) = 0 \quad \text{in } Q_T, \tag{2.4}$$

if for every $\phi \in C^1(Q_T)$, whenever $u - \phi$ attains a local maximum at $(x_0,t_0) \in Q_T$, then

$$\phi_t(x_0,t_0) + F\big(x_0,t_0,\phi_x(x_0,t_0)\big) \leqslant 0.$$

(2) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity super-solution of (2.4) if for every $\phi \in C^1(Q_T)$, whenever $u - \phi$ attains a local minimum at $(x_0,t_0) \in Q_T$, then

$$\phi_t(x_0,t_0) + F\big(x_0,t_0,\phi_x(x_0,t_0)\big) \geqslant 0.$$

(3) A function $u \in C(Q_T; \mathbb{R})$ is a viscosity solution of (2.4) if it is both a viscosity sub- and super-solution of (2.4).

(4) A function $u \in C(\bar{Q}_T; \mathbb{R})$ is a viscosity solution of the initial value problem (2.1) if u is a viscosity solution of (2.4) and $u(x,0) = u^0(x)$ in $\mathbb{R}$.

It is worth mentioning here that if a viscosity solution of a Hamilton–Jacobi equation is differentiable at a certain point, then it solves the equation there (see for instance [1]).

Definition 2.3 *(Viscosity solution: stationary case).* Let Ω be an open subset of $\mathbb{R}^n$ and let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous mapping. A function $u \in C(\Omega; \mathbb{R})$ is a viscosity sub-solution of

$$F\big(x, u(x), \nabla u(x)\big) = 0 \quad \text{in } \Omega, \tag{2.5}$$

if for any continuously differentiable function $\phi : \Omega \mapsto \mathbb{R}$ and any local maximum $x_0 \in \Omega$ of $u - \phi$, one has

$$F\big(x_0, u(x_0), \nabla\phi(x_0)\big) \leqslant 0.$$

Similarly, if at any local minimum point $x_0 \in \Omega$ of $u - \phi$, one has

$$F\big(x_0, u(x_0), \nabla\phi(x_0)\big) \geqslant 0,$$

then u is a viscosity super-solution. Finally, if u is both a viscosity sub-solution and a viscosity super-solution, then u is a viscosity solution.

We say that u is a viscosity solution of the Dirichlet problem (2.5) with $u = \zeta \in C(\partial\Omega)$ if:

(1) $u \in C(\bar{\Omega})$,
(2) u is a viscosity solution of (2.5) in Ω,
(3) $u = \zeta$ on $\partial\Omega$.

In fact, this definition is used for interpreting solutions of (1.9) in the viscosity sense. For a better understanding of the viscosity interpretation of boundary conditions of Hamilton–Jacobi equations, we refer the reader to [1, Section 4.2].

Now, we will proceed by giving some results concerning viscosity solutions of (2.1). In order to have existence and uniqueness, the Hamiltonian F will be restricted to the following conditions:

(F0) $F \in C(\mathbb{R} \times [0,T] \times \mathbb{R})$;
(F1) for each $R > 0$, there is a constant C_R such that:

$$\left|F(x,t,p) - F(y,t,q)\right| \leqslant C_R\left(|p-q| + |x-y|\right) \quad \forall (x,t,p), (y,t,q) \in \bar{Q}_T \times [-R,R];$$

(F2) there is a constant C_F such that for all $(t,p) \in [0,T] \times \mathbb{R}$ and all $x, y \in \mathbb{R}$, one has:

$$\left|F(x,t,p) - F(y,t,p)\right| \leqslant C_F|x-y|\left(1+|p|\right).$$

We use these conditions to write down some well-known results on viscosity solutions.

Theorem 2.4. *Under* (F0), (F1) *and* (F2), *if* $u^0 \in UC(\mathbb{R})$, *then* (2.1) *has a unique viscosity solution* $u \in UC_x(\bar{Q}_T)$ (*see* [7, *Section* 1] *for the precise definition of* $UC(\mathbb{R})$, $UC_x(\bar{Q}_T)$, *and for the proof of this theorem*).

Remark 2.5. In the case where the Hamiltonian has the form:

$$F(x,t,u) = g(x,t)f(u),$$

the following conditions:

$$\text{(V0) } f \in C_b^1(\mathbb{R};\mathbb{R}), \quad \text{(V1) } g \in C_b(\bar{Q}_T;\mathbb{R}) \quad \text{and} \quad \text{(V2)} g_x \in L^\infty(\bar{Q}_T),$$

imply (F0)–(F2) together with the boundedness of the Hamiltonian.

The next proposition reflects the behavior of viscosity solutions under additional regularity assumptions on u^0 and F.

Proposition 2.6 *(Additional regularity of the viscosity solution). Let* $F = gf$ *satisfy* (V0)–(V2). *If* $u^0 \in \mathrm{Lip}(\mathbb{R})$ *and* $u \in UC_x(\bar{Q}_T)$ *is the unique viscosity solution of* (2.1), *then* $u \in \mathrm{Lip}(\bar{Q}_T)$ (*see* [16, *Theorem* 3]).

Remark 2.7. It is worth mentioning that the space Lipschitz constant of the function u depends on C, where C appears in (F1) for $p = q$, and on the Lipschitz constant γ of the function u_0. While the time Lipschitz constant depends on the bound of the Hamiltonian.

2.2. Entropy solution: definition and properties

Definition 2.8 *(Entropy sub-/super-solution).* Let $F(x,t,v) = g(x,t)f(v)$ with $g, g_x \in L^\infty_{\mathrm{loc}}(Q_T;\mathbb{R})$ and $f \in C^1(\mathbb{R};\mathbb{R})$. A function $v \in L^\infty(Q_T;\mathbb{R})$ is an entropy sub-solution of (2.2) with bounded initial data $v^0 \in L^\infty(\mathbb{R})$ if it satisfies:

$$\int_{Q_T} \left[\eta_i\left(v(x,t)\right)\phi_t(x,t) + \Phi\left(v(x,t)\right)g(x,t)\phi_x(x,t) + h\left(v(x,t)\right)g_x(x,t)\phi(x,t)\right]dx\,dt + \int_{\mathbb{R}} \eta_i\left(v^0(x)\right)\phi(x,0)\,dx \geqslant 0, \tag{2.6}$$

$\forall \phi \in C_0^1(\mathbb{R} \times [0,T);\mathbb{R}_+)$, for any non-decreasing convex function $\eta_i \in C^1(\mathbb{R};\mathbb{R})$, $\Phi \in C^1(\mathbb{R};\mathbb{R})$ such that:

$$\Phi' = f'\eta_i', \quad \text{and} \quad h = \Phi - f\eta_i'. \tag{2.7}$$

An entropy super-solution of (2.2) is defined by replacing in (2.6) η_i with η_d; a non-increasing convex function. An entropy solution is defined as being both entropy sub- and super-solution. In other words, it verifies (2.6) for any convex function $\eta \in C^1(\mathbb{R};\mathbb{R})$.

Entropy solutions were first introduced by Kružkov [19] as the only physically admissible solutions among all weak (distributional) solutions to scalar conservation laws. These weak solutions lack the fact of being unique for it is easy to construct multiple weak solutions to Cauchy problems (2.2) (see for instance [21]). The next definition concerns classical sub-/super-solution to scalar conservation laws. This kind of solutions are easily shown to be entropy sub-/super-solutions.

Definition 2.9 *(Classical solution to scalar conservation laws).* Let $F(x,t,v) = g(x,t)f(v)$ with $g, g_x \in L^\infty_{\mathrm{loc}}(Q_T;\mathbb{R})$ and $f \in C^1(\mathbb{R};\mathbb{R})$. A function $v \in W^{1,\infty}(Q_T)$ is said to be a classical sub-solution of (2.2) with $v^0(x) = v(x,0)$ if it satisfies

$$v_t(x,t) + \big(F\big(x,t,v(x,t)\big)\big)_x \leqslant 0 \quad \text{a.e. in } Q_T. \tag{2.8}$$

Classical super-solutions are defined by replacing "$\leqslant$" with "$\geqslant$" in (2.8), and classical solutions are defined to be both classical sub- and super-solutions.

We move now to some classical results on entropy solutions.

Theorem 2.10 *(Kružkov's Existence Theorem). Let F, v^0 be given by Definition* 2.8*, and the following conditions hold*:

$$\text{(E0)}\ f \in C^1_b(\mathbb{R}), \quad \text{(E1)}\ g, g_x \in C_b(\bar{Q}_T) \quad \textit{and} \quad \text{(E2)}\ g_{xx} \in C(\bar{Q}_T),$$

then there exists an entropy solution $v \in L^\infty(Q_T)$ of (2.2) (*see* [19, *Theorem* 4]).

In fact, Kružkov's conditions for existence were given for a general flux function (see [19, Section 4] for details). However, in Subsection 5.4 of the same paper, a weak version of these conditions, that can be easily checked in the case $F(x,t,v) = g(x,t)f(v)$ and (E0)–(E2), is presented. Furthermore, uniqueness follows from the following comparison principle.

Theorem 2.11 *(Comparison Principle). Let F be given by Definition* 2.8 *with f satisfying* (E0)*, and g satisfies,*

$$\text{(E3)}\ g \in W^{1,\infty}(\bar{Q}_T).$$

Let u, $v \in L^\infty(Q_T)$ be two entropy sub-/super-solutions of (2.2) *with initial data $u^0, v^0 \in L^\infty(\mathbb{R})$ such that, $u^0(x) \leqslant v^0(x)$ a.e. in $\mathbb{R}$, then*

$$u(x,t) \leqslant v(x,t) \quad \textit{a.e. in } \bar{Q}_T.$$

The proof of this theorem can be adapted from [11, Theorem 3] with slight modifications. However, for the sake of completeness, we will present a sketch of the proof in Appendix A.

At this stage, we are ready to present a relation that sometimes holds between scalar conservation laws and Hamilton–Jacobi equations in one-dimensional space.

2.3. Entropy–viscosity relation

Formally, by differentiating (2.1) with respect to x and defining $v = u_x$, we see that (2.1) is equivalent to the scalar conservation law (2.2) with $v^0 = u^0_x$ and the same F. This equivalence of the two problems has been exploited in order to translate some numerical methods for hyperbolic conservation laws to methods for Hamilton–Jacobi equations. Moreover, several proofs were given in the one-dimensional case. The usual proof of this relation depends strongly on the known results about existence and uniqueness of the solutions of the two problems together with the convergence of the viscosity method (see [6,20,23]). Another proof of this relation could be found in [4] via the definition of viscosity/entropy inequalities, while a direct proof could also be found in [18] using the front tracking method. The case of a Hamiltonian of the form (2.3) is also treated even when $g(x,t)$ is allowed to be discontinuous in the (x,t) plane along a finite number of (possibly intersected) curves (see [25]). In our work, the above stated relation will be successfully used to get some gradient estimates on κ. To be more specific, we write down the precise statement of this relation: for every Hamiltonian/flux function $F = gf$ and every $u^0 \in \mathrm{Lip}(\mathbb{R})$, let

$$\mathcal{EV} = \{(\mathrm{V0}), (\mathrm{V1}), (\mathrm{V2}), (\mathrm{E0}), (\mathrm{E1}), (\mathrm{E2}), (\mathrm{E3})\},$$

in other words,

$$\mathcal{EV} = \left| \begin{array}{l} \text{The set of all conditions on } f \text{ and } g \text{ ensuring the} \\ \text{existence and uniqueness of a Lipschitz continuous viscosity} \\ \text{solution } u \in \mathrm{Lip}(\bar{Q}_T) \text{ of (2.1), and of an entropy} \\ \text{solution } v \in L^\infty(Q_T) \text{ of (2.2), with } v^0 = u^0_x \in L^\infty(\mathbb{R}). \end{array} \right. \tag{2.9}$$

Theorem 2.12 *(A link between viscosity and entropy solutions). Let $F = gf$ with $g \in C^2(\bar{Q}_T)$, $u^0 \in \mathrm{Lip}(\mathbb{R})$ and $\mathcal{EV}$ satisfied. Then,*

$$v = u_x \quad \text{a.e. in } Q_T.$$

Remark 2.13. In the multidimensional case this one-to-one correspondence no longer exists, instead the gradient $v = \nabla u$ satisfies formally a non-strict hyperbolic system of conservation laws (see [23,20]).

3. Main tools

Before proceeding with the proof of our main theorems, we have to introduce some essential tools that are the core of the "extension and restriction" method that we are going to use. For every $\epsilon > 0$, we build up an approximation function $f_\epsilon \in C^1_b(\mathbb{R})$ of the function $\frac{1}{x}$ defined by:

$$f_\epsilon(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \geqslant \epsilon, \\ \dfrac{2\epsilon - x}{\epsilon^2 + \epsilon^2(x-\epsilon)^2} & \text{otherwise.} \end{cases} \tag{3.1}$$

The function κ^0 given by (1.2) is extended to the real line $\mathbb{R}$ as follows:

$$\hat{\kappa}^0(x) = \begin{cases} \kappa^0(x) & \text{if } x \in [0,1], \\ (\|\rho^0_x\|_{L^\infty(I)} + \epsilon)(x-1) + \kappa^0(1) & \text{if } x \geqslant 1, \\ (\|\rho^0_x\|_{L^\infty(I)} + \epsilon)x + \kappa^0(0) & \text{if } x \leqslant 0, \end{cases} \tag{3.2}$$

where ρ^0 is defined in Theorem 1.2. Notice that $\hat{\kappa}^0 \in \mathrm{Lip}(\mathbb{R})$. Let ρ be the unique solution of the classical heat equation (1.10), we will extend the function ρ to $\bar{Q}_T$ and we will extract some of the properties of its extension.

Extension of ρ over $\mathbb{R} \times [0,T]$.

Consider the function $\hat{\rho}$ defined on $[0,2] \times [0,T]$ by:

$$\hat{\rho}(x,t) = \begin{cases} \rho(x,t) & \text{if } (x,t) \in \bar{I}_T, \\ -\rho(2-x,t) & \text{otherwise,} \end{cases} \tag{3.3}$$

this is just a C^1 antisymmetry of ρ with respect to the line $x = 1$. The continuation of $\hat{\rho}$ to $\mathbb{R} \times [0,T]$ is made by spatial periodicity of period 2. Always denote $\hat{\rho}^0 \in C^\infty(\mathbb{R})$ by:

$$\hat{\rho}^0(x) = \hat{\rho}(x,0). \tag{3.4}$$

A simple computation yields, for $(x,t) \in (1,2) \times (0,T)$:

$$\hat{\rho}_t(x,t) = -\rho_t(2-x,t) \quad \text{and} \quad \hat{\rho}_{xx}(x,t) = -\rho_{xx}(2-x,t),$$

and hence it is easy to verify that $\hat\rho|_{[1,2]\times[0,T]}$ solves (1.10) with I replaced with the interval $(1,2)$ and ρ^0 replaced with its symmetry with respect to the point $x=1$; the boundary conditions are unchanged and the regularity of the initial condition is conserved. To be more precise, we write down some useful properties of $\hat\rho$.

Regularity properties of $\hat\rho$. Let r and s are two positive integers such that $s\leqslant 2$. From the construction of $\hat\rho$ and the above discussion, we get the following:

$$\begin{aligned}
&\text{(i)}\quad \hat\rho_t \text{ and } \hat\rho_x \text{ are in } C(\mathbb{R}\times[0,T]),\\
&\text{(ii)}\quad \hat\rho=0 \quad \text{on } \mathbb{Z}\times[0,T],\\
&\text{(iii)}\quad \hat\rho_t=\hat\rho_{xx} \quad \text{on } (\mathbb{R}\setminus\mathbb{Z})\times(0,T),\\
&\text{(iv)}\quad \|\partial_t^r\partial_x^s\hat\rho(\cdot,t)\|_{L^\infty(\mathbb{R})}\leqslant C, \quad \forall t\in[0,T],
\end{aligned} \tag{3.5}$$

where C is a certain constant and the limitation $s\leqslant 2$ comes from the spatial antisymmetry. These conditions are valid thanks to the way of construction of the function $\hat\rho$ and to the maximum principle of the solution of the heat equation on bounded domains (see [2,10]).

Let

$$\hat g(x,t)=-\hat\rho_t(x,t)\hat\rho_x(x,t). \tag{3.6}$$

From the above discussion, it is worth noticing that this function is a Lipschitz continuous function in the x-variable. Consider the initial value problem defined by:

$$\begin{cases} u_t+\hat g f_\epsilon(u_x)=0 & \text{in } Q_T,\\ u(x,0)=\hat\kappa^0(x) & \text{in } \mathbb{R}. \end{cases} \tag{3.7}$$

This is a Hamilton–Jacobi equation with a Hamiltonian $F_\epsilon\in C(\bar Q_T\times\mathbb{R})$ defined by:

$$F_\epsilon(x,t,u)=\hat g(x,t)f_\epsilon(u).$$

From the regularity of $\hat\rho$ and f_ϵ, we can directly see that (V0)–(V2) are all satisfied. Moreover, since $\hat\kappa^0$ is a Lipschitz continuous function, we get the following proposition as a direct consequence of Theorem 2.4 and Proposition 2.6.

Proposition 3.1. *There exists a unique viscosity solution* $\hat\kappa\in \mathrm{Lip}(\bar Q_T)$ *of* (3.7).

The following lemmas will be used in the proof of Theorems 1.2 and 1.3.

Lemma 3.2 *(Entropy sub-solution). The function* $G_\epsilon(\hat\rho_x)$ *is an entropy sub-solution of*

$$\begin{cases} w_t+(\hat g f_\epsilon(w))_x=0 & \text{in } Q_T,\\ w(x,0)=w^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{3.8}$$

where the function G_ϵ *is given by* (1.13), *and* $w^0(x)=G_\epsilon(\hat\rho_x^0(x))$.

Proof. It is easily seen that $G_\epsilon(\hat\rho_x)\in W^{1,\infty}(Q_T)$ and that the function G_ϵ verifies:

$$\text{(G1) } G_\epsilon(x)\geqslant\epsilon>0,\quad \text{(G2) } G_\epsilon''\geqslant 0 \quad\text{and}\quad \text{(G3) } G_\epsilon(x)G_\epsilon'(x)=x.$$

Define for a.e. $(x,t)\in Q_T$, the scalar-valued quantity B by:

$$B(x,t)=\partial_t\big(G_\epsilon\big(\hat\rho_x(x,t)\big)\big)+\partial_x\big(F_\epsilon\big(x,t,G_\epsilon\big(\hat\rho_x(x,t)\big)\big)\big).$$

From (G1) and (G3), we have $f_\epsilon(G_\epsilon(\hat\rho_x))=1/G_\epsilon(\hat\rho_x)$ and we observe that:

$$\begin{aligned}
B&=G_\epsilon'(\hat\rho_x)\hat\rho_{xt}-\partial_x\left(\frac{\hat\rho_t\hat\rho_x}{G_\epsilon(\hat\rho_x)}\right)\\
&=G_\epsilon'(\hat\rho_x)\hat\rho_{xt}-\frac{\hat\rho_{xt}\hat\rho_x+\hat\rho_t\hat\rho_{xx}}{G_\epsilon(\hat\rho_x)}+\frac{G_\epsilon'(\hat\rho_x)\hat\rho_{xx}\hat\rho_t\hat\rho_x}{G_\epsilon^2(\hat\rho_x)}\\
&=\frac{G_\epsilon(\hat\rho_x)\hat\rho_x\hat\rho_{xt}-G_\epsilon(\hat\rho_x)\hat\rho_x\hat\rho_{xt}-G_\epsilon(\hat\rho_x)\hat\rho_t\hat\rho_{xx}+G_\epsilon'(\hat\rho_x)\hat\rho_{xx}\hat\rho_t\hat\rho_x}{G_\epsilon^2(\hat\rho_x)}\\
&=\frac{-\hat\rho_{xx}^2(G_\epsilon(\hat\rho_x)-G_\epsilon'(\hat\rho_x)\hat\rho_x)}{G_\epsilon^2(\hat\rho_x)}.
\end{aligned}$$

Hence, we get:

$$B = -\hat{\rho}_{xx}^2 G_\epsilon''(\hat{\rho}_x),$$

where the condition (G2) gives immediately that

$$B \leqslant 0 \quad \text{a.e. on } Q_T.$$

This proves that $G_\epsilon(\hat{\rho}_x)$ is a classical sub-solution of Eq. (3.8) and therefore an entropy sub-solution. □

Lemma 3.3 *(Entropy super-solution). Take* $0 < \epsilon < 1$. *Let* c_1 *and* c_2 *be two positive constants defined respectively by*:

$$c_1 = \|\hat{\rho}_{xx}\|_{L^\infty(Q_T)}^2 + \|\hat{\rho}_x\|_{L^\infty(Q_T)} \|\hat{\rho}_{tx}\|_{L^\infty(Q_T)} \quad \text{and} \quad c_2 = \left(\|\kappa_x^0\|_{L^\infty(I)} + 1\right)^2.$$

Then the function $\bar{S}$ *defined on* Q_T *by*:

$$\bar{S}(x,t) = \sqrt{2c_1 t + c_2} \tag{3.9}$$

is an entropy super-solution of (3.8) *with* $w^0(x) = \bar{S}(x,0) = \|\kappa_x^0\|_{L^\infty(I)} + 1$.

Proof. We remark that $\bar{S} \in W^{1,\infty}(Q_T)$, and for every $(x,t) \in Q_T$ we have:

$$\bar{S}(x,t) \geqslant \sqrt{c_2} = \|\kappa_x^0\|_{L^\infty(I)} + 1 \geqslant \epsilon,$$

and hence $f_\epsilon(\bar{S}(x,t)) = 1/\bar{S}(x,t)$. The regularity of the function $\bar{S}$ permits to inject it directly into the first equation of (3.8), thus we have for a.e. $(x,t) \in Q_T$:

$$\bar{S}_t - \left(\frac{\hat{\rho}_t \hat{\rho}_x}{\bar{S}}\right)_x = \frac{c_1}{\sqrt{2c_1 t + c_2}} - \frac{\hat{\rho}_{xx}^2 + \hat{\rho}_x \hat{\rho}_{tx}}{\sqrt{2c_1 t + c_2}} = \frac{c_1 - (\hat{\rho}_{xx}^2 + \hat{\rho}_x \hat{\rho}_{tx})}{\sqrt{2c_1 t + c_2}} \geqslant 0,$$

which proves that $\bar{S}$ is an entropy super-solution of (3.8).

Lemma 3.4 *(Differentiability property). Let* $u(x,t)$ *be a differentiable function with respect to* (x,t) *a.e. in* Q_T. *Define the set* M *by*:

$$M = \{x \in \mathbb{R};\ u \text{ is differentiable a.e. in } \{x\} \times (0,T)\},$$

then M *is dense in* $\mathbb{R}$.

Indeed, we have even that the set $\mathbb{R} \setminus M$ is of Lebesgue measure zero, and this can be easily shown using some elementary integration arguments.

Lemma 3.5. *Let* $\bar{c}$ *be an arbitrary real constant and take* $\psi \in \mathrm{Lip}(I;\mathbb{R})$ *satisfying*:

$$\psi_x \geqslant \bar{c} \quad \text{a.e. in } I.$$

If $\zeta \in C^1(I;\mathbb{R})$ *is such that* $\psi - \zeta$ *has a local maximum or local minimum at some point* $x_0 \in I$, *then*

$$\zeta_x(x_0) \geqslant \bar{c}.$$

Proof. Suppose that $\psi - \zeta$ has a local minimum at the point x_0; this ensures the existence of a certain $r > 0$ such that

$$(\psi - \zeta)(x) \geqslant (\psi - \zeta)(x_0) \quad \forall x;\ |x - x_0| < r.$$

We argue by contradiction. Assuming $\zeta_x(x_0) < \bar{c}$ leads, from the continuity of ζ_x, to the existence of $r' \in (0,r)$ such that

$$\zeta_x(x) < \bar{c} \quad \forall x;\ |x - x_0| < r'. \tag{3.10}$$

Let y_0 be a point such that $|y_0 - x_0| < r'$ and $y_0 < x_0$. Reexpressing (3.10), we get

$$(\zeta - \bar{c}x)_x(x) < 0 \quad \forall x \in (y_0, x_0),$$

and hence

$$\int_{y_0}^{x_0} \big[(\psi - \bar{c}x)_x(x) - (\zeta - \bar{c}x)_x(x)\big]\,dx > 0,$$

which implies that

$$(\psi - \zeta)(x_0) > (\psi - \zeta)(y_0),$$

and hence a contradiction. We remark that the case of a local maximum can be treated in a similar way. □

In the next lemma, we show that the spatial derivative of the function $\hat{\kappa}$ given by Proposition 3.1 is an entropy solution of (3.8). The reader can easily notice that this result would be a trivial consequence of Theorem 2.12 if the function $\hat{g}$ is sufficiently regular, which is not the case here. The following lemma shows that a similar result holds in the case $\hat{g} \in W^{1,\infty}(\bar{Q}_T)$.

Lemma 3.6. *The function $\hat{\kappa}_x \in L^\infty(Q_T)$ ($\hat{\kappa}$ is given by Proposition* 3.1) *is an entropy solution of* (3.8) *with initial data $w^0 = \hat{\kappa}_x^0 \in L^\infty(\mathbb{R})$.*

Proof. Let $\tilde{g}$ be an extension of the function $\hat{g}$ on $\mathbb{R}^2$ defined by:

$$\tilde{g}(x,t) = \begin{cases} \hat{g}(x,t) & \text{if } (x,t) \in \bar{Q}_T, \\ \hat{g}(x,T) & \text{if } t > T, \\ \hat{g}(x,0) & \text{if } t < 0. \end{cases} \tag{3.11}$$

Consider a sequence of mollifiers ξ^n in $\mathbb{R}^2$ and let $\tilde{g}^n = \tilde{g} * \xi^n$. Remark that, from the standard properties of the mollifier sequence, we have $\tilde{g}^n \in C^\infty(\mathbb{R}^2)$ and:

$$\tilde{g}^n \to \hat{g} \quad \text{uniformly on compacts in } \bar{Q}_T, \tag{3.12}$$

and

$$\tilde{g}^n_x \to \hat{g}_x \quad \text{in } L^p_{\text{loc}}(Q_T), \quad 1 \leqslant p < \infty, \tag{3.13}$$

together with the following estimates:

$$\|\partial_t^r \partial_x^s \tilde{g}^n\|_{L^\infty(\bar{Q}_T)} \leqslant \|\partial_t^r \partial_x^s \hat{g}\|_{L^\infty(\bar{Q}_T)} \quad \text{for } r, s \in \mathbb{N}, r + s \leqslant 1. \tag{3.14}$$

Now, take again the Hamilton–Jacobi equation (3.7) with $\hat{g}$ replaced with $\tilde{g}^n$:

$$\begin{cases} u_t + \tilde{g}^n f_\epsilon(u_x) = 0 & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = \hat{\kappa}^0(x) & \text{in } \mathbb{R}, \end{cases} \tag{3.15}$$

and notice that the above properties of the function $\tilde{g}^n$ enters us into the framework of Theorem 2.12. Thus, we have a unique viscosity solution $\tilde{\kappa}^n \in \text{Lip}(\bar{Q}_T)$ of (3.15) with initial condition $\hat{\kappa}^0$ whose spatial derivative $\tilde{\kappa}^n_x \in L^\infty(Q_T)$ is an entropy solution of the corresponding derived equation with initial data $\hat{\kappa}^0_x$. From Remark 2.7 and (3.14), we deduce that the sequence $(\tilde{\kappa}^n)_{n \geqslant 1}$ is locally uniformly bounded in $W^{1,\infty}(\bar{Q}_T)$ and that:

$$\|\tilde{\kappa}^n_x\|_{L^\infty(Q_T)} \leqslant \|\hat{\kappa}^0_x\|_{L^\infty(\mathbb{R})} + T\|\hat{g}_x\|_{L^\infty(Q_T)}\|f_\epsilon\|_{L^\infty(\mathbb{R})}. \tag{3.16}$$

Moreover, from (3.12), we use again the Stability Theorem of viscosity solutions [1, Theorem 2.3], and we obtain:

$$\tilde{\kappa}^n \to \hat{\kappa} \quad \text{locally uniformly in } \bar{Q}_T. \tag{3.17}$$

Back to the entropy solution, we write down the entropy inequality (see Definition 2.8) satisfied by $\tilde{\kappa}^n_x$:

$$\int_{Q_T} \big(\eta(\tilde{\kappa}^n_x)\phi_t + \Phi(\tilde{\kappa}^n_x)\tilde{g}^n\phi_x + h(\tilde{\kappa}^n_x)\tilde{g}^n_x\phi\big)\,dx\,dt + \int_{\mathbb{R}} \eta(\hat{\kappa}^0_x)\phi(x,0)\,dx \geqslant 0, \tag{3.18}$$

where η, Φ, h and ϕ are given by Definition 2.8. Taking (3.16) into consideration, we use a property of bounded sequences in $L^\infty(Q_T)$ (see [11, Proposition 3]) that guarantees the existence of a subsequence (call it again $\tilde{\kappa}_x^n$) so that, for any function $\psi \in C(\mathbb{R};\mathbb{R})$,

$$\psi(\tilde{\kappa}_x^n) \to U_\psi \quad \text{weak-}\star \text{ in } L^\infty(Q_T). \tag{3.19}$$

Furthermore, there exists $\mu \in L^\infty(Q_T \times (0,1))$ such that:

$$\int_0^1 \psi\big(\mu(x,t,\alpha)\big)\,d\alpha = U_\psi(x,t), \quad \text{for a.e. } (x,t) \in Q_T. \tag{3.20}$$

Applying (3.19) with ψ replaced with η, Φ and h respectively, and using (3.20), we get:

$$\begin{cases} \eta(\tilde{\kappa}_x^n(\cdot)) \to \displaystyle\int_0^1 \eta\big(\mu(\cdot,\alpha)\big)\,d\alpha & \text{weak-}\star \text{ in } L^\infty(Q_T),\\ \Phi\big(\tilde{\kappa}_x^n(\cdot)\big) \to \displaystyle\int_0^1 \Phi\big(\mu(\cdot,\alpha)\big)\,d\alpha & \text{weak-}\star \text{ in } L^\infty(Q_T),\\ h\big(\tilde{\kappa}_x^n(\cdot)\big) \to \displaystyle\int_0^1 h\big(\mu(\cdot,\alpha)\big)\,d\alpha & \text{weak-}\star \text{ in } L^\infty(Q_T). \end{cases} \tag{3.21}$$

This, together with (3.12), (3.13) permits to pass to the limit in (3.18) in the distributional sense, hence we get:

$$\int_{Q_T}\int_0^1 \Big(\eta\big(\mu(\cdot,\alpha)\big)\phi_t + \Phi\big(\mu(\cdot,\alpha)\big)\hat{g}\phi_x + h\big(\mu(\cdot,\alpha)\big)\hat{g}_x\phi\Big)\,dx\,dt\,d\alpha + \int_{\mathbb{R}} \eta(\hat{\kappa}_x^0)\phi(x,0)\,dx \geqslant 0. \tag{3.22}$$

In [11, Theorem 3], the function μ satisfying (3.22) is called an entropy process solution. It has been proved to be unique and independent of α. Although this result in [11] was for a divergence-free function $\hat{g} \in C^1(\bar{Q}_T)$, we remark that it can be adapted to the case of any function $\hat{g} \in W^{1,\infty}(\bar{Q}_T)$ (see for instance Remark 4.4 and the proof of [11, Theorem 3]). Using this, we infer the existence of a function $z \in L^\infty(Q_T)$ such that:

$$z(x,t) = \mu(x,t,\alpha), \quad \text{for a.e. } (x,t,\alpha) \in Q_T \times (0,1), \tag{3.23}$$

hence, z is an entropy solution of (3.8). We now make use of (3.23) and we apply equality (3.20) for $\psi(x) = x$ to obtain,

$$z = \text{weak-}\star \lim_{n\to\infty} \tilde{\kappa}_x^n \quad \text{in } L^\infty(Q_T). \tag{3.24}$$

From (3.24) and (3.17) we deduce that,

$$z(x,t) = \hat{\kappa}_x(x,t) \quad \text{a.e. in } Q_T,$$

which completes the proof of Lemma 3.6. □

4. Proof of Theorems 1.2 and 1.3

In this section, we will present the proof of our main results and we will end up by stating some results in the case $I = \mathbb{R}$.

4.1. Proof of Theorem 1.2

We claim that $\kappa = \hat{\kappa}|_{\bar{I}_T}$ is the required solution.

Boundary conditions. In order to recover the boundary conditions, given by (1.9), on $\partial I \times [0,T]$, we proceed as follows. Let M be the set defined by Lemma 3.4 and let $x \in M$. For every $t \in [0,T]$, we write:

$$\left|\hat\kappa(x,t)-\hat\kappa(x,0)\right| \leqslant \int_0^t \left|\hat\kappa_s(x,s)\right| ds \leqslant \int_0^t \left|F_\epsilon\big(x,s,\hat\kappa_x(x,s)\big)\right| ds \leqslant \int_0^t \big(\left|F_\epsilon\big(0,s,\hat\kappa_x(x,s)\big)\right| + C|x|\big)\, ds.$$

In these inequalities, we have used the fact that $\hat\kappa$ is a Lipschitz continuous viscosity solution of (3.7) and hence it verifies the equation in Q_T at the points where it is differentiable (see for instance [1]). Also, we have used the condition (F1) with $p = q$ and $C_R = C$; a constant independent of R. Now from (3.5)(ii), we deduce that:

$$\left|F_\epsilon\big(0,s,\hat\kappa_x(x,s)\big)\right| = \left|\hat\rho_x(0,s)\hat\rho_t(0,s) f_\epsilon\big(\hat\kappa_x(x,s)\big)\right| = 0, \quad \text{for a.e. } s \in (0,t),$$

and hence we get

$$\left|\hat\kappa(x,t)-\hat\kappa(x,0)\right| \leqslant C|x|t. \tag{4.1}$$

Since M is a dense subset of $\mathbb{R}$ (see Lemma 3.4), we pass to the limit in (4.1) as $x \to 0$ and the equality

$$\kappa(0,t) = \kappa(0,0) = \kappa^0(0) \quad \forall t \in [0,T]$$

holds. Similarly, we can verify that $\kappa(1,t) = \kappa(1,0) = \kappa^0(1)$ for all $t \in [0,T]$.

Inequality (1.14). We show a lower-bound estimate of $\hat\kappa_x$. A result of a lower-bound gradient estimate for first-order Hamilton–Jacobi equations was done in [22, Theorem 4.2]. However, this result holds for Hamiltonians $F(x,t,u)$ that are convex in the u-variable, using only the viscosity theory techniques. This is not the case here, and in order to obtain our lower-bound estimate, we need to use the viscosity/entropy theory techniques. In particular, we have the following: the extension $\hat\kappa^0$ of κ^0 outside the interval I is a linear extension of slope $\|\rho_x^0\|_{L^\infty(I)} + \epsilon$. This, together with (1.12) and (3.4) give:

$$\hat\kappa_x^0 \geqslant \sqrt{(\hat\rho_x^0)^2+\epsilon^2} = G_\epsilon(\hat\rho_x^0), \qquad \text{a.e. in } \mathbb{R}. \tag{4.2}$$

From Lemma 3.6, we know that $\hat\kappa_x$ is an entropy solution of (3.8). Also, from Lemma 3.2, we know that $G(\hat\rho_x)$ is an entropy sub-solution of (3.8). Since (4.2) holds, we use the Comparison Theorem 2.11 to get,

$$\hat\kappa_x \geqslant \sqrt{\hat\rho_x^2+\epsilon^2} = G_\epsilon(\hat\rho_x), \quad \text{a.e. in } \bar Q_T,$$

and hence

$$\kappa_x \geqslant G_\epsilon(\rho_x), \quad \text{a.e. in } \bar I_T.$$

The function κ is a viscosity solution of (1.9). Since κ is the restriction of $\hat\kappa$ on $\bar I_T$, $\hat\kappa^0$ and $\hat\rho$ have their automatic replacements κ^0 and ρ respectively on this subdomain. Hence, it is clear that $\kappa \in \mathrm{Lip}(\bar I_T)$ is a viscosity solution of:

$$\begin{cases} \kappa_t + g f_\epsilon(\kappa_x) = 0 & \text{in } I_T,\\ \kappa(x,0) = \kappa^0(x) & \text{in } I,\\ \kappa(0,t) = \kappa^0(0) \quad \text{and} \quad \kappa(1,t) = \kappa^0(1) & \forall 0 \leqslant t \leqslant T, \end{cases} \tag{4.3}$$

where $g(x,t) = -\rho_t(x,t)\rho_x(x,t)$. Let us show that κ is a viscosity solution of (1.9). Consider a test function $\phi \in C^1(I_T)$ such that $\kappa - \phi$ has a local minimum at some point $(x_0,t_0) \in I_T$. Proposition 2.6, together with inequality (1.14) gives that

$$\kappa(\cdot,t_0) \in \mathrm{Lip}(I) \quad \text{and} \quad \kappa_x(\cdot,t_0) \geqslant \epsilon \quad \text{a.e. in } I.$$

We make use of Lemma 3.5 with $\psi(\cdot) = \kappa(\cdot,t_0)$ and $\zeta(\cdot) = \phi(\cdot,t_0)$ to get,

$$\phi_x(x_0,t_0) \geqslant \epsilon. \tag{4.4}$$

Since κ is a viscosity super-solution of

$$\kappa_t - \rho_t\rho_x f_\epsilon(\kappa_x) = 0 \quad \text{in } I_T,$$

we have

$$\phi_t(x_0,t_0) - \rho_t(x_0,t_0)\rho_x(x_0,t_0) f_\epsilon\big(\phi_x(x_0,t_0)\big) \geqslant 0.$$

However, from (4.4), we get

$$\phi_t(x_0,t_0)\phi_x(x_0,t_0) - \rho_t(x_0,t_0)\rho_x(x_0,t_0) \geqslant 0,$$

and hence κ is a viscosity super-solution of

$$\kappa_t\kappa_x = \rho_x\rho_{xx} \quad \text{in } I_T. \tag{4.5}$$

In the same way, we can show that κ is a viscosity sub-solution of (4.5) and hence a viscosity solution.

Uniqueness among solutions verifying (1.14). We argue by contradiction. Since the function

$$\overline{H}(x,t,u) = g(x,t) f_\epsilon(u) \in C(\bar{I}_T \times \mathbb{R})$$

satisfies for a fixed $t \in (0,T)$:

$$\big|\overline{H}(x,t,u) - \overline{H}(y,t,u)\big| \leqslant C\big(|x-y|\big(1+|u|\big)\big)$$

for every $x, y \in (0,1)$ and $u \in \mathbb{R}$, we use [1, Theorem 2.8] to show that κ is the unique viscosity solution of (4.3). Suppose that $\kappa^1 \in \mathrm{Lip}(\bar{I}_T)$ is another viscosity solution of (1.9) verifying (1.14) and $\kappa^1 \neq \kappa$. It is easy to show, from Lemma 3.5, that such a function is again a viscosity solution of (4.3) and hence a contradiction. □

4.2. Proof of Theorem 1.3

Let $0 < \epsilon < 1$ be a fixed constant. Define $\kappa^{0,\epsilon} \in \mathrm{Lip}(I)$ by:

$$\kappa^{0,\epsilon}(x) = \kappa^0(x) + \epsilon x, \quad x \in (0,1). \tag{4.6}$$

From (1.6), it is clear that for a.e. $x \in I$ we have: $\kappa_x^{0,\epsilon} \geqslant G_\epsilon(\rho_x^0)$, and hence from Theorem 1.2, there exists a viscosity solution $\kappa^\epsilon \in \mathrm{Lip}(\bar{I}_T)$ of (1.9), unique among those satisfying (1.14). We will extract a subsequence of κ^ϵ that converges, in a suitable space, to the desired solution. Uniform bounds for the space/time gradients of κ^ϵ will play an important role in the determination of our subsequence.

Remark 4.1. It is worth noticing that the function κ^ϵ is obtained from the restriction of the function $\hat{\kappa}^\epsilon \in \mathrm{Lip}(\bar{Q}_T)$ on $\bar{I}_T$ (see the announcement of the proof of Theorem 1.2). From Lemma 3.6, we know that $\hat{\kappa}_x^\epsilon$ is an entropy solution of (3.8) with initial condition $w^0 = \hat{\kappa}_x^\epsilon(x,0)$ where $\hat{\kappa}^\epsilon(x,0)$ is given by (3.2) with κ^0 replaced by $\kappa^{0,\epsilon}$. Since $\hat{\kappa}_x^\epsilon(x,0) \leqslant \bar{S}(x,0)$, $\bar{S}$ given by (3.9), we use Lemma 3.3 together with Theorem 2.11 to get: $\hat{\kappa}_x^\epsilon \leqslant \bar{S} \leqslant C(T)$ a.e. in $\bar{Q}_T$, and hence:

$$\kappa_x^\epsilon \leqslant C(T) \quad \text{a.e. in } \bar{I}_T. \tag{4.7}$$

In order to obtain an ϵ-uniform upper-bound for κ_t^ϵ, we use directly the equation satisfied by κ^ϵ, namely:

$$\kappa_t^\epsilon\kappa_x^\epsilon - \rho_t\rho_x = 0 \quad \text{in } I_T, \tag{4.8}$$

and inequality (1.14) to get:

$$|\kappa_t^\epsilon| \leqslant \|\rho_{xx}^0\|_{L^\infty(I)} \quad \text{a.e. in } \bar{I}_T. \tag{4.9}$$

From (4.7) and (4.9), we obtain the boundedness of the sequence κ^ϵ in the space $W^{1,\infty}(\bar{I}_T)$, and hence by Ascoli's Theorem, there exists a subsequence of κ^ϵ that converges to $\kappa \in \mathrm{Lip}(\bar{I}_T)$ locally uniformly.

We claim that κ is the required solution.

Existence on I_T. Since $\kappa^\epsilon \to \kappa$ locally uniformly and since the Hamiltonian of (4.8) is independent of ϵ; indeed, for $X = (x,t)$, the Hamiltonian can be written as:

$$H^\epsilon(X, \nabla u) = u_t u_x - \rho_t(X)\rho_x(X),$$

we use the Stability Theorem (see [1, Theorem 2.3]) to conclude that κ is a viscosity solution of the limit equation:

$$\kappa_t\kappa_x = \rho_t\rho_x \quad \text{in } I_T.$$

Initial and boundary conditions. From (4.6), it is easy to see that $\kappa^{0,\epsilon} \to \kappa^0$ uniformly on I. Moreover, Theorem 1.2 guarantees that:

$$\kappa^\epsilon(0,t) = \kappa^{0,\epsilon}(0) = \kappa^0(0), \tag{4.10}$$

and

$$\kappa^\epsilon(1,t) = \kappa^{0,\epsilon}(1) = \kappa^0(1) + \epsilon, \tag{4.11}$$

for all $t \in [0,T]$. From (4.10), (4.11) and the pointwise convergence, up to a subsequence, of κ^ϵ to κ, we deduce that

$$\kappa(0,t) = \lim_{\epsilon\to 0} \kappa^\epsilon(0,t) = \kappa^0(0), \quad \forall t \in [0,T], \tag{4.12}$$

and

$$\kappa(1,t) = \lim_{\epsilon\to 0} \kappa^\epsilon(1,t) = \lim_{\epsilon\to 0}\big(\kappa^0(1)+\epsilon\big) = \kappa^0(1) \quad \forall t \in [0,T]. \tag{4.13}$$

Therefore, the initial and the boundary conditions are recovered.

Inequality (1.15). From (1.14), we have, for (x,t) and (y,t) close enough,

$$\frac{\kappa^\epsilon(y,t) - \kappa^\epsilon(x,t)}{y-x} \geqslant \big|\rho_x(x,t)\big| - \delta,$$

and hence, by the continuity of ρ_x, we obtain

$$\kappa_x^\epsilon(y,t) \geqslant \big|\rho_x(x,t)\big| - \delta \quad \text{for } y \in (x-r, x+r),\ r = r(\delta).$$

Therefore

$$\kappa_x \geqslant |\rho_x| - \delta \quad \forall \delta > 0,$$

and the inequality follows. □

At this stage, it is worth mentioning that the case of posing the problems (1.4) and (1.5) on the real line $\mathbb{R}$ instead of the bounded interval I (call these new problems (1.4′) and (1.5′) respectively), leads to similar results that can be proved along the same principles (we refer the interested reader to [17, Theorem 1.2, Theorem 1.6]). Let us just mention that this case is slightly less technical and the results that can be shown are:

Theorem 4.2 *(Existence and uniqueness of a viscosity solution). Let $T > 0$. Take $\kappa^0 \in \mathrm{Lip}(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$ as initial data that satisfy*:

$$\kappa_x^0 \geqslant \sqrt{(\rho_x^0)^2 + \epsilon^2} \quad \textit{a.e. in } \mathbb{R},$$

for some constant $\epsilon > 0$. Then, given the solution ρ of the heat equation (1.5′)*, there exists a viscosity solution $\kappa \in \mathrm{Lip}(\bar{Q}_T)$ of* (1.4′)*, unique among the viscosity solutions satisfying*:

$$\kappa_x \geqslant \sqrt{\rho_x^2 + \epsilon^2} \quad \textit{a.e. in } \bar{Q}_T.$$

Theorem 4.3 *(Existence of a viscosity solution, case $\epsilon = 0$). Let $T > 0$, $\kappa^0 \in \mathrm{Lip}(\mathbb{R})$ and $\rho^0 \in C_0^\infty(\mathbb{R})$. If the condition* (1.6) *is satisfied a.e. in $\mathbb{R}$, and ρ is the solution of* (1.5′)*, then there exists a viscosity solution $\kappa \in \mathrm{Lip}(\bar{Q}_T)$ of* (1.4′) *satisfying*:

$$\kappa_x \geqslant |\rho_x| \quad \textit{a.e. in } \bar{Q}_T.$$

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Appendix A. Sketch of the proof of Theorem 2.11

We will work on a variant of the entropy inequality (see for instance [19,11]) satisfied by u and its analogue satisfied by v. We write down these inequalities for clarity:

1. $u(x,t)$ satisfies:

$$\int_{Q_T} \big[\big(u(x,t)-k\big)^+\phi_t(x,t) + \operatorname{sgn}^+\big(u(x,t)-k\big)\big(f\big(v(x,t)\big)-f(k)\big)g(x,t)\phi_x(x,t) - \operatorname{sgn}^+\big(u(x,t)-k\big)f(k)g_x(x,t)\phi(x,t)\big]\,dx\,dt + \int_R \big(u^0(x)-k\big)^+\phi(x,0)\,dx \geqslant 0, \tag{A.1}$$

2. $v(y,s)$ satisfies:

$$\int_{Q_T} \big[\big(v(y,s)-k\big)^-\phi_s(y,s) + \operatorname{sgn}^-\big(v(y,s)-k\big)\big(f\big(v(y,s)\big)-f(k)\big)g(y,s)\phi_y(y,s) - \operatorname{sgn}^-\big(v(y,s)-k\big)f(k)g_y(y,s)\phi(y,s)\big]\,dy\,ds + \int_R \big(v^0(y)-k\big)^-\phi(y,0)\,dy \geqslant 0, \tag{A.2}$$

where $a^\pm = \frac{1}{2}(|a| \pm a)$ and $\operatorname{sgn}^\pm(x) = \frac{1}{2}(\operatorname{sgn}(x) \pm 1)$. We use the dedoubling variable technique of Kružkov (see [19]) and following the same steps of [11, Theorem 3], taking into consideration the new modifications arising from the fact that we are dealing with sub-/super-entropy solutions and the fact that $g \in W^{1,\infty}(\bar{Q}_T)$ is not a gradient-free function. The proof can be divided into three steps. Denote B_r by $B_r = \{x \in \mathbb{R}; |x| \leqslant r\}$ for any $r > 0$, $F^\pm(u,v) = \operatorname{sgn}^\pm(u-v)(f(u)-f(v))$,

$$y^\infty = \|y\|_{L^\infty(Q_T)} \quad \text{for every } y \in L^\infty(Q_T) \tag{A.3}$$

and

$$M_f = \max_{|x| \leqslant \max(u^\infty, v^\infty)} \big|f'(x)\big|. \tag{A.4}$$

In step 1, we prove that the initial conditions u^0, v^0 satisfy for any $a > 0$:

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau \int_{B_a} \big(u(x,t)-u^0(x)\big)^+\,dx\,dt = 0, \tag{A.5}$$

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau \int_{B_a} \big(v(x,t)-v^0(x)\big)^-\,dx\,dt = 0, \tag{A.6}$$

respectively.

In step 2, The following relation between u and v is shown:

$$\int_{Q_T} \big[\big(u(x,t)-v(x,t)\big)^+\psi_t + F^+\big(u(x,t),v(x,t)\big)g(x,t)\psi_x\big]\,dx\,dt \geqslant 0, \tag{A.7}$$

for every $\psi \in C_0^1(\mathbb{R} \times (0,T); \mathbb{R}_+)$.

After that, we define $A(t)$ for $0 < t < \min(T, \frac{a}{\omega})$ and $\omega = g^\infty M_f$, by:

$$A(t) = \int_{B_{a-\omega t}} \big(u(x,t)-v(x,t)\big)^+\,dx. \tag{A.8}$$

In step 3, we show that A is non-increasing a.e. in $(0, \min(T, \frac{a}{\omega}))$ and we deduce that

$$u(x,t) \leqslant v(x,t) \quad \text{a.e. in } Q_T.$$

Step 1: Proof of (A.5), (A.6). This is similar to Step 1 in [11, Theorem 3]. In fact, let $a > 0$ and $\tau \in \mathbb{R}$ such that $0 < \tau < T$. Consider the test function

$$\phi(x,t) = \psi(x)\xi^n(x-y)\gamma(t) \quad \text{where } \gamma(t) = \begin{cases} \dfrac{\tau - t}{\tau} & \text{if } 0 \leqslant t \leqslant \tau, \\ 0 & \text{if } t > \tau, \end{cases} \tag{A.9}$$

and $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) = 1$, $\forall x \in B_a$. Let y be a Lebesgue point of u^0. Upon plugging the test function ϕ and the constant $k = u^0(y)$ into (A.1); integrating the resulting relation over $\mathbb{R}$ with respect to y, we get similar terms to that in [11] with $|\cdot|$, $\mathrm{sgn}(\cdot)$ and $\mu(x,t,\alpha)$ replaced by $(\cdot)^+$, $\mathrm{sgn}^+(\cdot)$ and $u(x,t)$ respectively. However, the fact that our g is not a divergence-free function adds a new term which is:

$$T_{3n\tau} = -\int_0^\tau \int_{\mathbb{R}^2} \mathrm{sgn}^+\big(u(x,t) - u^0(y)\big) f\big(u^0(y)\big) g_x(x,t)\gamma(t)\psi(x)\xi^n(x-y)\,dx\,dy\,dt, \tag{A.10}$$

and can be treated in exactly the same way as $T_{2n\tau}$ (see Step 1 [11, Theorem 3] for the details).

Step 2: Proof of (A.7). We follow Step 2 of [11, Theorem 3]. Taking regularizations of ψ, it suffices to show (A.7) for $\psi \in C_0^\infty(Q_T;\mathbb{R}_+)$. We may assume without loss of generality that there is some $c > 0$ such that $\psi(x,t) = 0$ for $t \in (0,c) \cup (T-c,T)$. For $n > \frac{1}{c}$, let ξ^n be the usual mollifier sequence in $\mathbb{R}$ and consider the test function $\phi(x,t,y,s)$ defined for $(x,t), (y,s) \in Q_T$ by:

$$\phi(x,t,y,s) = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\xi^n(x-y)\xi^n(t-s).$$

Fix $(y,s) \in Q_T$ a Lebesgue point of v, and $(x,t) \in Q_T$ a Lebesgue point of u. We plug, on one hand, the test function $\phi(\cdot,\cdot,y,s)$ and the constant $k = v(y,s)$ into (A.1); integrate the resulting relation over Q_T with respect to (y,s). And on the other hand, we plug the test function $\phi(x,t,\cdot,\cdot)$ and the constant $k = u(x,t)$ into (A.2); integrate the resulting relation over Q_T with respect to (x,t). Upon performing the necessary change of variables, and making some elementary identity transformations in the integrands (which consist of adding and subtracting identical functions and arranging similar terms), we get:

$$\mathcal{X}_1^n + \mathcal{X}_2^n + \mathcal{X}_3^n + \mathcal{X}_4^n \geqslant 0, \tag{A.11}$$

where $\mathcal{X}_1^n$ and $\mathcal{X}_2^n$ are same as X_{1n} and X_{2n} from [11] with $\nu(x,t,\alpha)$, $\mu(x,t,\alpha)$ and F replaced by u, v and F^+ respectively. Thus, it is easy to see that:

$$\mathcal{X}_1^n \to \int_{Q_T} \big(u(x,t) - v(x,t)\big)^+ \psi_t(x,t)\,dx\,dt \quad \text{as } n \to \infty, \tag{A.12}$$

and

$$\mathcal{X}_2^n \to \int_{Q_T} F^+\big(u(x,t), v(x,t)\big) g(x,t)\psi_x(x,t)\,dx\,dt \quad \text{as } n \to \infty. \tag{A.13}$$

The remaining terms $\mathcal{X}_3^n$ and $\mathcal{X}_4^n$ can be written as follows:

$$\mathcal{X}_3^n = \int_{\mathcal{Q}_4} F^+\big(u(x^+,t^+), v(x^-,t^-)\big)\big(g(x^+,t^+) - g(x^-,t^-)\big)\psi(x,t)n\xi'(y)\xi(s)\,dx\,dt\,dy\,ds, \tag{A.14}$$

$$\begin{aligned}\mathcal{X}_4^n = \int_{\mathcal{Q}_4} &\mathrm{sgn}^+\big(u(x^+,t^+) - v(x^-,t^-)\big)\big[f\big(u(x^+,t^+)\big) g_x(x^-,t^-) \\ &- f\big(v(x^-,t^-)\big) g_x(x^+,t^+)\big]\psi(x,t)\xi(y)\xi(s)\,dx\,dt\,dy\,ds,\end{aligned} \tag{A.15}$$

where $x^+ = x + \frac{y}{2n}$, $x^- = x - \frac{y}{2n}$, $t^+ = t + \frac{s}{2n}$ and $t^- = t - \frac{s}{2n}$, taken for simplicity. These two terms will be treated independently. At this point, it is worth mentioning that we will frequently use the following lemma from [20]:

Lemma A.1. *If $\Gamma \in \mathrm{Lip}(\mathbb{R})$ satisfies $|\Gamma(u) - \Gamma(v)| \leqslant C_0|u - v|$, then the function*

$$H(u, v) = \mathrm{sgn}^+(u - v)\big(\Gamma(u) - \Gamma(v)\big)$$

satisfies $|H(u, v) - H(u', v')| \leqslant C_0(|u - u'| + |v - v'|)$ (see [19, Lemma 3]).

We now study the two terms $\mathcal{X}_3^n$ and $\mathcal{X}_4^n$. From the fact that $g \in W^{1,\infty}(\bar{Q}_T)$, we remark that for a.e. $(x, t, y, s) \in Q_T \times Q_T$, we have:

$$g(x^-, t^-) - g(x^+, t^+) = g_x(x^-, t^-)(-y/n) + g_t(x^-, t^-)(-s/n) + \mathrm{o}\left(\frac{1}{n}\right).$$

We also remark that the term $g_x(x^+, t^+)$ in $\mathcal{X}_4^n$ could be replaced by $g_x(x^-, t^-)$, since this adds a term that approaches 0 as n becomes large. This term will be omitted throughout what follows and we denote the new $\mathcal{X}_4^n$ by $\tilde{\mathcal{X}}_4^n$. From these two remarks, we rewrite $\mathcal{X}_3^n$ and $\tilde{\mathcal{X}}_4^n$ to get:

$$\mathcal{X}_3^n = \int\limits_{\mathcal{Q}_4} \mathrm{sgn}^+\big(u(x^+, t^+) - v(x^-, t^-)\big)\big(f\big(u(x^+, t^+)\big) - f\big(v(x^-, t^-)\big)\big) \\ \times \big(y g_x(x^-, t^-) + s g_t(x^-, t^-)\big)\psi(x, t)\xi'(y)\xi(s)\,dx\,dt\,dy\,ds + \mathcal{L}(n), \tag{A.16}$$

where $\mathcal{L}(n) \to 0$ as $n \to \infty$, and

$$\tilde{\mathcal{X}}_4^n = \int\limits_{\mathcal{Q}_4} \mathrm{sgn}^+\big(u(x^+, t^+) - v(x^-, t^-)\big)\big(f\big(u(x^+, t^+)\big) - f\big(v(x^-, t^-)\big)\big) \\ \times g_x(x^-, t^-)\psi(x, t)\xi(y)\xi(s)\,dx\,dt\,dy\,ds. \tag{A.17}$$

The term $\mathcal{L}(n)$ will also be omitted for simplification and we denote the new $\mathcal{X}_3^n$ by $\tilde{\mathcal{X}}_3^n$. Let $\mathcal{X}_{34}^n = \tilde{\mathcal{X}}_3^n + \tilde{\mathcal{X}}_4^n$, hence:

$$\mathcal{X}_{34}^n = \overbrace{\int\limits_{\mathcal{Q}_4} F^+\big(u(x^+, t^+), v(x^-, t^-)\big) g_x(x^-, t^-)\psi(x, t)\big(y\xi(y)\xi(s)\big)_y\,dx\,dt\,dy\,ds}^{\mathcal{X}_{34}^{1n}} \\ + \overbrace{\int\limits_{\mathcal{Q}_4} F^+\big(u(x^+, t^+), v(x^-, t^-)\big) g_t(x^-, t^-)\psi(x, t)\big(s\xi(y)\xi(s)\big)_y\,dx\,dt\,dy\,ds}^{\mathcal{X}_{34}^{2n}}. \tag{A.18}$$

In $\mathcal{X}_{34}^{1n}$ and $\mathcal{X}_{34}^{2n}$, the term $\psi(x, t)$ could be replaced by $\psi(x^-, t^-)$, for this also adds a term getting small when $n \to \infty$. We keep the same notations for $\mathcal{X}_{34}^{1n}$ and $\mathcal{X}_{34}^{2n}$. Since $y\xi(y)\xi(s)$ is a compactly supported smooth function in $\mathcal{Q}_4$, we have:

$$\int\limits_{\mathcal{Q}_4} F^+\big(u(x^-, t^-), v(x^-, t^-)\big) g_x(x^-, t^-)\psi(x^-, t^-)\big(y\xi(y)\xi(s)\big)_y\,dx\,dt\,dy\,ds = 0. \tag{A.19}$$

Moreover, since $F^+(u, v)$ is Lipschitz, we obtain:

$$\left|\mathcal{X}_{34}^{1n} - \int\limits_{\mathcal{Q}_4} F^+\big(u(x^-, t^-), v(x^-, t^-)\big) g_x(x^-, t^-)\psi(x^-, t^-)\big(y\xi(y)\xi(s)\big)_y\,dx\,dt\,dy\,ds\right| \\ \leqslant M_f (g_x)^\infty \psi^\infty \int\limits_{K_\psi}\int\limits_{B_1^2} \big|u(x^+, t^+) - u(x^-, t^-)\big|\,dx\,dt\,dy\,ds, \tag{A.20}$$

where K_ψ is the support of ψ. Therefore, by the Lebesgue Differentiation/Dominated Theorems, we deduce that the right-hand side of (A.20) tends to 0 as $n \to \infty$, hence we have:

$$\mathcal{X}_{34}^{1n} \to 0 \quad \text{as } n \to \infty. \tag{A.21}$$

In a similar way we can show that $\mathcal{X}_{34}^{2n} \to 0$ as $n \to \infty$. Passing to the limit as $n \to \infty$ in (A.11) yields (A.7), which concludes the proof of step 2.

Step 3: $u(x,t) \leqslant v(x,t)$ a.e. in Q_T. Following Step 3 of [11, Theorem 3], always taking into consideration the slight differences that are now clear from steps 1 and 2, we reach the following: if t_1 and t_2 are two Lebesgue points of the function A such that $0 < t_1 < t_2 < \min(T, \frac{a}{\omega})$, one have:

$$A(t_1) \geqslant A(t_2).$$

We move now to the goal of step 3. Using some elementary identities, we calculate for a.e. $(x,t) \in Q_T$:

$$\big(u(x,t) - v(x,t)\big)^+ \leqslant \big(u(x,t) - u^0(x)\big)^+ + \big(v(x,t) - v^0(x)\big)^- + \big(u^0(x) - v^0(x)\big)^+.$$

Since $u^0(x) \leqslant v^0(x)$ a.e. in $\mathbb{R}$, we get for a.e. $(x,t) \in Q_T$:

$$\big(u(x,t) - v(x,t)\big)^+ \leqslant \big(u(x,t) - u^0(x)\big)^+ + \big(v(x,t) - v^0(x)\big)^-. \tag{A.22}$$

Using (A.22) for $\tau \in (0,T)$, we get:

$$\begin{aligned} \frac{1}{\tau}\int_0^\tau A(t)\,dt &\leqslant \frac{1}{\tau}\int_0^\tau \int_{B_a} \big(u(x,t) - v(x,t)\big)^+\,dx\,dt \\ &\leqslant \frac{1}{\tau}\int_0^\tau \int_{B_a} \big(u(x,t) - u^0(x)\big)^+\,dx\,dt + \frac{1}{\tau}\int_0^\tau \int_{B_a} \big(v(x,t) - v^0(x)\big)^-\,dx\,dt. \end{aligned} \tag{A.23}$$

From (A.5), (A.6) and the passage to the limit as $\tau \to 0$ in (A.23), we deduce that,

$$\frac{1}{\tau}\int_0^\tau A(t)\,dt \to 0 \quad \text{as } \tau \to 0. \tag{A.24}$$

Thus, since A is a.e. non-increasing on $(0,\tau)$, and $A(t) \geqslant 0$ for a.e. $t \in (0, \min(T, \frac{a}{\omega}))$, one then has

$$A(t) = 0 \quad \text{for a.e. } t \in \left(0, \min\left(T, \frac{a}{\omega}\right)\right).$$

Since a is arbitrary, we deduce that,

$$u(x,t) \leqslant v(x,t) \quad \text{a.e. in } Q_T.$$

□

Remark 4.4. In [11], the entropy process solution $\mu(x,t,\alpha)$ was proved to be independent of α for a divergence-free function $g \in C^1(\bar{Q}_T)$. However, for the case of a general non-divergence-free function $g \in W^{1,\infty}(\bar{Q}_T)$, same result can be shown by adapting the same proof as in [11, Theorem 3] taking into account the slight modifications that could be deduced from the proof of Theorem 2.11. More precisely, the treatment of the two terms $\mathcal{X}_3^n$ and $\mathcal{X}_4^n$ in Step 2.

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