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A generalization of Aubry–Mather theory to partial differential equations and pseudo-differential equations

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Dedicated to Luis Caffarelli on the occasion of his 60th birthday

Abstract

We discuss an Aubry–Mather-type theory for solutions of non-linear, possibly degenerate, elliptic PDEs and other pseudo-differential operators.

We show that for certain PDEs and ΨDEs with periodic coefficients and a variational structure it is possible to find quasi-periodic solutions for all frequencies. This results also hold under a generalized definition of periodicity that makes it possible to consider problems in covers of several manifolds, including manifolds with non-commutative fundamental groups.

An abstract result will be provided, from which an Aubry–Mather-type theory for concrete models will be derived.

Keywords: Aubry–Mather theory; Quasi-periodic solutions; Calculus of variations; Comparison; Possibly degenerate and fractional operators; Subordination; Gradient flow

1. Introduction

The goal of this paper is to develop the beginning of an Aubry–Mather theory for some partial differential equations and pseudo-differential equations.

Roughly speaking, we establish that if the variational problem is:

- Symmetric under integer translations.
- Invariant under addition of integer constants to the unknown function.
- The gradient flow of the functional (in the case of PDEs, it will often be a semi-flow, but we will also use the more common name) is well defined, moderately regular and, more crucially, satisfies a weak comparison principle.
- The Euler–Lagrange equations are elliptic (in a rather weak sense).

Then, there are quasi-periodic equilibrium solutions for all frequencies.

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We also show that the invariance under integer translations can be generalized to invariance under the action of a suitable group, provided that we modify accordingly the definition of quasi-periodicity. In particular, we can answer a question of [58] and develop the theory of variational integrals in some manifolds whose fundamental group is non-Abelian.

Moreover, the quasi-periodic solutions we construct satisfy an extra geometric property, often called "Birkhoff property" even if it seems that they were emphasized and used first in [55]. This implies that, when reduced to the fundamental domain, the solutions give rise to a lamination.

In the Euclidean and non-degenerate settings, results similar to those proved in this paper were proved first in [58] (see also [61] and [62]) which brought to prominence the relation between Aubry–Mather theory and the variational formulation of elliptic PDEs. These results were extended in [5] and several papers have been recently appeared, which developed several features of Aubry–Mather-type theory for PDEs: see, in particular, [15,84,69–72,83,16,9,66,10,27] and [64].

In the proof presented here we make an important use of the gradient semi-flow of our functional. The idea of using the gradient semi-flow and its comparison properties for Aubry–Mather theory seems to have originated in [2] and was used to prove the classical result of Aubry–Mather in [40]. Extensions of this method to difference equations on lattices and on graphs were considered in [44,17,28,26,25]. The geometric framework of this paper is also similar to the one in [17]. The latter extensions made it clear that the locality of the interaction, which was assumed in several proofs of similar results, was only mildly important. This served as a motivation for including pseudo-differential operators in our formulation. The gradient semi-flow has also been used to prove other results of Aubry–Mather theory. In [78], for instance, there is a proof of the shadowing lemma in the context of dynamical systems.

The plan of this paper is as follows: In Section 2, we give a proof, using our method, of a result already proved by [58]. In Section 3, we formulate an abstract theorem whose proof follows exactly the proof of the particular case in Section 2.

Section 4 develops some analytical techniques related with the regularity of the gradient flow and with the comparison principle.

In Section 5, we show that our abstract theorem applies to several examples which are not covered by [58] since the manifold does not have an Abelian fundamental group and the variational principles are not local or degenerate.

In particular, Section 5.1 deals with degenerate operators of p-Laplacian type, while equations driven by the fractional Laplacian are considered in Section 5.2.

In Section 5.3, we show how the results can be extended to cover not only periodic variational problems in $\mathbb{R}^d$ but also problems in hyperbolic space and in other symmetric spaces. The main new ingredient for these geometric generalizations is an extension (taken from [17]) of the notion of quasi-periodic functions in $\mathbb{R}^d$ to situations where we have a space and an action of a well behaved (residually finite) discrete group. When the space is $\mathbb{R}^d$ and the group is $\mathbb{Z}^d$, we recover the classical notion of quasi-periodicity. As a matter of fact, in the terminology of [74], what we call here "quasi-periodic" could also be called "almost automorphic".

It should be clear from the preceding remarks, and even more so from the text, that this paper is somewhat open-ended. Aubry–Mather theory is very rich and it not only includes the existence result of quasi-periodic solutions, but also the existence of several quasi-periodic solutions of the same frequency, the non-existence of smooth solutions for large perturbations, the existence of connecting orbits, etc. To our knowledge, there are no complete generalizations to PDEs and ψDEs of these features. See, however, [5,15,69,83,16,9,28,27] and [65] for related results.

We also note that, besides the generalization presented here, there are several other possibilities in which the interaction between Aubry–Mather theory and partial differential equations can be carried out. For example [34,33] and [15] contain other points of view.

The gradient semi-flow also opens further possibilities, such as the construction of multiple solutions and the analysis of minimal versus non-minimal solutions (see [9] and [27]).

We remark that, since the main results of this paper are of geometric type, we did not try to minimize regularity assumptions. We also note that there are different variational principles which lead to the same equations. Notably, by adding an exact differential, one does not change the critical points, but one changes the minimizers (which, for instance, is a crucial observation in the construction of connecting orbits in [56] and [57]).

2. A first illustrative result

In this section we will give the proof of a result which will be very illustrative of the method we present here. The result itself is a particular case of results which have already been obtained in [58]. Later, we will show how the method presented here can be given an abstract formulation that follows, roughly, the same steps. The abstract theorem will have other instances which we discuss in subsequent sections.

A crucial ingredient in our treatment is the following definition, inspired by one that appeared in the classical Aubry–Mather theory in dynamical systems and that has been also used in [58,5,15,83] and [27].

Definition 2.1. We say that a function $u : \mathbb{R}^d \mapsto \mathbb{R}$ has the Birkhoff property (or that it is Birkhoff) when:

For a fixed $e \in \mathbb{Z}^d$, $\ell \in \mathbb{Z}$

$$u(x+e) - \big(u(x) + \ell\big) \tag{1}$$

does not change sign with x, i.e., when, for all values of x, the expression in (1) above is either $\geqslant 0$ or $\leqslant 0$ with the choice depending on e, ℓ but not on x.

Equivalently, if we consider the graph of the solution folded to the fundamental domain of $\mathbb{T}^{d+1}$, we obtain the set:

$$\mathcal{C} = \big\{\big(x, u(x+e) \bmod 1\big),\ x \in \mathbb{T}^d,\ e \in \mathbb{Z}^d\big\}$$

which consists of a countable set of manifolds that do not cross.

The main result of this section is the following:

Theorem 2.2. *Let $V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a C^2 function which satisfies*:

$$V(x+e, u) = V(x, u) \quad \forall e \in \mathbb{Z}^d,$$
$$V(x, u+\ell) = V(x, u) \quad \forall \ell \in \mathbb{Z}.$$

Then, for all $\omega \in \mathbb{R}^d$, the problem

$$\Delta u - \partial_2 V(x, u) = 0 \tag{2}$$

has a solution which satisfies the Birkhoff property and such that

$$u(x) - \omega \cdot x \in L^\infty\big(\mathbb{R}^d\big). \tag{3}$$

We will refer to ω in (3) as the frequency of the solution. Clearly, a function can only satisfy (3) for one frequency.

Remark 2.3. The reason why the result in Theorem 2.2 can be considered an analogue of the classical Aubry–Mather theory is discussed in [58] and [59]. Suffice it to note here that if we take $n = 1$, we are considering the motion in a time periodic potential, the time one map of this flow will be a twist mapping. Indeed, in [59] it is shown that the classical Aubry–Mather theorem on the existence of quasi-periodic invariant measures for twist mappings can be recovered from the consideration of continuous variational principles as above.

Remark 2.4. Notice that the above problem (2) is the Euler–Lagrange equation for the (formal) functional

$$\mathcal{S}(u) = \int_{\mathbb{R}^d} \frac{1}{2}\big|\nabla u(x)\big|^2 + V\big(x, u(x)\big)\,dx. \tag{4}$$

Of course, the variational functional above does not make sense and one has to interpret it as usual in the calculus of variations by considering variations with compact support and extending the integral only to compact sets which include the support of the variation (see, e.g., [15]). The kind of minimizers obtained by comparing with all the compactly supported variations are usually referred to with the name of “local” (or “class A”) minimizers. In this paper, we will be concerned mainly with solutions of the PDE associated to (4) (that is, with critical points of (4)) and not with minimizers. Minimizers of related variational problems have been considered in [15,69], and [83]. In general, rearrangement methods, such as the ones in [15], are needed both to construct minimizers and to select the ones with further geometric properties.

Remark 2.5. Note that in the second formulation of the Birkhoff property in Definition 2.1, the (closure of the) set $\mathcal{C}$ is a lamination, since it consists of leaves which do not cross. In fact, the strong maximum principle implies that they do not touch.

As it is shown in examples in [4], one can have that $\mathcal{C}$ is not dense in $\mathbb{T}^{d+1}$ even if the frequency is irrational. In classical Aubry–Mather theory, this is analogous to the well-known fact that one can have Aubry–Mather sets with irrational rotation number which are not invariant circles but Cantor sets.

If ω is Diophantine and V is sufficiently small in a sufficiently smooth norm, [60,46] and [47] used KAM theory to show that indeed $\mathcal{C}$ is dense in $\mathbb{T}^{d+1}$ and that, therefore, it can be extended uniquely to a foliation.

Remark 2.6. Note that the fact that (2) has solutions that grow only linearly is somewhat surprising since by comparing with the explicit solutions of $\Delta u + a = 0$, $a \neq 0$, one would expect that the growth is quadratic. This shows that the solutions u are rather equidistributed, so that the effective value of the term $\partial_2 V(x,u)$ is indeed zero. The same example shows that unless one uses the variational structure and the periodicity of V, there is no hope to obtain a similar result for general elliptic PDEs.

We also mention that, up to a space scaling, the kind of problems studied in this paper may be related to homogenization procedures in periodic media (see, e.g., [13]).

Remark 2.7. Related problems in a non-variational setting have been studied in [8].

2.1. Proof of Theorem 2.2

We will prove the result first for $\omega \in \frac{1}{N}\mathbb{Z}^d$ and show that the resulting u^ω satisfy some uniform a priori estimates. Then, given $\omega_n \in \mathbb{Q}^d$ with $\omega_n \to \omega^* \in \mathbb{R}^d$, by possibly passing to a subsequence, we can obtain that $u^{\omega_n} \to u^{\omega^*}$ which solves Eq. (2) and satisfies condition (3).

Definition 2.8. We denote by

$$\mathcal{B}_\omega = \{u \mid u \text{ is Birkhoff, } u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^d)\}.$$

We will consider this set as a subset of the affine space $\omega \cdot x + L^\infty(\mathbb{R}^d)$.

We note that $\mathcal{B}_\omega$ can also be characterized as the set of Birkhoff functions for which the choices of sign in (1) are the same as in $\omega \cdot x$. That is:

$$\begin{aligned} &u(x+e) - (u(x)+\ell) \geqslant 0 \quad \text{if } \omega \cdot e - \ell \geqslant 0,\\ &u(x+e) - (u(x)+\ell) \leqslant 0 \quad \text{if } \omega \cdot e - \ell \leqslant 0, \end{aligned} \tag{5}$$

for all $e \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}$. We also observe that the set $\mathcal{B}_\omega$ is closed under several natural topologies. Later, we will use that it is closed under L^2 convergence in all the cells $[0,N]^d + k$ with $N \in \mathbb{N}$, $k \in N\mathbb{Z}^d$.

Notice that $\mathcal{B}_\omega$ is not empty since $u(x) = \omega \cdot x$ belongs to it. Also, if $\omega \in \frac{1}{N}\mathbb{Z}^d$ and $u \in \mathcal{B}_\omega$, by taking $\ell = N\omega \cdot e$ in (5), we see that

$$u(x+Ne) = u(x) + N\omega \cdot e \quad \forall e \in \mathbb{Z}^d. \tag{6}$$

We will refer to locally bounded functions satisfying (6) as ω-periodic functions and denote the spaces of such functions as $\mathcal{P}_\omega$. It follows from (6) that if $u \in \mathcal{P}_\omega$, then $u(x) = \omega \cdot x + \hat{u}(x)$ with $\hat{u}(x)$ $N\mathbb{Z}^d$-periodic.

We will consider the functional

$$\mathcal{S}_N(u) = \int_{[0,N]^d} \frac{1}{2}|\nabla u|^2 + V(x,u(x))\,dx \tag{7}$$

defined on functions

$$u \in \mathcal{P}_\omega \cap H^1([0,N]^d) \equiv X.$$

The functional in (7) may be viewed as a regularization of the one in (4), since it avoids complications due to unbounded domains.

For the moment, we will proceed heuristically, to derive the gradient semi-flow. We will use the gradient semi-flow to construct our solutions and to establish properties about them.

It can be seen very easily that, on a H^1 dense subset of u and η,

$$\begin{aligned}\frac{d}{d\varepsilon}\mathcal{S}_N(u+\varepsilon\eta)|_{\varepsilon=0} &= \int\big[-\Delta u(x)+\partial_2 V\big(x,u(x)\big)\big]\eta(x)\,dx \\ &= \big\langle -\Delta u+\partial_2 V(\cdot,u),\eta\big\rangle_{L^2}.\end{aligned}$$

Hence, the gradient of $\mathcal{S}_N$ – with respect to the L^2 metric – is

$$\nabla\mathcal{S}_N(u) = -\Delta u+\partial_2 V(\cdot,u). \tag{8}$$

Inspired by the previous calculation of the gradient of the functional $\mathcal{S}_N$, we consider the gradient semi-flow

$$\frac{d}{dt}u = -\nabla\mathcal{S}_N(u),$$

which according to the steepest descent method will converge to a critical point. In the light of our calculation, the steepest descent equation reduces to:

$$\frac{\partial u}{\partial t} = \Delta u-\partial_2 V(\cdot,u). \tag{9}$$

We will collect a few properties of Eq. (9) on $\mathcal{P}_\omega$ and $\mathcal{B}_\omega$, in order to construct a solution of (2) and (3) satisfying some additional regularity properties.

The following result is well known. For example, it follows from Propositions 4.1 and 4.2 in [81, 15.4, p. 292 ff]. We will review the proof in Section 3. We will not attempt to optimize regularity at this stage, since for the final result, we can recover sharp regularity results by an approximation argument based on elliptic theory (as in Remark 2.18 below).

Lemma 2.9. *Assume that $V\in C^{k+3}$, with $k\geqslant 2$. Let u_0 be such that $u_0(x)-\omega\cdot x\in W^{k,p}(N\mathbb{T}^d)$, with $1\leqslant p\leqslant\infty$. Then, Eq.* (9) *with the initial conditions $u(0,x)=u_0(x)$ admits a unique solution $u(x,t)$ (or, for short, $u(t)$), which satisfies*

$$\begin{aligned}&u(x,\cdot)-\omega\cdot x\in C^0\big([0,T),W^{k,p}\big(N\mathbb{T}^d\big)\big)\quad\forall T>0,\\ &u(x,\cdot)-\omega\cdot x\in C^1\big([0,T),W^{k-2,p}\big(N\mathbb{T}^d\big)\big)\quad\forall T>0.\end{aligned}$$

Moreover, if $u_0\in\mathcal{P}_\omega$, then $u(t)\in\mathcal{P}_\omega$ for all times $t\geqslant 0$.

The result in Lemma 2.9 here above states that $u(t)$ is a continuous curve in $W^{k,p}$, that we can compute its derivatives in $W^{k-2,p}$ and that they have the values that make (9) hold.

As a consequence of the differentiability properties established in Lemma 2.9, we have:

Lemma 2.10. *Consider the setting of Lemma* 2.9. *Assume that $k\geqslant 2$ and $p\geqslant 2$. Then, $\mathcal{S}_N(u(t))$ is a Lipschitz function of t and we have*:

$$\frac{d}{dt}\mathcal{S}_N\big(u(t)\big) = -\int\limits_{N\mathbb{T}^d}\big(-\Delta u+\partial_2 V(\cdot,u)\big)^2\leqslant 0. \tag{10}$$

Proof. Denote

$$K(u)=\frac{1}{2}\int\limits_{N\mathbb{T}^d}|\nabla u|^2.$$

Clearly, it is a differentiable function when considered as a function in $W^{k,p}$ and, if $\eta\in C^\infty(N\mathbb{T}^d)$,

$$DK(u)\eta = \int\limits_{N\mathbb{T}^d} \nabla u \nabla \eta = -\int\limits_{N\mathbb{T}^d} \Delta u \eta.$$

Let also

$$\Gamma(u) = \int\limits_{N\mathbb{T}^d} V(\cdot, u).$$

We have that

$$V\big(x, u(x) + \eta(x)\big) = V\big(x, u(x)\big) + \partial_2 V\big(x, u(x)\big)\eta(x) + R(x),$$

where

$$\big|R(x)\big| \leqslant \frac{1}{2}\|V\|_{C^2}\big|\eta(x)\big|^2.$$

Since

$$\int\limits_{N\mathbb{T}^d} \big|R(x)\big| \leqslant C\|\eta\|_{L^2}^2,$$

it follows that Γ is differentiable as a function in L^2 and

$$D\Gamma(u)\eta = \int\limits_{N\mathbb{T}^d} \partial_2 V(\cdot, u)\eta,$$

for any $\eta \in C^\infty(N\mathbb{T}^d)$.

Accordingly, we obtain that $\mathcal{S}_N = K + \Gamma$ is differentiable in $W^{k,p}$ and that

$$D\mathcal{S}_N(u)\eta = \int\limits_{N\mathbb{T}^d} \big(-\Delta u + \partial_2 V(\cdot, u)\big)\eta, \tag{11}$$

for any $\eta \in L^2(N\mathbb{T}^d)$.

If we take into account that the solutions of (9) are in $C^1([0, T], W^{k-2,p})$, thanks to Lemma 2.9, and we exploit formula (11) for the derivative with $\eta = \partial_t u$, we obtain (10). $\square$

Lemma 2.11. *Let $\tilde{u}(t)$ be any solution of* (9) *produced in Lemma* 2.9 *and let*

$$u(t) = \tilde{u}(t) - \left[\frac{1}{|N\mathbb{T}^d|}\int\limits_{N\mathbb{T}^d} \tilde{u}(t)\,dx\right],$$

where $[\cdot]$ here above denotes the integer part.

Then, we can find a sequence t_n, and $u^*_\omega \in (\omega \cdot x + L^2(N\mathbb{T}^d)) \cap \mathcal{P}_\omega$ such that

$$u(t_n) \to u^*_\omega \text{ in } L^2$$

and u^*_ω solves (2).

Proof. From the fact that $\mathcal{S}_N(u(t))$ is differentiable in t and that

$$\frac{d}{dt}\mathcal{S}_N \leqslant 0,$$

we conclude that $\mathcal{S}_N(u(t))$ remains uniformly bounded for all $t \geqslant 0$. Since V is bounded, we gather that $\|\nabla u(t)\|_{L^2}$ remains uniformly bounded. By Poincaré Inequality, we conclude that $u(t)$ is contained in an L^2 compact set.

Since $\mathcal{S}_N$ is clearly bounded from below, we can find a sequence of times t_n such that

$$\frac{d}{dt}\mathcal{S}_N\big(u(t)\big)\bigg|_{t=t_n} \to 0.$$

Since we have shown that $u(t)$ lies in an L^2 compact set, by passing to a subsequence, if necessary, we can assume $u(t_n) \to u^*_\omega$ in L^2.

Since $\partial_2 V$ is Lipschitz, we conclude that

$$\left\| \partial_2 V\left(\cdot, u(t_n)\right) - \partial_2 V\left(\cdot, u^*_\omega\right) \right\|_{L^2} \to 0.$$

Since the Laplacian is an L^2 closed operator, we obtain:

$$\Delta u^*_\omega - \partial_2 V\left(\cdot, u^*_\omega\right) = 0,$$

as desired. □

Note that Lemma 2.11 establishes the existence result in Theorem 2.2 (and, precisely, formulas (2) and (3)) when $\omega \in \frac{1}{N}\mathbb{Z}^d$ for any $N \in \mathbb{N}$. We remark, in particular, that (3) is fulfilled due to the fact that $u(t) \in \mathcal{P}_\omega$, by Lemma 2.9.

Now, we want to obtain enough control on the solutions produced in such a way that when $\omega_i \to \omega^*$, then $u^*_{\omega_i} \to u^*_\omega$ (perhaps after passing to a subsequence) and that the resulting u^*_ω also solves (2) and the satisfies the growth condition (3). Such passage to the limit will be established by showing that the solutions satisfy some form of equicontinuity and equiboundedness. The equicontinuity will be a very easy consequence of elliptic estimates (see Lemma 2.17 here below). The equiboundedness will be a consequence of the Birkhoff property as claimed in Theorem 2.2.

Hence, we will turn our attention to proving that the solutions produced by the argument above are Birkhoff (see Lemma 2.16 below). Since the Birkhoff property depends on comparing a function with its translates, it is natural to investigate how these properties interact with the flow in Lemma 2.9.

Also, we will find it convenient to consider the evolution constructed in Lemma 2.9 to act on functions defined on the whole of $\mathbb{R}^d$. Given that we are considering only solutions in $\mathcal{P}_\omega$, we identify them with their periodic extension to $\mathbb{R}^d$.

Following is the very well-known comparison principle.

Lemma 2.12. *In the conditions of Lemma* 2.9*, if we have* $u(0) \geqslant v(0)$*, then* $u(t) \geqslant v(t)$ $\forall t \geqslant 0$.

Proof. A proof of this result can be found in almost any textbook in PDEs. See, for example, [85]. See also [27] for further details. We will provide here a different proof of Lemma 2.12, which admits several generalizations and which is based on the following result:

Proposition 2.13. *Denote by* Φ_t *the semi-flow of* $\frac{\partial u}{\partial t} = \Delta u$ *and by* ψ_t *the semi-flow of* $\frac{\partial u}{\partial t} = -\partial_2 V(\cdot, u)$*. Let* $u^\star$ *be a solution of* (9)*. Then,*

$$u^\star(t) = \lim_{n\to\infty} (\psi_{t/n}\Phi_{t/n})^n u^\star(0). \tag{12}$$

Formula (12) is a non-linear generalization of Trotter product formula for semigroups (which in turn generalizes a formula of Lie for finite dimensional Lie groups). A proof of this formula with precise hypothesis, which are verified in our case, taking the limit in (12) in the L^∞ sense, can be found in [81, Proposition 5.1, p. 310] (see also Propositions 5.2 and 5.3 there), which we reproduce as Proposition 4.5 later in this paper for the convenience of the reader. The original proof of formula (12) is in [19].

As it is well known, Φ_t and Ψ_t satisfies comparison principles.

Proposition 2.14. *If* $u(0) \geqslant v(0)$ *and* $u(0) \not\equiv v(0)$*, then* $\Phi_t u(0) > \Phi_t v(0)$ *for all* $t > 0$.

If $u(0) > v(0)$*, then* $\psi_t u(0) > \psi_t v(0)$.

Proof. The proof of the comparison principle for the heat equation follows easily from the fact that the evolution is given by convolution with the heat kernel, which is positive. For Ψ_t, note that when we fix x, the equation satisfied by $u(t,x)$ is an ODE solutions of ODEs cannot cross, so that if $u(t,x) > v(t,x)$ for $t = 0$, it has to remain bigger for later t. □

Combining Propositions 2.13 and 2.14 (and taking care of some trivial cases such as $t = 0$, $u(0) = v(0)$), we complete the proof of Lemma 2.12. □

As an important corollary of Lemma 2.12, we obtain the following:

Corollary 2.15. *Let the conditions of Lemma* 2.9 *hold. Let* $u(0) \in \mathcal{B}_\omega$. *Then,* $u(t) \in \mathcal{B}_\omega$.

Proof. The idea of the proof goes as follows. The comparison principle in Proposition 2.14 tells us that the semi-flow preserves the order. Also, by the uniqueness result in Lemma 2.9, the semi-flow of a translate is the translate of the semi-flow.

Let us now give more details on this proof, in order to introduce some notation that emphasizes the essential geometric properties, which will be generalized in the sequel. We introduce the operators

$$(\mathcal{C}_k u)(x) = u(x+k) \quad k \in \mathbb{Z}^d,$$
$$(\mathcal{R}_\ell u)(x) = u(x) + \ell \quad \ell \in \mathbb{Z}$$

and we denote the evolution in Lemma 2.9 by Ψ_t. Then,

$$\begin{aligned} &\Psi_t \mathcal{C}_k = \mathcal{C}_k \Psi_t, \\ &\Psi_t \mathcal{R}_\ell = \mathcal{R}_\ell \Psi_t \end{aligned} \tag{13}$$

by the invariance of (9) and the uniqueness in Lemma 2.9.

Note also that the Birkhoff condition can be expressed concisely by saying that for all $k \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}$ we have

$$\mathcal{C}_k u \prec \mathcal{R}_\ell u, \tag{14}$$

where $\prec$ is either $\leqslant$ or $\geqslant$.

By Lemma 2.12, if (14) holds, we have that

$$\Psi_t \mathcal{C}_k u \prec \Psi_t \mathcal{R}_\ell u$$

and, by (13), that

$$\mathcal{C}_k \Psi_t u \prec \mathcal{R}_\ell \Psi_t u.$$

That is, $\Psi_t u$ is also Birkhoff.

Moreover, using (5), we obtain that $\Psi_t u \in \mathcal{B}_\omega$. □

Lemma 2.16. *The solution* u^*_ω *produced in Lemma* 2.11 *can be assumed to be in* $\mathcal{B}_\omega$.

Proof. Consider the semi-flow starting with the initial conditions $u(x) = \omega \cdot x \in \mathcal{B}_\omega$. Then, by the previous corollary $u(t) \in \mathcal{B}_\omega$. In the proof of Lemma 2.11 we constructed a subsequence $u(t_n)$ which converges to u^*_ω in $L^2(N\mathbb{T}^d)$.

When considering functions defined on $\mathbb{R}^d$, this translates into $u(t_n) \to u^*_\omega$ in $L^2([0,N]^d + k)$ for all $k \in N\mathbb{Z}^d$. The set $\mathcal{B}_\omega$ is clearly closed in this topology. □

Hence, in virtue of Lemma 2.16, we have established all the claims in Theorem 2.2 for ω rational.

Now we start to consider the passage to the limit to irrational frequencies. We will find it important that the functions thus produced satisfy regularity estimates which are independent of the frequency considered.

The following result is a consequence of the standard Schauder-type elliptic estimates (see, e.g., [80, Theorem 11.1, p. 379] or [39, Theorem 6.2, p. 90]):

Lemma 2.17. *Assume that* u^*_ω *solves* (2). *Then*

$$\|D^{r+2} u^*_\omega\|_{L^\infty(N\mathbb{T}^d)} \leqslant K \|V\|_{C^{r+1+\varepsilon}(N\mathbb{T}^d)},$$

where K *is independent of* N.

We call attention to the fact that the elliptic estimates of the previous lemma are local. They are independent of N (that is, the period of the fundamental domain), because

$$\|V\|_{C^\ell(N\mathbb{T}^d)} = \|V\|_{C^\ell(\mathbb{T}^d)},$$

due to the periodicity of V. Let $\{\omega_n\} \subset \mathbb{Q}^d$. Denote by u^{ω_n} the solutions of (2) produced by applying the argument above.

We note that, since $u + \ell$ solves (2) if so does u, we can assume that

$$u^{\omega_n}(0) \in [0, 1]. \tag{15}$$

Given any $k \in \mathbb{Z}^d$, since $u^{\omega_n} \in \mathcal{B}_{\omega_n}$ we conclude that $u^{\omega_n}(x+k) - u^{\omega_n}(x) - \ell$ has the same sign than $\omega_n \cdot k - \ell$, independently of x.

Given (15), we conclude that $u^{\omega_n}(k)$ has to lie in an interval with integer endpoints and of length 2 which depends only on ω_n.

Since $|u^{\omega_n}(k) - \omega_n \cdot k| \leqslant 2$, for all $k \in \mathbb{Z}^d$, taking into account that for all $x \in \mathbb{R}^d$ we can find $k \in \mathbb{Z}^d$ such that $|k - x| \leqslant \sqrt{d}$, applying Lemma 2.17 for $r = 0$, we obtain that:

$$\left|u^{\omega_n}(x) - \omega_n \cdot x\right| \leqslant \left(K + |\omega_n|\right)\sqrt{d} + 2. \tag{16}$$

In particular, if $\omega_n \to \omega$, the sequence u^{ω_n} is equibounded in $C^0(B_R)$ for any $R > 0$. Since by Lemma 2.17 it is also equibounded in $C^2(B_R)$, we obtain that, perhaps passing to a subsequence,

$$u^{\omega_n} \xrightarrow{C^0_{\mathrm{loc}}} u^*. \tag{17}$$

Since the Laplacian is a closed operator under C^0_{loc} limits, we obtain that the limit u^* also satisfies (2). Since the set of Birkhoff functions is closed under pointwise limits, we obtain that u^* is also Birkhoff. It also follows that $u^* \in \mathcal{B}_\omega$, ending the proof of Theorem 2.2. □

Remark 2.18. Note that the approximation procedure here above is only carried out on the solutions of (2) which are rather regular. We do not need to control the approximations of the heat flow.

The same approximation argument works to show that if we have $V^{(n)} \xrightarrow{C^1} V$ we can obtain that the solutions of (2) for $V^{(n)}$ have a limit that is a solution for V.

This allows us to obtain the theorem for $V \in C^1$ even if the estimates for the heat flow require that V is more regular.

3. An abstract formulation of Aubry–Mather theorem for PDEs and ΨDEs

In this section, we will examine the proof of Theorem 2.2 presented above and find a general framework that allows to give an abstract version of that result. As a matter of fact, this section will be developed at the level of generality of operators in Banach spaces and we will not mention specifically PDEs and ΨDEs.

We will also develop an abstraction of the notion of quasi-periodicity that can be applied to variational problems in other manifolds.

Later we will see that, besides the main motivating example of Theorem 2.2, there are other examples of interest that can be proved following the same strategy.

3.1. Generalization of the notions of periodicity and quasi-periodicity

Our first task will be to generalize the notion of periodicity. We will describe a set up that is well suited for Aubry–Mather theory and which was introduced in [17]. We will show that this allows us to develop an analogous theory to the one developed so far for $\mathbb{T}^d$ and to establish existence of laminations of solutions for other manifolds whose fundamental group contains a residually finite subgroup.

We see that the main property of $\mathbb{R}^d$ and $\mathbb{Z}^d$ we used in the proof of Theorem 2.2 are the following:

(i) $\mathbb{Z}^d$ has subgroups $G_N \equiv N\mathbb{Z}^d$ of finite index. These subgroups satisfy:

(i.1) $\mathbb{R}^d/G_N$ is compact.

(i.2) As $N \to \infty$, $\mathbb{R}^d/G_N \to \mathbb{R}^d$. This convergence happens in the sense that any compact set of $\mathbb{R}^d$ is contained in $\mathbb{R}^d/G_N$ for sufficiently large N.

(ii) There are functions $\phi_\omega : \mathbb{R}^d \to \mathbb{R}$ (in the previous example, $\phi_\omega(x) = \omega \cdot x$) with the property that, given any $k \in \mathbb{Z}^d$, we have

$$\phi_\omega(x+k) = \phi_\omega(x) + \omega \cdot k.$$

Note that the constant term $\omega \cdot k$ considered as a function of k is a character of the group $\mathbb{Z}^d$, that is, $\omega \cdot (k+k') = \omega \cdot k + \omega \cdot k'$.

Note also that the operators $\mathcal{C}_k$ can be considered as an action on the space of functions, induced, of course, by an action on the lattice.

There are several groups – including the fundamental groups of hyperbolic manifolds – for which the essential parts of this set up still hold.

We recall the following definition (which is well known from group theory: see, e.g., [20]):

Definition 3.1. We say that a group is residually finite when, given any element different from the identity, we can find a subgroup of finite index which does not contain it.

We say that a function φ from a group G to the reals is a cocycle when

$$\varphi(gg') = \varphi(g) + \varphi(g') \quad \forall g, g' \in G.$$

The space of cocycles is a vector space, which we will denote by $H^1(G, \mathbb{R})$.

The following result is a combination of Propositions 1 and 2 from [17]. We refer to this paper for the proof.

Proposition 3.2. *Let $\tilde{M}$ be a manifold. Let G be a countable, finitely generated group acting on $\tilde{M}$. Assume that M, the fundamental domain of the G action, is a compact manifold.*

Then,

(i) *If G is residually finite, there is a sequence G_i of groups of finite index such that:*

$$G_i \supset G_{i+1}, \quad M/G_i \subset M/G_{i+1} \quad \text{and} \quad \bigcup_i M/G_i = \tilde{M}.$$

(ii) *Given any cocycle φ, there is sequence of cocycles φ_n such that*

$$\varphi_n \to \varphi$$

for the generators and, hence, for all the elements of the group, and such that for each of the φ_n we can find a group G_n of finite index such that φ_n takes only integer values on G_n, that is, $\varphi_n(G_n) \subset \mathbb{Z}$.

Cocycles as the ones in the last statement of Proposition 3.2, that is for which there exists a finite index subgroup $\tilde{G}$ for which $\varphi(\tilde{G}) \subset \mathbb{Z}$, are called *rational cocycles*.

When the group that we are considering acts on a manifold in such a way that the fundamental domain is compact (e.g., when the group is the fundamental group of a compact manifold) it is natural to think of the cocycles as functions defined in the manifold.

Indeed, take any continuous function u_0 in the fundamental domain of the action and then consider the extension $\tilde{u}$ defined in $\tilde{M}$ by

$$u(g \cdot x) = u_0(x) + \varphi(g). \tag{18}$$

This function does not need to be continuous, but we can construct a smooth function which has bounded C^k norm and which is at a bounded C^0 distance.

The usual linear functions correspond to choosing $u_0(x) = \omega \cdot x$ in the fundamental domain.

Note also that, even if the extensions of cocycles are not unique, all the extensions of cocycles obtained in the manner described above lead to functions that differ only in a L^∞ function.

In the context of actions of groups on manifolds, we can also define the notion of Birkhoff property, by using the following notation. Given an element $g \in G$ we define $\mathcal{C}_g$ as the action on functions defined by

$$(\mathcal{C}_g u)(x) = u(g \cdot x) \tag{19}$$

where we have denoted by $g \cdot x$ the action of the group on $\tilde{M}$.

Similarly, we define R_ℓ for $\ell \in \mathbb{Z}$ by:

$$(R_\ell u)(x) = u(x) + \ell. \tag{20}$$

Then, we define the Birkhoff property in the following way:

Definition 3.3. Let $\tilde{M}$ be a manifold and G be a group acting on it.

We say that a function $u : \tilde{M} \mapsto \mathbb{R}$ is Birkhoff when given any $g \in G$, $\ell \in \mathbb{Z}$, we have

$$\mathcal{C}_g u(x) \prec \mathcal{R}_\ell u(x) \quad \forall x \in \tilde{M}, \tag{21}$$

where $\prec$ denotes either $\leqslant$ or $\geqslant$ and the choice of which symbol to chose depends on k, l but not on x.

The interpretation of Definition 3.3 is that if we move the graph of u by transforming the arguments according to the action of G and the range by integer translations, then these hypersurfaces in $\tilde{M} \times \mathbb{R}$ do not intersect.

We remark that a cocycle can (and, in the sequel, implicitly, will) be extended to a smooth, Birkhoff function. In particular, by assuming the existence of cocycles, we have that the set of Birkhoff functions is non-empty.

Remark 3.4. We now list some important examples in which our set up applies:

A particularly important case is

E1 When M is a compact manifold, $\tilde{M}$ is a cover of M, and G is a finitely generated, residually finite subgroup of the fundamental group $\Pi_1(M)$ acting on $\tilde{M}$ by Deck transformations.

Of course, some major subcases of the above are:

E1.1 The case in which $\tilde{M}$ is the Abelian cover of M and G is the maximal Abelian subgroup of $\Pi_1(M)$.
E1.2 The case in which $\tilde{M}$ is the universal cover of M and $G = \Pi_1(M)$.

The assumption that the fundamental group is residually finite is verified in many examples, such as the torus and the hyperbolic manifolds (see [20]).

Another interesting example is when

E2 G is a free group.

For instance, if G is the free group generated by two elements (say, a and b), we can consider the following cocycles of G. Given $g = a_1^{e_1} b_1^{f_1} \ldots a_k^{e_k} b_k^{f_k} \in G$, with $e_i, f_i \in \mathbb{N}$, we set $A(g) = e_1 + \cdots + e_k$ and $B(g) = f_1 + \cdots + f_k$. Then, given $\alpha, \beta \in \mathbb{R}$, one may considers cocycles of the type

$$\varphi_{\alpha,\beta}(g) = \alpha A(g) + \beta B(g).$$

In the above scenario, given a manifold M, we can construct solutions of the Euler–Lagrange equations that "behave under transformations in G as cocycles do". Note that the choice of the group is chiefly arbitrary (though, of course, if G does not have any cocycle our results are void). A precise result will be stated in Theorem 5.12.

For subsequent applications, we will use the cocycles of G. Roughly speaking, we will associate a solution to each cocycle. Hence, in some sense, we will obtain "more solutions the more cocycles G has". On the other hand, since some solutions may be symmetries under G, the results will be "sharper" if G possesses "few" cocycles.

Even if we will not explicitly use it later, we now show that there is close relation between Birkhoff functions and cocycles. The following result also generalizes the classical theory of the rotation number for circle maps.

Proposition 3.5. *Assume that u is Birkhoff. Then, the following limits exist*:

$$\varphi(g) = \lim_{n\to\infty} \frac{1}{n}\big(u\big(g^n \cdot x\big) - u(x)\big). \tag{22}$$

Moreover, the limits are reached uniformly in x. The function φ obtained above satisfies:

$$\begin{aligned} &\varphi(g'g) = \varphi(gg'); \qquad \varphi(\mathrm{Id}) = 1; \\ &\textit{If} \quad g'g = gg', \quad \textit{we have } \varphi(gg') = \varphi(g) + \varphi(g'). \end{aligned} \tag{23}$$

Proof. This proof is done in [17, p. 654]. We reproduce it, since we will use some of the ingredients.

By the Birkhoff property, given any g we can find $l^-(g) \leqslant l^+(g) \in \mathbb{Z}$ such that

$$\mathcal{R}_{l^-(g)}u \leqslant \mathcal{C}_g u \leqslant \mathcal{R}_{l^+(g)}u. \tag{24}$$

Moreover, we can assume that l^- (resp. l^+) is the largest (resp. smallest) integer for which (24) holds. In this case,

$$l^+(g) \leqslant l^-(g) + 1. \tag{25}$$

Therefore, there is a point x with

$$\mathcal{R}_{l^-(g)}u(x) \leqslant \mathcal{C}_g u(x) \leqslant \mathcal{R}_{l^-(g)+1}u(x).$$

Notice also that

$$l^+(gg') \leqslant l^+(g) + l^+(g')$$

and analogously

$$l^-(gg') \geqslant l^-(g) + l^-(g').$$

Therefore, by a subadditivity argument (see, e.g., Proposition 10.1 in [15]),

$$\varphi(g) = \lim_n \frac{l^+(g^n)}{n} \tag{26}$$

exists. By (25), we obtain that

$$\varphi(g) = \lim_n \frac{l^-(g^n)}{n}. \tag{27}$$

By the subadditivity of l^+, we have that

$$\begin{aligned} \varphi(gg') &= \lim_n \frac{1}{n} l^+\big((gg')^n\big) = \lim_n \frac{1}{n} l^+\big(g(gg')^{n-1}g'\big) \\ &\leqslant \lim_n \frac{1}{n}\big(l^+(g) + l^+\big((gg')^{n-1}\big) + l^+(g')\big) \\ &\leqslant \varphi(g'g). \end{aligned}$$

Hence,

$$\varphi(gg') = \varphi(g'g).$$

What is more, by (24),

$$\frac{u(g^n \cdot x) - u(x)}{n} \in \left[\frac{l^-(g^n)}{n}, \frac{l^+(g^n)}{n}\right].$$

Consequently,

$$\lim_n \frac{1}{n}\big(u\big(g^n\cdot x\big)-u(x)\big)=\varphi(g)$$

and that the limit is reached uniformly in x, due to (26) and (27). □

Remark 3.6. A function φ as in (23) is sometimes called a quasi-cocycle. We do not know whether the notions of cocycles and quasi-cocycles are equivalent. We will see in Theorem 3.7 that, given any cocycle φ, one can construct a Birkhoff solution u for which φ "plays the role of a rotation number". As a partial counterpart, Proposition 3.5 says that any Birkhoff function u has a quasi-cocycle φ which "plays the role of a rotation number".

3.2. Formulation of an abstract version of Theorem 2.2

Now, we turn our attention to the formulation of an abstract version of the variational scheme in Theorem 2.2. We will first abstract the argument in terms of existence of a gradient semi-flow enjoying certain regularity properties as well as a comparison principle, and apriori regularity properties of the equilibrium solutions.

The existence of flows and their regularity properties can still be discussed for a while at the level of functional analysis (and, in fact, at the level of the theory of semi-groups and their generators) without discussing concrete models. In this section, we will discuss the abstract set up and the functional analysis that can be used to verify the properties of the set up. Then, we will show how these abstract properties can be verified in concrete models: for this, several useful analytic tools will be collected in Section 4, and they will be applied in several concrete cases in Section 5.

We point out that the arguments in Section 2 use several steps, which we abstract as:

H1 We are considering a problem in a manifold $\tilde{M}$. This manifold is endowed with a smooth action by a finitely generated, residually finite group G. The fundamental domain M of the action is compact.
Note that the action of G on M by $x\to gx$ induces an action in spaces of functions given by

$$u(\cdot)\to(\mathcal{C}_g u)(\cdot)\equiv u(g\cdot).$$

H2 The problem has a variational structure compatible with the action $\mathcal{C}$ on functions and with the action $\mathcal{R}$ of the group $\mathbb{Z}$ given by

$$(\mathcal{R}_\ell u)(x)\equiv u(x)+\ell.$$

This assumption will be made more precise in H2.1, H2.2 and H2.3 here below. We assume that our problem is the Euler Lagrange equation of a variational problem $\mathcal{S}(u)$. In order to include in our theory general equations, such as equations involving pseudo-differential operators, we will not assume that the variational problem is the integral of a function.
In the case that we are dealing with PDEs, the functional will be of the form

$$\mathcal{S}(u)=\int F\big(x,u(x),\nabla u(x)\big)\,dx, \tag{28}$$

but more general functionals are allowed.
As usual in the calculus of variations, we will not need that $\mathcal{S}(u)$ is a well defined functional – that is, it may diverge when the integral is computed over the whole space. In the case that our functional is not local, the precise meaning of the Euler–Lagrange will need to be made precise in the following points. We will also assume that $\mathcal{S}(\mathcal{C}_g\mathcal{R}_l u)=\mathcal{S}(u)$ in a formal sense. In the case that we are dealing with local functionals as in (28), we can just assume that:

$$F\big(x,u\circ g(x)+\ell,(\nabla u\circ g)(x)\big)J_g(x)=F\big(g^{-1}x,u(x)+\ell,Dg\big(g^{-1}x\big)\nabla u(x)\big) \tag{29}$$

so that the change of variable gives formally the invariance. For the ΨDE case, which does not have a local variational principle, a simple characterization of invariance of the variational principle such as (29) is not possible.
The exact meaning of such formal invariance is that we can define the functional in the quotients of the manifold under the group action. In that case, the functional is a well defined functional. This will be made precise in the following:

H2.1 The variational principle is a well defined functional for smooth enough functions that are periodic restricted to a fundamental domain of the period. We refer to these variational principles on a compact quotient as reduced variational principles.

H2.2 The reduced variational principle is invariant under the G action and under the addition of constants.

H2.3 For every subgroup H of finite index (and, therefore by hypothesis H1, such that the action by H has compact fundamental domain), the reduced variational principle is bounded from below. Note that we allow that these lower bounds depend on the subgroup H.

Points H2.1 and H2.2 mean that, given any subgroup of finite index H, we can define $\mathcal{S}_H(u)$ on C^r functions of $\tilde{M}/H$. This is a well defined functional with Euler–Lagrange equations, which we will write as

$$(\nabla \mathcal{S}_H)(u) = 0.$$

The way that we can define the Euler–Lagrange equations on the manifold is through and inductive limit process. We have that if $H \subset H'$ are two subgroups of finite index, we can consider $\tilde{M}/H' \subset \tilde{M}/H$ and, hence, any function in $\tilde{M}/H$ as a function in $\tilde{M}/H'$. We will need that if a function u is H-periodic and it satisfies $\nabla S_H(u) = 0$, then, it also satisfies $\nabla S_{H'} = 0$. Of course, this is obvious for local variational principles but it is not automatic for the non-local problem for ΨDEs.

This will allow us to define the notion of a solution to the Euler–Lagrange equations as the existence of a sequence of subgroups of finite index

$$H_i \subset H_{i+1}$$

such that

$$\nabla \mathcal{S}_{H_i}(u) = 0.$$

We will see that this notion agrees with the usual notion for local operators.

H3 The gradient of the reduced variational principle generates a semi-flow Ψ_t. Moreover, this semi-flow satisfies:

H3.1 It is defined for all times when the given data are in a closed subspace of C^0 functions.

H3.2 For any fixed u, $\mathcal{S}_H \circ \Psi_t(u)$ is a continuously differentiable function of t for $t > 0$. Moreover

$$\frac{d}{dt}\mathcal{S}_H \circ \Psi_t(u) = -\left\| \nabla \mathcal{S}_H \circ \Psi_t(u) \right\|_{L^2}^2 \leqslant 0.$$

H3.3 There exist sequences $k_n \in \mathbb{Z}$, $t_n \in \mathbb{R}$ such that $t_n \to +\infty$ and $\Psi_{t_n}(u) - k_n \to u^\star$ a.e. $x \in \tilde{M}/H$, with $\nabla \mathcal{S}_H(u^\star) = 0$.

H3.4 Ψ_t admits a weak comparison principle.

In the applications we have in mind, the verification of these properties will come from the theory of parabolic equations.

Finally, we will include some hypotheses about the behavior of the solutions of the Euler–Lagrange equations:

H4 The equilibrium solutions of the reduced variational principle have a $C^{r+\alpha}_{\mathrm{loc}}$ norm which is bounded independently of which is the subgroup we are considering, for suitable $r \in \mathbb{N}$ and $\alpha > 0$.

H5 The gradient of the functional $\mathcal{S}$ is a closed operator under convergence in C^r_{loc} in the domain and L^2_{loc} in the range. That is, if $u_n \xrightarrow{C^r_{\mathrm{loc}}} \tilde{u}$ and $\nabla \mathcal{S}_H(u_n) \xrightarrow{L^2_{\mathrm{loc}}} \tilde{v}$, then $\nabla \mathcal{S}_H(\tilde{u}) = \tilde{v}$.

Note that hypotheses H1 and H2 are about the set up of the problem, which in practical cases are easy to verify (they only require inspection). Hypothesis H3 is fulfilled once the solutions of the gradient semi-flow for $\mathcal{S}_H$ are regular enough, and checking this requires some methods from the theory of evolution equations (for instance, in the case of Theorem 2.2, they are verified by parabolic equation methods). Note that parts H3.1–H3.3 are rather abstract properties of the evolution that can often be established at the functional analysis level for very broad classes of equations. Property H3.4 – the existence of a comparison principle – is much more geometric in nature.

Property H4 deals with the equilibrium solutions and it is usually established using methods from elliptic regularity theory. Notice also that H4 is the only uniform assumption in the subgroup H.

Property H5 is a rather harmless assumption. For linear operators it can be proved very often using functional analysis methods. In the case that the equilibrium equations are second order elliptic equations (as in [58] and [5]), one can also use the theory of viscosity solutions to show that the passage to the C^0 limit is also a viscosity solution (see [14, Proposition 4.11, p. 38]).

One should also notice that properties H4 and H5 are very similar in spirit to the usual Palais–Smale conditions, which allow to take limits of solutions of approximate variational problems.

In fact, as we will see, conditions H4 and H5 are only needed to deal with irrational cocycles.

If we are only interested in rational cocycles, H5 is not needed and H4 may be weakened to

H4′ For any finite index subgroup H, the equilibrium solutions of the reduced variational principle in $\tilde{M}/H$ are continuous.

A detailed statement of the abstract result will be given in Theorem 3.7. See also Remark 3.10 for additional comments.

The verification of some of the hypotheses can also be reduced to some other abstract hypotheses. For example, it is customary in PDE theory to obtain the existence of evolutions defined for all time and regularity from abstract semigroup theory assuming properties of the generator. The verification of the comparison principle can be reduced, via Proposition 2.13, to the hypothesis that the semi-flow is the sum of two semi-flows admitting comparison and satisfying the hypothesis of the product formula.

As we will now see, under these hypotheses, the argument we have presented in Section 2 goes through without too many changes. We thus have the following generalization of Theorem 2.2:

Theorem 3.7. *Consider a manifold $\tilde{M}$ endowed with an action of a discrete group G. Consider a formal variational principle $\mathcal{S}$ defined on functions from $\tilde{M}$ to $\mathbb{R}$.*

Assume that the variational principle and that the group action satisfy the hypotheses H1–H5 *above.*

Then, for every cocycle φ of the group G, we can find a critical point u for $\mathcal{S}$ such that

$$u(x) - \varphi(x) \in L^\infty(\tilde{M}). \tag{30}$$

Moreover, u is Birkhoff in the sense of Definition 3.3.

In case only hypotheses H1–H3 *and* H4′ *are fulfilled, the above result holds true for rational cocycles.*

Proof. The proof is just going over the proof of Theorem 2.2 and verifying that all the main steps are captured by some assumption in H1–H5 (or in H1–H3 and H4′).

The first step is:

Lemma 3.8. *Given any subgroup of finite index $\hat{G}$, and any cocycle φ such that $\varphi(\hat{G}) \subset \mathbb{Z}$, we can find a critical point u of $\mathcal{S}_{\hat{G}}$ in $\mathcal{B}_\varphi$. Moreover, u is uniformly bounded in C^r.*

Then, a density argument will establish a result analogous to Lemma 3.8 for all the cocycles.

We will refer to cocycles taking integer values in subgroups of finite index as rational cocycles.

Proof. Let φ be a rational cocycle with $\varphi(\hat{G}) \subset \mathbb{Z}$. We will consider the set of functions such that

$$u(g \cdot x) = u(x) + \varphi(g) \quad \forall g \in \hat{G}.$$

On this set of functions we can define the reduced variational principle $\mathcal{S}_{\hat{G}}$. Notice that these functions can be considered as functions in the manifold $\tilde{M}/\hat{G}$. Since $\hat{G}$ is a subgroup of finite index, $\tilde{M}/\hat{G}$ is a finite cover of M/G and is, therefore, compact.

By assumption H3, we can define a semi-flow associated to the steepest descent

$$\frac{d}{dt}\Psi_t = -\nabla \mathcal{S}_{\hat{G}} \circ \Psi_t.$$

We start this semi-flow on the cocycle function φ and we denote the solution of this problem by $u(t)$ (or by $u(p,t)$, if we want to emphasize its dependence on the point $p \in \tilde{M}/\hat{G}$).

By assumption H3.2, $\mathcal{S}_{\hat{G}}(u(t))$ as a function of the real variable t is differentiable and decreasing. Since $t \mapsto \mathcal{S}_{\hat{G}}(u(t))$ is bounded from below and decreasing, we can find a sequence $t_n \to \infty$ such that

$$\frac{d}{dt}\mathcal{S}_{\hat{G}}\big(u(t_n)\big) \to 0.$$

By assumption H3.3, we can assume that $u(t_n) - k_n \to \tilde{u}$, with $\nabla\mathcal{S}_{\hat{G}}(\tilde{u}) = 0$.

This establishes Lemma 3.8 except for the fact that the solution this produced is Birkhoff.

This will be a fairly easy consequence of the following

Proposition 3.9. *Assume* H3.1, H3.4 *and* H2.2. *Then, the gradient semi-flow preserves the set of Birkhoff functions* (*as defined in Definition* 3.3).

Proof. The proof of this proposition is quite simple. Note that if we have (21) for some k, ℓ $\mathcal{C}_k u \prec \mathcal{R}_\ell u$, then, the comparison principle H3.4 implies that for all $t \geqslant 0$, $\Psi_t \mathcal{C}_k u \prec \Psi_t \mathcal{R}_\ell u$.

Now note that, by the conditions of symmetry, we see that the equation is invariant under the action of $\mathcal{C}_k$, R_ℓ. Therefore, by the uniqueness of solutions, the semi-flow commutes with these symmetries. Hence, $\mathcal{C}_k \Psi_t u \prec \mathcal{R}_\ell \Psi_t u$. Therefore, $\Psi_t u$ is Birkhoff. □

Using Proposition 3.9, we have that $u(t_n)$ are Birkhoff. Hence, so are $u(t_n) + k_n$. The passage to the limit also preserves the set of Birkhoff functions. This finishes the proof of Lemma 3.8 (and of Theorem 3.7 in case only H1–H3 hold). □

To finish the proof of Theorem 3.7 (in case the whole set of assumptions H1–H5 holds), given any cocycle φ, we approximate it by a sequence $\{\varphi_n\}$ of rational cocycles (see Proposition 3.2).

Applying Lemma 3.8 we can produce solutions u^{φ_n} which are φ_n-periodic.

We can assume, by adding appropriate integers to the u^{φ_n}, which does not affect the fact that they are critical points of the functional, that $u^{\varphi_n}(p) \in [0, 1]$.

By assumption H4, we have that u^{φ_n} are uniformly differentiable.

Since $u^{\varphi_n}(gp) - (u^{\varphi_n}(p) + \ell)$ and $\varphi_n(g) - \ell$ have the same sign, we see that $u^{\varphi_n}(gp)$ has to belong to an interval of length 2 and with rational endpoints. Since φ_n converges to φ over the generators of the group (and therefore uniformly on bounded sets of g), we have that the above intervals have to converge as $n \to \infty$.

By the uniform continuity given by assumption H4, the uniform bounds on $u^{\varphi_n}(gp)$ and the fact that the action has compact fundamental domain, we obtain that the sequence u^{φ_n} is equibounded and equicontinuous on compact sets.

Hence, we can extract a subsequence which converges uniformly on compact sets. By assumption H5, the limit will also be a solution of the Euler–Lagrange equation. Since the set of Birkhoff functions is closed under uniform convergence on compact sets, we see that the limit will also be Birkhoff and it will also be at a finite distance of the cocycle φ. This ends the proof of Theorem 3.7. □

Remark 3.10. A statement analogous to Theorem 3.7 may be obtained only under conditions H1–H3: in this case, the bound in (30) gets replaced by

$$\big|u(g \cdot x) - u(x) - \varphi(g)\big| \leqslant 1$$

for any $x \in \tilde{M}$ and $g \in G$.

4. Some tools from functional analysis

In this section, we discuss some functional analysis tools that can help in the verification of the abstract hypotheses in concrete cases. Later, we will have to show how the concrete models that we have in mind verify the hypothesis we introduce here.

All the tools included in this section are in the literature. Some of them are quite well known among the practitioners, but others may have become forgotten. Hence, we will not present all the proofs of the results, but limit ourselves to making precise statements, which will be needed in the verification in concrete models, and give references for the proofs.

4.1. Verification of existence and regularity of the gradient flow and regularity of equilibrium solutions

The verification of the assumptions H3 and H4 in Section 3.2 (out of other more accessible assumptions) follows, in many concrete cases, from standard PDE theory. Roughly speaking, H3 involves parabolic theory and H4 involves elliptic theory.

We have found useful for our purposes the books [41,76,75,81] and [37] for the parabolic theory, and the books [81] and [37] for the elliptic theory. We refer to these books for references to the original literature. Of course, using the more advanced theory of fully non-linear equations one can consider other problems. References for this theory are [14] and [48].

The following two results come from classical parabolic theory. We will use them in the verification of the abstract hypotheses in concrete models.

Theorem 4.1. *Let H be a Hilbert space, $D \subset H$ a linear subspace dense in H. Let A be a linear (possibly unbounded) operator $A : D \to H$ which is m-accretive, that is*

$$\begin{aligned} &(-Au, u) \geqslant 0 \quad \forall u \in D \quad \textit{and} \\ &(-A + \mathrm{Id})D = H. \end{aligned} \tag{31}$$

Then, given $u_0 \in H$ it is possible to find a unique $u(t) \in C^0([0,\infty), H) \cap C^1((0,\infty), H)$ solving

$$\frac{d}{dt}u(t) = Au(t); \quad u(0) = u_0. \tag{32}$$

If we introduce the linear operator e^{tA} by $e^{tA}u_0 = u(t)$, where $u(t)$ is the solution above, we have that for every $t > 0 \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$, the operator e^{tA} is in the domain of A^n and

$$\|A^n e^{tA}\| \leqslant \left(\frac{n}{t\sqrt{2}}\right)^n. \tag{33}$$

Therefore, $u(t) \in C^0([0,\infty), H) \cap C^\infty((0,\infty), H)$.

Proof. This is the celebrated Hille–Phillips–Yoshida Theorem. See, for instance, I.5 of [76] or Theorem 1.2, p. 95 of [37]. In particular, the existence of u, the estimate in (33) and the regularity of u are discussed, respectively, in Corollaries 5.3, 5.4 and 5.5 of [76]. □

Remark 4.2. If, for some $a \in \mathbb{R}$, we have

$$(-Au, u) \geqslant a(u,u) \tag{34}$$

then, $(A - a)$ is also m-accretive, and so we can define $e^{(A-a)t}$. It is easy to check that $e^{(A-a)t} = e^{at}e^{-At}$.

If we apply (33) to $A - a$, we obtain, under the hypothesis (34), that

$$\|A^n e^{tA}\| \leqslant \left(\frac{n}{t\sqrt{2}}\right)^n e^{-at}. \tag{35}$$

Moreover, the use of interpolation spaces (see, e.g., p. 275 in [80], and also [45] and [53]) allows us to extend formula (33) to any positive real power of the operator A. For instance, for any $\alpha, T > 0$, we have that

$$\|(-A)^\alpha e^{tA}\| \leqslant C_{\alpha,T}\left(\frac{\alpha}{t}\right)^\alpha, \tag{36}$$

for any $t \in (0, T]$, for a suitable $C_{\alpha,T} > 0$.

To prove this, one may consider the domain $D(A)$ of the operator A and (as done, for instance, on p. 33 of [53]) endow such a space with the "graph norm"

$$\|u\|_{D(A)} = \|u\|_H + \|Au\|_H.$$

Then, one considers the operator $B = (-A)^m$, where $\alpha \in [m-1, m)$ and $m \in \mathbb{N}$ and exploits Proposition 2.2.15 and Definition 1.1.1 of [53] to deduce that

$$\|u\|_{D(B^{\alpha/m})} \leqslant \mathrm{const}\|u\|_H^{1-\alpha/m}\|u\|_{D(B)}^{\alpha/m},$$

for any $u \in D(B)$. Therefore, from (33),

$$\begin{aligned}
\left\|(-A)^\alpha e^{tA} f\right\|_H &= \left\|B^{\alpha/m} e^{tA} f\right\|_H \\
&\leqslant \left\|e^{tA} f\right\|_{D(B^{\alpha/m})} \\
&\leqslant \mathrm{const}\left\|e^{tA} f\right\|_H^{1-\alpha/m}\left\|e^{tA} f\right\|_{D(B)}^{\alpha/m} \\
&= \mathrm{const}\left\|e^{tA} f\right\|_H^{1-\alpha/m}\left(\left\|e^{tA} f\right\|_H + \left\|Be^{tA} f\right\|_H\right)^{\alpha/m} \\
&= \mathrm{const}\left\|e^{tA} f\right\|_H^{1-\alpha/m}\left(\left\|e^{tA} f\right\|_H + \left\|A^m e^{tA} f\right\|_H\right)^{\alpha/m} \\
&\leqslant \mathrm{const}\,\|f\|_H^{1-\alpha/m}\left(\left(1+(m/t)^m\right)\|f\|_H\right)^{\alpha/m} \\
&\leqslant \mathrm{const}(\alpha/t)^\alpha\|f\|_H,
\end{aligned}$$

where the constants here may depend on α and T, proving (36).

Also, from (36), it is possible to obtain short-time estimates (as the ones in (41) below). Namely, we fix $\alpha \in (0,1)$ and we consider the "graph space" $Y = D((-A)^\alpha)$ endowed with the "graph norm"

$$\|u\|_Y = \|u\|_H + \left\|(-A)^\alpha u\right\|_H.$$

Then, we gather from (36) that

$$\left\|e^{tA}\right\|_{\mathcal{L}(H,Y)} \leqslant C_{\alpha,T}\left(1+\left(\frac{\alpha}{t}\right)^\alpha\right),$$

for any $t \in (0,T]$, for a suitable constant $C_{\alpha,T}$. Furthermore, using again (36) (and the fact that $\alpha \in (0,1)$),

$$\begin{aligned}
\left\|e^{tA} - \mathrm{Id}\right\|_{\mathcal{L}(Y,H)} &\leqslant \sup_u \frac{\int_0^t \|Ae^{sA}u\|_H\,ds}{\|u\|_H + \|(-A)^\alpha u\|_H} \\
&\leqslant \sup_u \frac{\int_0^t \|(-A)^{1-\alpha}e^{sA}(-A)^\alpha u\|_H\,ds}{\|(-A)^\alpha u\|_H} \\
&\leqslant \sup_v \frac{\int_0^t \|(-A)^{1-\alpha}e^{sA}v\|_H\,ds}{\|v\|_H} \\
&\leqslant \mathrm{const}\int_0^t \left(\frac{1-\alpha}{s}\right)^{1-\alpha} ds \\
&= \mathrm{const}\cdot t^\alpha.
\end{aligned}$$

In the case of fractional powers of the Laplacian, related estimates may be obtained by Fourier analysis (see Proposition 5.9 below).

Analogously, to study non-linear problems, as in [41], it is convenient to interpret (33) and (35) for even n as stating that the operator e^{-tA} is bounded from the Hilbert space H to the Hilbert space Y obtained by completing the domain of A^n endowed with the inner product

$$\langle u, v\rangle_Y = \langle u, v\rangle_H + \left\langle A^n u, A^n v\right\rangle_H.$$

Once we have defined these spaces for any even integer n, one can use interpolation to define them for $n \in \mathbb{R}^+$ (see, e.g., 4.2 in [80]).

In case that we take $A = \Delta$, we obtain the classical Sobolev spaces.

The control of the non-linear terms is easily done using the following so-called Duhamel formula.

Proposition 4.3. *Let H, Y be Hilbert spaces $Y \subset H$. Let $N : H \to Y$ be a locally Lipschitz function. Assume that for some $0 \leqslant \alpha < 1$*

$$\left\| e^{tA} \right\| \leqslant C t^{-\alpha} \quad \forall t > 0. \tag{37}$$

A function $u(t) \in C^0([0,T), H) \cap C^1((0,T), H)$ is a solution of

$$\frac{d}{dt} u(t) = A u(t) + N\big(u(t)\big), \quad u(0) = u_0 \tag{38}$$

if and only if it is a solution of

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} N\big(u(s)\big)\, ds. \tag{39}$$

Moreover, for T sufficiently small, we can find a unique solution $u(t) \in C^0([0,T), H)$ of (39)*. Such T can be estimated from below in terms of $\|u_0\|_H$. Also, $u \in C^0((0,T), Y)$.*

Proof. See [81, p. 272]. The basic idea is to show that the right-hand side of (39) is a contraction operator for T small.

Once we have a solution of (39) in $C^0([0,T), H)$, it follows from (37) that the right-hand side of (39) is in $C^0((0,T), Y)$. □

In the applications we have in mind, this basic result can be extended in several ways:

Proposition 4.4. *Assume that A is m-accretive in all Sobolev spaces $W^{s,2}$, for $s \leqslant s^*$ and that N is Lipschitz in all the spaces $W^{s,2}$ for $s \leqslant s^*$. Then, the solution produced in Proposition* 4.3 *is $C^0((0,T), W^{s^*,2})$.*

Proof. We can take H to be L^2 and then apply Proposition 4.3 to conclude existence for short time with $Y = W^{\alpha,2}$ with α any number smaller than the order of the linear operator.

If N is such that it maps $W^{\alpha,2}$ into itself, we can see that the right-hand side of (39) belongs to $W^{2\alpha,2}$.

The process can be iterated to conclude that $u(t) \in C^0((0,T), W^{n\alpha,2})$ provided that N is Lipschitz in the spaces $W^{i\alpha,2}$, $i = 1, \dots, n$. □

In the applications we have discussed in detail, the operator N is a composition operator. The regularity of the composition of a C^r functions considered as an operator among Sobolev spaces is a consequence of Moser estimates (see, e.g., [81, 13.3] and, for a more comprehensive study, [3] and [24]).

Since the time of existence depends only on the Y norm, in the situations when the Y norm remains bounded by apriori bounds, we have existence for all times.

4.2. Verification of comparison

In this subsection, we collect several functional analysis tools which can be used to verify the comparison principle H3.4.

Of course, the literature on comparison results for parabolic equations is quite extensive and we cannot hope to make it justice here. We just mention that the classic [68] is still very useful for us.

In this subsection, we will just discuss two functional analysis methods that can be used to reduce the comparison results to simpler ones: namely, product formulas and subordination identities.

4.2.1. Product formulas

The product formula that we have found useful is the non-linear analogue of the Trotter product formula of [19]. We reproduce the version of [81, Proposition 5.1, p. 310].

This formula is applied to the study of equations of the form

$$\frac{d}{dt}u = Lu + X(u); \quad u(0) = u_0, \tag{40}$$

where L is a linear operator and X is a vector field.

Proposition 4.5. *Let V, W be Banach spaces of l-tuples of functions for which e^{tL} satisfies the estimates*

$$\|e^{tL}\|_{\mathcal{L}(V)} \leqslant e^{ct}, \qquad \|e^{tL}\|_{\mathcal{L}(W,V)} \leqslant Ct^{-\gamma}, \qquad \|e^{tL} - \mathrm{Id}\|_{\mathcal{L}(V,W)} \leqslant Ct^{\delta}, \tag{41}$$

for $0 < t < T$, with some $\delta > 0$, $0 < \gamma < 1$.

Let X be a vector field generating a flow $\mathcal{F}_X^t$ on $\mathbb{R}^l$, satisfying, for $\|f\| \leqslant C_1$,

$$\|\mathcal{F}_X^t(f)\| \leqslant C_2 \tag{42}$$

and, for $\|f\| \geqslant C_1$,

$$\|\mathcal{F}_X^t(f)\| \leqslant e^{ct}\|f\|, \tag{43}$$

for any $0 \leqslant t \leqslant T$. Assume also that

$$\mathcal{X} : V \to V \quad \text{and} \quad \mathcal{G} : V \times V \to \mathcal{L}(W) \cap \mathcal{L}(V) \tag{44}$$

are bounded, where

$$\begin{aligned} &(\mathcal{X} f)(x) = X(f(x)), \\ &[\mathcal{G}(f,g)](x) = Y(f(x), g(x)) = \int_0^1 DX(sg(x) + (1-s)f(x))\,ds. \end{aligned} \tag{45}$$

Let $u_0 \in V$, and let $u(t) \in C^0([0,T], V)$ be a solution of (40). We define $v(t)$ by

$$\begin{aligned} &v_k = \left(e^{(1/n)L}\mathcal{F}_X^{1/n}\right)^n(u_0), \\ &v(t) = e^{sL}\mathcal{F}_X^s v_k \end{aligned} \tag{46}$$

for $t = k/n + s$ and $0 \leqslant s < 1/n$. Then,

$$\|v(t) - u(t)\|_V \leqslant C(\|u_0\|_V)n^{-\delta}. \tag{47}$$

One consequence of Proposition 4.5 is that, if both the semigroup e^{tL} and $\mathcal{F}_X^t$ satisfy comparison, then so does the evolution generated by (40). In this sense, Proposition 4.5 is an abstract version of Proposition 2.13.

Remark 4.6. Notice that only a weak comparison principle is needed in this paper to obtain solutions of our problem.

4.2.2. Subordination identities

Subordination identities are another tool that proves useful in the verification of comparison in several occasions, such as the case of fractional operators.

Subordination identities come from Laplace transform identities. The following one comes from [67] (see also section IX.11 in [87]):

Proposition 4.7. *Let $0 < \lambda < 1$ and let x be a non-negative number. Then, we have*:

$$e^{-x^{\lambda}t} = \int_0^{\infty} e^{-st^{1/\lambda}x}\phi_\lambda(s)\,ds, \tag{48}$$

where ϕ_λ is a suitable non-negative function.

When $\lambda = 1/2$, formula (48) can be made particularly explicit and is the Bochner subordination identity (see, e.g., [80] formula (5.22), p. 219, and formula (A.21), p. 264):

Proposition 4.8. *Let x be a non-negative number. We have*:

$$\begin{aligned} e^{-tx^{1/2}} &= (1/2)t\pi^{-1/2}\int_0^\infty e^{-sx}e^{-t^2/4s}s^{-3/2}\,ds \\ &= (1/2)\pi^{-1/2}\int_0^\infty e^{-sxt^2}e^{-1/4s}s^{-3/2}\,ds. \end{aligned} \tag{49}$$

Remark 4.9. By comparing (48) and (49), one sees that

$$\phi_{1/2}(s) = (1/2)\pi^{-1/2}e^{-1/4s}s^{-3/2}. \tag{50}$$

The following argument deduces Proposition 4.7 from the particular case in Proposition 4.8 and it gives an alternative to the proof in [67].

Due to a scaling argument, we may and do assume $t = 1$ for proving Proposition 4.7. First of all, by applying Proposition 4.8 repeatedly, we get Proposition 4.7 for $\lambda = 1/2^m$.

Moreover, using the fact that if $a = \int_0^\infty \phi_a(s)e^{-s}\,ds$, $b = \int_0^\infty \phi_b(s)e^{-s}\,ds$, then:

$$a \cdot b = \int_0^\infty \phi_{ab}(s)e^{-s}\,ds$$

where

$$\phi_{ab}(s) = \int_0^s \phi_a(s-\tau)\phi_b(\tau)\,d\tau,$$

we obtain that if two functions can be represented as the Laplace transform of a non-negative function so can the product.

Therefore, we obtain that e^{-x^λ} is the Laplace transform of a positive function whenever

$$\lambda = \sum_{i=0}^N 2^{-m_i} \tag{51}$$

and $m_i \in \mathbb{N}$.

We also recall the well known inverse Laplace transform formula (see, e.g., [22] for the basics of the Laplace transform theory).

If $F(x) = \int_0^\infty f(u)e^{-xu}\,du$, then

$$f(u) = \int_{c-i\infty}^{c+i\infty} F(x)e^{xu}\,dx. \tag{52}$$

If we take

$$F_\lambda(x) = \exp\left(-x^\lambda\right),$$

for $0 < \lambda < 1$ and $c > 0$, setting $x = c + iy$, we get that

$$e^{(c+iy)u} = e^{xu}.$$

On the other hand, if we choose the branch of x^λ which takes real values when x is real – as is implicit in the previous discussion – we have that

$$(c+iy)^\lambda \approx |y|^\lambda\bigl(\cos(\lambda\pi/2) + i\sin(\lambda\pi/2)\bigr) + O\bigl(y^{\lambda'}\bigr), \quad \lambda' < \lambda,$$

for large $|y|$. Accordingly,

$$\left|e^{-(c+iy)^{\lambda}}\right| \approx e^{-\cos(\lambda\pi/2)|y|^{\lambda}},$$

for large $|y|$. Hence, when

$$F_{\lambda}(x) = \exp\left(-x^{\lambda}\right),$$

the integral in (52) converges uniformly when $\lambda \in (0, 1)$ ranges over a compact set of values.

We also note that when $\lambda_n \to \lambda$, $x^{\lambda_n} \to x^{\lambda}$ uniformly on compact sets of the right hand plane.

The two previous observations imply that as $\lambda_n \to \lambda$, for a fixed $x \in \mathbb{R}$,

$$f_{\lambda_n}(x) \to f_{\lambda}(x). \tag{53}$$

We remark that, in fact, with only slightly more work, one can show that they converge uniformly on compact subsets of x, but we will not need it here.

By the arguments above, given $0 < \lambda < 1$, we chose a sequence $0 < \lambda_n < 1$ contained in the set in (51) and such that $\lambda_n \to \lambda$. We had argued that for all the λ_n, $f_{\lambda_n}(x) \geqslant 0$. By (53), we have proved Proposition 4.7.

Notice that the gist of these formulas is that we can express $e^{-x^{\lambda}t}$ as a superposition of e^{-tx} with positive coefficients. If we interpret x to be an operator and e^{-tx} to denote the semigroup generated by it, we obtain formally the semigroup generated by x^{λ} can be obtained by superposing the semigroup generated by x with positive coefficients.

In particular, if the semigroup generated by x is positive preserving, the semigroup generated by x^{λ} will also be positive preserving.

To justify these formal considerations, we note that it is easy to verify that, if x indeed defines a semigroup, then, the right-hand side of (48) is also a semigroup. In many cases, depending on the asymptotic properties of the semigroup e^{-sx}, it is possible to justify that the semigroup is continuous at 0. By the Hille–Phillips theorem (see, e.g., [43] and [86]) this semigroup should have a generator. This can be taken as the definition of x^{λ}.

For example, for m-accretive self-adjoint operators on Hilbert spaces, we can use the functional calculus to show that indeed formulas (48) and (49) do define a good operator and that this definition agrees with that of the functional calculus. In particular, interesting representation formulas and analytic bounds may be obtained from (48) and (49), as we show in Lemma 4.10 and Proposition 4.11 here below.

Lemma 4.10. *Let A be a self-adjoint operator on a Hilbert space with non-negative spectrum and let $\lambda \in (0, 1)$. Then,*

$$e^{-tA^{\lambda}} = \int_0^{\infty} e^{-st^{1/\lambda}A}\phi_{\lambda}(s)\,ds. \tag{54}$$

Proof. The Spectral theorem (see, e.g., Theorem VIII.6 in [73]) gives that

$$A = \int_0^{+\infty} \sigma\,dE(\sigma),$$

for suitable projection-valued measures $E(\sigma)$ and that

$$g(A) = \int_0^{+\infty} g(\sigma)\,dE(\sigma),$$

for any Borel function g. In particular, taking $g(r) = e^{-tr^{\lambda}}$ and $g(r) = e^{-st^{1/\lambda}r}$ and using Proposition 4.7, we obtain

$$e^{-tA^{\lambda}} = \int_0^{+\infty} e^{-t\sigma^{\lambda}}\,dE(\sigma)$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-st^{1/\lambda}\sigma} \phi_\lambda(s)\, ds\, dE(\sigma)$$

$$= \int_0^{+\infty} e^{-st^{1/\lambda}A} \phi_\lambda(s)\, ds,$$

which gives (54). □

Note that the subordination identity not only gives us the existence of the semigroup given by the roots, but also establishes asymptotic bounds similar to those used in Proposition 4.5, as next result shows:

Proposition 4.11. *Let A and λ be as in Lemma* 4.10. *Assume that e^{-tA} satisfies*

$$\|e^{-tA}\|_{\mathcal{L}(X,Y)} \leqslant Ct^{-\alpha}, \tag{55}$$

for some $\alpha \geqslant 0$. Then, for every $m \in \mathbb{N}$, $e^{-tA^{1/2^m}}$ is also bounded in the same spaces and it satisfies the bounds

$$\|e^{-tA^{1/2^m}}\|_{\mathcal{L}(X,Y)} \leqslant Ct^{-\alpha 2^m}, \tag{56}$$

for a suitable $C > 0$, possibly depending on d and α. In particular, when $\alpha = 0$ and $X = Y$, i.e., when e^{-tA} is a continuous semigroup in the Banach space X, then so is $e^{-tA^{1/2}}$.

Proof. Note that taking norms inside (54) and exploiting (50) and (55), we obtain:

$$\|e^{-tA^{1/2}}\|_{\mathcal{L}(X,Y)} \leqslant (1/2)t\pi^{-1/2} \int_0^{\infty} e^{-t^2/4s} Cs^{-\alpha} s^{-3/2}\, ds. \tag{57}$$

It is easy to check that, for $t > 0$ the integral converges, since $\alpha > -1/2$. Using the change of variables $u = s/t^2$, we obtain that the integral in the right-hand side of (57) is $\tilde{C}t^{-2\alpha}$. The desired result follows by induction over m. □

5. Some applications of the abstract theorem

In this section we discuss some applications of Theorem 3.7.

5.1. Degenerate elliptic operators

We now extend Theorem 2.2 to possibly degenerate elliptic operators. Though more general operators may be dealt with using similar techniques, we will focus on the standard p-Laplacian operator, with $p \geqslant 2$. We recall that

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \tag{58}$$

for any smooth function u. As usual, when u is not smooth, say $u \in W^{1,p}$, then (58) is interpreted in the weak distributional sense (see [52,31] and [42] for further details on p-Laplace equations).

The main result of this section is the following:

Theorem 5.1. *Let $p \in [2, +\infty)$. Let $V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a C^2 function which satisfies*:

$$V(x+e, u) = V(x, u) \quad \forall e \in \mathbb{Z}^d,$$
$$V(x, u+\ell) = V(x, u) \quad \forall \ell \in \mathbb{Z}.$$

Then, for all $\omega \in \mathbb{R}^d$, the problem

$$\Delta_p u - \partial_2 V(x, u) = 0$$

has a solution which satisfies the Birkhoff property and such that

$$u(x) - \omega \cdot x \in L^\infty(\mathbb{R}^d).$$

Since p-Laplacian-type equations are a delicate subject, we need some preliminary result for the proof of Theorem 5.1. We first point out a regularity result:

Lemma 5.2. *Let $T > 0$. Suppose that $u - \omega \cdot x \in W^{1,p}(N\mathbb{T}^d \times [0,T]) \cap C(N\mathbb{T}^d \times [0,T])$ weakly satisfies*

$$\partial_t u(x,t) = \Delta_p u(x,t) - \partial_2 V(x, u(x,t)). \tag{59}$$

Then, $\nabla u \in C^\alpha_{\mathrm{loc}}$, for some $\alpha > 0$ and the C^α-norm of ∇u in any subdomain of $\{t > 1\}$ is bounded uniformly by a quantity which only depends on N, on $\|\nabla u(x,0)\|_{L^p(N\mathbb{T}^d)}$ and on the structural constants.

Proof. That $\nabla u \in C^\alpha_{\mathrm{loc}}$ follows from [30]. Let us now explicitly bound its norm.

We write

$$E(t) := \int_{N\mathbb{T}^d} \frac{1}{p}\left|\nabla u(x,t)\right|^p + V(x, u(x,t))\,dx.$$

By differentiating, as done in Lemma 2.10, it follows from (59) that $E' \leqslant 0$. Thence,

$$\begin{aligned}\int_{N\mathbb{T}^d} \frac{1}{p}\left|\nabla u(x,t)\right|^p dx &\leqslant E(t) + \|V\|_{L^\infty}\left|N\mathbb{T}^d\right| \\ &\leqslant E(0) + \|V\|_{L^\infty}\left|N\mathbb{T}^d\right| \\ &\leqslant \int_{N\mathbb{T}^d} \frac{1}{p}\left|\nabla u(x,0)\right|^p dx + 2\|V\|_{L^\infty}\left|N\mathbb{T}^d\right|.\end{aligned} \tag{60}$$

We will denote the latter quantity by C_0.

By Theorem 5.1 in page 238 of [30], we know that

$$\left|\nabla u(x_o,t_o)\right| \leqslant C \int_{t_o-1/2}^{t_o} \int_{N\mathbb{T}^d} \left|\nabla u(x,t)\right|^p dx\,dt$$

for any $x_o \in N\mathbb{T}^d$ and $t_o > 9/10$, where $C > 0$ is a suitable constant.

So, by (60),

$$\left|\nabla u(x_o,t_o)\right| \leqslant CC_0$$

for any $t_o > 9/10$.

We can therefore make use of Theorem 1.1′ on page 256 of [30] (taking the quantity μ there to be simply CC_0 here). Such result then bounds the C^α seminorm of ∇u in terms of C_0, in any subdomain of $\{t > 1\}$. □

Following is a modification of Lemma 3.1 on page 160 of [30]:

Lemma 5.3. *Let $T > 0$. Let $f : \mathbb{R}^d \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a bounded function, uniformly Lipschitz in its last variable. Suppose that $u - \omega \cdot x \in W^{1,p}(N\mathbb{T}^d \times [0,T]) \cap C(N\mathbb{T}^d \times [0,T])$ weakly satisfies*

$$\partial_t u(x,t) \leqslant \Delta_p u(x,t) + f(x,t,u(x,t)) \tag{61}$$

and that $v - \omega \cdot x \in W^{1,p}(N\mathbb{T}^d \times [0,T]) \cap C(N\mathbb{T}^d \times [0,T])$ weakly satisfies

$$\partial_t v(x,t) \geqslant \Delta_p v(x,t) + f(x,t,v(x,t)). \tag{62}$$

Suppose that

$$u(x,0) \leqslant v(x,0). \tag{63}$$

Then, $u(x,t) \leqslant v(x,t)$ *for any* $t \in [0,T]$.

Proof. Fix $\epsilon > 0$, to be taken arbitrarily small in the sequel. Let $w := u - v$ and $W := \max\{w, 0\}$. Note that $0 \leqslant W \in W^{1,p}(N\mathbb{T}^d \times [0,T])$. We also define

$$\Phi(t) := \frac{1}{2} \int_{N\mathbb{T}^d} W^2(x,t)\,dx.$$

Note that, by (63), $w(x,0) \leqslant 0$ and so

$$\Phi(0) = 0. \tag{64}$$

Also, using Lemma 4.4 on page 13 of [30],

$$\big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big) \cdot \nabla(u-v) \geqslant 0$$

and therefore

$$\int_{N\mathbb{T}^d} \big(\big|\nabla u(x,t)\big|^{p-2}\nabla u(x,t) - \big|\nabla v(x,t)\big|^{p-2}\nabla v(x,t)\big) \cdot \nabla W(x,t)\,dx \geqslant 0 \tag{65}$$

for any $t \in (0,T)$.

Exploiting (61), (62) and (65), we obtain

$$\begin{aligned}
\Phi'(t) &= \int_{N\mathbb{T}^d} W(x,t)\partial_t W(x,t)\,dx \\
&= \int_{N\mathbb{T}^d} W(x,t)\partial_t\big(u(x,t) - v(x,t)\big)\,dx \\
&\leqslant \int_{N\mathbb{T}^d} \nabla W(x,t) \cdot \big(\big|\nabla v(x,t)\big|^{p-2}\nabla v(x,t) - \big|\nabla u(x,t)\big|^{p-2}\nabla u(x,t)\big)\,dx \\
&\quad + \int_{N\mathbb{T}^d} W(x,t)\big(f\big(x,t,u(x,t)\big) - f\big(x,t,v(x,t)\big)\big)\,dx \\
&\leqslant 0 + C \int_{N\mathbb{T}^d} W(x,t)\big|u(x,t) - v(x,t)\big|\,dx \\
&= C \int_{N\mathbb{T}^d} W^2(x,t)\,dx
\end{aligned}$$

for a suitable $C > 0$.

Consequently, if $\bar{C} := 2C$,

$$\Phi'(t) \leqslant \bar{C}\Phi(t) + \epsilon.$$

Since $\Phi \geqslant 0$ by construction, this means that

$$\partial_t \log\big(\bar{C}\Phi(t) + \epsilon\big) \leqslant \bar{C}$$

and so, by integrating and making use of (64),

$$\bar{C}\Phi(t) \leqslant \epsilon e^{\bar{C}T}$$

for any $t \in [0, T]$.

Since ϵ may be taken arbitrarily small, we conclude that Φ is identically zero, which gives the desired claim. □

We now complete the proof of Theorem 5.1. This will be accomplished via Theorem 3.7.

To this end, we need to check conditions H1–H5. Conditions H1 and H2 are obvious, just taking the manifold to be $\mathbb{R}^d$, the fundamental domain to be the torus, the group action to be the integer translation and the variational principle to be

$$\mathcal{S}(u) = \int \frac{1}{p}|\nabla u|^p + V(x,u)\,dx. \tag{66}$$

The existence of the degenerate parabolic semi-flow is assured by Théorème 1.1 on page 156 of [52], thus H3.1 is satisfied. By differentiating, as done in Lemma 2.10, we obtain H3.2.

The fact that H3.3 holds may be obtained as follows. We take k_n to be the integer part of $\Psi_{t_n}(u)(0)$. In this way $|\Psi_{t_n}(u)(0) - k_n| \leqslant 1$. Therefore, Lemma 5.2 gives that $\Psi_{t_n}(u) - k_n$ is locally equicontinuous and equibounded together with its derivatives. Accordingly, H3.3 holds.

Lemma 5.3 implies that H3.4 holds.

The regularity results of [29] or [82] give H4, with $r := 1$ and a suitable $\alpha \in (0, 1)$.

Since only first derivatives of solutions are involved in the gradient of the functional in (66), the fact that $r = 1$ gives H5.

The above arguments yield that all the hypotheses H1–H5 are satisfied in the p-Laplacian setting. Making use of Theorem 3.7, we thus end the proof of Theorem 5.1.

5.2. *Fractional powers of the Laplacian*

In this section, we will consider some generalizations of Theorem 2.2 to fractional powers of the Laplacian instead of the Laplacian (see, e.g., [49] and [79] for the basic properties of the fractional Laplacian operator).

A representative result is the following:

Theorem 5.4. *Let $V : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a C^∞ function which satisfies*:

$$V(x+e,u) = V(x,u) \quad \forall e \in \mathbb{Z}^d,$$
$$V(x,u+\ell) = V(x,u) \quad \forall \ell \in \mathbb{Z}^d.$$

Let $0 < \lambda < 1$. Then, for all $\omega \in \mathbb{Q}^d$, the problem

$$(-\Delta)^\lambda(u - \omega\cdot x) + \partial_2 V(x,u) = 0 \tag{67}$$

has a solution such that

$$u(x) - \omega\cdot x \in L^\infty(\mathbb{R}^d). \tag{68}$$

Moreover, the solution claimed above can be assumed to have the Birkhoff property.

Remark 5.5. Equations similar to (67) for $\lambda = 1/2$ appear in several fields where the $(-\Delta)^{1/2}$ operator plays a role. For examples in the ultrarelativistic limit of quantum mechanics, see [35]. See also [54] and [21] for related problems in the theory of quasi-geostrophic flow. Similar equations also arise in the theory of water waves: see, e.g., [23] and [63]. For this latter application, we note that the operator $(-\Delta)^{1/2}$ appears frequently as an approximation to the Dirichlet to Neumann operator.

The operator $(-\Delta)^{1/2}$ also plays an important role in the wave equation approach to inverse spectral problems (see [32]) and in the thin obstacle problem (see [12]).

Several applications to phase transition problems driven by either the fractional Laplacian or non-local interactions have recently appeared in the literature: see, for instance, [1,7,18,11,38,77] and references therein.

For further motivation, see also [6].

Remark 5.6. The method of proof works for more complicated problems such as

$$Lu + \partial_2 V(x,u) = 0, \tag{69}$$

where L is an operator obtained by from the Laplacian by repeated taking of roots and combination with positive numbers For example, we could consider:

$$L = \left(2(-\Delta)^{1/2} + 17(-\Delta)^{4/3}\right)^{1/4} + (-\Delta)^{1/16}.$$

The result that we will need is the following.

Proposition 5.7. *$e^{-t(-\Delta)^\lambda}$ is positive preserving in L^2.*

Proof. By (54), applied with $A = -\Delta$,

$$e^{-t(-\Delta)^\lambda} = \int_0^\infty e^{st^{1/\lambda}\Delta}\phi_\lambda(s)\,ds.$$

Then, the result follows from the fact that $e^{t\Delta}$ is positive preserving and $\phi_\lambda \geqslant 0$. □

Remark 5.8. We also note that in [79] one can find explicit realizations of the semigroup $e^{-t(-\Delta)^{1/2}}$. This reference also contains a different proof of the positivity of this semigroup. It is shown that $P(t,x) = e^{-t(-\Delta)^{1/2}} f(x)$ is the solution of $\Delta + \partial_t^2 P = 0$ with boundary values at $t = 0$ given by f. Therefore, the positivity preserving of the semigroup is implied by the maximum principle of the Laplacian in one dimension more.

With the goal of checking (41) in the case of the fractional Laplacian, we now point out some Fourier analysis estimates on the fractional Laplacian operator, quite related to the ones on p. 193 of [51]. Since we supposed $\omega \in \mathbb{Q}^d$, say $\omega \in \frac{1}{N}\mathbb{Z}^d$, such analysis will involve periodic functions on $N\mathbb{T}^d$. For typographical reasons, we denote the Fourier transform of a function $v \in L^2(\mathbb{R}^d)$ equivalently by $\hat{v}$ and by $\mathcal{F}(v)$.

Note that, if $v \in W^{2\lambda,2}(N\mathbb{T}^d)$,

$$\hat{v}_k = \frac{1}{N^d}\int_{N\mathbb{T}^d} v(x)e^{-2\pi i x\cdot k/N}\,dx \quad \text{and}$$
$$v(x) = \sum_{k\in\mathbb{Z}^d} v_k e^{2\pi i x\cdot k/N},$$

it follows that

$$\partial_j v(x) = \sum_{k\in\mathbb{Z}^d} \frac{2\pi i k_j}{N}\hat{v}_k e^{2\pi i x\cdot k/N}$$

and so

$$\mathcal{F}\big((-\Delta)^\lambda v\big)_k = \text{const}\left|\frac{k}{N}\right|^{2\lambda}\hat{v}_k. \tag{70}$$

Proposition 5.9. *For any $t > 0$,*

$$\left\|e^{-(-\Delta)^\lambda t}\right\|_{\mathcal{L}(L^2(N\mathbb{T}^d))} + \left\|e^{-(-\Delta)^\lambda t}\right\|_{\mathcal{L}(W^{\lambda,2}(N\mathbb{T}^d))} \leqslant C,$$
$$\left\|e^{-(-\Delta)^\lambda t}\right\|_{\mathcal{L}(L^2(N\mathbb{T}^d), W^{\lambda,2}(N\mathbb{T}^d))} \leqslant C\left(1 + \frac{1}{\sqrt{t}}\right) \quad \textit{and}$$
$$\left\|e^{-(-\Delta)^\lambda t} - \mathrm{Id}\right\|_{\mathcal{L}(W^{\lambda,2}(N\mathbb{T}^d), L^2(N\mathbb{T}^d))} \leqslant C\sqrt{t},$$

for a suitable constant $C > 0$, possibly depending on N.

Proof. From (70), we have

$$\begin{aligned}\big\|e^{-(-\Delta)^\lambda t}v\big\|^2_{L^2(N\mathbb{T}^d)} &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big|\mathcal{F}\big(e^{-(-\Delta)^\lambda t}v\big)(k)\big|^2\\ &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big|\hat{v}_k e^{-\text{const}|k|^{2\lambda}t}\big|^2\\ &\leqslant \text{const}\,\|v\|^2_{L^2(N\mathbb{T}^d)}.\end{aligned}$$

Analogously,

$$\begin{aligned}\big\|e^{-(-\Delta)^\lambda t}v\big\|^2_{W^{\lambda,2}(N\mathbb{T}^d)} &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big(1+|k|^\lambda\big)^2\big|\mathcal{F}\big(e^{-(-\Delta)^\lambda t}v\big)(k)\big|^2\\ &= \text{const}\sum_{k\in\mathbb{Z}^d}\big(1+|k|^\lambda\big)^2\big|\hat{v}_k e^{-\text{const}|k|^{2\lambda}t}\big|^2\\ &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big(1+|k|^\lambda\big)^2|\hat{v}_k|^2\\ &= \text{const}\,\|v\|^2_{W^{\lambda,2}(N\mathbb{T}^d)}.\end{aligned}$$

The first estimate in Proposition 5.9 then follows. Also, the function

$$[0,+\infty)\ni\theta\mapsto\theta e^{-\theta t}$$

reaches its maximum for $\theta=1/t$ and so

$$\theta e^{-\theta t}\leqslant\frac{1}{et}$$

for any $\theta\geqslant 0$. Accordingly,

$$\begin{aligned}\big\|e^{-(-\Delta)^\lambda t}v\big\|^2_{W^{\lambda,2}(N\mathbb{T}^d)} &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big(1+|k|^\lambda\big)^2\big|\mathcal{F}\big(e^{-(-\Delta)^\lambda t}v\big)(k)\big|^2\\ &= \text{const}\sum_{k\in\mathbb{Z}^d}\big(1+|k|^\lambda\big)^2\big|\hat{v}_k e^{-|\xi|^{2\lambda}t}\big|^2\\ &\leqslant \text{const}\bigg(\sum_{k\in\mathbb{Z}^d}|\hat{v}_k|^2+\sum_{k\in\mathbb{Z}^d}|k|^{2\lambda}e^{-2|k|^{2\lambda}t}|\hat{v}_k|^2\bigg)\\ &\leqslant \text{const}\bigg(1+\frac{1}{t}\bigg)\|v\|^2_{L^2(N\mathbb{T}^d)},\end{aligned}$$

which yields the second estimate in Proposition 5.9.

We now observe that

$$1-e^{-\tau}\leqslant\min\{\tau,1\}$$

and so

$$\begin{aligned}\big\|e^{-(-\Delta)^\lambda t}v-v\big\|^2_{L^2(N\mathbb{T}^d)} &\leqslant \text{const}\sum_{k\in\mathbb{Z}^d}\big|\mathcal{F}\big(e^{-(-\Delta)^\lambda t}v\big)(k)-\hat{v}_k\big|^2\\ &= \text{const}\sum_{k\in\mathbb{Z}^d}\big(1-e^{-|k|^{2\lambda}t}\big)^2|\hat{v}_k|^2\\ &\leqslant \text{const}\sum_{|k|^{2\lambda}t\geqslant 1}|\hat{v}_k|^2+\sum_{|k|^{2\lambda}t<1}|k|^{4\lambda}t^2|\hat{v}_k|^2\\ &\leqslant \text{const}\sum_{|k|^{2\lambda}t\geqslant 1}|k|^{2\lambda}t|\hat{v}_k|^2+\sum_{|k|^{2\lambda}t<1}|k|^{2\lambda}t|\hat{v}_k|^2\end{aligned}$$

$$\begin{aligned}&\leqslant \operatorname{const} t \sum_{k\in\mathbb{Z}^d} \big(1+|k|^\lambda\big)^2 |\hat{v}_k|^2\\ &= \operatorname{const} t \|v\|^2_{W^{\lambda,2}},\end{aligned}$$

which ends the proof of Proposition 5.9. □

Following is the weak comparison principle that we need for our purposes:

Proposition 5.10. *Let $f \in C^1(\mathbb{R}^{d+1})$ and suppose that both u_1 and u_2 are in $L^2(N\mathbb{T}^d)$ and satisfy*

$$\partial_t u_i(x,t) = -(-\Delta)^\lambda u_i(x,t) + f\big(x, u_i(x,t)\big) \quad i = 1,2,$$

for any $x \in \mathbb{R}^d$ and $t \in [0,T]$. Suppose that $u_1(x,0) \leqslant u_2(x,0)$ for any $x \in \mathbb{R}^d$. Then, $u_1(x,t) \leqslant u_2(x,t)$ for any $x \in \mathbb{R}^d$ and $t \in [0,T]$.

Proof. We will make use of Proposition 4.5. For this, we set $l = d+1$ and consider the vector field on $\mathbb{R}^l$ given by

$$X : (x,r) \in \mathbb{R}^d \times \mathbb{R} \mapsto \big(0, f(x,r)\big) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

We also consider the Banach (and, in fact, Hilbert) spaces $V = \mathbb{R}^d \times W^{\lambda,2}$ and $W = \mathbb{R}^d \times L^2$. We define the linear operator L on W as

$$L : (x,u) \in \mathbb{R}^d \times L^2 \mapsto \big(0, -(-\Delta)^\lambda\big).$$

Then, $e^{tL} = (0, e^{-t(-\Delta)^\lambda})$, and so estimates (41) are satisfied, thanks to Proposition 5.9.

Furthermore, the flow $\mathcal{F}^t_X$ generated by X is, in this case,

$$\mathcal{F}^t_X(x_0,r_0) = \big(x_0, r(t)\big)$$

where $r(t)$ is the solution of the ODE

$$\begin{cases} \dot{r}(t) = f\big(x_0, r(t)\big),\\ r(0) = r_0.\end{cases}$$

Note that

$$\big\|\mathcal{F}^t_X(x_0,r_0)\big\| \leqslant \big\|(x_0,r_0)\big\| + \|f\|_{L^\infty} t$$

and so it satisfies (42) and (43).

Also, if $\mathcal{X}$ and $\mathcal{G}$ are as in (45), we have that

$$\mathcal{X}(x,u) = \big(0, f\big(x, u(x,t)\big)\big)$$

for any $(x,u) \in \mathbb{R}^d \times L^2$ and that $\mathcal{G}$ sends $(x', u', x'', u'') = (v', v'') \in V \times V$ to the linear operator $\mathcal{G}(v', v'')$ on W given by

$$\big[\mathcal{G}(v'v''), v'''\big] = \left(0, \int_0^1 x''' \cdot \partial_x f(\star) + u'''(x)\partial_r f(\star)\, ds\right),$$

for any $v''' = (x''', u''') \in W$ where "$\star$" above is short for

$$\big(sx' + (1-s)x'', su'(x) + (1-s)u''(x)\big).$$

Then, $\mathcal{X}$ and $\mathcal{G}$ are as requested in (44).

Moreover, if we define $U_i(x,t) = (x, u_i(x,t))$ for $i = 1,2$, we have that U_i is a solution of (40). We define $v^i(t)$ as in (46) (obviously, by replacing $u_0(x)$ there with $U_i(x,0)$).

Then, we deduce from (47) that

$$v^i \text{ approaches } U_i \text{ in } V \text{ when } n \to +\infty. \tag{71}$$

We know observe that $\mathcal{F}_X^t$ satisfies the comparison principle (because of the Cauchy Uniqueness Theorem for ODEs) and so does e^{tL} (because of Proposition 5.7) and therefore $v^1 \leqslant v^2$. Consequently, possibly taking a subsequence which converges almost everywhere, we gather from (71) that $U_1 \leqslant U_2$ and so that $u_1 \leqslant u_2$. $\square$

We now recall the following compact embedding of $W^{\lambda,2}(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$:

Lemma 5.11. *Let $C > 0$. Consider a family $\mathcal{F}$ of functions in $L^2(N\mathbb{T}^d)$ such that*

$$\sup_{v\in\mathcal{F}} \sum_{k\in\mathbb{Z}^d} k^{2\lambda} |\hat{v}_k|^2 \leqslant C. \tag{72}$$

Then, given any sequence $v^{(j)} \in \mathcal{F}$, we have that the sequence

$$v^{(j)} - \left[\frac{1}{|N\mathbb{T}^d|} \int\limits_{N\mathbb{T}^d} v^j(x)\,dx\right] \tag{73}$$

is compact in $L^2(N\mathbb{T}^d)$, where $[\cdot]$ here above denotes the integer part.

Proof. The proof is a variation of the standard diagonal trick. For this, we first note that

$$\left|\hat{v}_0^{(j)} - \left[\frac{1}{|N\mathbb{T}^d|} \int\limits_{N\mathbb{T}^d} v^j(x)\,dx\right]\right| = \left|\hat{v}_0^{(j)} - \left[\hat{v}_0^{(j)}\right]\right| \leqslant 1,$$

hence we may and do assume that

$$\lim_{j\to+\infty} \hat{v}_0^{(j)} - \left[\frac{1}{|N\mathbb{T}^d|} \int\limits_{N\mathbb{T}^d} v^j(x)\,dx\right] = \hat{v}_0^{\infty},$$

for a suitable $\hat{v}_0^{\infty}$, with $|\hat{v}_0^{\infty}| \leqslant 1$.

Fix now $M > 0$. By (72),

$$\sum_{0<|k|\leqslant M} \left|\hat{v}_k^{(j)}\right|^2 \leqslant C.$$

That is, the sequence $\{\hat{v}_k^{(j)}, 0 < |k| \leqslant M\}$ is bounded in a finite dimensional space. So, for $0 < |k| \leqslant M$, we can consider subsequences $\hat{v}_k^{\phi_M(j)}$ such that $\hat{v}_k^{\phi_{M+1}(j)}$ is a subsequence of $\hat{v}_k^{\phi_M(j)}$ and

$$\lim_{j\to+\infty} \hat{v}_k^{\phi_M(j)} = \hat{v}_k^{\infty},$$

for any $0 < |k| \leqslant M$ and suitable $\hat{v}_k^{\infty}$.

We now show that, if $\tilde{v}^{(j)}$ is the sequence in (73), then the sequence $\tilde{v}^{\phi_j(j)}$ converges in $L^2(\mathbb{T}^d)$ to the function

$$v^{\infty}(x) = \sum_{k\in\mathbb{Z}^d} \hat{v}_k^{\infty} e^{ik\cdot x}.$$

Indeed,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} \left|\hat{v}_k^{\infty}\right|^2 &= \left|\hat{v}_0^{\infty}\right|^2 + \lim_{M\to+\infty} \sum_{\substack{k\in\mathbb{Z}^d\\ 0<|k|\leqslant M}} \left|\hat{v}_k^{\infty}\right|^2 \\
&= \left|\hat{v}_0^{\infty}\right|^2 + \lim_{M\to+\infty} \lim_{j\to+\infty} \sum_{\substack{k\in\mathbb{Z}^d\\ 0<|k|\leqslant M}} \left|\hat{v}_k^{\phi_j(j)}\right|^2 \leqslant 1 + C,
\end{aligned}$$

in force of (72), thence $v^{\infty} \in L^2(N\mathbb{T}^d)$.

Also, using again (72), for any $M > L > 0$,

$$L^{2\lambda} \sum_{L+1\leqslant|k|\leqslant M} \left|\hat{v}_k^{(j)}\right|^2 \leqslant \sum_{L+1\leqslant|k|\leqslant M} |k|^{2\lambda}\left|\hat{v}_k^{(j)}\right|^2 \leqslant C$$

and so

$$L^{2\lambda} \sum_{L+1\leqslant|k|\leqslant M} \left|\hat{v}_k^{\infty}\right|^2 \leqslant C.$$

Accordingly,

$$\sum_{L+1\leqslant|k|} \left(\left|\hat{v}_k^{(j)}\right| + \left|\hat{v}_k^{\infty}\right|\right)^2 \leqslant 3 \sum_{L+1\leqslant|k|} \left(\left|\hat{v}_k^{(j)}\right|^2 + \left|\hat{v}_k^{\infty}\right|^2\right) \leqslant \frac{8C}{L^{2\lambda}}.$$

Thus, fixed any $L \in \mathbb{N}$,

$$\begin{aligned}
\lim_{j\to+\infty} \left\|\tilde{v}^{\phi_j(j)} - v^{\infty}\right\|^2_{L^2(N\mathbb{T}^d)} &= \lim_{j\to+\infty} \left|\hat{\tilde{v}}_0^{\phi_j(j)} - \hat{v}_0^{\infty}\right|^2 + \sum_{\substack{k\in\mathbb{Z}^d\\ k\neq 0}} \left|\hat{v}_k^{\phi_j(j)} - \hat{v}_k^{\infty}\right|^2 \\
&\leqslant 0 + \frac{8C}{L^{2\lambda}} + \lim_{j\to+\infty} \sum_{\substack{k\in\mathbb{Z}^d\\ 0<|k|\leqslant L}} \left|\hat{v}_k^{\phi_j(j)} - \hat{v}_k^{\infty}\right|^2 \\
&= 0 + \frac{8C}{L^{2\lambda}} + 0.
\end{aligned}$$

Since L may be taken as large as we wish, we conclude that $\tilde{v}^{\phi_j(j)}$ approaches v^{∞} in $L^2(\mathbb{T}^d)$. □

We now complete the proof of Theorem 5.4. To this purpose, we set $v(x) = u(x) - \omega \cdot x$, with $\omega \in \frac{1}{N}\mathbb{Z}^d$, and we seek $v : N\mathbb{T}^d \to \mathbb{R}$. Note that the fractional Laplacian operator $(-\Delta)^{\lambda}$ is well defined on $W^{2\lambda,2}(N\mathbb{T}^d) \subset L^2(N\mathbb{T}^d)$ by Fourier series.

Given the above setting, we consider the functional

$$\mathcal{S}_N(v) = \frac{1}{2}\left\|(-\Delta)^{\lambda/2} v\right\|^2_{L^2(N\mathbb{T}^d)} + \int_{[0,N]^d} V\big(x, v(x) + \omega\cdot x\big)\,dx.$$

Notice that critical points of $\mathcal{S}$ satisfies (67) since $(-\Delta)^{\lambda/2}$ is self-adjoint.

Also, H1 and H2.1 are here obviously satisfied. Moreover, if $v \in L^2(N\mathbb{T}^d)$, we have that $\mathcal{F}(v+1)_k = \mathcal{F}(v)_k = \hat{v}_k$ for any $k \in \mathbb{Z}\setminus\{0\}$ and so

$$\mathcal{F}\big((-\Delta)^{\lambda}(v+1)\big)_k = \big(|k|^{2\lambda}\mathcal{F}(v+1)_k\big) = \big(|k|^{2\lambda}\hat{v}_k\big) = \mathcal{F}\big((-\Delta)^{\lambda} v\big)_k$$

for any $k \in \mathbb{Z}^d$ and so

$$(-\Delta)^{\lambda}(v+1) = (-\Delta)^{\lambda} v.$$

Consequently, the invariance property in H2.2 is satisfied in this case. Of course, $\mathcal{S}_N$ is bounded from below, since V is smooth enough, thus fulfilling condition H2.3.

Besides,

$$\big\langle(-\Delta)^{\lambda} v, v\big\rangle_{L^2(\mathbb{T}^d)} = \big\langle(-\Delta)^{\lambda/2} v, (-\Delta)^{\lambda/2} v\big\rangle_{L^2(\mathbb{T}^d)} \geqslant 0 \tag{74}$$

for any $v \in W^{2\lambda,2}(\mathbb{T}^d)$, since the fractional Laplacian is self-adjoint.

Furthermore, given any $f \in L^2(\mathbb{T}^d)$, if we define v by

$$\hat{v}_k = \frac{\hat{f}_k}{1 + |k|^{2\lambda}}$$

we have that $v \in W^{2\lambda,2}(\mathbb{T}^d)$ and

$$\big((-\Delta)^\lambda + \mathrm{Id}\big)v = f.$$

The latter argument and (74) imply that the operator $-(-\Delta)^\lambda$ satisfies (31) with $D = W^{2\lambda,2}(\mathbb{T}^d)$ and $H = L^2(\mathbb{T}^d)$, so it is m-accretive and thus it generates a heat flow, thanks to Theorem 4.1. Then, the gradient of $\mathcal{S}_N$ generates a semi-flow by Proposition 4.3 (which may be applied here due to the first estimate in Proposition 5.9) and this argument gives condition H3.1. Condition H3.2 is obtained by the standard argument in Lemma 2.10.

Assumption H3.3 is fulfilled in the light of the following argument. Since the fractional Laplacian is self-adjoint,

$$\frac{d}{dt}\mathcal{S}_N\big(v(t)\big) = -\big\|(-\Delta)^{\lambda/2}v + \partial_2 V(x, v + \omega\cdot x)\big\|_{L^2(N\mathbb{T}^d)} \leqslant 0$$

and so $\mathcal{S}_N(v(t)) \leqslant \mathcal{S}_N(v(0))$. This yields that

$$\sum_{k\in\mathbb{Z}^d} k^{2\lambda}\big|\hat{v}_k(t)\big|^2 \leqslant C_1\big(N, v(0)\big)$$

for a suitable $C_1(N, v(0))$ independent of t. Then, recalling Lemma 5.11, by possibly subtracting the integer part of its average to $v(t)$, we have that, for $t \to +\infty$, up to subsequences, v is compact in L^2. Therefore, since the fractional Laplacian is self-adjoint, $v(t)$ converges pointwise, as $t \to +\infty$, up to subsequences, to a suitable $v^\star$ which solves $(-\Delta)^{\lambda/2}v^\star + \partial_2 V(x, v^\star + \omega\cdot x) = 0$. This shows that condition H3.3 is satisfied.

Also, condition H3.4 is warranted by Proposition 5.10.

Condition H4′ is checked by Fourier analysis on (70), which gives $W^{2\lambda,2}$ estimates, and standard bootstrap.

Thus, the proof of Theorem 5.4 is ended by the use of Theorem 3.7.

5.3. Quasi-periodic solutions in manifolds with residually finite fundamental group

In this section, we show how the results of [58] can be extended to other manifolds than the torus and, in particular, to some manifolds with a non-Abelian fundamental group. This answers a question that was posed in [58].

We will establish:

Theorem 5.12. *Let M be a compact Riemannian manifold. Let G be a finitely generated subgroup of the fundamental group $\Pi_1(M)$.*

Let $\tilde{M}$ be the universal cover associated to G, i.e. $M = \tilde{M}/G$. Denote by gx the action of G on $\tilde{M}$ by Deck transformations.

Let $F : TM \times \mathbb{R} \to \mathbb{R}$ be a C^3 function which we can write in local coordinates as $F(x, u, p)$.

Consider the formal variational principle

$$\mathcal{S}(u) = \int_{\tilde{M}} F\big(x, u(x), \nabla u(x)\big)\,dx \tag{75}$$

whose Euler–Lagrange equations are

$$\nabla\mathcal{S}(u) = -\operatorname{div}\big(\nabla_p F\big(x, u(x)\nabla u(x)\big) + \nabla_u F\big(x, u(x)\nabla u(x)\big) = 0. \tag{76}$$

Assume:

- (i) *The group G is residually finite.*
- (ii) $F(x, u+\ell, p) = F(x, u, p)$, $\forall \ell \in \mathbb{Z}$.
- (iii) *The quadratic form F_{pp} is positive definite and there is a lower bound for it which is independent of x and u.*

Then, for every cocycle φ of G, we can find a $u : \tilde{M} \to \mathbb{R}$ solving (76) *and such that*

$$u(\cdot) - \varphi(\cdot) \in L^\infty(\tilde{M}).$$

Moreover, u is Birkhoff.

Proof. We will need to verify the abstract framework of Theorem 3.7. Note that, in the particular case that $F(x,u,p) = Q(x)[p,p] + V(x,u)$ where Q is a quadratic form and V is a function $V(x,u+\ell) = V(x,u)$, the variational equations are such that the higher order term is linear. Then, the theory discussed in Section 4 is enough to verify the hypothesis of Theorem 3.7. In the full generality considered in Theorem 5.12 we will need to use a part of the theory of fully non-linear equations.

For our purposes, we only require existence, uniqueness and small amounts of regularity. Moreover, our coefficients and functions being periodic are quite well behaved. Moreover, since our equations are gradient flows, the energy estimates are automatic. Hence, the required properties will be within easy reach of textbook theory, such as [81,36] and [50].

Clearly, the geometric hypothesis H1 and H2 are verified. The variational principle, being local is invariant under Deck transformations. In particular, H2.2 is a consequence of assumption (ii) in Theorem 5.12.

We recall that the assumption that the manifold M has a finite fundamental group includes in particular, the tori considered in [58] and [5], manifolds of negative curvature and other manifolds such as Heisenberg groups and products of the above, etc.

Hypothesis H3 about existence of the gradient flow and its regularity and comparison properties for the case at hand fits into the classical theory of parabolic equations explained in different versions of varying sophistication in [81] 15.7, 15.8 and 15.9 (p. 327 ff.) We just summarize the argument as developed in the above source, to which we refer for references to the original literature.

We call attention to the fact that the only term of (76) involving derivatives of second order is (in local coordinates)

$$\sum_{i,j} -F_{p_i,p_j}(x,u,\nabla u)\partial_{x_j}\partial_{x_j}u. \tag{77}$$

Hence, the equation for the gradient semi-flow is quasilinear.

Recall that we are working in a compact manifold and note that the *"strong parabolicity condition"* (see [81, (7.2), p. 327]) is assumption (iii) of Theorem 5.12 here.

The classical theory of quasilinear equations starts by establishing local existence and uniqueness using a Galerkin method. The uniform parabolicity is used to show that, for short enough time, the Galerkin approximations converge to a solution which is unique. Of crucial importance is the fact that the solution persists if it is regular enough.

The following is Proposition 8.2 of [81, p. 339]:

Proposition 5.13. *Assume that F is as indicated and $s > d/2 + 1$. Then, provided $u_0 \in H^s$, the equation*

$$\partial_t u = \nabla S(u), \quad u(0) = u_0$$

has a unique solution in $C^0((0,T], H^s(M)) \cap C^1((0,T], H^{s-2}(M))$.

If also

$$\sup_{t\in[0,T]} \|u\|_{C^{1+r}} < \infty$$

for some $r > 0$, then, it is possible to find a $T' > T$ for which the same conclusions hold.

The proof of Theorem 5.12 is thus ended by the following argument. Assumption (iii) in Theorem 5.12 gives that the operator in (77) is m-accretive and so its heat flow satisfies H3.1, according to Theorem 4.1. Condition H3.2 follows from the standard arguments in Lemma 4.1.

The fact that the energy decreases on the semi-flow $u(t)$ implies a control of the L^2 norm of the gradient $\nabla u(t)$ which is independent of t and then, by Poincaré Inequality, a control of the L^2 norm of $u(t)$ (up to subtracting the integer part of the average): this and standard compactness arguments yield H3.3.

Condition H3.4 follows from Proposition 4.5 and H4 is warranted by elliptic regularity.

Then, exploiting Theorem 3.7, we conclude the proof of Theorem 5.12. □

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