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Traveling waves in a one-dimensional heterogeneous medium

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Abstract

We consider solutions of a scalar reaction–diffusion equation of the ignition type with a random, stationary and ergodic reaction rate. We show that solutions of the Cauchy problem spread with a deterministic rate in the long time limit. We also establish existence of generalized random traveling waves and of transition fronts in general heterogeneous media.

1. Introduction

We consider solutions to the equation

$$u_t = \Delta u + f(x,u,\omega), \quad x \in \mathbb{R}, \tag{1.1}$$

where $f(x,u,\omega)$ is a random ignition-type non-linearity that is stationary with respect to translation in x. The function f has the form $f(x,u,\omega) = g(x,\omega) f_0(u)$. Here, $f_0(u)$ is an ignition-type non-linearity with an ignition temperature $\theta_0 \in (0,1)$: $f_0(u)$ is a Lipschitz function, and, in addition,

$$f_0(u) = 0 \quad \text{for } u \in [0,\theta_0] \cup \{1\}, \qquad f_0(u) > 0 \quad \text{for } u \in (\theta_0, 1), \qquad f_0'(1) < 0.$$

The reaction rate $g(x,\omega)$, $x \in \mathbb{R}$, is a stationary, ergodic random field defined over a probability space $(\Omega, \mathbb{P}, \mathcal{F})$: there exists a group $\{\pi_x\}$, $x \in \mathbb{R}$, of measure-preserving transformations acting ergodically on $(\Omega, \mathbb{P}, \mathcal{F})$ such that $g(x+h,\omega) = g(x, \pi_h \omega)$. We suppose that $g(x,\omega)$ is almost surely Lipschitz continuous with respect to x and that there are deterministic constants $g^{\min}$, $g^{\max}$ such that

$$0 < g^{\min} \leqslant g(x,\omega) \leqslant g^{\max} < \infty$$

holds almost surely. Thus, we have

$$f^{\min}(u) \leqslant f(x,u,\omega) \leqslant f^{\max}(u),$$

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where $f^{\min}(u) = g^{\min} f_0(u)$ and $f^{\max}(u) = g^{\max} f_0(u)$ are both ignition-type non-linearities with the same ignition temperature. We assume that the probability space $\Omega = C(\mathbb{R}; [g^{\min}, g^{\max}])$ and that $\mathcal{F}$ contains the Borel σ-algebra generated by the compact open topology (the topology of locally uniform convergence) on $C(\mathbb{R}; [g^{\min}, g^{\max}])$.

We are interested in the following two issues: first, how do solutions of the Cauchy problem for (1.1) with a compactly supported non-negative initial data spread in the long time limit? Second, do there exist special solutions of (1.1) that generalize the notion of a traveling front in the homogeneous case?

It is well known since the pioneering work by Ya. Kanel [14] that in the uniform case:

$$u_t = \Delta u + f(u) \tag{1.2}$$

with an ignition-type non-linearity $f(u)$, all solutions with the initial data $u_0(x) = u(0, x)$ in a class $I \subset C_c(\mathbb{R})$, $0 \leqslant u_0(x) \leqslant 1$, propagate with the same speed c^* in the sense that

$$\lim_{t\to+\infty} u(t, ct) = 0 \quad \text{for } |c| > c^*, \tag{1.3}$$

and

$$\lim_{t\to+\infty} u(t, ct) = 1 \quad \text{for } |c| < c^*. \tag{1.4}$$

The initial data is restricted to the class I to preclude the possibility of the so-called quenching phenomenon where $u \to 0$ uniformly in x as $t \to \infty$. In particular, I contains functions that are larger than $\theta_0 + \varepsilon$ on a sufficiently large interval, depending on $\varepsilon > 0$. The constant c^* above is the speed of the unique traveling wave solution $u(t, x) = U(x - c^*t)$ of (1.2):

$$-c^*U' = U'' + f(U), \qquad U(-\infty) = 1, \qquad U(+\infty) = 0.$$

As far as heterogeneous media are concerned this result has been extended to the periodic case: J. Xin [29,30], and H. Berestycki and F. Hamel [2] have shown that when the function $f(x, u)$ is periodic in x, Eq. (1.1) admits special solutions of the form $u(t, x) = U(x - c^*t, x)$, called pulsating fronts, which are periodic in the second variable and satisfy

$$U(s, x) \to 1 \quad \text{as } s \to -\infty, \quad \text{and} \quad U(s, x) \to 0 \quad \text{as } s \to +\infty.$$

H. Weinberger [28] has proved that solutions with general non-negative compactly supported initial data spread with the speed c^* in the sense of (1.3)–(1.4), though the spreading rates to the left and right may now be different.

The purpose of the present paper is to extend the result of [28] to the stationary random ergodic case, and show that special solutions which generalize the notion of a pulsating front to random media exist.

Deterministic spreading rates

Our first result concerns the asymptotic behavior of solutions to the Cauchy problem for (1.1) with compactly supported initial data. We show that for sufficiently large initial data the solution develops two diverging fronts that propagate with a deterministic asymptotic speed. Specifically, we prove the following.

Theorem 1.1. *Let $w(t, x, \omega)$ solve (1.1) with compactly supported deterministic initial data $w_0(x)$, $0 \leqslant w_0(x) \leqslant 1$. Let $h \in (\theta_0, 1)$ and suppose that $w_0 \geqslant h$ on an interval of size $L > 0$. There exist deterministic constants $c_-^* < 0 < c_+^*$ such that for any $\varepsilon > 0$, the limits*

$$\lim_{t\to\infty} \inf_{c\in[c_-^*+\varepsilon, c_+^*-\varepsilon]} w(t, ct, \omega) = 1$$

and

$$\lim_{t\to\infty} \sup_{c\in(-\infty, c_-^*-\varepsilon]\cup[c_+^*+\varepsilon, \infty)} w(t, ct, \omega) = 0$$

hold almost surely with respect to $\mathbb{P}$, if L is sufficiently large. The constants c_-^*, c_+^* are independent of h and L.

The condition that L be sufficiently large is necessary only to exclude the possibility of uniform convergence to zero [14].

Using large deviation techniques, Freidlin, and Freidlin and Gärtner (see [8, Section 7.4], [7,9–11]) proved a similar asymptotic result in the case that $f_0(u)$ is of KPP-type satisfying $f(u) \leqslant f'(0)u$ (e.g. $f_0(u) = u(1-u)$). Moreover, the asymptotic speed can be identified by a variational principle that arises from the linearized problem at $u = 0$. This asymptotic spreading result has been extended recently to time-dependent random media in [24,25]. The problem with a KPP non-linearity also admits homogenization, both in the periodic [20] and random [15,19] cases. However, in all aforementioned papers, the KPP condition $f(u) \leqslant f'(0)u$ seems to be essential, and the techniques do not extend to the present case where f vanishes when u is close to zero. To the best of our knowledge Theorem 1.1 is the first result on the deterministic spreading rates of solutions of reaction–diffusion equations with a non-KPP non-linearity in a random medium.

Random traveling waves

Two generalizations of the notion of a traveling front in a uniform medium for general (non-periodic) inhomogeneous media were proposed. Shen in [27], and Berestycki and Hamel in [3,4] have introduced generalized transition fronts (called wave-like solutions in [27]) — these are global in time solutions that, roughly speaking, have an interface which "stays together" uniformly in time. On the other hand, H. Matano has defined a generalized traveling wave as a global in time solution whose shape is "a continuous function of the current environment" [22]. These notions are not equivalent: there exist transition fronts of the KPP equation with constant coefficients that are not traveling waves in the usual sense (and hence not generalized traveling waves in the sense of Matano as there is only one environment in the case of a uniform medium and thus only one solution profile) [12,13].

Matano's definition was formalized by W. Shen in [27] as follows.

Definition 1.2. (See [27, Definition 2.2].) A solution $\tilde{w}(t,x,\omega) : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ of (1.1) is called a **random traveling wave** if the following hold:

(i) For almost every $\omega \in \Omega$, $\tilde{w}(t,x,\omega)$ is a classical solution of (1.1) for all $t \in \mathbb{R}$.
(ii) The function $\tilde{w}(0,x,\omega)$ is measurable with respect to ω.
(ii) $0 < \tilde{w}(0,x,\omega) < 1$, $\forall x \in \mathbb{R}$.
(iii) $\lim_{x\to+\infty} \tilde{w}(0,x,\omega) = 0$.
(iv) $\lim_{x\to-\infty} \tilde{w}(0,x,\omega) = 1$.
(v) There exists a measurable function $\tilde{X}(t,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$\tilde{w}(t,x,\omega) = \tilde{w}\big(0, x - \tilde{X}(t,\omega), \pi_{\tilde{X}(t,\omega)}\omega\big).$$

The function $W(x,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ defined by $W(x,\omega) = \tilde{w}(0,x,\omega)$ is said to **generate** the random traveling wave.

The random function $W(x,\omega)$ is the profile of the wave in the moving reference frame defined by the current front position $\tilde{X}(t,\omega)$. In the pioneering paper [27], Shen has established some general criteria for the existence of a traveling wave in ergodic spatially and temporally varying media and also proved some important properties of the wave. In particular, as shown in [27] (see [27, Theorem B]), Definition 1.2 of a random traveling wave generalizes the notion of a pulsating traveling front in the periodic case. More precisely, if f is actually periodic in x, then a random traveling wave solution of (1.1) is a pulsating traveling front solution in the sense of [2,29,30].

However, the only example provided in [27] where the results of [27] ensure existence of a random traveling wave is a bistable reaction–diffusion equation of the form

$$u_t = u_{xx} + (1-u)(1+u)\big(u - a(t)\big),$$

where $a(t)$ is a stationary ergodic random process. As far as we are aware, no other examples of such traveling waves in non-periodic media have been exhibited. In this paper we construct a Matano–Shen traveling wave in a spatially varying ergodic random medium for (1.1) with an ignition-type non-linearity.

Theorem 1.3. *There exists a random traveling wave solution* $\tilde{w}(t,x,\omega)$ *of* (1.1) *which is increasing monotonically in time*:

$$\tilde{w}_t(t,x,\omega) > 0 \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}.$$

Moreover, the interface $\tilde{X}(t,\omega)$ *satisfies*

(i) $\tilde{w}(t,\tilde{X}(t,\omega),\omega) = \theta_0$ *for all* $t \in \mathbb{R}$, *and*
(ii) $\tilde{X}(t+h,\omega) > \tilde{X}(t,\omega)$ *for all* $t \in \mathbb{R}$ *and* $h > 0$, *and*

(iii)
$$\lim_{t\to\infty} \frac{\tilde{X}(t,\omega)}{t} = c_+^* \tag{1.5}$$

holds almost surely, where c_+^* *is the same constant as in Theorem* 1.1.

Monotonicity of the wave and the fact that the interface is moving to the right is the direct analog of the corresponding properties of the periodic pulsating fronts.

Since $\tilde{X}(t,\omega)$ is increasing in t, we may define its inverse $\tilde{T}(x,\omega): \mathbb{R}\times\Omega \to \mathbb{R}$ by

$$x = \tilde{X}\big(\tilde{T}(x,\omega),\omega\big). \tag{1.6}$$

This may be interpreted as the time at which the interface reaches the position $x \in \mathbb{R}$. The following corollary says that the statistics of the profile of the wave as the wave passes through the point ξ are invariant with respect to ξ:

Corollary 1.4. *The function* $\tilde{w}(\tilde{T}(\xi,\omega), x+\xi,\omega)$ *is stationary with respect to shifts in* ξ.

This is a direct analog of the corresponding property of a pulsating front in a periodic medium: the profile of a pulsating front at the time $T(\xi)$ it passes a point ξ is periodic in ξ.

We believe the present article gives the first construction of such a wave in a spatially random medium. To construct the wave, we use a dynamic approach from [27] combined with some analytical estimates needed to show that the construction produces a non-trivial result.

Generalized transition fronts

Our last result concerns existence of the transition fronts for (1.1) in the sense of Berestycki and Hamel, and Shen, in general heterogeneous (non-random) media with the reaction rate uniformly bounded from below and above. Let us recall first the definition of a generalized transition wave.

Definition 1.5. A global in time solution $\tilde{v}(t,x)$, $t\in\mathbb{R}$, $x\in\mathbb{R}$, of (1.1) is called a transition wave if for any $h,k \in (0,1)$ with $h > k$, we have

$$0 \leqslant \theta_k^+(t,\omega) - \theta_h^-(t,\omega) \leqslant C \tag{1.7}$$

for all $t \in \mathbb{R}$, where

$$\begin{aligned} \theta_h^-(t,\omega) &= \sup\big\{x\in\mathbb{R} \,\big|\, \tilde{v}(t,x',\omega) > h,\ \forall x' < x\big\},\\ \theta_k^+(t,\omega) &= \inf\big\{x\in\mathbb{R} \,\big|\, \tilde{v}(t,x',\omega) < k,\ \forall x' > x\big\} \end{aligned} \tag{1.8}$$

and $C = C(h,k)$ is a constant independent of t and ω.

Roughly speaking, a transition wave is a global in time solution for which there are uniform, global-in-time bounds on the width of the interface. Basic properties of transition waves were investigated in [3,4].

Theorem 1.6. *Let* $f(x,u)$ *be a non-linearity such that* $g^{\min} f_0(u) \leqslant f(x,u) \leqslant g^{\max} f_0(u)$, *with the constants* $g^{\min} > 0$, $g^{\max} < +\infty$ *and* $f_0(u)$ *an ignition-type non-linearity. Then there exists a transition front solution* $u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$, *of* (1.1) *which is monotonically increasing in time*: $u_t(t,x) > 0$. *In addition, there exists a unique point* $X(t)$ *such that* $u(t,X(t)) \equiv \theta_0$, *and a constant* $p > 0$ *so that* $u_x(t,X(t)) < -p$ *for all* $t \in \mathbb{R}$.

As this paper was written we learned about the concurrent work by A. Mellet, J.-M. Roquejoffre, and Y. Sire [23], in which Theorem 1.6 is also proved.

Let us point out that all of the results in this paper extend to the case of a bistable-type non-linearity, under certain restrictions. Specifically, we may let f have the form $f(x,u) = g(x) f_0(u)$ with $f_0(0) = f_0(\theta_0) = f_0(1) = 0$, $f_0(u) < 0$ for $u \in (0, \theta_0)$, $f_0(u) > 0$ for $u \in (\theta_0, 1)$, and $f_0'(1) < 0$. Under the additional condition that

$$\int_0^1 f^{\min}(u)\,du = \int_0^1 \Big(\min_{x\in\mathbb{R}} f(x,u)\Big)\,du > 0, \tag{1.9}$$

all of the results in Theorems 1.1, 1.3, and 1.6 apply. This condition is necessary to preclude the phenomenon of wave-blocking, which can occur with a spatially-dependent bistable-type non-linearity (for example, see the work Lewis and Keener [17]). In particular, under the condition (1.9), one can modify our argument to construct the time-monotonic solutions that are the building blocks for the generalized transition fronts.

The paper is organized as follows. In Section 2, we study solutions to (1.1) that are monotone increasing in time and prove Theorem 1.6. The main ingredients in the proof are Propositions 2.3 and 2.5 which show that the interface (the region where $\varepsilon < u < 1 - \varepsilon$, for some $\varepsilon > 0$) may not be arbitrarily wide and must move forward with an instantaneous speed that is bounded above and below away from zero. These estimates are also used later in the proof of the asymptotic spreading and in the construction of the random traveling waves. In Section 3 we prove Theorem 1.1, first for monotone increasing solutions and then for general compactly supported data. In Section 4 we construct the random traveling wave and prove Theorem 1.3 and Corollary 1.4.

Throughout the paper we denote by C and K universal constants that depend only on the constants $g^{\min}$ and $g^{\max}$, and the function $f_0(u)$.

2. Existence of a generalized transition front

Monotonic in time solutions

In this section we prove Theorem 1.6. The generalized transition wave is constructed as follows. We consider a sequence of solutions $u_n(t,x)$ of (1.1) (with $f(x,u,\omega)$ replaced by $f(x,u)$ as in the statement of the theorem) defined for $t \geqslant -n$, with the Cauchy data

$$u_n(t=-n,x) = \zeta\big(x - x_0^n\big), \qquad \zeta(x) := \max\big(\hat{\zeta}(x), 0\big). \tag{2.1}$$

The choice of the initial shift x_0^n is specified below while the function $\hat{\zeta}(x)$ is positive on an open interval and is a sub-solution for (1.1):

$$-\hat{\zeta}''(x) = f^{\min}\big(\hat{\zeta}(x)\big) \leqslant f(x,\hat{\zeta}), \tag{2.2}$$

with $f^{\min}(u) = g^{\min} f_0(u) \leqslant f(x,u)$. It is constructed as follows. For a given $h_0 \in (\theta_0, 1)$ and $x \in \mathbb{R}$, let $\hat{\zeta}(x)$ satisfy

$$-\hat{\zeta}''(x) = f^{\min}\big(\hat{\zeta}(x)\big), \qquad \hat{\zeta}(0) = h_0, \qquad \hat{\zeta}'(0) = 0,$$

with the convention that $f^{\min}(u) = 0$ for $u < 0$ above. To fix ideas we may set $h_0 = (1+\theta_0)/2$ in (2.2). Let us define $z_1 = \min\{x > 0 \mid \hat{\zeta}(x) = \theta_0\}$ and $z_2 = \min\{x > 0 \mid \hat{\zeta}(x) = 0\}$. The function $\hat{\zeta}$ satisfies the following elementary properties:

- $\hat{\zeta}(-x) = \hat{\zeta}(x)$ for all $x \in \mathbb{R}$,
- $0 \leqslant \hat{\zeta}(x) \leqslant h_0 = \hat{\zeta}(0)$ for all $x \in [-z_2, z_2]$,
- $\hat{\zeta}(x)$ is strictly concave for $x \in (-z_1, z_1)$,
- $\hat{\zeta}(-z_2) = \hat{\zeta}(z_2) = 0$.

As in [21,26] it follows that $u_n(t,x) \geqslant u_n(-n,x)$ for $t \geqslant -n$, and $u_n(t,x)$ is monotonically increasing in time to $\bar{u} \equiv 1$.

Lemma 2.1. *Let $u_n(t,x)$ solve* (1.1) *with initial data* (2.1) *at time $t=-n$. Then, $u_n(t,x)$ is strictly increasing in t:*

$$\frac{\partial u_n}{\partial t}(t,x) > 0 \quad \text{for all } t > -n, \tag{2.3}$$

and, moreover,

$$\lim_{t\to\infty} u_n(t,x) = 1 \quad \text{locally uniformly in } x. \tag{2.4}$$

Proof. Since

$$-\zeta_{xx} \leqslant f(x,\zeta),$$

the maximum principle implies that $u_n(t,x) \geqslant u_n(-n,x) = \zeta(x)$ for all $t > -n$. Applying the maximum principle to the function $w(t,x) = u_n(t+\tau,x) - u_n(t,x)$, for $\tau > 0$ fixed, we see that, as $w(-n,x) \geqslant 0$, we have $w(t,x) > 0$ for all $t > -n$; thus, u_n is monotonically increasing in time and (2.3) holds.

Since u_n is monotone in t, the limit $\bar{u}(x) = \lim_{t\to\infty} u_n(t,x)$ exists and satisfies

$$\bar{u}_{xx} = -f(x,\bar{u}), \quad 0 < \bar{u}(x) \leqslant 1, \qquad \max_x \bar{u} > h_0 > \theta_0. \tag{2.5}$$

Note that $\bar{u}_{xx} = 0$ on the set $\{\bar{u} < \theta_0\}$, so $\bar{u}$ is linear there. It is easy to see that this implies that this set must be empty because of the lower bound $\bar{u} \geqslant 0$ and the fact that $\max_x \bar{u} > h_0 > \theta_0$. Hence, we have $\bar{u} \geqslant \theta_0$.

Now, (2.5) implies that $\bar{u}$ is concave. Since $\theta_0 \leqslant \bar{u} \leqslant 1$, this implies $\bar{u}$ is constant, so that $f(x,\bar{u}) = -\bar{u}_{xx} \equiv 0$. This fact and $\max_x \bar{u} > h_0$ implies that $\bar{u} \equiv 1$. The local uniformity of the limit follows from standard regularity estimates for u. □

The initial shift

The initial shift x_0^n is normalized by requiring that

$$u_n(0,0) = \theta_0. \tag{2.6}$$

Lemma 2.2. *There exists x_0^n so that $u_n(t,x)$ satisfies* (2.6) *and* $\lim_{n\to+\infty} x_0^n = -\infty$. *Moreover, there exists N_0 and $\varepsilon > 0$ so that $x_0^n < -\varepsilon n$ for all $n > N_0$.*

Proof. Let $v_n(t,x;y)$ be the solution of (1.1) with the initial data $v_n(t=-n,x;y) = \zeta(x-y)$ — we are looking for x_0 such that $v_n(0,0;x_0) = \theta_0$. Note that (2.3) implies $v_n(0,0;0) > \theta_0$. In addition, the function $\psi(x) = \exp(-\lambda(x-ct))$ is a super-solution for (1.1) provided that

$$c\lambda \geqslant \lambda^2 + Mg^{\max}, \tag{2.7}$$

with the constant $M > 0$ chosen so that $f_0(u) \leqslant Mu$. Let us choose $\lambda > 0$ and $c > 0$ sufficiently large so that (2.7) holds. The maximum principle implies that there exists a constant $C > 0$ so that

$$v_n(t,x;y) \leqslant C\exp\bigl\{-\lambda\bigl(x-y-c(t+n)\bigr)\bigr\},$$

and thus

$$v_n(0,0;y) \leqslant C\exp\bigl\{\lambda(y+cn)\bigr\} \leqslant \frac{\theta_0}{2}$$

for $y < -cn-K$ with a constant $K > 0$. By continuity of $v_n(0,0;y)$ as a function of y there exists $x_0 \in (-cn-K,0)$ such that $v_n(0,0;x_0) = \theta_0$. In order to see that $x_0^n \to -\infty$ as $n \to +\infty$, observe that $v_n(t,x;y) \geqslant w_n(t,x;y)$, where $w_n(t,x;y)$ is the solution of the Cauchy problem

$$\frac{\partial w_n}{\partial t} = \frac{\partial^2 w_n}{\partial x^2} + g^{\min} f_0(w_n), \qquad w_n(-n,x;y) = \zeta(x-y). \tag{2.8}$$

Note that if y stays uniformly bounded from below as $n \to +\infty$: $y \geqslant K$ for all n, then, as in (2.4), $w_n(0,0;y) \to 1$, which contradicts (2.6), thus $x_0^n \to -\infty$ as $n \to +\infty$. The refined estimate $x_0^n < -\varepsilon n$ follows the results of [26] on the exponential in time convergence of the solution of (2.8) to a sum of two traveling waves of (2.8) moving with a positive speed $c^{\min} > 0$ to the right and left, respectively. In particular, this implies that if $y > -nc^{\min}/2$ then for n sufficiently large we have $w_n(0, 0; y) > (1 + \theta_0)/2 > \theta_0$ which is a contradiction. □

The standard elliptic regularity estimates imply that the sequence of functions $u_n(t, x)$ is uniformly bounded, together with its derivatives, so that along a suitable subsequence the limit $u(t, x) = \lim_{k\to+\infty} u_{n_k}(t, x)$ is a global in time and space, monotonically increasing solution to (1.1). The main remaining difficulty is to show that $u(t, x)$ has the correct limits as $x \to \pm\infty$ and its "interface width" is uniformly bounded in time so that it is indeed a transition front in the sense of Berestycki and Hamel. The rest of the proof of Theorem 1.6 is based on the following estimates for any solution of the Cauchy problem (1.1) with the initial data $u(0, x) = \zeta(x - x_0)$, with any $x_0 \in \mathbb{R}$ (we set here the initial time $t_0 = 0$ for convenience).

The interface width estimate

For $h \in (\theta_0, 1)$ and $k \in (0, \theta_0)$, let $X_h^l(t)$ and $X_k^r(t)$ be defined by

$$X_h^l(t) = \max\bigl\{x > x_0 \bigm| u(t, x') > h,\ \forall x' \in [x_0, x)\bigr\},$$
$$X_k^r(t) = \min\bigl\{x > x_0 \bigm| u(t, x') < k,\ \forall x' \in (x, \infty)\bigr\}. \tag{2.9}$$

Our goal is to show that the width of the front can be bounded by a universal constant depending only on $f^{\max}$ and $f^{\min}$.

Proposition 2.3. *Let $u(t, x)$ be a solution of* (1.1) *with the initial data $u(0, x) = \zeta(x - x_0)$ for some $x_0 \in \mathbb{R}$. For any $h \in (\theta_0, 1)$ and $k \in (0, \theta_0]$, there are constants $K_h \geqslant 0$ and $C \geqslant 0$ depending only on h, k, $f^{\max}$ and $f^{\min}$ such that for any $t > K_h$ we have $u(t, x_0) > h$, and*

$$0 < X_k^r(t) - X_h^l(t) \leqslant C < +\infty \tag{2.10}$$

for all $t > K_h$. We can take $K_h = 0$ for $h \in (\theta_0, h_0)$.

Let us note that the time delay K_h is introduced simply because initially the solution may be below h everywhere so that $X_h^l(t)$ is not defined for small times.

The interface steepness bound

The next crucial estimate provides a lower bound for the steepness of the interface. First, we use the following lemma to define the interface location.

Lemma 2.4. *Let $u(t, x)$ be a solution of* (1.1) *with the initial data $u(0, x) = \zeta(x - x_0)$ for some $x_0 \in \mathbb{R}$. For all $t > 0$, there exists a continuous function* (*the right interface*) *$X(t)$, $t \geqslant 0$, monotonically increasing in t and satisfying*: *$x_0 < X(t)$, $u(t, X(t)) = \theta_0$ and $u(t, x) < \theta_0$ for all $x > X(t)$, $u(t, x) > \theta_0$ for all $x \in (x_0, X(t))$.*

Proof. This follows from the strict monotonicity of u with respect to time and the maximum principle which precludes $X(t)$ from having jumps since $f(x, u) = 0$ for $0 \leqslant u \leqslant \theta_0$. □

Proposition 2.5. *Let $u(t, x)$ be a solution of* (1.1) *with the initial data $u(0, x) = \zeta(x - x_0)$ for some $x_0 \in \mathbb{R}$. Then the following hold.*

(i) *There are constants $p > 0$ and $\tau_0 \geqslant 0$ depending only on $g^{\max}$, $g^{\min}$, and the function f_0 such that*

$$u_x\bigl(t, X(t)\bigr) < -p \tag{2.11}$$

for all $t > 0$, and

$$u\bigl(t, x + X(t)\bigr) \leqslant \theta_0 e^{-px} \tag{2.12}$$

for all $x > 0$ and $t \geqslant \tau_0$.

(ii) *There exists a constant* $\delta > 0$ *depending only on* $g^{\max}$, $g^{\min}$, *and the function* f_0 *such that*

$$u_t\big(t, X(t)\big) > \delta \tag{2.13}$$

for all $t > 1$. *Moreover, for any* $t_1 > 0$, *there are constants* $H > 0$ *and* $L > 0$ *such that*

$$0 < L < \dot{X}(t) < H < +\infty \quad \text{for all } t \geqslant t_1. \tag{2.14}$$

The constants L *and* H *depend only on* t_1, $g^{\min}$, $g^{\max}$ *and the function* f_0.

The end of the proof of Theorem 1.6

Theorem 1.6 is an immediate consequence of Propositions 2.3 and 2.5. Consider the sequence of functions $u_n(t,x)$ defined for $t \geqslant -n$ as solutions of the Cauchy problem for (1.1) with the initial data (2.1) and x_0^n fixed by normalization (2.6). As we have mentioned above, the standard elliptic regularity estimates imply that there exists a subsequence $n_k \to +\infty$ so that $u_{n_k}(t,x)$ converge locally uniformly, together with its derivatives, to a limit $u(t,x)$ which is a global in time and space solution to (1.1), monotonically increasing in time. Moreover, the interface locations $X_n(t)$ converge to $X(t)$ such that $u(t, X(t)) = \theta_0$, $0 < L < \dot{X}(t) < H$, and $X(0) = 0$. The normalization (2.6) implies that $u(0,0) = \theta_0$, and, in addition, the bounds (2.11)–(2.13) hold for the limit $u(t,x)$. The upper bound (2.12) implies immediately that $u(t, x + X(t)) \to 0$ as $x \to +\infty$ uniformly in t. It remains only to check that $u(t, x + X(t)) \to 1$ as $x \to -\infty$ uniformly in t. To this end, assume that there exists $\varepsilon_0 > 0$ and a sequence of points $t_m \in \mathbb{R}$, and $x_m \to -\infty$ such that $u(t_m, x_m + X(t_m)) < 1 - \varepsilon_0$. This, however, contradicts (2.10) with $k = \theta_0$ and $h = 1 - \varepsilon_0$. Therefore, $u(t,x)$ is, indeed, a transition wave.

A convenient way to restate some of the properties of the functions $u_n(t,x)$ we will need later is as follows. Let us define the class of admissible non-linearities

$$\mathcal{G} = \big\{ f(x,u) = g(x) f_0(u)\colon\ g^{\min} \leqslant g(x) \leqslant g^{\max},\ g(x) \in C(\mathbb{R}) \big\}.$$

Lemma 2.6. *Given* $0 < g^{\min} \leqslant g^{\max} < +\infty$ *and* $f_0(u)$, *there exist* $p > 0$ *and a non-increasing in* x *function* $v(x)$, *such that* $v(0) = \theta_0$, $v'(0) < -p$,

$$\lim_{x \to -\infty} v(x) = 1, \tag{2.15}$$

$$\lim_{x \to +\infty} v(x) = 0, \tag{2.16}$$

and the following holds: *given any solution* $u_n(t,x)$ *of* (1.1) *with* $f(x,u) \in \mathcal{G}$, *and with the initial data* (2.1) *which satisfies the normalization* (2.6), *and for any* $R > 0$, *we have*

$$u_n\big(t, x + X_n(t)\big) \geqslant v(x), \quad \forall x \in [-R, 0], \tag{2.17}$$

$$u_n\big(t, x + X_n(t)\big) \leqslant v(x), \quad \forall x \in [0, \infty], \tag{2.18}$$

for all $t \geqslant 0$, *if* n *is sufficiently large, depending only on* R, $g^{\min}$, *and* $g^{\max}$. *The function* $v(x)$ *depends only on the constants* $g^{\max}$ *and* $g^{\min}$, *and the function* $f_0(u)$.

Proof. Setting $v(x) = \theta_0 e^{-px}$ for $x \geqslant 0$, with $p > 0$ as in Proposition 2.5 we see that the upper bound (2.18) follows from (2.12), and (2.16) is obviously satisfied, as well as a strictly negative upper bound for $v'(0)$.

In order to define $v(x)$ for $x < 0$ we consider a solution of (1.1) with $f \in \mathcal{G}$, which satisfies (2.6), and with initial data as in (2.1), and choose $t_1 = 1$ and find the corresponding L as in Proposition 2.5, so that $\dot{X}_n(t) \geqslant L$ for $t \geqslant -n+1$. Now, $X_n(t) \geqslant x_n^0 + L(t + n - 1)$, and thus for $t \geqslant 0$, $n \geqslant N_R = 1 + R/L$, and $x \in [-R, 0]$ we have

$$x + X_n(t) \geqslant x_n^0 + L(t + n - 1) - R \geqslant x_n^0.$$

For $h \in [\theta_0, 1)$, let $X_{n,h}^l(t)$ be defined by (2.9). We use the convention that $X_{n,h}^l(t) = -\infty$ if $u_n(t, x_n^0) < h$. For any $h' \in [\theta_0, 1)$, $X_{n,h}^l(t)$ is finite for all $t > 0$ and for all $h \in [\theta_0, h']$, if $n > N(h')$ is sufficiently large, depending only on $g^{\min}$ and h'. This follows directly from Proposition 2.3. Now, for $x < 0$ and $n \geqslant 1$, define

$$v_n(x; f) = \sup\Big\{h \in [\theta_0, 1)\colon \sup_{t \geqslant 0}\big(X_n(t) - X^l_{n,h}(t)\big) \leqslant |x|\Big\}. \tag{2.19}$$

We indicate above explicitly the dependence of v_n on the non-linearity $f(x, u)$. Then we set

$$v(x) = \inf_{f \in \mathcal{G}} \inf_{n \geqslant 1+|x|/L} v_n(x; f).$$

The set of possible values of h over which the supremum is taken in (2.19) contains θ_0. Therefore, $v(x) \geqslant \theta_0$ for all $x < 0$. From (2.19) it is easy to see that $v_n(x; f)$ is non-increasing in x (for $x < 0$) for each $f \in \mathcal{G}$, and $v_n(0; f) = \theta_0$. Hence $v(x)$ is also non-increasing in x and $v(0) = \theta_0$.

Next we show that $v(x) \to 1$ as $x \to -\infty$. For any $h \in [\theta_0, 1]$, Proposition 2.3 implies that for all $f \in \mathcal{G}$ we have

$$\sup_{t \geqslant 0}\big(X_n(t) - X^l_{n,h}(t)\big) < C(h)$$

for some finite constant $C(h)$, depending only on $g^{\min}$ and $g^{\max}$, provided that $n > N(h)$, which ensures that $u_n(0, x^0_n) \geqslant h$. So, for x such that both $x < -C(h)$ and $1 + |x|/L > N(h)$, we have $v(x) > h$. Since h may be chosen arbitrarily close to 1, it follows that $\lim_{x\to-\infty} v(x) = 1$.

Finally, in order to see that (2.17) holds, fix $R > 0$ and $f \in \mathcal{G}$, and let $u_n(t, x)$ be the solution of the corresponding Cauchy problem. By definition of v,

$$v(x) \leqslant v_n(x; f) \quad \text{for all } x \in [-R, 0],$$

provided that $n \geqslant 1 + R/L$. Therefore,

$$X_n(t) - X^l_{n,h}(t) \leqslant -x \tag{2.20}$$

for all $h \in [\theta_0, v(x)]$, $n \geqslant 1 + R/L$, and all $t \geqslant 0$. Hence, $X^l_{n,h}(t) \geqslant x + X_n(t) \geqslant x^n_0$ so that

$$u_n\big(t, x + X_n(t)\big) \geqslant h \quad \text{for all } h \in [\theta_0, v(x)], \tag{2.21}$$

and for all $t \geqslant 0$ and $n \geqslant 1 + R/L$. This proves (2.17). □

Bounds for the location of level sets

In order to finish the proof of Theorem 1.6 it remains to prove Propositions 2.3 and 2.5. We need first to establish some simple bounds on the location of the level sets of the function $u(t, x)$. Let $c^{\min}$ and $c^{\max}$ be the unique speeds of the traveling wave solutions of the constant coefficient equations

$$-cq_x = q_{xx} + f^{\min}(q), \qquad q(-\infty) = 1, \qquad q(+\infty) = 0,$$

and

$$-cq_x = q_{xx} + f^{\max}(q), \qquad q(-\infty) = 1, \qquad q(+\infty) = 0,$$

respectively. The next lemma will allow us to relate the position $X^l_h(t)$ to $X^l_{h'}(t - s)$ with $s > 0$ and $h' < h$ — this allows us to control the width of the front in the back, where u is close to 1.

Lemma 2.7. *Let $\delta > 0$ and let $0 \leqslant u(t, x) \leqslant 1$ satisfy (1.1) for $t > 0$. Suppose, in addition, that $u(0, x) > \delta + \theta_0$ for all $x \in [x_L, x_R]$. If $\sigma = |x_R - x_L|$ is sufficiently large, depending on δ and $f^{\min}$, then for any $h \in (\theta_0, 1)$ there are constants $\beta > 0$ and $\tau_1 > 0$, such that*

$$X^l_h(t) \geqslant x_R + c^{\min}t - \beta \tag{2.22}$$

for all $t \geqslant \tau_1$. The constants β and τ_1 depend only on h, δ, σ, and $f^{\min}$.

Proof. This follows from the comparison principle and the stability results of [26]. Specifically, if σ is sufficiently large, consider the function $v(t, x)$ which solves the equation

$$v_t = \Delta v + f^{\min}(v)$$

with the initial data

$$v(0,x) = (\delta + \theta_0)\chi_{[x_L,x_R]}(x)$$

at $t = 0$. Then, as we have mentioned, the results of [26] imply that v converges as $t \to +\infty$ to a pair of traveling waves moving to the left and the right with speed $c^{\min} > 0$. The convergence is exponentially fast. Therefore, after some time τ_1, which depends on h and on the convergence rate, $v(t,x) \geqslant h$ on the set $[x_R, x_R + c^{\min}t - \beta]$, for some constant $\beta > 0$. The maximum principle implies that $u(t,x) \geqslant v(t,x)$ and (2.22) follows. □

Corollary 2.8. *Let $h \in (\theta_0, 1)$. Let $u(t,x)$ be as in Propositions* 2.3 *and* 2.5. *There are constants $\beta \geqslant 0$, $\tau_2 \geqslant 0$ depending only on h and $f^{\min}$ such that*

$$X_h^l(t) \geqslant X_h^l(t_1) + \max\bigl(0, c^{\min}(t - t_1) - \beta\bigr) \tag{2.23}$$

for all $t \geqslant t_1 \geqslant \tau_2$.

Proof. Let $\delta = h - \theta_0$ and let σ be sufficiently large, as required by Lemma 2.7. Since $u \geqslant v$ where v solves $v_t = \Delta v + f^{\min}(v)$ with the same initial data, there is a time $\tau_2 > 0$ depending only on $f^{\min}$ and h such that $u(t,x) \geqslant v(t,x) \geqslant h$ on the interval $[x_0, x_0 + \sigma]$, for all $t \geqslant \tau_2$. So, $X_h^l(t) \geqslant x_0 + \sigma$ is well-defined and increasing for $t \geqslant \tau_2$. The bound now follows from Lemma 2.7 with $x_L = x_0$, $x_R = X_h^l(t_1)$, and replacing $t = 0$ with t_1. □

Lemma 2.9. *Suppose that $u(0,x) \leqslant Ce^{-c^{\max}(x-x_0)}$ for all $x \in \mathbb{R}$. Then there is a constant $\eta > 0$ depending only on $f^{\max}$ and C such that*

$$X(t) \leqslant x_0 + c^{\max}t + \eta, \quad \forall t > 0. \tag{2.24}$$

Proof. This follows from the comparison principle and the stability results of [26]. □

Corollary 2.8 and Lemma 2.9 immediately imply that

$$c^{\min} \leqslant \liminf_{t\to\infty} \frac{X(t)}{t} \leqslant \limsup_{t\to\infty} \frac{X(t)}{t} \leqslant c^{\max}. \tag{2.25}$$

The proof of Proposition 2.5(i): (2.11)

We begin the proof of Proposition 2.5 with the proof of (2.11). The strategy of this proof is a version of the sliding method [5]. Suppose $\varepsilon \in (0, c^{\min})$ is sufficiently small so that initially at time $t = 0$ we have

$$u_x\bigl(0, X(0)\bigr) \leqslant -\varepsilon\theta_0 \tag{2.26}$$

and

$$u\bigl(0, x + X(0)\bigr) \leqslant \theta_0 e^{-\varepsilon x} \quad \text{for all } x \geqslant 0. \tag{2.27}$$

The good times

Let us define the set of good times G when we can control the decay of the solution ahead of the front by an exponential:

$$G = \bigl\{t \geqslant 0\colon u\bigl(t, x + X(t)\bigr) \leqslant \theta_0 e^{-\varepsilon x} \text{ for all } x \geqslant 0\bigr\}. \tag{2.28}$$

Note that if $t \in G$ then $u_x(t, X(t)) \leqslant -\varepsilon\theta_0$ and thus both (2.11) and (2.12) hold. Our goal is to show that $G = [\tau_0, +\infty)$ for $\varepsilon > 0$ sufficiently small, and τ_0 is a universal constant.

Given $s \in G$ and $y > 0$ set

$$\psi(t,x;s,y) = \theta_0 e^{-\varepsilon(x - \varepsilon(t-s) - y - X(s))}.$$

The difference $w = \psi - u$ satisfies $w_t \geqslant w_{xx}$ in the region $R = \{(t,x)\colon\ x > X(t),\ t \geqslant s\}$ and $w(s,x) > 0$ for $x \geqslant X(s)$. Therefore, for $t - s$ small we have $w(t,x) > 0$ for $x > X(t)$. On the other hand, as $\liminf_{t\to+\infty} X(t)/t \geqslant c^{\min} > \varepsilon$, there exist a time t and $x > X(t)$ so that $w(t,x) < 0$. Let us define the first time when ψ and u touch:

$$\tau_{y,s} = \sup\bigl\{t > s\colon\ w(\tau,x) > 0 \text{ for all } x \geqslant X(\tau) \text{ and all } \tau \in [0,t)\bigr\}. \tag{2.29}$$

Note that $\tau_{y,s} > s$ for all $y > 0$. The maximum principle implies that the only point where the functions u and ψ can touch is at the boundary, that is, at $x = X(\tau_{y,s})$, where

$$u\bigl(\tau_{y,s}, X(\tau_{y,s})\bigr) = \psi\bigl(\tau_{y,s}, X(\tau_{y,s}); s, y\bigr) = \theta_0,$$

so that, in particular,

$$X(\tau_{y,s}) - \varepsilon(\tau_{y,s} - s) - y - X(s) = 0.$$

In addition, we have $w(\tau_{y,s}, x + X(\tau_{y,s})) \geqslant 0$ for all $x > 0$. It follows that

$$u\bigl(\tau_{y,s}, x + X(\tau_{y,s})\bigr) \leqslant \psi\bigl(\tau_{y,s}, x + X(\tau_{y,s}); s, y\bigr) = \theta_0 e^{-\varepsilon x}$$

and thus $\tau_{y,s} \in G$ is a "good" time for any $y > 0$.

The bad times

Now, suppose that the set $B = [0,\infty) \setminus G$ of "bad" times is not empty. Note that $t \in B$ if and only if there exists $x > 0$ so that

$$u\bigl(t, x + X(t)\bigr) > \theta_0 e^{-\varepsilon x},$$

and thus B is open. Hence, B is an at most countable union of disjoint open intervals $\{(t_j, \tilde{t}_j)\}_{j=1}^{\infty}$, with $t_j, \tilde{t}_j \in G$. Observe that

$$t'_j := \lim_{y\downarrow 0} \tau_{y,t_j} \geqslant \tilde{t}_j.$$

Indeed, as $\tau_{y,t_j} > t_j$ for all $y > 0$, otherwise we could find $y > 0$ so that $t_j < \tau_{y,t_j} < \tilde{t}_j$, which would be a contradiction since $\tau_{y,t_j} \in G$ for all $y > 0$. Since G is closed, $t'_j \in G$.

Let us enlarge B to

$$B' = \bigcup_{j=1}^{\infty} \bigl(t_j, t'_j\bigr) \supseteq B.$$

We claim that the average front speed on each time interval $[t_j, t'_j]$ is small:

$$X(t) - X(t_j) \leqslant \varepsilon(t - t_j), \quad \forall t \in \bigl[t_j, t'_j\bigr]. \tag{2.30}$$

Indeed, for any $t \in (t_j, t'_j)$, any $x \geqslant X(t)$, and any $y > 0$ we have

$$u(t,x) < \psi(t,x;t_j,y) = \theta_0 e^{-\varepsilon(x - \varepsilon(t-t_j) - y - X(t_j))}.$$

Passing to the limit $y \downarrow 0$ we obtain

$$u(t,x) \leqslant \psi(t,x;t_j,0) = \theta_0 e^{-\varepsilon(x-\varepsilon(t-t_j)-X(t_j))} \quad \text{for all } t \in \bigl[t_j, t'_j\bigr],\ x \geqslant X(t)\ . \tag{2.31}$$

Evaluating this inequality at $(t, X(t))$, we obtain (2.30).

The key estimate we will need below is given by the following lemma.

Lemma 2.10. *There exists a constant $K > 0$ which depends only on $f_0(u)$, $g^{\min}$ and $g^{\max}$ so that for all j we have $|t'_j - t_j| \leqslant K$.*

We postpone the proof of this lemma for the moment and finish the proof of (2.11) first.

A lower bound for the slope at the front

Using Lemma 2.10 we may prove the lower bound (2.11) for the slope at the front. Note that for each "good" time $t \in G$ this bound holds automatically, so we need only to look at "bad" times $t \in B$. More generally, consider a time $t \in B'$ so that $t \in (t_j, t_j')$ for some j. The function $u(t,x)$ is convex in x for $x > X(t)$ — this is because $u_{xx} = u_t > 0$ in this region. Therefore, for all $l > 0$ we have

$$u_x\big(t, X(t)\big) \leqslant \frac{u(t, X(t)+l) - u(t, X(t))}{l}.$$

Let us choose $l > 0$ so that $\psi(t, X(t)+l; t_j, 0) = \theta_0/2$, then, according to (2.31) we have

$$u\big(t, X(t)+l\big) \leqslant \psi\big(t, X(t)+l; t_j, 0\big) = \frac{\theta_0}{2}, \tag{2.32}$$

and thus

$$u_x\big(t, X(t)\big) \leqslant -\frac{\theta_0}{2l}. \tag{2.33}$$

However, the distance l can be computed explicitly:

$$X(t) + l - \varepsilon(t - t_j) - X(t_j) = \frac{\log 2}{\varepsilon},$$

and thus

$$l < \varepsilon\big(t_j' - t_j\big) + \frac{\log 2}{\varepsilon}.$$

Lemma 2.10 and (2.33) imply now that for $t \in B'$ we have an estimate

$$u_x\big(t, X(t)\big) \leqslant -\frac{\theta_0}{2(\varepsilon(t_j' - t_j) + \varepsilon^{-1}\log 2)} \leqslant -\frac{\theta_0}{2(\varepsilon K + \varepsilon^{-1}\log 2)}, \tag{2.34}$$

which is nothing but (2.11).

The proof of Lemma 2.10

Step 1 *(Reducing to large times).* First, we note that if $0 \leqslant t_j \leqslant T$ then t_j' is bounded from above. From Corollary 2.8, we have for $t_j' \geqslant \tau_2$,

$$X\big(t_j'\big) \geqslant x_0 + c^{\min}\big(t_j' - \tau_2\big) - \beta,$$

while for $0 \leqslant t_j \leqslant T$,

$$X\big(t_j'\big) \leqslant X(t_j) + \varepsilon\big(t_j' - t_j\big) \leqslant X(T) + \varepsilon t_j' \leqslant x_0 + c^{\max}T + \eta + \varepsilon t_j',$$

from Lemma 2.9. It follows that $t_j' - t_j \leqslant t_j' \leqslant (c^{\max}T + \eta + \beta + c^{\min}\tau_2)/(c^{\min} - \varepsilon)$ if $0 \leqslant t_j \leqslant T$. Hence, for a constant T to be chosen, we will assume that $t_j \geqslant T$ for the rest of the proof of Lemma 2.10.

Step 2 *(Forming a large plateau).* Our next goal is to show that a large plateau develops behind the front sufficiently fast. Without loss of generality, we assume $t_j \geqslant T \geqslant 1$, for a constant T to be chosen. Because t_j is a "good" time, we have $u_x(t_j, X(t_j)) \leqslant -\varepsilon\theta_0$. Elliptic regularity implies that there exists a constant M so that $|u_{xx}| \leqslant M$ for all $t \geqslant 1$. Thus, at $t = t_j$ we have a lower bound for $u(t,x)$ immediately behind the front:

$$u(t_j, x) \geqslant \theta_0 - p\big(x - X(t_j)\big) - \frac{M}{2}\big(x - X(t_j)\big)^2,$$

with $p = -\varepsilon\theta_0$. Now define $\delta := \frac{p^2}{4M}$ and let σ_δ be the corresponding constant in Lemma 2.7. Evaluating this inequality at $x_j = X(t_j) - p/M$ we obtain

$$u\left(t_j, X(t_j) - \frac{p}{M}\right) \geqslant \theta_0 + \frac{p^2}{2M} = \theta_0 + 2\delta.$$

Since $u(t,x)$ is monotonic in t, we conclude that

$$u(t,x_j) \geqslant \theta_0 + 2\delta \quad \text{for all } t \geqslant t_j \text{ and } x_j = X(t_j) - \frac{p}{M}.$$

Now take T sufficiently large so that $X(T) - p/M - 10\sigma_\delta > 0$. By Corollary 2.8, T may be chosen to depend only on $g^{\min}$, $g^{\max}$, and f_0. Therefore, for $t \geqslant t_j \geqslant T$ the function $u(t,x)$ satisfies the following differential inequality and boundary conditions on the interval $(x_j - 10\sigma_\delta, x_j) \subset [0,\infty)$:

$$u_t - u_{xx} \geqslant 0, \qquad u(t, x_j - 10\sigma_\delta) \geqslant \theta_0, \qquad u(x_j) \geqslant \theta_0 + 2\delta.$$

It follows that there exists a time $\tau_\delta > 0$ which depends only on δ so that $u(t_j + \tau_\delta, x) \geqslant \theta_0 + \delta$ for all $x \in (x_j - \sigma_\delta, x_j)$. By Lemma 2.7 this forces the interface to move forward for $t \geqslant s_j = t_j + \tau_\delta$ at the speed of at least $c^{\min}$:

$$X(t) \geqslant c^{\min}(t - s_j) + X(t_j) - p/M - \beta \tag{2.35}$$

for all $t \geqslant s_j$.

Step 3 *(Plateau catching up with the front).* We claim that there exists a constant K so that

$$t'_j - t_j \leqslant K(1 + \tau_\delta). \tag{2.36}$$

Indeed, according to (2.30), the average front speed on the interval (t_j, t'_j) is smaller than ε. Combining this with (2.35), which says that on the interval $[s_j, t'_j]$ the average speed cannot be too small, leads to

$$X(t_j) + \varepsilon\big(t'_j - t_j\big) \geqslant X\big(t'_j\big) \geqslant c^{\min}\big(t'_j - t_j - \tau_\delta\big) + X(t_j) - p/M - \beta.$$

Thus, (2.36) holds in that case as well. Now, the conclusion of Lemma 2.10 follows. Note that since $\varepsilon < c^{\min}$ is arbitrary, the constant K in that lemma can indeed be chosen to depend only on $f_0(u)$, $g^{\min}$ and $g^{\max}$.

An upper bound for the front speed

The lower bound (2.11) on the slope of the front implies an upper bound on its speed in Proposition 2.5(ii). Indeed, since $u_x(t, X(t)) \leqslant -p < 0$ for all $t \geqslant 0$, regularity estimates imply that for any $t_1 > 0$ and any $t \geqslant t_1$,

$$\dot{X}(t) = -\frac{u_t(t, X(t))}{u_x(t, X(t))} \leqslant \sup_{t \geqslant t_1} \frac{\|u_t(\cdot, t)\|_\infty}{p} \leqslant H, \tag{2.37}$$

with a constant $H > 0$, depending on t_1, $g^{\max}$, $g^{\min}$, and f_0.

The proof of Proposition 2.5(ii)

We may now prove (2.13), the lower bound on $u_t(t, X(t))$. The standard elliptic regularity estimates imply that for any $t_1 > 0$, there exists a constant $M > 0$ so that $\|u_{xx}(t,\cdot)\|_\infty < M$ for all $t > t_1$. Therefore, we have for $x < X(t)$, using (2.11):

$$\begin{aligned} u(t,x) &\geqslant u\big(t, X(t)\big) + u_x\big(t, X(t)\big)\big(x - X(t)\big) - \frac{1}{2}M\big(x - X(t)\big)^2 \\ &\geqslant \theta_0 - p\big(x - X(t)\big) - \frac{1}{2}M\big(x - X(t)\big)^2 \end{aligned} \tag{2.38}$$

for all $t > t_1$. For $h = \theta_0 + p^2/2M$ and $x = X(t) - p/M$, this gives us

$$u(t,x) \geqslant \theta_0 + \frac{p^2}{2M} = h. \tag{2.39}$$

Since u is monotone in time, this implies that there is $t_2 \geqslant t_1$ such that for all $t \geqslant t_2$,

$$X_h^l(t) \geqslant X(t) - p/M. \tag{2.40}$$

The time gap between t_1 and t_2 may be needed to allow $u(t,x)$ go above the value h on the interval between the points x_0 and $X(t_1) - p/M$. However, this constant t_2 depends only on $f^{\min}$ and $f^{\max}$. Indeed, Corollary 2.8 implies that for $t_2 \geqslant \tau_2$

$$X_h^l(t_2) \geqslant x_0 + \max\bigl(0, c^{\min}(t_2 - \tau_2) - \beta\bigr) \tag{2.41}$$

where τ_2 and β depend only on $f^{\min}$, $f^{\max}$, and h. At the same time, Lemma 2.8 tells us that

$$X(t_1) \leqslant x_0 + c^{\max} t_1 + \eta. \tag{2.42}$$

So, if $t_2 \geqslant \tau_2 + (c^{\min})^{-1}(\beta + c^{\max}t_1 + \eta)$, (2.40) holds for all $t \geqslant t_2$.

Now let $C = p/M$. Let τ_2 be as in Corollary 2.8. Suppose $t \geqslant t_1 \geqslant \tau_2$. Then for all $s \in [t_1, t]$, $|\dot{X}(s)| \leqslant H$, so

$$X(t_1) \in \bigl[X(t) - H(t - t_1), X(t)\bigr].$$

The constant H depends only on τ_2. It follows from Corollary 2.8 that there is a constant β, independent of $t_1 \geqslant \tau_2$, such that

$$X_h^l(t_1 + \Delta t) \geqslant X_h^l(t_1) + \max\bigl(0, c^{\min}(\Delta t) - \beta\bigr) \tag{2.43}$$

for all $\Delta t > 0$. So, if we choose, $\Delta t = (C + \beta)/c^{\min}$, (2.43) implies

$$X_h^l(t_1 + \Delta t) \geqslant X_h^l(t_1) + C \geqslant X(t_1).$$

The last inequality follows from (2.40) and our choice of C. Therefore, we have

$$u\bigl(t_1 + \Delta t, X(t_1)\bigr) \geqslant h,$$

and by the Mean Value Theorem there must be a point $t_2 \in [t_1, t_1 + \Delta t]$ such that

$$u_t\bigl(t_2, X(t_1)\bigr) \geqslant (h - \theta_0)/(\Delta t),$$

since $u(t_1, X(t_1)) = \theta_0$.

Now, let $t \geqslant \tau_2 + 2\Delta t$ and set $t_1 = t - 2\Delta t \geqslant \tau_2$ (recall that Δt is defined independently of t_1). Thus there exists a point $x_2 \in [X(t) - 2H\Delta t, X(t)]$ and $t_2 \in [t - 2\Delta t, t - \Delta t]$ such that $u_t(t_2, x_2) \geqslant r > 0$, where $r = (h - \theta_0)/(\Delta t)$.

The function $q(t,x) = u_t(t,x)$ satisfies a PDE of the form

$$q_t = \Delta q + V(x,t)q$$

with $q \geqslant 0$ and $\|V\|_\infty < \infty$. The Harnack inequality [16] implies that there is K depending only on H, Δt, and $\|V\|_\infty$ such that

$$q\bigl(t, X(t)\bigr) \geqslant K \sup_{\substack{x_2 \in [X(t) - 2H\Delta t, X(t)] \\ t_2 \in [t - 2\Delta t, t - \Delta t]}} q(t_2, x_2) \geqslant Kr > 0.$$

Therefore, there is $\delta = Kr$ depending only on $g^{\min}$, $g^{\max}$, and f_0 such that $u_t(t, X(t)) \geqslant \delta$ for all $t \geqslant \tau_2 + 2\Delta t$. Since $u_t(t, X(t)) > 0$ for all $t > 0$, this implies (2.13). Finally, the lower bound $\dot{X}(t) > L > 0$ now follows from (2.11), (2.13), the first equality in (2.37), and the elliptic regularity estimates for u.

The proof of Proposition 2.5(i): (2.12)

In order to finish the proof of Proposition 2.5 it remains only to prove the upper bound (2.12). It is a consequence of the lower bound $\dot{X}(t) \geqslant L > 0$ in (2.14). Let us recall the definitions of the set G of "good" times and the "catching-up" times $\tau_{y,s}$ introduced in the proof of (2.11): see (2.28) and (2.29).

Let η be as in Lemma 2.9, and set $y = \eta + 1$ and let $\tau_0 = \tau_{y,0} \in G$. Lemma 2.9 tells us that $X(t) \leqslant x_0 + c^{\max}t + \eta \leqslant x_0 + \eta + 1$ if $t \leqslant (c^{\max})^{-1}$. It follows that $\tau_0 \geqslant (c^{\max})^{-1}$. There are constants, L, H such that $0 < L \leqslant \dot{X}(t) \leqslant H$ for all $t \geqslant (c^{\max})^{-1}$. This implies $X((c^{\max})^{-1} + s) \geqslant x_0 + sL \geqslant x_0 + \eta + 1 + \varepsilon((c^{\max})^{-1} + s)$, if $\varepsilon < L/2$ and $s \geqslant (c^{\max})^{-1} + 2(\eta + 1)/L$. Hence $(c^{\max})^{-1} \leqslant \tau_0 \leqslant (c^{\max})^{-1} + 2(\eta + 1)/L$.

As $\dot{X}(t)$ is uniformly positive for $t \geqslant \tau_0$, it follows that for $\varepsilon \in (0, c^{\min})$ sufficiently small, $\varepsilon < L/2$, the function $\tau_{y,s}$ is a continuous function of y for each $s \geqslant \tau_0$ fixed, and, moreover, $\lim_{y \downarrow 0} \tau_{y,s} = s$ for all s. Furthermore, as the front speed is bounded from above, $\lim_{y \to +\infty} \tau_{y,s} = +\infty$ for all such s. We also recall that $\tau_{y,s} \in G$ for all $y > 0$ and $s \geqslant 0$. It follows that $[\tau_0, +\infty) \subset G$ and thus (2.12) holds for all $t \geqslant \tau_0$.

The proof of Proposition 2.3

The exponential bound (2.12) implies that for any $k \in (0, \theta_0)$, the distance

$$0 < X_k^r(t) - X(t) \leqslant \frac{1}{p} |\log k - \log \theta_0|$$

is bounded uniformly in time. Combining this with (2.40), we see that there is a constant C_1 such that for any $h \in (\theta_0, \theta_0 + p^2/2M]$ and any $k \in (0, \theta_0)$, we have

$$X_k^r(t) - X_h^l(t) \leqslant C_1 \tag{2.44}$$

for all $t \geqslant t_2$. Recall that t_2 depends only on $g^{\max}$, $g^{\min}$ and the function f_0.

Now suppose that $h \in (\theta_0 + p^2/2M, 1]$. By comparing u with the function v solving $v_t = v_{xx} + f^{\min}(v)$, we see that for t_2 larger, if necessary (depending only on h, $f^{\min}$, and $f^{\max}$), we have $u(t_2, x_0) > h$. Thus, $X_h^l(t)$ is well-defined for $t \geqslant t_2$. Let $\delta = p^2/2M$, $\gamma = \theta_0 + \delta$, and define $d_0 > 0$ by

$$d_0 = \sup_{t \in [0, t_2 + \tau_1]} \big(X_\gamma^l(t) - X_h^l(t)\big) \leqslant X(t_2 + \tau_1) - x_0 \leqslant C_2 + c^{\max}(t_2 + \tau_1) < +\infty$$

where τ_1 is the constant from Lemma 2.7 with $x_L = x_0$ and $x_R = X_\gamma^l(t_2)$. If necessary, we may take t_2 to be larger so that $|x_R - x_L| = \sigma$ is sufficiently large according to Lemma 2.7. Now for any $t \geqslant t_2 + \tau_1$, we apply Lemma 2.7 with the starting time $t_0 = t - \tau_1$, $\delta = p^2/2M$ and $x_R = X_\gamma^l(t - \tau_1) \geqslant X_\gamma^l(t_2)$. We conclude that

$$X_\gamma^l(t) - X_h^l(t) \leqslant X_\gamma^l(t) - X_\gamma^l(t - \tau_1) - c^{\min}\tau_1 + \beta.$$

As we have already shown in the proof of Proposition 2.5(i), we have $\dot{X}(t) \leqslant H$ for all $t > t_1$. Therefore, since $t - \tau_1 \geqslant t_2 \geqslant t_1$, we have

$$\begin{aligned}
X_\gamma^l(t) - X_h^l(t) &\leqslant X_\gamma^l(t) - X_\gamma^l(t - \tau_1) - c^{\min}\tau_1 + \beta \leqslant X(t) - X_\gamma^l(t - \tau_1) - c^{\min}\tau_1 + \beta \\
&= \big(X(t) - X(t - \tau_1)\big) + \big(X(t - \tau_1) - X_\gamma^l(t - \tau_1)\big) - c^{\min}\tau_1 + \beta \\
&\leqslant H\tau_1 + \big(X(t - \tau_1) - X_\gamma^l(t - \tau_1)\big) - c^{\min}\tau_1 + \beta \leqslant H\tau_1 + C_1 - c^{\min}\tau_1 + \beta.
\end{aligned}$$

This holds for all $t \geqslant t_2 + \tau_1$. Therefore, for any $t \geqslant 0$, we have

$$X_\gamma^l(t) - X_h^l(t) \leqslant d_0 + H\tau_1 + C_1 - c^{\min}\tau_1 + \beta$$

where the constants t_2, τ_1, H, d_0, C_1, $c^{\min}$, and β depend only on $g^{\max}$, $g^{\min}$ and the function f_0. So, the conclusion of Proposition 2.3 holds.

3. Asymptotic spreading for the Cauchy problem

Spreading of monotonically increasing in time solutions

Now we return to Eq. (1.1) with a random reaction term, and we prove Theorem 1.1. We first prove the result for monotone increasing solutions. Consider the solution to (1.1) with initial data $u_0(x, \omega) = \zeta(x + z_1)$ at time $t = 0$. Recall from the definition of the function ζ that $\zeta(z_1) = \theta_0$ and $\zeta(x) < \theta_0$ for $x > z_1$. Hence, we have $u_0(0, \omega) = \theta_0$. The initial data looks like a bump-function with the right interface at the origin. For each realization $\omega \in \Omega$ of the random medium, the following hold:

- The solution $u(t, x, \omega)$ is strictly monotone increasing in t and all the estimates of Section 2 hold $\mathbb{P}$-a.s.
- The function $X^+(t, \omega)$ defined by $u(t, X^+(t, \omega), \omega) = \theta_0$ and $X^+ \geqslant 0$ is well-defined and continuous. This defines uniquely the position of the right-moving interface.
- There are positive constants $C_{\min}$, $C_{\max}$, independent of ω such that for $t > 1$ we have $C_{\min} \leqslant \dot{X}^+(t, \omega) \leqslant C_{\max}$.
- For any $\xi \geqslant 0$, the time at which "the interface reaches ξ", denoted by $T(\xi, \omega)$, is well-defined:

$$\xi = X^+\big(T(\xi, \omega), \omega\big). \tag{3.1}$$

The first claim above follows from Lemma 2.1, the second one is a consequence of the maximum principle and monotonicity of $u(t,x,\omega)$ in time. The last two claims are implied by (2.14). Similarly, we may define the position $X^-(t,\omega)$ of the left-moving interface by $u(t,X^-(t,\omega),\omega)=\theta_0$ and $X^-(t,\omega)\leqslant -2z_1$ for $t\geqslant 0$.

The following proposition is a version of Theorem 1.1 for such monotonically increasing in time solutions. We will then use a comparison argument to generalize this result to arbitrary non-negative compactly supported initial data as claimed in Theorem 1.1.

Proposition 3.1. *There are non-random constants $c_+^*\in[c^{\min},c^{\max}]$ and $c_-^*\in[-c^{\max},-c^{\min}]$ such that*

$$\lim_{t\to\infty}\frac{X^+(t,\omega)}{t}=c_+^*, \tag{3.2}$$

$$\lim_{t\to\infty}\frac{X^-(t,\omega)}{t}=c_-^* \tag{3.3}$$

hold almost surely with respect to $\mathbb{P}$, and in $L^1(\Omega,\mathbb{P})$. For any $\varepsilon>0$,

$$\lim_{t\to\infty}\inf_{c\in[c_-^*+\varepsilon,c_+^*-\varepsilon]}u(t,ct,\omega)=1 \tag{3.4}$$

and

$$\lim_{t\to\infty}\sup_{c\in(-\infty,c_-^*-\varepsilon]\cup[c_+^*+\varepsilon,\infty)}u(t,ct,\omega)=0 \tag{3.5}$$

hold almost surely with respect to $\mathbb{P}$, and in $L^1(\Omega,\mathbb{P})$.

Proof of Proposition 3.1

First, we explain that $X^+(t,\omega)$ is $\mathcal{F}$-measurable for each t. Let m be a positive integer. For each m define the set of points $\{x_j^m\}=2^{-m}\mathbb{Z}$. For m and t fixed, let

$$A_j^m=\{\omega\in\Omega \mid u(t,x,\omega)\leqslant\theta_0,\ \forall x\geqslant x_j^m\}.$$

This is an $\mathcal{F}$-measurable set, since it is a closed set in $C(\mathbb{R};[g^{\min},g^{\max}])$ (in the uniform convergence norm). Define the random variable

$$\eta^m(\omega)=\min_j\bigl(x_j^m\chi_{A_j^m}(\omega)\bigr) \tag{3.6}$$

where χ is the characteristic function. Since there are countably many terms in the minimization, this is an $\mathcal{F}$-measurable random variable. By definition, $X^+(t,\omega)\leqslant\eta^m(\omega)\leqslant X^+(t,\omega)+2^{-m}$. Also, η^m is non-increasing in m. Therefore

$$X^+(t,\omega)=\lim_{m\to\infty}\eta^m(\omega) \tag{3.7}$$

and this must be $\mathcal{F}$-measurable, since the limit of a sequence of measurable functions is also measurable.

Next, we prove (3.2) by using the sub-additive ergodic theorem. Let us drop the superscript and denote $X(t,\omega)=X^+(t,\omega)$. Given a positive integer $m\in\mathbb{N}$, let $u^{(m)}(t,x,\omega)$ be the solution to (1.1) for $t\geqslant 0$ with shifted initial data $u^{(m)}(x,0,\omega)=\zeta(x+z_1-m)$ — its right interface is located initially at $x=m$. Let $X_m(t,\omega)\geqslant m$, $t\geqslant 0$, denote the position of the corresponding right-moving interface: $u^{(m)}(t,X_m(t,\omega),\omega)=\theta_0$. By Proposition 2.5, $X_m(t,\omega)$ satisfies the same properties as $X(t,\omega)$, listed above. For $\xi\geqslant m$, let $T_m(\xi,\omega)\geqslant 0$ denote the inverse of $X_m(t,\omega)$: $u^{(m)}(T_m(\xi,\omega),\xi)=\theta_0$.

Now, for a pair of non-negative integers $m,n\in\mathbb{N}$, $n\geqslant m$, define the family of random variables

$$q_{m,n}(\omega)=T_m(n,\omega)$$

which is the first time the interface hits the position n, when started from position m. It is easy to see that for any integer $h\geqslant 1$, the following translation invariance holds:

$$q_{m+h,n+h}(\omega) = q_{m,n}(\pi_h \omega). \tag{3.8}$$

The key observation in the proof of Proposition 3.1 is the following "near-sub-additivity" lemma.

Lemma 3.2. *There exists a constant $\alpha > 0$ independent of ω such that*

$$q_{m,r}(\omega) \leqslant q_{m,n}(\omega) + q_{n,r}(\omega) + \alpha \tag{3.9}$$

holds for all pairs of integers $0 \leqslant m < n < r$.

We postpone the proof of this lemma for the moment and proceed with the proof of Proposition 3.1. Using Lemma 3.2 we now show that there is a non-random constant $\bar{q}$ such that the limit

$$\lim_{n\to\infty} \frac{1}{n} q_{0,n}(\omega) = \bar{q}$$

holds almost surely. Lemma 3.2 shows that the family $\{q_{n,m}\}$ is "almost" sub-additive. In order to turn it into a truly sub-additive family define a new family

$$\hat{q}_{m,n} = q_{m,n} + \beta(n-m)^{1/2}$$

with β sufficiently large to be chosen. The point here is that $\hat{q}_{m,n}$ is a sub-linear correction of $q_{m,n}$. It also preserves translation invariance of $q_{m,n}$: for any integer $h > 0$, we have, using (3.8):

$$\hat{q}_{m+h,n+h}(\omega) = q_{m+h,n+h}(\omega) + \beta(n-m)^{1/2} = q_{m,n}(\pi_h\omega) + \beta(n-m)^{1/2} = \hat{q}_{m,n}(\pi_h\omega).$$

Let $\alpha > 0$ be as in (3.9) and choose $\beta > 4\alpha$. Then for any integers $0 \leqslant m < n < r$ the following elementary inequality holds:

$$\alpha + \beta(r-m)^{1/2} - \beta(r-n)^{1/2} - \beta(n-m)^{1/2} \leqslant 0$$

since $r - n \geqslant 1$ and $n - m \geqslant 1$. Lemma 3.2 implies that with this choice of β the family $\hat{q}_{m,n}$ is sub-additive: for any integers $0 \leqslant m < n < r$ we have

$$\begin{aligned}\hat{q}_{m,r} &= q_{m,r} + \beta(r-m)^{1/2} \leqslant q_{m,n} + q_{n,r} + \alpha + \beta(r-m)^{1/2}\\ &= \hat{q}_{m,n} + \hat{q}_{n,r} + \left(\alpha + \beta(r-m)^{1/2} - \beta(r-n)^{1/2} - \beta(n-m)^{1/2}\right) \leqslant \hat{q}_{m,n} + \hat{q}_{n,r}.\end{aligned}$$

Corollary 2.8 implies that $\hat{q}_{m,r}$ is at most linear: $0 \leqslant \hat{q}_{m,r} \leqslant C(1 + (m - r))$ for some constant $C > 0$. As the group π_n acts ergodically on Ω, we can apply the sub-additive ergodic theorem (see, e.g. [18]) to conclude that

$$\lim_{n\to\infty} \frac{1}{n}\hat{q}_{0,n} = \inf_{n>0} \frac{1}{n}\mathbb{E}[\hat{q}_{0,n}] = \bar{q} \tag{3.10}$$

holds almost surely, where $\bar{q}$ is a deterministic constant. By definition of $\hat{q}$, this implies that

$$\lim_{n\to\infty} \frac{1}{n} q_{0,n} = \bar{q}$$

also holds almost surely. Since $q_{0,n} = T(n,\omega)$ and $X(t)$ is increasing in t, it is easy to see that, as a consequence,

$$\lim_{t\to\infty} \frac{X(t,\omega)}{t} = (\bar{q})^{-1} := c_+^*$$

holds almost surely. The fact that $c_+^* \in [c^{\min}, c^{\max}]$ follows from (2.25). This proves (3.2), and the proof of (3.3) is identical.

The fact that limits (3.4) and (3.5) hold is an immediate consequence of (3.2), (3.3) and the fact that the width of the interface is bounded by a universal constant, as stated in Proposition 2.3. This completes the proof of Proposition 3.1.

The proof of Theorem 1.1

Now, we use comparison arguments to extend Proposition 3.1 to the case of any non-negative deterministic initial data with a sufficiently large compact support. By "sufficiently large", we mean large enough so that the solution does not converge uniformly to zero (extinction). Lemma 2.1 implies that the condition $u_0(x) \geqslant \zeta(x - x_0)$ with some $x_0 \in \mathbb{R}$ is sufficient to guarantee that extinction does not occur.

Let $w_0(x)$ be compactly supported with $0 \leqslant w_0 \leqslant 1$ and deterministic. Suppose that

$$w_0(x) \geqslant \zeta(x - x_0)$$

for some $x_0 \in \mathbb{R}$ and let $w(t, x, \omega)$ solve (1.1) with initial data $w_0(x)$. For each $t > 0$, let $X^+(t, \omega)$ be the largest real number satisfying $w(t, X^+(t, \omega), \omega) = \theta_0$.

If $u(t, x, \omega)$ solves the equation with initial data $u(0, x, \omega) = \zeta(x - x_0) \leqslant w_0(x)$, Proposition 3.1 applies to $u(t, x, \omega)$, and the maximum principle implies that $w(t, x, \omega) \geqslant u(t, x, \omega)$. Therefore, $w(t, x, \omega)$ satisfies

$$\lim_{t\to\infty} \inf_{c\in[c_-^*+\varepsilon, c_+^*-\varepsilon]} w(t, ct, \omega) \geqslant 1.$$

Since $w \leqslant 1$ for all $t \geqslant 0$, this implies the first bound of Theorem 1.1.

For the other bound, observe that for every realization ω we have $\max_{x\in\mathbb{R}} w(t = 1, x, \omega) < c_0 < 1$ with a deterministic constant c_0. The estimates in the previous section imply that there is a finite time $\tau > 0$ depending only on the properties of f such that

$$w(t = 1, x, \omega) \leqslant u(t = 1 + \tau, x, \omega), \quad \forall x \in \mathbb{R}.$$

Then the maximum principle implies that $w(s, x, \omega) \leqslant u(s + \tau, x, \omega)$ for all $s \geqslant 1$. This implies

$$\lim_{t\to\infty} \sup_{c\in(-\infty, c_-^*-\varepsilon]\cup[c_+^*+\varepsilon, \infty)} w(t, ct, \omega) \leqslant 0.$$

Since $w \geqslant 0$ for all t, this completes the proof of Theorem 1.1.

The proof of Lemma 3.2

Translation invariance (3.8) implies that it suffices to prove that (3.9) holds for $m = 0$. We first show that there is an integer $K > 0$ independent of ω such that for all r, n a "delayed" version

$$q_{0,r}(\omega) \leqslant q_{0,n}(\omega) + q_{n-j,r}(\omega) \tag{3.11}$$

holds for $j = \min(K, n)$. Let $h = \max_x \zeta(x) \in (\theta_0, 1)$ and define $X_h^l(t)$ as in (2.9). By Proposition 2.3, there is a constant $C > 0$, independent of ω such that

$$X_h^l(t) \geqslant X(t, \omega) - C. \tag{3.12}$$

Now let K be the smallest integer greater than $C + z_2 + z_1$ (recall that $\zeta(x) = 0$ for all $|x| \geqslant z_2$). First, (3.11) obviously holds for $n \leqslant K$ as for such n it becomes

$$q_{0,r}(\omega) \leqslant q_{0,n}(\omega) + q_{n-n,r}(\omega) = q_{0,n}(\omega) + q_{0,r}(\omega),$$

which is true since $q_{0,n}(\omega) \geqslant 0$.

If $n \geqslant K$ then (3.12) implies that

$$u\big(T(n, \omega), x, \omega\big) \geqslant h, \quad \forall x \in (-z_1, n - C) \subseteq (-z_1, n - K + z_2).$$

On the other hand, we have

$$\zeta\big(x + z_1 - (n - K)\big) = 0 \quad \text{for } x \notin (-z_1, n - K + z_2).$$

Therefore, if $n \geqslant K$, we have

$$u\big(T(n, \omega), x, \omega\big) \geqslant \zeta\big(x + z_1 - (n - K)\big) = u^{(n-K)}(0, x, \omega) \quad \text{for all } x \in \mathbb{R}.$$

Since the equation is invariant with respect to t, the maximum principle implies that for any $s \geqslant 0$,

$$u\big(T(n,\omega)+s,x,\omega\big) \geqslant u^{(n-K)}(s,x,\omega),$$

thus $X(T(n,\omega)+s,\omega) \geqslant X_{n-K}(s)$. Now setting $s = T_{n-j}(r,\omega) = T_{n-K}(r,\omega)$ we see that

$$X\big(T(n,\omega)+T_{n-K}(r,\omega)\big) \geqslant X_{n-K}\big(T_{n-K}(r,\omega)\big) = r.$$

Since X is increasing in t, this implies $T(r,\omega) \leqslant T(n,\omega) + T_{n-K}(r,\omega)$ which establishes (3.11) for $n \geqslant K$. Thus, the claim holds for all $n > 0$.

Using the fact that $u^{(n-j)}$ is monotone in t and the estimates of the previous section, one can show that there is a constant $\alpha > 0$ independent of n and ω such that

$$u^{(n-j)}(t,x,\omega) \geqslant \zeta(x+z_1-n), \quad \forall x \in \mathbb{R},\ t \geqslant \alpha,$$

where $j = \min(K,n)$ is bounded independent of n and ω. This and the maximum principle imply that

$$u^{(n-j)}(\alpha+s,x,\omega) \geqslant u^{(n)}(s,x,\omega), \quad \forall x \in \mathbb{R},\ s \geqslant 0.$$

Thus, we have

$$q_{n-j,r}(\omega) \leqslant q_{n,r}(\omega) + \alpha.$$

This inequality and (3.11) imply the desired result:

$$q_{m,r}(\omega) \leqslant q_{m,n}(\omega) + q_{n,r}(\omega) + \alpha.$$

This finishes the proof of Lemma 3.2.

4. Random traveling waves

Now we use the results of the previous sections to construct a random traveling wave solution to Eq. (1.1) and prove Theorem 1.3 and Corollary 1.4.

4.1. Construction of the traveling wave

The starting point comes from the proof of Theorem A(1) in [27]. We consider a family $\tilde{u}_n(t,x,\omega)$ of solutions of the Cauchy problem (1.1) with the initial data $\tilde{u}_n(t=-n,x,\omega) = \zeta^s(x-\tilde{x}_0^n(\omega))$. Here ζ^s is the step function:

$$\zeta^s(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geqslant 0, \end{cases}$$

and the shift $\tilde{x}_0^n(\omega)$ is fixed by the normalization, as in (2.6)

$$\tilde{u}_n(0,0,\omega) = \theta_0, \qquad \tilde{u}_n(0,x,\omega) < \theta_0 \quad \text{for } x > 0.$$

In this section we denote with tilde objects related to solutions with step-like initial data, while those without tilde correspond to those arising from bump-like initial data.

The random initial shift $\tilde{x}_0^n(\omega)$ is measurable with respect to $\mathcal{F}$ and is uniquely defined. The existence and uniqueness of $\tilde{x}_0^n(\omega)$ follow from the fact that if $y_1 < y_2$, the comparison principle implies that the solution to (1.1) with initial data $\zeta^s(x-y_1)$ must be below the solution with initial data $\zeta^s(x-y_2)$. Therefore, for fixed n, the front position at time $t=0$ is a monotonic function of the shift, and the maximum principle implies that it is continuous. Then, using arguments similar to those in the proof of Lemma 2.2 one can show that there must be a unique $\tilde{x}_0^n(\omega) \in [-cn, cn]$ such that the normalization condition is satisfied, if $c > 0$ is sufficiently large.

The measurability of $\tilde{u}_n$ and $\tilde{x}_0^n$ may be proved as in [27, Theorem A(1)]. For the readers' convenience we sketch the proof now. For each n, let $w(t,x,\omega;y)$ solve (1.1) for $t > -n$ with initial data $w(t=-n,x,\omega) = \zeta^s(x-y)$. Let $\eta_n(y,\omega)$ denote the largest real number satisfying $w(0,\eta_n,\omega) = \theta_0$. For each y, $\eta_n(y,\omega)$ is $\mathcal{F}$-measurable. This may be proved as in the case of $X^+(t,\omega)$ in Section 3. Now we vary y, and we wish to choose y as a measurable function of ω so that $\eta_n(y,\omega) = 0$. For each positive integer k define $\{y_l^k\} = 2^{-k}\mathbb{Z}$. Let r be a positive integer, and define

$$A_l^{k,r} = \{\omega \in \Omega \mid |\eta(y_l^k, \omega)| \leqslant 1/r\}.$$

This is an $\mathcal{F}$-measurable set since $\eta(y, \cdot)$ is $\mathcal{F}$-measurable. Then we set

$$\hat{x}_0^n(\omega) = \lim_{r\to\infty} \lim_{k\to\infty} \min_l \left(y_l^k \chi_{A_l^{k,r}}(\omega)\right). \tag{4.1}$$

Notice that

$$\min_l \left(y_l^k \chi_{A_l^{k,r}}(\omega)\right)$$

is $\mathcal{F}$-measurable, being the infimum of a countable set of measurable functions, and it is non-increasing in k and non-decreasing in r. Thus, the limits in (4.1) exist and $\hat{x}_0^n(\omega)$ is measurable. The continuity of $\eta(y, \omega)$ with respect to y and the uniqueness of $\tilde{x}_0^n$ imply that $\hat{x}_0^n(\omega) = \tilde{x}_0^n(\omega)$. So, $\tilde{x}_0^n$ is $\mathcal{F}$-measurable.

The measurability of $\tilde{u}_n$ now follows from the measurability of $\tilde{x}_0^n$. Specifically, for fixed n and t, the function $\tilde{u}_n$ may be expressed as a composition of measurable maps:

$$\tilde{u}_n(t, \cdot, \omega) = G_2 \circ G_1(\omega) \tag{4.2}$$

where $G_1(\omega) : (\Omega, \mathcal{F}) \to (\mathbb{R} \times \Omega, \mathcal{B} \times \mathcal{F})$ is the measurable map $G_1(\omega) = (\tilde{x}_0^n(\omega), \omega)$ and $G_2(y, \omega) : (\mathbb{R} \times \Omega, \mathcal{B} \times \mathcal{F}) \to C(\mathbb{R}; [0, 1])$ is the measurable map defined by solution of (1.1) with initial data $\zeta^s(x - y)$ (shifted by y) at time $t = -n$. Here $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$.

Now, for $\tilde{x}_0^n(\omega)$ defined in this way, we wish to take a limit $n \to +\infty$ to construct a global-in-time solution. That is, we wish to define

$$\tilde{w}(t, x, \omega) = \lim_{n\to+\infty} \tilde{u}_n(t, x, \omega), \tag{4.3}$$

and show that this is a traveling wave solution. The existence of a measurable limit, converging locally uniformly, and satisfying the PDE follows from Shen [27] (see proof of Theorem A(1)) and regularity estimates. A key observation in [27], is that the convergence (4.3) holds as $n \to +\infty$, not just along a particular subsequence n_k. This is because the functions $\tilde{u}_n$ satisfy the following monotonicity relation at $t = 0$:

$$\begin{aligned} &\tilde{u}_n(0, x, \omega) > \tilde{u}_m(0, x, \omega) \quad \text{if } x < 0, \\ &\tilde{u}_n(0, x, \omega) < \tilde{u}_m(0, x, \omega) \quad \text{if } x > 0, \end{aligned} \tag{4.4}$$

almost surely, for any $m > n$. Therefore, the function $\tilde{w}(t, x, \omega)$ is measurable in ω. However, the difficulty is that the limit might be trivial: one may obtain $\tilde{w}(t, x, \omega) \equiv \theta_0$ for all x and t. Here is where we invoke the results of the previous sections.

Uniform limits at infinity

Using Proposition 2.3 and the estimates of Section 2, we can show that the limit $\tilde{w}$ must be non-trivial.

Lemma 4.1. *Let $\tilde{w}(t, x, \omega)$ be constructed as above. Then we have*

$$\begin{aligned} &\lim_{x\to\infty} \sup_{\omega\in\Omega} \tilde{w}(t = 0, x, \omega) = 0, \\ &\lim_{x\to-\infty} \inf_{\omega\in\Omega} \tilde{w}(t = 0, x, \omega) = 1. \end{aligned} \tag{4.5}$$

Proof. We prove (4.5) by comparing the functions $\tilde{u}_n(t, x, \omega)$ with functions $u_n(t, x, \omega)$ defined as follows. For each n, let $u_n(t, x, \omega)$ denote the solution of (1.1) with initial data $\zeta(x - x_0^n)$ at time $t = -n - 1$ (note that u_n starts at time $t = -n - 1$, and not at $t = -n$). The function $\zeta(x)$ is the bump-like sub-solution used in Section 2, so the solution $u_n(t, x, \omega)$ is strictly monotone increasing in t and the estimates of Section 2 apply to u_n. The point $x_0^n = x_0^n(\omega)$ is a random shift depending on n. For such initial data, let $X_n^+(t; x_0^n, \omega)$ be defined as in Lemma 2.4. The random shift $x_0^n(\omega)$ is chosen so that $X_n^+(0; x_0^n, \omega) = 0$ for all $n \in \mathbb{N}$, $\omega \in \Omega$. This is the same normalization as applied to $\tilde{u}_n(t, x, \omega)$. Existence of the shift $x_0^n(\omega)$ for each realization ω follows from Lemma 2.2.

Having defined the function $x_0^n(\omega)$, one can show that for each $t > -n$, there exists a unique point $\xi_n(t,\omega)$ such that

$$\tilde{u}_n(t,x,\omega) > u_n(t,x,\omega) \quad \text{if } x < \xi_n(t,\omega),$$
$$\tilde{u}_n(t,x,\omega) < u_n(t,x,\omega) \quad \text{if } x > \xi_n(t,\omega).$$

That is, the graphs of the two solutions $\tilde{u}_n$ and u_n intersect at time t only at the point $x = \xi_n(t,\omega)$. This may be proved as in Lemma 4.6 of [27] using the results of Angenent [1] and the maximum principle. Here we sketch the argument. Recall that despite the suggestive notation we have initialized $u_n(t,x,\omega)$ at time $t = -n-1$ so that at time $t = -n$, we have $0 < u_n(t=-n,x,\omega) < 1$ everywhere. Therefore, using the approximation argument employed in the proof of Lemma 4.6 of [27], one may argue as if the graphs of $u_n(t=-n,x,\omega)$ and $\tilde{u}_n(t=-n,x,\omega) = \zeta^s(x-\tilde{x}_0^n)$ intersect at only one point. Since the function $q = \tilde{u}_n - u_n$ satisfies a PDE of the form

$$q_t = \Delta q + V(t,x)q$$

with $\|V\|_\infty < \infty$, Theorems A and B of [1] show that the zero set of the function $q(t,x)$ is discrete and cannot increase. Therefore, the graphs of $\tilde{u}_n$ and u_n have only one intersection point for all $t > -n$. We have chosen x_0^n and $\tilde{x}_0^n$ so that at $t = 0$, the graphs intersect at $x = 0$: $\tilde{u}_n(0,0,\omega) = \theta_0 = u_n(0,0,\omega)$ almost surely. Therefore, $\xi_n(0,\omega) = 0$, and both

$$\tilde{u}_n(0,x,\omega) > u_n(0,x,\omega), \quad x < 0,$$

and

$$\tilde{u}_n(0,x,\omega) < u_n(0,x,\omega), \quad x > 0,$$

must hold, $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$.

Passing to the limit $n \to +\infty$, we see that for $x < 0$ we have a lower bound for $\tilde{w}(0,x,\omega)$:

$$\tilde{w}(0,x,\omega) \geqslant \liminf_{n\to+\infty} u_n(0,x,\omega) := v^-(x,\omega).$$

It follows that from Lemma 2.6 that $v^-(x,\omega)$ has a deterministic lower bound

$$\lim_{x\to-\infty} v^-(x,\omega) \geqslant \lim_{x\to-\infty} v(x) = 1,$$

which holds for all realizations ω. Similarly, for $x > 0$, we have an upper bound for $\tilde{w}(0,x,\omega)$:

$$\tilde{w}(0,x,\omega) \leqslant \limsup_{n\to+\infty} u_n(0,x,\omega) := v^+(x,\omega)$$

and, once again, by Lemma 2.6, $v^+(x,\omega)$ has a deterministic upper bound:

$$\lim_{x\to+\infty} v^+(x,\omega) \leqslant \lim_{x\to+\infty} v(x) = 0,$$

that holds for all ω. This proves that (4.5) holds uniformly in ω. □

The translation property

We have now shown that $\tilde{w}(t,x,\omega)$ satisfies properties (i)–(iv) in the definition of a random traveling wave. Since the limit $\tilde{w}(t,x,\omega)$ is non-trivial, the position of the interface $\tilde{X}(t,\omega)$ may be defined at time t:

$$\tilde{X}(t,\omega) = \max\{x \in \mathbb{R} \mid \tilde{w}(t,x,\omega) = \theta_0\}. \tag{4.6}$$

The measurability of $\tilde{X}(t,\omega)$ may be proved as in the case of $\tilde{x}_0^n(\omega)$.

Finally we show that the translation property (v) holds. The argument here is similar to that in [27]; we sketch details for the readers' convenience. Notice that we have not needed to assume that the index n is an integer. In fact, we may assume $n \in [1,\infty)$. The key observation that leads to property (v) is that for any $m \geqslant 0$,

$$\tilde{u}_n\big(m, x+\theta_n(m,\omega), \omega\big) = \tilde{u}_{n+m}\big(0, x, \pi_{\theta_n(m,\omega)}\omega\big) \tag{4.7}$$

must hold. Here, $\theta_n(m,\omega)$ is the position of the interface at time $t=m$, when the solution is initialized at time $t=-n$ (with initial data $\zeta^s(x-\tilde{x}_0^n)$). One may think of $\pi_{\theta_n(m,\omega)}\omega$ as the "current environment" associated with the "current location" of the interface (i.e. $\theta_n(m,\omega)$) at time $t=m$. If at time $t=m$ the interface is at $x=\theta_n(m,\omega)$, then in the coordinate system shifted by $\theta_n(m,\omega)$ the interface is at the origin. So if we simply shift x by $\theta_n(m,\omega)$ and t by m, equality (4.7) follows from the definition of $\tilde{u}_n$ and $\tilde{u}_{n+m}$, the fact that $f(x+\theta_n(m,\omega),u,\omega)=f(x,u,\pi_{\theta_n(m,\omega)}\omega)$, and the fact that $\tilde{x}_0^n$ and $\tilde{x}_0^{n+m}$ are uniquely defined. In particular, the function

$$v(t,x,\pi_{\theta_n(m,\omega)}\omega):=\tilde{u}_n\big(t+m,x+\theta_n(m,\omega),\omega\big) \tag{4.8}$$

satisfies the shifted equation

$$v_t=\Delta v+f\big(x+\theta_n(m,\omega),v,\omega\big)=\Delta v+f(x,v,\pi_{\theta_n(m,\omega)}\omega) \tag{4.9}$$

with initial data $v(t=-n-m,x,\pi_{\theta_n(m,\omega)}\omega)=\zeta^s(x-\tilde{x}_0^n(\omega)+\theta_n(m,\omega))$. Since $\tilde{x}_0^{n+m}(\pi_{\theta_n(m,\omega)}\omega)$ is uniquely defined, this is the same initial value problem solved by $\tilde{u}_{n+m}(t,x,\pi_{\theta_n(m,\omega)}\omega)$. Therefore, uniqueness implies $v=\tilde{u}_{n+m}$. So, (4.7) holds.

By definition of $\tilde{w}$ and $\tilde{X}$, $\theta_n(m,\omega)\to\tilde{X}(m,\omega)$ as $n\to\infty$, and the left-hand side of (4.7) converges to

$$\lim_{n\to\infty}\tilde{u}_n\big(m,x+\theta_n(m,\omega),\omega\big)=\tilde{w}\big(m,x+\tilde{X}(m,\omega),\omega\big). \tag{4.10}$$

We claim that as $n\to\infty$ the right-hand side of (4.7) converges to $\tilde{w}(0,x,\pi_{\tilde{X}(m,\omega)}\omega)$. To see this, we express the right-hand side of (4.7) in the reference frame corresponding to $\tilde{X}(m,\omega)$. Let $\omega_m=\pi_{\tilde{X}(m,\omega)}\omega$ and define

$$z_{n+m}(t,x,\omega_m)=\tilde{u}_{n+m}\big(t,x+\tilde{X}(t,\omega)-\theta_n(m,\omega),\pi_{\theta_n(m,\omega)}\omega\big).$$

Then z_{n+m} satisfies

$$z_t=\Delta z+f\big(x+\tilde{X}(t,\omega)-\theta_n(m,\omega),z,\pi_{\theta_n(m,\omega)}\omega\big)=\Delta z+f(x,z,\omega_m) \tag{4.11}$$

with initial condition $z_{n+m}(t=-n-m,x,\omega_m)=\zeta^s(z-z_0^n)$ where $z_0^n=\tilde{X}(t,\omega)-\theta_n(m,\omega)-\tilde{x}_0^{n-m}$. However, the function $\tilde{u}_{n+m}(t,x,\omega_m)$ satisfies the same equation (4.11) with initial condition $\tilde{u}_{n+m}(t=-n-m,x,\omega_m)=\zeta^s(z-\tilde{x}_0^n(\omega_m))$. In general, $z_0^n\neq\tilde{x}_0^n(\omega_m)$, but the maximum principle still implies that at time $t=0$ either $z_{n+m}(0,x,\omega_m)>\tilde{u}_{n+m}(0,x,\omega_m)$ for all x, or $z_{n+m}(0,x,\omega_m)<\tilde{u}_{n+m}(0,x,\omega_m)$ for all x. However, at time $t=0$, $\tilde{u}_{n+m}(0,0,\omega_m)=\theta_0$, and $z_{n+m}(0,\theta_n(m,\omega)-\tilde{X}(t,\omega),\omega_m)=\theta_0$. Since $\lim_{n\to\infty}|\theta_n(m,\omega)-\tilde{X}(t,\omega)|=0$, one can use the maximum principle to show that in the limit $n\to\infty$, the two functions coincide:

$$\lim_{n\to\infty}z_{n+m}(t,x,\omega_m)=\lim_{n\to\infty}\tilde{u}_{n+m}(t,x,\omega_m)$$

for all x and t, as in Lemma 4.5(2) of [27], since they both converge to θ_0 at the point $x=0$, $t=0$. By definition of $\tilde{w}$, the right-hand side at $t=0$ is simply

$$\lim_{n\to\infty}\tilde{u}_{n+m}(0,x,\omega_m)=\tilde{w}(0,x,\pi_{\tilde{X}(m,\omega)}\omega).$$

This proves the claim (4.10) and establishes the translation property

$$\tilde{w}(0,x,\pi_{\tilde{X}(m,\omega)}\omega)=\tilde{w}\big(m,x+\tilde{X}(m,\omega),\omega\big).$$

This completes the construction of the traveling wave.

For later use, let us note that the preceding proof shows that the function $W(x,\omega)=\tilde{w}(0,x,\omega)$ satisfies

$$W(x,\omega)\geqslant v(x),\quad \forall x<0,$$
$$W(x,\omega)\leqslant v(x),\quad \forall x>0,$$

where $v(x)$ is deterministic and defined in Lemma 2.6. Therefore, the translation property (v) implies that

$$\tilde{w}(t,x+\tilde{X}(t),\omega)\geqslant v(x),\quad \forall x<0,$$
$$\tilde{w}(t,x+\tilde{X}(t),\omega)\leqslant v(x),\quad \forall x>0,$$

also holds.

Traveling waves and generalized transition waves

Let us point out that an alternative way to establish existence of a traveling wave is to use the bump functions $u_n(t,x,\omega)$ and pass to the limit along a subsequence $n_k(\omega) \to +\infty$ to obtain a non-trivial transition front $u(t,x,\omega)$ in the sense of Berestycki and Hamel. Theorem A of [27] shows that a traveling wave will exist if there exists such a generalized transition front for each realization. However, it may be necessary to take the limit along a different subsequence $n_k(\omega)$ for each ω. This may result in a transition wave $u(t,x,\omega)$ that may not be measurable. The advantage of using a shift of the step function $\zeta^s(x)$ is that the sequence is monotone in the sense of (4.4) and the limit (4.3) may be taken as $n \to +\infty$. Therefore, the limit is measurable.

4.2. *Properties of the traveling wave*

Now, we finish the proof of Theorem 1.3 — it remains to show that the interface location $\tilde{X}(t)$ is a strictly increasing function and that the limit in (1.5) exists and is deterministic. First, we show that

$$\lim_{t\to\infty} \frac{\tilde{X}(t,\omega)}{t} = c_+^* \tag{4.12}$$

almost surely with respect to $\mathbb{P}$, where c_+^* is the deterministic right spreading rate defined in Theorem 1.1. Using Theorem 1.1 and the comparison principle, it is easy to show that

$$\liminf_{t\to\infty} \frac{\tilde{X}(t,\omega)}{t} \geqslant \liminf_{t\to\infty} \frac{X(t,\omega)}{t} = c_+^*,$$

with probability one, since we may construct compactly supported initial data that fits below each realization of the profile $W(x,\omega)$.

A super-solution for the traveling wave

For an upper bound, we construct a super-solution related to a construction in [6]. Let $u_n(t,x,\omega)$ be the same family of monotone increasing solutions constructed in the proof of Theorem 1.3. Let $q \in (0,\theta_0/3)$ and set $h = 1-q$. For $v(x)$ defined as in Lemma 2.6, let $y_h = v^{-1}(h) < 0$ (i.e. $v(y_h) = h$). Pick $n \in \mathbb{N}$ sufficiently large so that Lemma 2.6 holds with $R = -y_h$. Therefore, by Lemma 2.6, we have

$$\begin{aligned} &u_n\big(t, x + X_n(t), \omega\big) \geqslant v(x), \quad \forall x \in [y_h, 0],\\ &u_n\big(t, x + X_n(t), \omega\big) \leqslant v(x), \quad \forall x > 0, \end{aligned} \tag{4.13}$$

for all $t \geqslant 0$. For a function $\gamma(t)$ to be chosen, define

$$\bar{u}_n(t,x,\omega) = \begin{cases} \min(1, u_n(\gamma(t),x,\omega) + q), & x > X_n(\gamma(t)) - y_h,\\ 1, & x \leqslant X_n(\gamma(t)) - y_h. \end{cases} \tag{4.14}$$

The function $\gamma(t)$ will be chosen so that $\gamma(0) > 0$ and $\gamma'(t) > 1$. We want to pick $\gamma(t)$ so that $\bar{u}$ is a super-solution for $t \geqslant 0$. By construction, $\bar{u}$ now has a wave-like profile, and $\bar{u} = 1$ for x sufficiently negative.

If $u_n(\gamma(t),x,\omega) \geqslant h$ or $x < X_n(\gamma(t)) - y_h$, then $\bar{u}_n(t,x,\omega) = 1 \geqslant \tilde{w}(t,x,\omega)$. On the other hand, if $u_n(\gamma(t), x,\omega) \leqslant h$ and $x \geqslant X_n(\gamma(t)) - y_h$, then $\bar{u}(t,x,\omega) \leqslant 1$ and

$$\frac{\partial \bar{u}_n}{\partial t} - \frac{\partial^2 \bar{u}_n}{\partial x^2} - f(x,\bar{u}_n) = \big(\gamma'(t) - 1\big)\frac{\partial u_n}{\partial t} + \big[f(x,u_n) - f(x,\bar{u}_n)\big]. \tag{4.15}$$

Now we show that the right-hand side of (4.15) can be made non-negative for $x \geqslant X_n(\gamma(t)) - y_h$, so that $\bar{u}_n$ is a super-solution in this region.

By the properties of f, there exists $s \in (0,\theta_0/3)$ such that $f(x,u) - f(x,\bar{u}) \geqslant 0$ wherever $1 - s \leqslant u \leqslant \bar{u} \leqslant 1$. Note that such an s may be chosen independently of q and h. For such an s fixed, (4.13) and the properties of v imply that there is $\beta > 0$ such that

$$\big\{x \in [y_h,\infty) \;\big|\; u_n\big(\gamma(t), x + X_n\big(\gamma(t)\big),\omega\big) \in [s, 1-s]\big\} \subset [-\beta,\beta] \tag{4.16}$$

for all $t \geqslant 0$. By Proposition 2.5, there is $\delta > 0$ such that

$$\frac{\partial u_n}{\partial t}\big(\gamma(t), X_n\big(\gamma(t)\big), \omega\big) > \delta.$$

This and the Harnack inequality imply that there is $\varepsilon > 0$ such that

$$\frac{\partial u_n}{\partial t}\big(\gamma(t), x + X_n\big(\gamma(t)\big), \omega\big) > \varepsilon, \quad \forall x \in [-\beta, \beta],\ t \geqslant 0. \tag{4.17}$$

Now, if $x \in [X_n(\gamma(t)) - y_h, X_n(\gamma(t)) - \beta]$, then by (4.16) we have $\bar{u}_n(t,x) \geqslant u_n(\gamma(t), x) \geqslant 1 - s$, so $f(x, \bar{u}_n) \leqslant f(x, u_n)$, the last term on the right side of (4.15) is non-negative and thus (4.15) implies that in this interval

$$\frac{\partial \bar{u}_n}{\partial t} - \frac{\partial^2 \bar{u}_n}{\partial x^2} - f(x, \bar{u}_n) = \big(\gamma'(t) - 1\big)\frac{\partial u_n}{\partial t} \geqslant 0, \tag{4.18}$$

since $\gamma'(t) \geqslant 1$.

If $x \in [X_n(\gamma(t)) + \beta, +\infty)$, then $u_n(\gamma(t), x) \leqslant s$, so $\bar{u}_n(t,x) \leqslant s + q < \theta_0$. Hence $f(x, u_n) = f(x, \bar{u}_n) = 0$ in this region, so again (4.18) holds.

Finally, if $x \in [X_n(\gamma(t)) - \beta, X_n(\gamma(t)) + \beta]$, the right side of (4.15) can be bounded below using (4.17) by

$$\partial_t \bar{u} - \bar{u}_{xx} - f(x, \bar{u}) \geqslant \big(\gamma'(t) - 1\big)\varepsilon + \big[f(x,u) - f(x,\bar{u})\big] \geqslant \big(\gamma'(t) - 1\big)\varepsilon - Kq$$

where $K > 0$ is the Lipschitz constant for f. So if we choose $\gamma'(t) = 1 + Kq/\varepsilon$, the right side is non-negative. For $\gamma(t)$ chosen in this way, we see that $\bar{u}_n$ is a super-solution wherever $\bar{u} < 1$, for all $t \geqslant 0$. Since u_n is monotone increasing in t, we may also choose $\gamma(0)$ sufficiently large so that

$$\bar{u}_n(0, x, \omega) \geqslant \tilde{w}(0, x, \omega).$$

Therefore, the maximum principle implies that $\bar{u}_n(t, x, \omega) \geqslant \tilde{w}(t, x, \omega)$ for all $t \geqslant 0$. Hence, we have

$$\limsup_{t\to\infty} \frac{\tilde{X}(t)}{t} \leqslant \limsup_{t\to\infty} \frac{X_n(\gamma(t))}{t} = \limsup_{t\to\infty} \frac{X_n(\gamma(t))}{\gamma(t)}\frac{\gamma(t)}{t} = c_+^*\left(1 + \frac{Kq}{\varepsilon}\right) = c_+^*\left(1 + \frac{K(1-h)}{\varepsilon}\right).$$

Since h can be chosen to be arbitrarily close to 1, the right side can be made arbitrarily close to c_+^*. Note that s and β can be chosen independently of h, so that the parameter ε does not become small as $h \uparrow 1$. This proves the upper bound and establishes (4.12).

Monotonicity of the right interface

We now prove the last claim of Theorem 1.3 — that the interface $\tilde{X}(t)$ always moves to the right.

Lemma 4.2. *For almost every $\omega \in \Omega$, the function $\tilde{X}(t, \omega)$ is differentiable and strictly increasing in t.*

Proof. The maximum principle and the fact that $f(x,u) = 0$ for $u \leqslant \theta_0$ implies that $\tilde{X}$ cannot have jumps to the right:

$$\limsup_{h\to 0^+} \tilde{X}(t + h, \omega) \leqslant \tilde{X}(t, \omega). \tag{4.19}$$

To see that $\tilde{X}$ is continuous and differentiable, note that

$$\theta_0 = \tilde{w}\big(t, \tilde{X}(t), \omega\big) \tag{4.20}$$

for all t. The function $W(x, \omega) = \tilde{w}(0, x, \omega)$ satisfies

$$W(x, \omega) > v(x) \quad \text{if } x < 0,$$
$$W(x, \omega) < v(x) \quad \text{if } x > 0,$$

$\mathbb{P}$-almost surely, and $v_x(0) < -p$ for some constant $p > 0$. Therefore, we have

$$W_x(0, \omega) = \tilde{w}_x\big(t, \tilde{X}(t), \omega\big) < -p < 0.$$

The Implicit Function Theorem applied to (4.20) implies that there is a C^1 function $Y(t)$ such that $\theta_0 = \tilde{w}(Y(t+h), t+h, \omega)$ for h sufficiently small, and $Y(t) = \tilde{X}(t)$. This, combined with the definition (4.6) and (4.19), implies that $\tilde{X}(t)$ is continuous and that we may differentiate (4.20) to obtain

$$\tilde{X}'(t,\omega) = -\frac{\tilde{w}_t(t, \tilde{X}(t,\omega), \omega)}{\tilde{w}_x(t, \tilde{X}(t,\omega), \omega)} < \infty.$$

This may also be written as

$$\tilde{X}'(t,\omega) = -\frac{W_{xx}(0, \pi_{\tilde{X}(t,\omega)}\omega) + f(0, W(0, \pi_{\tilde{X}(t,\omega)}\omega), \pi_{\tilde{X}(t,\omega)}\omega)}{W_x(0, \pi_{\tilde{X}(t,\omega)}\omega)}.$$

We have already shown that there is a set of full measure $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$, and $\tilde{X}(t,\omega)/t \to c_+^* \geqslant c^{\min} > 0$ for all $\omega \in \Omega_0$ as $t \to +\infty$. If $\tilde{X}(t)$ is not strictly increasing in time, there are $t_1, t_2 \in \mathbb{R}$ such that $t_2 > t_1$ and $\tilde{X}(t_1, \omega_0) = \tilde{X}(t_2, \omega_0)$ for some $\omega_0 \in \Omega_0$. Then

$$\tilde{w}(t_1, x, \omega_0) = W\big(x - \tilde{X}(t_1,\omega_0), \pi_{\tilde{X}(t_1,\omega_0)}\omega_0\big) = W\big(x - \tilde{X}(t_2,\omega_0), \pi_{\tilde{X}(t_2,\omega_0)}\omega_0\big) = \tilde{w}(t_2, x, \omega_0)$$

holds for all $x \in \mathbb{R}$. Hence, the function $\tilde{w}(t, x, \omega_0)$ is periodic in t. This contradicts the fact that $\tilde{X}(t,\omega)/t \to c_+^* > 0$ for all $\omega \in \Omega_0$. Therefore, $\tilde{X}(t+h,\omega) > \tilde{X}(t,\omega)$ for all $t \in \mathbb{R}$, $h > 0$, $\omega \in \Omega_0$. The proof of Theorem 1.3 is now complete. □

Proof of Corollary 1.4

This follows immediately from the definition of $\tilde{X}$ and $\tilde{T}$:

$$\tilde{w}\big(\tilde{T}(\xi,\omega), x+\xi, \omega\big) = W\big(x + \xi - \tilde{X}\big(\tilde{T}(\xi,\omega),\omega\big), \pi_{\tilde{X}(\tilde{T}(\xi,\omega),\omega)}\omega\big) = W(x+\xi-\xi, \pi_\xi\omega) = W(x, \pi_\xi\omega).$$

The last term on the right side is stationary with respect to shifts in ξ since the action of π is measure-preserving.

Monotonicity of the wave in time

The next lemma is a consequence of the monotonicity of the interface in time.

Lemma 4.3. *For any $h > 0$, $\tilde{w}(t+h, x, \omega) > \tilde{w}(t, x, \omega)$ holds for all $x \in \mathbb{R}$ and all $t \in \mathbb{R}$, $\mathbb{P}$-almost surely.*

Proof. Fix $h > 0$. Due to the translation property of the wave, we have

$$\tilde{X}(t+h,\omega) = \tilde{X}(t,\omega) + \tilde{X}(h, \pi_{\tilde{X}(t,\omega)}\omega), \tag{4.21}$$

and thus

$$\tilde{w}(t,x,\omega) = \tilde{w}\big(0, x - \tilde{X}(t,\omega), \pi_{\tilde{X}(t,\omega)}\omega\big),$$

and

$$\begin{aligned}\tilde{w}(t+h,x,\omega) &= \tilde{w}\big(0, x - \tilde{X}(t+h,\omega), \pi_{\tilde{X}(t+h,\omega)}\omega\big)\\ &= \tilde{w}\big(0, x - \tilde{X}(t,\omega) - \tilde{X}(h, \pi_{\tilde{X}(t,\omega)}\omega), \pi_{\tilde{X}(t,\omega)+\tilde{X}(h,\pi_{\tilde{X}(t,\omega)}\omega)}\omega\big)\\ &= \tilde{w}\big(0, x - \tilde{X}(t,\omega) - \tilde{X}(h, \pi_{\tilde{X}(t,\omega)}\omega), \pi_{\tilde{X}(h,\pi_{\tilde{X}(t,\omega)}\omega)}\pi_{\tilde{X}(t,\omega)}\omega\big) = \tilde{w}\big(h, x - \tilde{X}(t,\omega), \pi_{\tilde{X}(t,\omega)}\omega\big).\end{aligned}$$

Hence, it suffices to prove the result for $t = 0$. By definition of the wave, we have

$$\tilde{w}(h,x,\omega) - \tilde{w}(0,x,\omega) = \lim_{n\to\infty} \tilde{u}_n(h,x,\omega) - \tilde{u}_n(0,x,\omega). \tag{4.22}$$

The translation property implies that

$$\tilde{u}_n(h,x,\omega) = \tilde{u}_n\big(0, x - \tilde{X}(h,\omega), \pi_{\tilde{X}(h,\omega)}\omega\big).$$

The function

$$v_n(t,x,\omega)=\tilde u_n\big(t,x-\tilde X(h,\omega),\pi_{\tilde X(h,\omega)}\omega\big)$$

satisfies

$$\partial_t v=\Delta v+f\big(x-\tilde X(h,\omega),v,\pi_{\tilde X(h,\omega)}\omega\big)=\Delta v+f(x,v,\omega),$$

which is the same equation as satisfied by $\tilde u_n(t,x,\omega)$. Moreover, at the initial time $t=-n$, we have

$$v_n(t=-n,x,\omega)=\zeta^s\big(x-z_n(\omega)\big),$$

where $z_n(\omega)=\tilde X(h,\omega)+\tilde x_0^n(\pi_{\tilde X(h,\omega)}\omega)$. Observe that if $z_n(\omega)\leqslant \tilde x_0^n(\omega)$, then

$$\zeta^s\big(x-z_n(\omega)\big)=v_n(t=-n,x,\omega)\leqslant \tilde u_n(t=-n,x,\omega)=\zeta^s\big(x-\tilde x_0^n(\omega)\big). \tag{4.23}$$

In this case, the maximum principle would imply that $v_n(0,x,\omega)\leqslant \tilde u_n(0,x,\omega)$ for all $x\in\mathbb{R}$.

We also know that $v_n(0,\tilde X(h,\omega),\omega)=\theta_0$ — this follows from the definition of v_n. However, $\tilde u_n(0,x,\omega)<\theta_0$ for all $x>0$. Therefore, since $\tilde X(h,\omega)>0$ (by Lemma 4.2), inequality (4.23) cannot hold, so we must have

$$z_n(\omega)>\tilde x_0^n(\omega),$$

or, equivalently

$$\tilde x_0^n(\pi_{\tilde X(h,\omega)}\omega)>\tilde x_0^n(\omega)-\tilde X(h,\omega).$$

The maximum principle implies that $v_n(0,x,\omega)>\tilde u_n(0,x,\omega)$ for all $x\in\mathbb{R}$, which means that

$$\tilde u_n\big(0,x-\tilde X(h,\omega),\pi_{\tilde X(h,\omega)}\omega\big)-\tilde u_n(0,x,\omega)>0$$

for all x and n. Then from (4.22) we see that

$$\begin{aligned}\tilde w(h,x,\omega)-\tilde w(0,x,\omega)&=\lim_{n\to\infty}\tilde u_n(h,x,\omega)-\tilde u_n(0,x,\omega)\\&=\lim_{n\to\infty}\tilde u_n\big(0,x-\tilde X(h,\omega),\pi_{\tilde X(h,\omega)}\omega\big)-\tilde u_n(0,x,\omega)\geqslant 0\end{aligned}$$

for all $x\in\mathbb{R}$. Then the maximum principle implies strict inequality: $\tilde w(h,x,\omega)>\tilde w(0,x,\omega)$ for all $x\in\mathbb{R}$. □

This completes the proof of Theorem 1.3.

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