On asymptotic stability in energy space of ground states of NLS in 2D

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Abstract
We transpose work by K. Yajima and by T. Mizumachi to prove dispersive and smoothing estimates for dispersive solutions of the linearization at a ground state of a Nonlinear Schrödinger equation (NLS) in 2D. As an application we extend to dimension 2D a result on asymptotic stability of ground states of NLS proved in the literature for all dimensions different from 2.

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1. Introduction
We consider even solutions of a NLS
\[ iu_t + \Delta u + \beta(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \quad u(0, x) = u_0(x). \] (1.1)
We assume:

(H1) $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$;
(H2) there exists a $p_0 \in (1, \infty)$ such that for every $k = 0, 1$,
\[ \left| \frac{d^k}{dv^k} \beta(v^2) \right| \lesssim |v|^{p_0-k-1} \text{ if } |v| \geq 1; \]

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Theorem 1.1. Let \( u_0 \in \mathcal{O} \) such that \( \Delta u - \omega u + \beta(u^2)u = 0 \) admits a \( C^1 \)-family of ground states \( \phi_\omega(x) \) for \( \omega \in \mathcal{O} \);

(4) \( \frac{d}{da} \| \phi_{\omega_0} \|_{L^2(\mathbb{R})}^2 > 0 \) for \( \omega \in \mathcal{O} \);

(5) Let \( L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - 2\beta'(\phi_\omega^2) \phi_\omega^2 \) be the operator whose domain is \( H^2_{\text{rad}}(\mathbb{R}^2) \). We assume that \( L_+ \) has exactly one negative eigenvalue and that it has no (radial) kernel. By [27] the \( \omega \to \phi_\omega \in H^1(\mathbb{R}^2) \) is \( C^2 \) and by [38,13,14] (H4)–(H5) yields orbital stability of the ground state \( e^{i\omega t}\phi_\omega(x) \).

Here we investigate asymptotic stability. We need some dispersive estimates. Thanks to the work by [22], it is quite clear how to transpose to dimension 2 the higher dimensional arguments in [10]. Here we consider the case of dimension 2.

(b) If \( \omega \) is large, with the generalizations contained in [10].

Consider the Pauli matrices \( \sigma_j \) and the linearization \( H_\omega \) given by:

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};
\]

\[
H_\omega = \sigma_3[-\Delta + \omega - \beta(\phi_\omega^2) - \beta'(\phi_\omega^2) \phi_\omega^2] + i\beta'(\phi_\omega^2) \phi_\omega^2 \sigma_2.
\]

Then we assume:

(7) Let \( H_\omega \) be the linearized operator around \( e^{i\omega t}\phi_\omega \), see (1.2). \( H_\omega \) has a positive simple eigenvalue \( \lambda(\omega) \) for \( \omega \in \mathcal{O} \) whose corresponding eigenfunctions are even functions. There exists an \( N \in \mathbb{N} \) such that \( N\lambda(\omega) < \omega < (N+1)\lambda(\omega) \).

(8) The Fermi Golden Rule (FGR) holds (see Hypothesis 4.2 in Section 4).

(9) The point spectrum of \( H_\omega \) consists of 0 and \( \pm \lambda(\omega) \). The points \( \pm \omega \) are not resonances.

Then we prove:

Theorem 1.1. Let \( \omega_0 \in \mathcal{O} \) and \( \phi_\omega(x) \) be a ground state in a family of ground states \( \phi_\omega \). Let \( u(t, x) \) be a solution to (1.1). Assume (H1)–(H9). In particular assume the (FGR) in Hypothesis 4.2. Then, there exist an \( \epsilon_0 > 0 \) and a \( C > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \) and for any \( u_0 \) with \( \| u_0 - e^{i\theta_0} \phi_{\omega_0} \|_{H^1} < \epsilon \), there exist \( \omega_+ \in \mathcal{O} \), \( \theta \in C^1(\mathbb{R}; \mathbb{R}) \), \( \| h_{\infty} \|_{H^1} \leq C \epsilon \) and \( |\omega_+ - \omega_0| \leq C \epsilon^2 \) such that

\[
\lim_{t \to +\infty} \| u(t, \cdot) - e^{i\theta(t)} \phi_{\omega_0} - e^{i\Delta t} h_{\infty} \|_{H^1} = 0.
\]

Theorem 1.1 is the two dimensional version of Theorem 1.1 [10]. The one dimensional version is in [7]. We recall that results of the sort discussed here were pioneered by Soffer and Weinstein [29], see also [24], followed by Buslaev and Perelman [3,4], about 15 years ago. In this decade these early works were followed by a number of results [5,8,9,15,21–23,25,29–31,33–36]. It was heuristically understood that the rate of the leaking of energy from the so called “internal modes” into radiation, is small and decreasing when \( N \) increases, producing technical difficulties in the closure of the nonlinear estimates. For this reason prior to Gang Zhou and Sigal [12], the literature treated only the case when \( N = 1 \) in (H6), [12] sheds light for \( N > 1 \). The results in [12] deal with all spatial dimensions different from 2 under the so called Fermi Golden Rule (FGR) hypothesis, [10,7] strengthen [12] by considering initial data in \( H^1 \), by showing that the (FGR) hypothesis is a consequence of what looks generic condition, Hypothesis 4.2 below, if (H8) is assumed. [10] treats also the case when there are many eigenvalues and Hypothesis 4.2 is replaced by a more stringent hypothesis which is a natural generalization of the (FGR) hypothesis in [12]. The same result with many eigenvalues case can be proved also here and in [7], but we skip for simplicity the proof. We recall that Mizumachi [21], resp. [22], extends to dimension 1, resp. 2, the results in [15] valid for small solitons obtained by bifurcation from ground states of a linear equation, while [20] extends in 2D the result in [30], [7] transposes [21] to the case of large solitons, with the generalizations contained in [10]. Here we consider the case of dimension 2. Thanks to the work by [22], it is quite clear how to transpose to dimension 2 the higher dimensional arguments in [10]. The nonlinear arguments in [10] are not sensitive to the dimension except for the lack in 2D of the endpoint Strichartz estimate. Mizumachi [22] shows how to replace it with an appropriate smoothing estimate of Kato type. The estimate and its proof are suggested by [22]. In order to complete the proof of Theorem 1.1 we need some dispersive estimates on the linearization \( H_\omega \) which in spatial dimension 2 are not yet proved in the literature. The main technical task of this paper is the transposition to \( H_\omega \) of the proof of \( L^p \) boundedness of wave operators of Schrödinger operators in dimension 2 due to Yajima [40]. We use the following notation. We set \( H_0(\omega) = \sigma_3(-\Delta + \omega) \); given normed spaces
X and Y we denote by \( B(X, Y) \) the space of operators from X to Y and given \( L \in B(X, Y) \) we denote by \( \|L\|_{X,Y} \) or by \( \|L\|_{B(X,Y)} \) its norm. We prove:

**Proposition 1.2.** Assume the hypotheses of Theorem 1.1. The following limits are well defined isomorphism, inverse of each other:

\[
Wu = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_0(\omega)} u \quad \text{for any } u \in L^2,
\]

\[
Zu = \lim_{t \to +\infty} e^{itH_0(\omega)} e^{-itH_\omega} u \quad \text{for any } u \in L^2_c(H_\omega) \text{ (defined in Section 2)}.
\]

For any \( p \in (1, \infty) \) and any \( k \) the restrictions of \( W \) and \( Z \) to \( L^2 \cap W^{k,p} \) extend into operators such that for \( C(\omega) < \infty \) semicontinuous in \( \omega \)

\[
\|W\|_{W^{k,p}(\mathbb{R}^2),W^{k,p}(H_\omega)} + \|Z\|_{W^{k,p}(H_\omega),W^{k,p}(\mathbb{R}^2)} < C(\omega)
\]

with \( W_c(\omega) \) the closure in \( W^{k,p}(\mathbb{R}^2) \) of \( W^{k,p}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \).

We will set \( L^{2,s} \) and \( H^{m,s} \)

\[
\|u\|_{L^{2,s}} = \|(x)^s u\|_{L^2} \quad \text{and} \quad \|u\|_{H^{m,s}} = \|(x)^s u\|_{H^m},
\]

where \( m \in \mathbb{N}, s \in \mathbb{R} \) and \( \langle x \rangle = (1 + |x|^2)^{1/2} \). For \( f(x) \) and \( g(x) \) column vectors, their inner product is \( \langle f, g \rangle = \int_{\mathbb{R}^2} f(x) \cdot g(x) \, dx \). The adjoint \( H^* \) is defined by \( \langle Hf, g \rangle = \langle f, H^*g \rangle \). Given an operator \( H \), its resolvent is \( R_H(z) = (H - z)^{-1} \). We will write \( R_0(z) = (-\Delta - z)^{-1} \). We write \( \|g(t,x)\|_{L^p} = \|g(t,x)\|_{L^p} \) and \( \|g(t,x)\|_{L^p} = \|g(t,x)\|_{L^p} \).

2. Linearization, modulation and set up

We will use the following classical result, [38,13,14], see also [7]:

**Theorem 2.1.** Suppose that \( e^{it\phi_\omega(x)} \) satisfies (H4). Then \( \exists \epsilon > 0 \) and \( A_0(\omega) > 0 \) such that for any \( \|u(0,x) - \phi_\omega\|_{H^1} < \epsilon \) we have for the corresponding solution \( \inf \|u(t,x) - e^{it\phi_\omega(x - x_0)}\|_{H^1} \gamma (r) \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^2 \) < \( A_0(\omega)\) \( \epsilon \).

We can write the ansatz \( u(t,x) = e^{i\Theta(t)}(\phi_{\omega(t)}(x) + r(t,x)), \Theta(t) = \int_0^t \omega(s) \, ds + \gamma(t) \), Inserting the ansatz into the equation we get

\[
ir_t = -\Delta r + \omega(t) \gamma(t) r - \beta'(\phi_{\omega(t)}^2) \phi_{\omega(t)}^2 \gamma(t) r - \beta'(\phi_{\omega(t)}^2) \phi_{\omega(t)}^2 \gamma(t) r + \gamma(t) \phi_{\omega(t)}
\]

\[
i \omega(t) \, \partial_{\omega(t)} \phi_{\omega(t)} + \gamma(t) r + O(r^2).
\]

We set \( \gamma = \gamma(t) \), \( \gamma = \gamma(t, \omega, \phi_\omega) \), and we rewrite the above equation as

\[
i R_t = H_\omega + \sigma_3 \gamma R + \sigma_3 \gamma R - i \gamma \partial_{\omega(t)} \phi_{\omega(t)} + O(\gamma^2).
\]

Set \( H_0(\omega) = \sigma_3(-\Delta + \omega) \) and \( V(\omega) = H_\omega - H_0(\omega) \). The essential spectrum is

\[
\sigma_e = \sigma_e(H_0(\omega)) = \sigma_e(H_0(\omega)) = (-\infty, -\omega] \cup [\omega, +\infty),
\]

0 is an isolated eigenvalue. Given an operator \( L \) we set \( N_g(L) = \bigcup_{j \geq 1} \ker(L^j) \). [37] implies that, if \{ \} means span, \( N_g(H^*_\omega) = \{ \Phi, \sigma_3 \phi_{\omega(t)} \} \), \( \lambda(\omega) \) has corresponding real eigenvector \( \xi(\omega) \), which can be normalized so that \( \langle \xi, \sigma_3 \xi \rangle = 1 \), \( \sigma_3 \xi(\omega) \) generates \( \ker(H_\omega + \lambda(\omega)) \). The function \( (\omega, x) \in \mathcal{O} \times \mathbb{R} \to \xi(\omega, x) \) is \( C^2; |\xi(\omega, x)| < ce^{-\omega|x|^2} \) for fixed \( c > 0 \) and \( a > 0 \) if \( \omega \in K \subset \mathcal{O}, \) \( K \) compact. \( \xi(\omega, x) \) is even in \( x \) since by assumption we are restricting ourselves in the category of such functions. We have the \( H_\omega \), invariant Jordan block decomposition

\[
L^2 = N_g(H_\omega) \oplus \left( \bigoplus_{j \geq 1} \ker(H_\omega + \lambda(\omega)) \right) \oplus L^2_c(H_\omega) = N_g(H_\omega) \oplus N_g(H_\omega)
\]
where we set \( L^2_c(H_\omega) = \{ N_\epsilon (H^s_\omega) \oplus \bigoplus_{\pm} \ker (H^s_\omega \mp \lambda (\omega)) \}^{-1} \). We can impose

\[
R(t) = (\dot{z}\xi + \bar{z}\sigma_1 \dot{\xi} + f(t)) \in \left[ \sum_{\pm} \ker (H_{\omega(t)} \mp \lambda (\omega(t))) \right] \oplus L^2_c(H_{\omega(t)}).
\]  

(2.2)

The following claim admits an elementary proof which we skip:

**Lemma 2.2.** There is a Taylor expansion at \( R = 0 \) of the nonlinearity \( O(R^2) \) in (2.1) with \( R_{m,n}(\omega, x) \) and \( A_{m,n}(\omega, x) \) real vectors and matrices rapidly decreasing in \( x \):

\[
O(R^2) = \sum_{2 \leq m+n \leq 2N+1} R_{m,n}(\omega) z^m \bar{z}^n + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n A_{m,n}(\omega) f + O(f^2 + |z|^{2N+2}).
\]

In terms of the frame in (2.2) and the expansion in Lemma 2.2, (2.1) becomes

\[
i f = (H_{\omega(t)} + \sigma_3 \dot{\gamma}) f + \sigma_3 \dot{\gamma} \Phi (\omega) - i \dot{\omega} \partial_\omega \Phi (t) + \left( \dot{z}\lambda (\omega) - i \bar{z}\right) \dot{\xi} (\omega)
- \left( \bar{z}\lambda (\omega) + i \bar{z}\right) \sigma_1 \dot{\xi} (\omega) + \sigma_3 \dot{\gamma} \left( \dot{z}\xi + \bar{z}\sigma_1 \dot{\xi} \right) - i \dot{\omega} (z\partial_\omega \xi + \bar{z}\sigma_1 \partial_\omega \xi)
+ \sum_{2 \leq m+n \leq 2N+1} z^m \bar{z}^n R_{m,n}(\omega) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n A_{m,n}(\omega) f + O(f^2) + O_{\text{loc}} (|z|^{2N+2}).
\]

(2.3)

where by \( O_{\text{loc}} \) we mean that the there is a factor \( \chi(x) \) rapidly decaying to 0 as \( |x| \to \infty \). By taking inner product of the equation with generators of \( N_\epsilon (H^s_\omega) \) and \( \ker (H^s_\omega - \lambda) \) we obtain modulation and discrete modes equations:

\[
i \dot{\omega} \frac{d}{d\omega} \| \phi_\omega \|^2 = \left( \sigma_3 \dot{\gamma} (z\xi + \bar{z}\sigma_1 \dot{\xi}) - i \dot{\omega} (z\partial_\omega \xi + \bar{z}\sigma_1 \partial_\omega \xi) + \sum_{m+n=2} \bar{z}^n R_{m,n}(\omega)
+ \left( \sigma_3 \dot{\gamma} + i \dot{\omega} \partial_\omega P_c + \sum_{m+n=1} \bar{z}^n A_{m,n}(\omega) \right) f + O(f^2) + O_{\text{loc}} (|z|^{2N+2}) \right), \Phi.
\]

(4.4)

3. Spacetime estimates for \( H_\omega \)

We need analogues of Lemmas 2.1–2.3 and Corollary 2.1 in [22]. We call admissible all pairs \((p, q)\) with \(1/p = 1/2 - 1/q\) and \(2 \leq q < \infty\). We set \((p', q') = (p/(p - 1), q/(q - 1))\). In the lemmas below we assume that the \( H_\omega \) of the form (1.2) for which hypotheses (H3)–(H5), (H7) and (H9) hold.

**Lemma 3.1** *(Strichartz estimate).* There exists a positive number \( C = C(\omega) \) upper semicontinuous in \( \omega \) such that for any \( k \in \{0, 2\} \):

(a) for any \( f \in L^2_c(\omega) \) and any admissible all pairs \((p, q)\),

\[
\| e^{-i t H_\omega} f \|_{L^p \dot{W}^{k, q}} \leq C \| f \|_{H^k};
\]

(b) for any \( g(t, x) \in S(\mathbb{R}^2) \) and any couple of admissible pairs \((p_1, q_1) (p_2, q_2)\) we have

\[
\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) \, ds \right\|_{L^p \dot{W}^{k, q_1}} \leq C \| g \|_{L^{p_1} \dot{W}^{k_1, q_1}}.
\]

Lemma 3.1 follows immediately from Proposition 1.2 since \( W \) and \( Z \) intertwine \( e^{-it H_\omega} P_c(H_\omega) \) and \( e^{-it H_0} \).
Lemma 3.2. Let $s > 1$. \( \exists C = C(\omega) \) upper semicontinuous in $\omega$ such that:

(a) for any $f \in S(\mathbb{R}^2)$,
\[
\left\| e^{-itH_\omega} P_c(\omega) f \right\|_{L^2_{tL^2_x}} \leq C \| f \|_{L^2_x};
\]

(b) for any $g(t, x) \in S(\mathbb{R}^2)$
\[
\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) \, ds \right\|_{L^2_{t} L^2_{x}} \leq C \| g \|_{L^2_{t} L^2_{x}}.
\]

Notice that (b) follows from (a) by duality.

Lemma 3.3. Let $s > 1$. \( \exists C = C(\omega) \) as above such that for all $g(t, x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$:
\[
\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) \, ds \right\|_{L^2_{t} L^2_{x}} \leq C \| g \|_{L^2_{t} L^2_{x}}.
\]

As a corollary from Christ and Kiselev [6], Lemmas 3.2 and 3.3 imply:

Lemma 3.4. Let $(p, q)$ be an admissible pair and let $s > 1$. \( \exists C = C(\omega) \) as above such that for all $g(t, x) \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$:
\[
\left\| \int_0^t e^{-i(t-s)H_\omega} P(\omega) g(s, \cdot) \, ds \right\|_{L^p_t L^q_x} \leq C \| g \|_{L^2_{t} L^2_{x}},
\]

Lemma 3.5. Consider the diagonal matrices $E_+ = \text{diag}(1, 0)$, $E_- = \text{diag}(0, 1)$. Set $P_\pm(\omega) = Z(\omega) E_\pm W(\omega)$ with $Z(\omega)$ and $W(\omega)$ the wave operators associated to $H_\omega$. Then we have for $u \in L^2_c(H_\omega)$
\[
P_+(\omega) u = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_{-M}^{M} \left[ R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon) \right] u \, d\lambda,
\]
\[
P_-(-\omega) u = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \lim_{M \to +\infty} \int_{-M}^{-\omega} \left[ R_{H_\omega}(\lambda + i\epsilon) - R_{H_\omega}(\lambda - i\epsilon) \right] u \, d\lambda.
\]

and for any $s_1$ and $s_2$ and for $C = C(s_1, s_2, \omega)$ upper semicontinuous in $\omega$, we have
\[
\left\| (P_+(\omega) - P_-(-\omega) - P_c(\omega)\sigma_3) f \right\|_{L^2_{t} L^2_{x}} \leq C \| f \|_{L^2_{t} L^2_{x}}.
\]

Proof. Formulas (1) hold with $P_\pm(\omega)$ replaced by $E_\pm$ and $H_\omega$ replaced by $H_0$ and for any $u \in L^2(\mathbb{R}^2)$. Applying $W(\omega)$ we get (1) for $H_\omega$. Estimate (2) follows by the proof of inequality (3) in Lemma 5.12 [7] which is valid for all dimensions. \( \square \)

4. Proof of Theorem 1.1

We restate Theorem 1.1 in a more precise form:

Theorem 4.1. Under the assumptions of Theorem 1.1 we can express
\[
u(t, x) = e^{i\Theta(t)} \left( \phi_{\omega(t)}(x) + \sum_{j=1}^{2N} p_j(z, \bar{z}) A_j(x, \omega(t)) + h(t, x) \right)
\]
with \( p_j(z, \bar{z}) = O(z) \) near 0, with \( \lim_{t \to +\infty} \omega(t) \) convergent, with \( |A_j(x, \omega(t))| \leq C e^{-a|x|} \) for fixed \( C > 0 \) and \( a > 0 \), \( \lim_{t \to +\infty} \omega(t) = 0 \), and for fixed \( C > 0 \)

\[
\|z(t)\|_{L^2(t)}^{N+1} + \|h(t, x)\|_{L^2(t)}^{H^1} \leq C. 
\]

Furthermore, there exists \( h_\infty \in H^1(\mathbb{R}, \mathbb{C}) \) such that

\[
\lim_{t \to \infty} e^{it} \omega(x) dS + i\gamma(t) h(t) - e^{it} h_\infty \|_{H^1} = 0. 
\]

The proof of Theorem 4.1 consists in a normal forms expansion and in the closure of some nonlinear estimates. The normal forms expansion is exactly the same of [10,7], in turn adaptations of [12].

4.1. Normal form expansion

We repeat [10]. We pick \( k = 1, 2, \ldots, N \) and set \( f = f_k \) for \( k = 1 \). The other \( f_k \) are defined below. In the ODE’s there will be error terms of the form

\[
E_{\text{ODE}}(k) = O(|z|^2 N + 2) + O(z^N + f_k) + O(f_k^2) + O(\beta(|f_k|^2) f_k). 
\]

In the PDE’s there will be error terms of the form

\[
E_{\text{PDE}}(k) = O_{\text{loc}}(|z|^2 N + 2) + O_{\text{loc}}(z f_k) + O_{\text{loc}}(f_k^2) + O(\beta(|f_k|^2) f_k). 
\]

In the right-hand sides of Eqs. (2.3)–(2.4) we substitute \( \dot{\gamma} \) and \( \dot{\omega} \) using the modulation equations. We repeat the procedure a sufficient number of times until we can write for \( k = 1 \) and \( f_1 = f \)

\[
i \hat{\omega} \frac{d|\phi_0|^2}{d\omega} = \left(\sum_{m+n=2}^{2N+1} z^m \bar{z}^n A_{m,n}(k)(\omega) + \sum_{m+n=1}^{N} z^m \bar{z}^n A_{m,n}(k)(\omega) f_k + E_{\text{ODE}}(k), \Phi(\omega)\right),
\]

\[
i \dot{z} - \lambda z = \{\text{same as above} , \sigma_3 \xi(\omega)\},
\]

\[
i \hat{\gamma} f_k = (H_\omega + \sigma_3 \dot{\gamma}) f_k + E_{\text{PDE}}(k) + \sum_{k+1 \leq m+n \leq N+1} z^m \bar{z}^n R_{m,n}(k)(\omega),
\]

with \( A_{m,n}(k), R_{m,n}(k) \) and \( A_{m,n}(\omega, x) \) real exponentially decreasing to 0 for \( |x| \to \infty \) and continuous in \( (\omega, x) \). Exploiting \( |(m-n)\lambda(\omega)| < \omega \) for \( m + n \leq N, m \geq 0, n \geq 0 \), we define inductively \( f_k \) with \( k \leq N \) by

\[
f_{k-1} = - \sum_{m+n=k} z^m \bar{z}^n R_{m,n}(m-n) \lambda(\omega) R_{m,n-1}^{(m-1)}(\omega) + f_k. 
\]

Notice that if \( R_{m,n-1}^{(m-1)}(\omega, x) \) is real exponentially decreasing to 0 for \( |x| \to \infty \), the same is true for \( R_{m,n}(m-n) \lambda(\omega) R_{m,n-1}^{(m-1)}(\omega) \) by \( |(m-n)\lambda(\omega)| < \omega \). By induction \( f_k \) solves the above equation with the above notifications. Now we manipulate the equation for \( f_{N} \). We fix \( \omega_1 = \omega_0 \). We write

\[
i \hat{\gamma} P_c(\omega_1) f_N - \left\{H_{\omega_1} + (\dot{\gamma} + \omega - \omega_1)(P_+ (\omega_1) - P_-(\omega_1))\right\} P_c(\omega_1) f_N
\]

\[
= + P_c(\omega_1) \tilde{E}_{\text{PDE}}(N) + \sum_{m+n=N+1} z^m \bar{z}^n P_c(\omega_1) R_{m,n}(N)(\omega), 
\]

where we split \( P_c(\omega_1) = P_+ (\omega_1) + P_- (\omega_1) \) with \( P_\pm (\omega_1) \), see Lemma 3.5, where \( P_\pm (\omega_1) \) are the projections in \( \sigma_\epsilon (H_{\omega_1}) \cap \{\lambda: \pm \lambda \geq \omega_1\} \) and with

\[
\tilde{E}_{\text{PDE}}(N) = E_{\text{PDE}}(N) + \sum_{m+n=N+1} z^m \bar{z}^n (R_{m,n}(N)(\omega) - R_{m,n}(N)(\omega_1)) + \varphi(t, x) f_N, 
\]

\[
\varphi(t, x) := (\dot{\gamma} + \omega - \omega_1)(P_+(\omega_1)\sigma_3 - \left(P_+(\omega_1) - P_-(\omega_1)\right)) f_N + (V(\omega) - V(\omega_1)) f_N
\]

\[
+ (\dot{\gamma} + \omega - \omega_1)(P_c(\omega) - P_c(\omega_1))\sigma_3 f_N. 
\]
By Lemma 3.5 for $C_N(\omega_1)$ upper semicontinuous in $\omega_0$, $\forall N$ we have
\[
\| (x)^N (P_+ (\omega_1) - P_- (\omega_1) - P_c (\omega_1) \sigma_3) f \|_{L^2} \leq C_N(\omega_1) \| (x)^{-N} f \|_{L^2}.
\] (4.3)
The term $\varphi(t, x)$ in (4.2) can be treated as a small cutoff function. We write
\[
f_N = - \sum_{m+n=N+1} z^m \bar{z}^n R_{H_{\omega_1}} ((m-n) \lambda_1 (\omega_1) + i0) P_c (\omega_1) R_{m,n}^{(N)} (\omega_1) + f_{N+1}.
\] (4.4)
Then
\[
i \partial_t P_c (\omega_1) f_{N+1} = (P_{\omega_1} + (\dot{\gamma} + \omega - \omega_1) (P_+ (\omega_1) - P_- (\omega_1))) P_c (\omega_1) f_{N+1} + \sum_{\pm} O(\epsilon |z|^{N+1}) R_{H_{\omega_1}} ((\pm(N+1) \lambda (\omega_1) + i0) R_{\pm} (\omega_1) + P_c (\omega_1) \tilde{E}_{PDE}(N)
\] (4.5)
with $R_+ = R_{N+1,0}^{(N)}$ and $R_- = R_{0,N+1}^{(N)}$ and $\tilde{E}_{PDE}(N) = \tilde{E}_{PDE}(N) + O_{\text{loc}} (\epsilon z^{N+1})$, where we have used that $(\omega - \omega_1) = O(\epsilon)$ by Theorem 2.1. Notice that $R_{H_{\omega_1}} ((\pm(N+1) \lambda (\omega_1) + i0) R_{\pm} (\omega_1)) \in L^\infty$ do not decay spatially. In the ODE’s with $k = N$, by the standard theory of normal forms and following the idea in Proposition 4.1 [5], see [10] for details, it is possible to introduce new unknowns
\[
\tilde{\omega} = \omega + q(\omega, z, \tilde{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \{ f_N, a_{mn}(\omega) \},
\]
\[
\tilde{z} = z + p(\omega, z, \tilde{z}) + \sum_{1 \leq m+n \leq N} z^m \bar{z}^n \{ f_N, b_{mn}(\omega) \},
\] (4.6)
with $p(\omega, z, \tilde{z}) = \sum p_{m,n}(\omega) z^m \bar{z}^n$ and $q(z, \tilde{z}) = \sum q_{m,n}(\omega) z^m \bar{z}^n$ polynomials in $(z, \tilde{z})$ with real coefficients and $O(|z|^2)$ near 0 such that we get
\[
i \dot{\tilde{\omega}} = \{ E_{PDE}(N), \Phi \},
\]
\[
i \dot{\tilde{z}} - \lambda (\omega) \tilde{z} = \sum_{1 \leq m \leq N} a_m(\omega) |z|^2 \tilde{z} + \{ E_{ODE}(N), \sigma_3 \xi \},
\] (4.7)
with $a_m(\omega)$ real. Next step is to substitute $f_N$ using (4.4). After eliminating by a new change of variables $\tilde{z} = \tilde{z} + p(\omega, \tilde{z}, \tilde{\omega})$ the resonant terms, with $p(\omega, \tilde{z}, \tilde{\omega}) = \sum p_{m,n}(\omega) z^m \bar{z}^n$ a polynomial in $(z, \tilde{z})$ with real coefficients $O(|z|^2)$ near 0, we get
\[
i \dot{\tilde{\omega}} = \{ E_{PDE}(N), \Phi \},
\]
\[
i \dot{\tilde{z}} - \lambda (\omega) \tilde{z} = \sum_{1 \leq m \leq N} \hat{a}_m(\omega) |z|^2 \tilde{z} + \{ E_{ODE}(N), \sigma_3 \xi \}
\]
\[
- |z|^2 \{ \hat{A}_{0,N}^{(N)} (\omega) R_{H_{\omega_1}} ((N+1) \lambda (\omega_1) + i0) P_c (\omega_0) R_{N+1,0}^{(N)} (\omega_1), \sigma_3 \xi (\omega) \}
\]
\[
+ |z|^2 \{ \hat{A}_{0,N}^{(N)} (\omega) f_{N+1}, \sigma_3 \xi (\omega) \}
\] (4.8)
with $\hat{a}_m$, $\hat{A}_{0,N}^{(N)}$ and $R_{N+1,0}^{(N)}$ real. By $\frac{1}{x-i0} = PV \frac{1}{x} + i \pi \delta_0 (x)$ and by an elementary use of the wave operators, we can denote by $\Gamma (\omega, \omega_1)$ the quantity
\[
\Gamma (\omega, \omega_1) = 3 (|\hat{A}_{0,N}^{(N)} (\omega) R_{H_{\omega_1}} ((N+1) \lambda (\omega_1) + i0) P_c (\omega_1) R_{N+1,0}^{(N)} (\omega_1), \sigma_3 \xi (\omega) |)
\]
\[
= \pi (|\hat{A}_{0,N}^{(N)} (\omega) \delta (H_{\omega_1} - (N+1) \lambda (\omega_1)) P_c (\omega_1) R_{N+1,0}^{(N)} (\omega_1), \sigma_3 \xi (\omega) |).
\]
Now we assume the following:

**Hypothesis 4.2.** There is a fixed constant $\Gamma > 0$ such that $|\Gamma (\omega, \omega)| > \Gamma$.

By continuity and by Hypothesis 4.2 we can assume $|\Gamma (\omega, \omega_1)| > \Gamma / 2$. Then we write
\[
\frac{d}{dt} |z|^2 = -\Gamma (\omega, \omega_1) |z|^{2N+2} + 3 (|\hat{A}_{0,N}^{(N)} (\omega) f_{N+1}, \sigma_3 \xi (\omega) | z^{N+1} ) + 3 (|E_{ODE}(N), \sigma_3 \xi (\omega) | \tilde{z}).
\] (4.9)
4.2. Nonlinear estimates

By an elementary continuation argument, the following a priori estimates imply inequality (1) in Theorem 4.1, so to prove (1) we focus on:

Lemma 4.4. There are fixed constants $C_0$ and $C_1$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ if we have

$$
\|\tilde{\psi}\|_{L_t^{N+1}L_x^{2N+2}} \leq 2C_0 \epsilon \quad \text{and} \quad \|f\|_{L_t^\infty H^1 \cap L_x^3 W^{1,6} \cap L_x^{2p_0} W^{1,2p_0} \cap L_x^2 H^{1-s}} \leq 2C_1 \epsilon
$$

(4.10)

then we obtain the improved inequalities

$$
\|f\|_{L_t^\infty H^1 \cap L_x^3 W^{1,6} \cap L_x^{2p_0} W^{1,2p_0} \cap L_x^2 H^{1-s}} \leq C_1 \epsilon,
$$

(4.11)

$$
\|\tilde{\psi}\|_{L_t^{N+1}L_x^{2N+2}} \leq C_0 \epsilon.
$$

(4.12)

Proof. Set $\ell(t) := \gamma + \omega - \omega_1$. First of all, we have:

Lemma 4.5. Let $g(0, x) \in H^1_0 \cap L^2_0(\omega_1)$ and let $\omega(t)$ be a continuous function. Consider $ig_x = (H_{\omega_1} + \ell(t)(P_+ (\omega_0) - P_- (\omega_0)))g + P_+ (\omega_1) F$. Then for a fixed $C = C(\omega_1, s)$ upper semicontinuous in $\omega_1$ and $s > 1$ we have

$$
\|g\|_{L_t^\infty H^1 \cap L_x^3 W^{1,6} \cap L_x^{2p_0} W^{1,2p_0}} \leq C(\|g(0, x)\|_{H^1} + \|F\|_{L_t^1 H^1 + L_t^2 H^1,s}).
$$

Lemma 4.4 follows easily from Lemmas 3.1–3.4 and

$$
P_{\pm}(\omega_1)g(t) = e^{-itH_{\omega_1}} e^{-i\int_0^t \ell(\tau) d\tau} P_{\pm}(\omega_1) g(0) - i \int_0^t e^{-i(t-\tau)H_{\omega_1}} e^{i\int_0^\tau \ell(\tau) d\tau} P_{\pm}(\omega_1) F(s) ds.
$$

Lemma 4.5. Consider Eq. (4.1) for $f_N$ and assume (4.10). Then we can split $\tilde{\psi}_{\text{PDE}}(N) = X + O(f^3_N)$ and $O(f^3_N) \leq \epsilon^3$.

Proof of Lemma 4.5. In the error terms for $k = N$ at the beginning of Section 4.1 we can write

$$
\tilde{\psi}_{\text{PDE}}(N) = O(\epsilon) \psi(x) f_N + O_{\text{loc}}(|z|^{N+2}) + O_{\text{loc}}(zf_N) + O_{\text{loc}}(f^2_N) + O(f^3_N)
$$

with $\psi(x)$ a rapidly decreasing function, $p_0$ the exponent in (H2) and with $O(f^3_N)$ relevant only for $p_0 > 3$. Denoting $X$ the sum of all terms except the last one, setting $f = f_N$, by (4.10) we have:

(1) $\|O(\epsilon) \psi(x) f\|_{L_t^2 H^1_x} \lesssim \epsilon \|f\|_{L_t^2 H^1_x} \lesssim \epsilon^2$;

(2) $\|O_{\text{loc}}(zf)\|_{L_t^2 H^1_x} \lesssim \|z\|_{L_t^\infty} \|f\|_{L_t^2 H^1_x} \lesssim \epsilon^2$;

(3) $\|O_{\text{loc}}(f^2)\|_{L_t^2 H^1_x} \lesssim \|f\|_{L_t^2 H^1_x}^2 \lesssim \epsilon^2$.

This yields $\|X\|_{L_t^2 H^1_x} \lesssim \epsilon^2$. To bound the remaining term observe:

(4) $\|f\|_{L_t^2 H^1_x} \lesssim \|f\|_{L_t^3 W^{1,6}_x} \lesssim \epsilon^3$;

(5) $\|O(f^{p_0})\|_{L_t^2 H^1_x} \lesssim \|f\|_{L_t^{p_0} W^{1,2p_0}_x} \|f\|_{L_t^{p_0-1} W^{1,2p_0}_x} \lesssim \epsilon^p_0$, where in the last step we use $\|f\|_{L_t^{p_0-1} W^{1,2p_0}_x} \lesssim \|f\|_{L_t^{p_0} W^{1,2p_0}_x}^{1-\alpha} \|f\|_{L_t^\infty}^{\alpha} \lesssim \epsilon^{p_0}$, for some $0 < \alpha < 1$ by $p_0 > 3$, interpolation and Sobolev embedding. □
Proof of (4.11). Recall that \( f_N \) satisfies Eq. (4.1) whose right-hand side is \( P_c(\omega_1) \tilde{E}_{\text{PDE}}(N) + O_{\text{loc}}(z^{N+1}) \). In addition to Lemma 4.5 we have the estimate \( \|O_{\text{loc}}(z^{N+1})\|_{L^1_t H^{1.1}_x} \lesssim \|z\|_{L^1_t H^{N+1}} \lesssim 2C_0\varepsilon \). So by Lemmas 3.1–3.4, for some fixed \( c_2 \) we get schematically

\[
\|f_N\|_{L^1_{\infty} H^1_t \cap L^2_{\tilde{t}0} W^{1,6}_x \cap L^2_{\tilde{t}0} W^{1,2p_0}_x} \lesssim 2c_2C_0\varepsilon + \varepsilon + O(\varepsilon^2)
\]

where \( \varepsilon \) comes from initial data, \( O(\varepsilon^2) \) from all the nonlinear terms save for the \( R^{(N)}_{m,n}(\omega_0)z^m\bar{z}^n \) terms which contribute the \( 2c_2C_0\varepsilon \). Let now \( f_N = g + h \) with

\[
i g_t = \{ H_{\omega_1} + \ell(t) (P_+ - P_-) \} g + X + O_{\text{loc}}(z^{N+1}), \quad g(0) = f_N(0),
\]
\[
i h_t = \{ H_{\omega_1} + \ell(t) (P_+ - P_-) \} h + O(f^3_N) + O(f^{p_0}_N), \quad h(0) = 0
\]

in the notation of Lemma 4.5. Then, by Lemmas 3.2 and 3.3 and by the estimates in Lemma 4.5 we get \( \|g\|_{L^1_t H^{1.1}_x} \lesssim 2C_0\varepsilon + O(\varepsilon^2) + c_0\varepsilon \) for a fixed \( c_0 \). Finally,

\[
\int_0^\infty \|e^{-i(t-s)H_{\omega_1}} e^{\pm i \int_0^t \ell(\tau) d\tau} (O(f^3_N) + O(f^{p_0}_N))(s)\|_{L^1_t H^{1.1}_x} \lesssim \int_0^\infty \|O(f^3_N) + O(f^{p_0}_N))(s)\|_{H^{1.1}_x} \lesssim \varepsilon^3.
\]

So if we set \( C_1 \approx 2C_0 + c_0 + 1 \) we obtain (4.11). We need to bound \( C_0 \).

Proof of (4.12): We first need:

**Lemma 4.6.** We can decompose \( f_{N+1} = h_1 + h_2 + h_3 + h_4 \) with a fixed large \( M > 0 \):

1. \( \|h_1\|_{L^2_t L^2_{x,M}} \lesssim O(\varepsilon^2) \);
2. \( \|h_2\|_{L^2_t L^2_{x,M}} \lesssim O(\varepsilon^2) \);
3. \( \|h_3\|_{L^2_t L^2_{x,M}} \lesssim O(\varepsilon^2) \);
4. \( \|h_4\|_{L^2_t L^2_{x,M}} \lesssim c(\omega_1)\varepsilon \) for a fixed \( c(\omega_1) \) upper semicontinuous in \( \omega_1 \).

**Proof of Lemma 4.6.** We set

\[
i \partial_t h_1 = (H_{\omega_1} + \ell(t)(P_+ - P_-))h_1,
\]
\[
h_1(0) = \sum_{m+n=N+1} R_{\omega_1}((m-n)\lambda(\omega_1) + i0) R^{(N)}_{m,n}(\omega_1)z^m(0)\bar{z}^n(0).
\]

We get \( \|h_1\|_{L^2_t L^2_{x,M}} \lesssim c(\omega_1)|z(0)|^2 \sum \|R^{(N)}_{m,n}(\omega_1)\|_{L^2_{x,M}} = O(\varepsilon^2) \) by the following lemma:

**Lemma 4.7.** There is a fixed \( s_0 \) such that for \( s > s_0 \),

\[
\|e^{-iH_{\omega_1} t} R_{\omega_1}(A + i0) P_c(\omega) \varphi\|_{L^2_{x} L^2_{t,-s}} < C_s(\Lambda, \omega) \|\varphi(x)\|_{L^2_x},
\]
\[
\left\| \int_0^t e^{-iH_{\omega_1}(t-\tau)} R_{\omega_1}(A + i0) P_c(\omega) g(\tau) d\tau \right\|_{L^2_{x} L^2_{t,-s}} < C_s(\Lambda, \omega) \|g(t,x)\|_{L^2_{x} L^2_{t}}
\]

with \( C_s(\Lambda, \omega) \) upper semicontinuous in \( \omega \) and in \( \Lambda > \omega \).

Let us assume Lemma 4.7 for the moment, for the proof see Section 9. We set \( h_2(0) = 0 \) and

\[
i \partial_t h_2 = (H_{\omega_1} + \ell(t)(P_+ - P_-))h_2 + O(\varepsilon z^{N+1}) R_{\omega_1}((N + 1)\lambda(\omega_1) + i0) R^{(N)}_{N+1,0}(\omega_0)
\]
\[
+ O(\varepsilon z^{N+1}) R_{\omega_1}(-(N + 1)\lambda(\omega_1) + i0) R^{(N)}_{0,N+1}(\omega_1).
\]
Then we have $h_2 = h_{21} + h_{22}$ with $h_{2j} = \sum_\pm h_{2j\pm}$ with

$$h_{21\pm}(t) = \int_0^t e^{-iH_{\omega_1}(t-s)} e^{\pm i t^j \ell(t)} e^{\ell(t) dt} P_\pm O(\epsilon z^{N+1}) R_{H_{\omega_1}}((N+1)\lambda(\omega_1) + i0) R^{(N)}_{N, 0, N+1}(\omega_1) ds$$

and $h_{22\pm}$ defined similarly but with $R_{H_{\omega_0}}(-(N+1)\lambda(\omega_1) + i0) R^{(N)}_{0, N+1}$. Now by (4.13) we get

$$\|h_{2\pm}(t)\|_{L^2_t L^2_x}^{2N+2} \leq C \epsilon \|z\|_{H^1_t L^2_x}^{N+1} + o(\epsilon^2).$$

Continuation of proof of Lemma 4.3. We integrate (4.9) in time. Then by Theorem 2.1 and by Lemma 4.4 we get, for $A_0$ an upper bound of the constants $A_0(\omega)$ of Theorem 2.1,

$$\|\hat{z}\|_{L^2_t L^2_x}^{2N+2} \leq A_0 \epsilon^2 + \epsilon \|\hat{z}\|_{H^1_t L^2_x}^{N+1} + o(\epsilon^2).$$

Then we can pick $C_0 = (A_0 + 1)$ and this proves that (4.10) implies (4.12). Furthermore $\hat{z}(t) \to 0$ by $\frac{d}{dt} \hat{z}(t) = O(\epsilon)$. □

As in [10, 7] in the above argument we did not use the sign of $\Gamma(\omega, \omega_0)$. With the same argument in [10, 7] one can prove

**Corollary 4.8.** If Hypothesis 4.2 holds, then $\Gamma(\omega, \omega) > \Gamma$.

The proof that, for $\int h_N(t) = (h(t), \bar{h}(t))$, $h(t)$ is asymptotically free for $t \to \infty$, is similar to the analogous one in [10] and we skip it.

5. Limiting absorption principle and $L^2$ theory for $H_\omega$

In Sections 5–7 we prove Proposition 1.2. We start emphasizing two consequences of hypothesis (H9), in particular (b) clarifies the absence of resonance at $\pm \omega$:

(a) $H_\omega$ has no eigenvalues in $[\omega, +\infty) \cup (-\infty, -\omega]$;

(b) if $g \in W^{2, \infty}(\mathbb{R}^2, \mathbb{C}^2)$ satisfies $H_\omega g = og$ or $H_\omega g = -og$ then $g = 0$.

Because of the fact that $H_\omega$ is not a symmetric operator, we need some preparatory work to show that in fact $H_\omega$ is diagonalizable in the continuous spectrum. This work is done in Section 5 which ends with a formula for the wave operator $W$ which is the basis to develop in Sections 6 and 7 a transposition of the work of Yajima [40].

We first need a preliminary on Schrödinger operators. We will denote by $q(x)$ a real valued function with: $q(x) \geq 0$ with $q(x) > 0$ at some points; $q(x) \in C^0_0(\mathbb{R}^2)$. We set $h_q = -\Delta + q(x)$. Then we have:

**Lemma 5.1.** Let $\mathbb{C}_+ = \{ z \in \mathbb{C} : \Re z > 0 \}$. Suppose $q(x) = 0$ for $r \geq r_0 > 0$. Then we have the following facts.

1. There exists $s_0 > 0$ and $C_0 > 0$ such that for $s \geq s_0$, $R_{h_q}(z)$ extends into a function $z \to R_{h_q}^+ (z)$ which is in $(L^\infty \cap C^0)(\mathbb{C}_+, B(L^{2-s}, L^{2-s})).$
For any \( n_0 \in \mathbb{N} \) there exists \( s_0 > 0 \) such that for any \( a_0 > 0 \) there is a choice of \( C > 0 \) such that for \( n \leq n_0 \)
\[
\left\| \frac{d^n}{dz^n} R_{h_q} (z) : L^{2,s}(\mathbb{R}^2) \rightarrow L^{2,-s}(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{4}(1+n)} \quad \forall z \in \mathbb{C}_+ \cap \{ z : |z| \geq a_0 \}.
\]

(3) The same argument can be repeated for \( \mathbb{C}_- = \{ z \in \mathbb{C} : \exists \varepsilon < 0 \} \) and \( R_{h_q}^-(z) \).

Claim (2) follows from [1] and [16] and claim (3) follows along the lines of the previous two claims. In view of (2), it is enough to prove (1) for \( z \approx 0 \). For \( z = re^{i\theta} \) with \( \theta \in (-\pi, \pi) \) let \( \sqrt{r} = \sqrt{r}e^{i\theta/2} \). With this convention for \( z \notin [0, \infty) \) for \( R_0(z) = (-\Delta - z)^{-1} \) we have
\[
R_0(z) = \frac{1}{2\pi} K_0(\sqrt{-z}\,|x|) \ast \frac{i}{4} H_0^+(i \sqrt{-z}\,|x|) \ast = -\frac{i}{4} H_0^-(i \sqrt{-z}|x|) \ast
\]
for the Macdonald function \( K_0 \) and the Hankel functions \( H_0^\pm \). We set \( G_0 = -\frac{1}{2\pi} \log |x| \ast \), \( P_0f = \int_{\mathbb{R}^2} f \, dx \). We have for \( M(z) = (1 + \sqrt{q} R_0(z) \sqrt{q}) \)
the identity
\[
R_{h_q} = R_0(z) - R_0(z) \sqrt{q} M^{-1}(z) \sqrt{q} R_0(z).
\]
(4)
From the expansion at 0 in \( \mathbb{C}_+ \) of \( H^+_0 \) and by the argument in Lemma 5 [26] we have in \( B(L^{2,s}, L^{2,-s}) \), for \( s \) sufficiently large,
\[
R_0(z) = c(z) P_0 - G_0 + O(-z \log \sqrt{-z}), \quad c(z) = \frac{i}{4} - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \log(\sqrt{-z}/2).
\]
(5)
Consider the projections in \( L^2(\mathbb{R}^2) \), \( P = \sqrt{q}(\cdot, \sqrt{q})/\|q\|_{L^1} \) and \( Q = 1 - P \). Let \( T = 1 + \sqrt{q} G_0 \sqrt{q} \). Then \( QTQ \) is invertible in \( QL^2(\mathbb{R}^2) \). Denote its inverse in \( QL^2(\mathbb{R}^2) \) by \( D_0 = (QTQ)^{-1} \). Consider the operator in \( L^2 = PL^2 \oplus QL^2 \) defined by
\[
S = \begin{bmatrix}
P & -PTQD_0Q \\
-QD_0Q^TP & QD_0QTPTQD_0Q
\end{bmatrix}
\]
and \( h(z) = \|q\|_{L^1} c(z) + \text{trace}(PQT - PTQD_0QTPTQD_0) \). Then by [26]
\[
R_{h_q}(z) = R_0(z) - h^{-1}(z) R_0(z) \sqrt{q} S \sqrt{q} R_0(z) - R_0(z) \sqrt{q} Q D_0 Q \sqrt{q} R_0(z) - R_0(z) \sqrt{q} O(-z \log \sqrt{-z}) \sqrt{q} R_0(z).
\]
(6)
By direct computation
\[
h^{-1}(z) R_0(z) \sqrt{q} S \sqrt{q} R_0(z) = \frac{c^2(z)}{h(z)} \langle \cdot, 1 \rangle \sqrt{q} S \sqrt{q} \langle \cdot, 1 \rangle + \frac{c(z)}{h(z)} \langle \cdot, 1 \rangle \sqrt{q} S \sqrt{q} G_0 + \frac{c(z)}{h(z)} G_0 \sqrt{q} S \sqrt{q} \langle \cdot, 1 \rangle
\]
\[
+ \frac{c(z)}{h(z)} G_0 \sqrt{q} S \sqrt{q} G_0 + O(-z \log \sqrt{-z}),
\]
where all terms, except the first on the right-hand side, admit continuous extension in \( \mathbb{C}_+ \) at 0. We have \( \langle \cdot, 1 \rangle \sqrt{q} S \sqrt{q} \langle \cdot, 1 \rangle = \|q\|_{L^1} P_0 \) and so by (5)
\[
R_0(z) - \frac{c^2(z)}{h(z)} \|q\|_{L^1} P_0
\]
admits continuous extension in \( \mathbb{C}_+ \) at 0. By direct computation
\[
R_0(z) \sqrt{q} Q D_0 Q \sqrt{q} R_0(z) = G_0 \sqrt{q} Q D_0 Q \sqrt{q} G_0 + O(-z \log \sqrt{-z})
\]
admits continuous extension in \( \mathbb{C}_+ \) at 0. So \( R_{h_q}(z) \) admits continuous extension in \( \mathbb{C}_+ \) at 0, and so on all \( \mathbb{C}_+ \).

A consequence of Lemma 5.1 is the \( h_q \) smoothness in the sense of Kato [19] of multiplication operators involving rapidly decreasing functions \( \psi \):
Lemma 5.2. Let \( \psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,q}(\mathbb{R}^2) \) for \( s \gg 1 \) and \( q \) as in Lemma 5.1. Then the multiplication operator \( \psi \) is \( h_q \) smooth, that is, for a fixed \( C > 0 \)

\[
\int_\mathbb{R} \| \psi R_{h_q}(\lambda + i \varepsilon)u \|_2^2 \, d\lambda < C \| u \|_2^2 \quad \text{for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.
\]

This follows from one of the characterizations of \( H \) smoothness in the case \( H \) is selfadjoint, see Theorem 5.1 [19], specifically from the fact that by Lemma 5.1 we have that for \( \psi \), \( \psi_1 \in L^\infty \cap L^{2,q} \) for \( s \gg 1 \) there is a number \( C > 0 \) such that for all \( z \neq 0 \) we have \( \| \psi_1 R_{h_q}(z) \|_{L^2,L^2} < C \).

We consider now \( H_q = \sigma_0(-\Delta + q + \omega) \) and consider our linearization \( H_{\omega} \). Write \( H_{\omega} = H_q + (V_{\omega} - \sigma_3 q) \), and factorize \( V_{\omega} - \sigma_3 q = B^*A \) with \( A, B \) smooth \( |\partial^\beta A(x)| + |\partial^\beta B(x)| < C e^{-\alpha|x|} \) \( \forall x \), for some \( \alpha, C > 0 \) and for \( |\beta| \leq N_0, N_0 \) sufficiently large. We have \( \sigma_1 H_q = -H_q \sigma_1, \sigma_1 H_{\omega} = -H_{\omega} \sigma_1 \). We choose the factorization \( B^*A \) so that \( \sigma_1 B^* = -B^* \sigma_1, \sigma_1 A = A \sigma_1 \). By these equalities \( \sigma_1 R_{H_q}(z) = -R_{H_q}(-z) \sigma_1 \) and \( \sigma_1 R_{H_{\omega}}(z) = -R_{H_{\omega}}(-z) \sigma_1 \), so in some of the estimates below it is enough to consider \( z \in \mathbb{C}_{++} \) with \( \mathbb{C}_{++} = \{ z : \Re z > 0, \Im z > 0 \} \).

Lemma 5.3. For \( z \in \mathbb{C}_{++} \) the function \( R^+_{H_q}(z) \) is well defined and satisfies the following properties:

1. There exists \( s_0 > 0 \) and \( C_0 > 0 \) such that for \( s \geq s_0 \) the function \( z \mapsto R^+_{H_q}(z) \) is in \( (L^\infty \cap C^0)(\mathbb{C}_{++}, B(L^{2,q}, L^{2,s})). \)
2. For any \( n_0 \in \mathbb{N} \) there exists \( s_0 > 0 \) such that for any \( a_0 > 0 \) there is a choice of \( C > 0 \) such that for \( n \leq n_0 \) and \( \forall z \in \mathbb{C}_{++} \cap \{ z : \dist(z, \pm \omega) \geq a_0 \} \)

\[
\left\| \frac{d^n}{dz^n} R^+_{H_q}(z) : L^{2,q}(\mathbb{R}^2) \rightarrow L^{2,s}(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{2}(1+n)}.
\]

3. For any \( \psi(x) \in L^\infty(\mathbb{R}^2) \cap L^{2,q}(\mathbb{R}^2) \) for \( s \gg 1 \) the multiplication operator \( \psi \) is \( h_q \) smooth, that is, for a fixed \( C > 0 \)

\[
\int_\mathbb{R} \| \psi R_{H_q}(\lambda + i \varepsilon)u \|_2^2 \, d\lambda < C \| u \|_2^2 \quad \text{for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.
\]

4. Analogous statements hold for \( z \in \mathbb{C}_{--} \) and the function \( R^-_{H_q}(z) \).

Lemma 5.4. Fix an exponentially decreasing bounded function \( \psi \). For \( z \in \mathbb{C}_{++} \) the function \( AR^+_{H_q}(z)\psi \) extends into a function \( AR^+_{H_q}(z)\psi \) for \( z \in \mathbb{C}_{++} \cap \sigma_d(H_{\omega}) \) with the following properties:

1. \( \forall a_0 > 0 \exists C_0 > 0 \) such that for \( X_{a_0} = \mathbb{C}_{++} \cap \{ z : \dist(z, \sigma_d(H_{\omega})) \geq a_0 \} \)

\[
AR^+_{H_{\omega}}(z) \psi \in \big( L^\infty \cap C^0 \big)(X_{a_0}, B(L^2, L^2));
\]

2. For any \( n_0 \in \mathbb{N} \) there exists \( s_0 > 0 \) such that for any \( a_0 > 0 \) there is a choice of \( C > 0 \) such that for \( n \leq n_0 \) and \( \forall z \in X_{a_0} \cap \{ z : \dist(z, \pm \omega) \geq a_0 \} \)

\[
\left\| \frac{d^n}{dz^n} AR^+_{H_{\omega}}(z) \psi : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2) \right\| \leq C_0(z)^{-\frac{1}{2}(1+n)}.
\]

3. There is a constant \( C > 0 \) such that

\[
\int \| AR_{H_{\omega}}(\lambda + i \varepsilon)u \|_2^2 \, d\lambda \leq C \| u \|_2^2 \quad \text{for all } u \in L^2(\mathbb{R}^2) \text{ and } \varepsilon \neq 0.
\]

4. Analogous statements hold for \( z \in \mathbb{C}_{--} \) and the function \( R^-_{H_{\omega}}(z) \).
Proof. Let us write $Q_q^+(z) = AR_{H_q}(z)B^*$ and for $z \in \mathbb{C}_+$

$$AR_{H_q}(z) = (1 + Q_q^+(z))^{-1} AR_{H_q}(z).$$

(5)

By Lemma 5.3 we have $\lim_{z \to \infty} \|Q_q^+(z)\|_{L^2,L^2} = 0$. By analytic Fredholm theory $1 + Q_q^+(z)$ is not invertible only at the $z \in \mathbb{C}_+$ where ker$(1 + Q_q^+(z)) \neq 0$. This set has 0 measure in $\mathbb{R}$. By Lemma 2.4 [11] if at some $z \neq \pm \omega$ we have ker$(1 + Q_q^+(z)) \neq 0$, then $z$ is an eigenvalue. By hypothesis there are no eigenvalues in $\sigma_r(H_\omega)$. Hence we get claim (2).

Lemma 5.5. If ker$(1 + Q_q^+(\omega)) \neq 0$ then there exists $g \in W^{2,\infty}(\mathbb{R}^2)$ with $g \neq 0$ such that $H_\omega g = \omega g$.

Let us assume Lemma 5.5. By hypothesis such $g$ does not exist. This yields (1). By (5), claim (4) Lemma 5.4 and Neumann expansion we get (4). Next apply (5) to $u \in L_c(H_\omega)$. $AR_{H_\omega}(z)u$ is an analytic function in $z$ with values in $L^2(\mathbb{R}^2)$ for $z$ near any isolated eigenvalue $z_0$ of $H_\omega$ because the natural projection of $u$ in $N_g(H_\omega - z_0)$ is 0. Away from isolated eigenvalues of $H_\omega$, $(1 + Q_q^+(z))^{-1}$ is uniformly bounded. Hence (3) in Lemma 5.3 implies (3) in Lemma 5.4.

Proof of Lemma 5.5. Let $0 \neq \tilde{g} \in \text{ker}(1 + Q_q^+(\omega))$. Then

$$B^* \tilde{g} + (V_\omega - q)R_{H_q}(\omega)B^* \tilde{g} = 0.$$ 

Set $g = R_{H_q}(\omega)B^* \tilde{g}$. Then $Ag = -\tilde{g}$ and so $g \neq 0$. By $g + R_{H_q}(\omega)(V_\omega - q)g = 0$ we have $g \in H^2_{\text{loc}}(\mathbb{R}^2)$ and $H_\omega g = \omega g$. We want now to show that $g \in L^\infty(\mathbb{R}^2)$, contrary to the hypotheses. We have $g = (g_1, g_2)$ with $g_2 = (\Delta - q - 2\omega)^{-1}(B^* \tilde{g})_2$, where $B^* \tilde{g} \in L^{2,s}(\mathbb{R}^2)$ for any $s$, so $g_2 \in H^2(\mathbb{R}^2)$. We have $g_1 = R_{H_q}^+(0)(B^* \tilde{g})_1$ with $g_1 \in L^{2,-s}(\mathbb{R}^2)$ for sufficiently large $s$. We split $Q^2_s = L^{2,1-s}_r \oplus (L^{1,2-s}_r)^\perp$ where $L^{2,1-s}_r$ are the radial functions and we are considering the standard pairing $L^{2,s}_r \times L^{2,-s}_r \to \mathbb{C}$ given by $\int_{\mathbb{R}^2} f(x)q(x)dx$. We decompose $g_1 = g_{1r} + g_{1nr}$ with $g_{1r} \in L^{2,-i}$ and $g_{1nr} \in (L^{2,-i})^\perp$. In $(L^{2,-i})^\perp \to (L^{2,-s})^\perp$ we have $R_{H_q}^+(0) = G_0 - G_0q(1 + Q_0qQ)^{-1}G_0$ with $Q = 1 - P$, for $P = P_0q_0, q_0 = c_0^{-1}, c_0 = \int_{\mathbb{R}^2} qdx, P_0u = \int_{\mathbb{R}^2} u dx$. Then

$$g_{1nr} = G_0(B^* \tilde{g})_{1nr} - G_0q(1 + Q_0qQ)^{-1}G_0(B^* \tilde{g})_{1nr}$$

and by asymptotic expansion for $|x| \to \infty$ we conclude that for some constants

$$a_x \in g_{1nr} - a = \frac{b_1x_1 + b_2x_2}{|x|^2} = O(|x|^{-1-a+\epsilon})$$

for some $\epsilon > 0$. Finally we look at $g_{1r}$. We can consider solutions $\phi(r)$ and $\psi(r)$ of $h_\omega u = 0$ with: $\phi(0) = 1$ and $\phi_x(0) = 0; \psi(0) = 1$ and $|\psi(r)|$ bounded for $r \geq r_0, \psi(r_0) \approx c \log r$ with $c \neq 0$ for $r \to 0$. In terms of these two functions the kernel of $R_{H_q}^+(0)$ in $L^2((0, \infty), dr)$ is

$$R_{H_q}^+(0)(r_1, r_2) = \begin{cases} \frac{\phi(r_1)\psi(r_2)}{W(r_2)} & \text{if } r_1 < r_2, \\ \frac{\phi(2)\psi(r_1)}{W(r_2)} & \text{if } r_1 > r_2, \end{cases}$$

with $W(r) = [\phi(\cdot), \psi(\cdot)](r) = c/r$ for some $c \neq 0$. We have

$$g_{1r}(r) = c^{-1}\psi(r) \int_0^r \phi(s)(B^* \tilde{g})_{1r}(s)s ds + c^{-1}\phi(r) \int_r^\infty \psi(s)(B^* \tilde{g})_{1r}(s)s ds.$$

Then for $r \geq r_0$,

$$|g_{1r}(r)| \leq |c^{-1}\psi(r) \int_0^r |\phi(t)(B^* \tilde{g})_{1r}(t)|t dt + |c^{-1}\phi(r) \int_r^\infty |\psi(t)(B^* \tilde{g})_{1r}(t)|t dt$$

$$\leq \|\log(x)\|_{L^{2,1-s}(\mathbb{R}^2)} \|B^* \tilde{g}\|_{L^{2,1-s}(\mathbb{R}^2)} + \log(2 + r)\|B^* \tilde{g}\|_{L^{2,1-s}(\mathbb{R}^2)} = O(1).$$
Then we conclude that we have a nonzero $g \in H^2_{\mathrm{loc}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ such that $H_0 g = og$. But this is contrary to the nonresonance hypothesis. \hfill \square

Analogous to Lemma 5.4 is:

**Lemma 5.6.** Fix an exponentially decreasing bounded function $\psi$. For $z \in \mathbb{C}_+$ the function $BR_{H_0}(z)\psi$ extends into a function $BR_{H_0}^+(z)\psi$ for $z \in \mathbb{C}_+ \sigma_d(H_0)$ with the following properties:

1. For any $a_0 > 0$ there exists $C_0 > 0$ such that $BR_{H_0}^+(z)\psi \in L^\infty(X_{a_0}, B(L^2, L^2))$ where

$$X_{a_0} = \overline{C}_+ \cap \{z : \mathrm{dist}(z, \sigma_d(H_0)) \geq a_0\}.$$

2. For any $n_0 \in \mathbb{N}$ there exists $s_0 > 0$ such that for any $a_0 > 0$ there is a choice of $C > 0$ such that for $n \leq n_0$ and $\forall z \in X_{a_0} \cap \{z : \mathrm{dist}(z, \pm \omega) \geq a_0\}$,

$$\left\| \left( \begin{array}{c} BR_{H_0}^+(z)\psi \end{array} \right) \right\|_{L^2(\mathbb{R}^2)} \leq C_0(z)^{-\frac{1}{2}(1+n)}.$$

3. There is a constant $C > 0$ such that

$$\int \|BR_{H_0}(\lambda + i\varepsilon)u\|_2^2 d\lambda \leq C \|u\|_2^2$$

for all $u \in L^2_c(H_0^*)$ and $\varepsilon \neq 0$.

4. Analogous statements hold for $z \in \mathbb{C}_-$ and the function $R_{H_0}^-(z)$.

From [19, Section 2] we conclude:

**Lemma 5.7.** There are isomorphisms $\tilde{W} : L^2 \to L^1_\omega(H_0)$ and $\tilde{Z} : L^1_\omega(H_0) \to L^2$, inverses of each other, defined as follows:

for $u \in L^2$, $v \in L^2_c(H_0^*)$,

$$\langle \tilde{W}u, v \rangle = \langle u, v \rangle + \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_0}(\lambda + i\varepsilon)u, BR_{H_0}(\lambda + i\varepsilon)v \rangle d\lambda;$$

for $u \in L^2_c(H_0)$, $v \in L^2$,

$$\langle \tilde{Z}u, v \rangle = \langle u, v \rangle + \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \langle AR_{H_0}(\lambda + i\varepsilon)u, BR_{H_0}(\lambda + i\varepsilon)v \rangle d\lambda.$$

We have $H_0 \tilde{W} = \tilde{W} H_q$ and $H_q \tilde{Z} = \tilde{Z} H_0$, $e^{itH_0} \tilde{W} = \tilde{W} e^{itH_q}$ and $e^{itH_q} \tilde{Z} = \tilde{Z} e^{itH_0} P_c(H_0)$. The operators $\tilde{W}$ and $\tilde{Z}$ depend continuously on $A$ and $B^+$ and can be expressed as

$$\tilde{W}u = \lim_{t \to +\infty} e^{itH_0} e^{-itH_q} u \quad \text{for any } u \in L^2;$$

$$\tilde{Z}u = \lim_{t \to +\infty} e^{itH_q} e^{-itH_0} u \quad \text{for any } u \in L^2(H_0).$$

In particular we remark:

**Lemma 5.8.** We have for $C(\omega)$ upper semicontinuous in $\omega$ and

$$\|e^{-itH_0}g\|_2 \leq C(\omega)\|g\|_2$$

for any $g \in L^2_c(H_0)$.

Having proved that $e^{-itH_0} P_c(H_0)$ are bounded in $L^2$, we want to relate $H_0$ to $H_0 = \sigma_3(-\Delta + \omega)$. Write $H = H_0 + V_{\omega}, V_\omega = B^* A$. We have $\sigma_1 H_0 = -H_0 \sigma_1, \sigma_1 H_0 = -H_0 \sigma_1$. We choose the factorization of $V_\omega$ so that $\sigma_1 B^* = B^* \sigma_1, \sigma_1 A = -A \sigma_1$. By these equalities $\sigma_1 R_{H_0}(z) = -R_{H_0}(-z) \sigma_1$ and $\sigma_1 R_{H_0}(z) = -R_{H_0}(-z) \sigma_1$. We have the following result about existence and completeness of wave operators:
Lemma 5.9. The following limits are well defined:

1. \( W u = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_0} u \) for any \( u \in L^2 \),
2. \( Z u = \lim_{t \to +\infty} e^{itH_0} e^{-itH_\omega} u \) for any \( u \in L^2_c(H_\omega) \).

\( W(L^2) = L^2_c(H_\omega) \) is an isomorphism with inverse \( Z \).

**Proof.** The existence of \( P_c(H_\omega) \circ W \) follows from Cook’s method and Lemma 5.8. By an elementary argument \( W u \in L^2_c(H_\omega) \) for any \( u \in L^2 \), so \( W = P_c(H_\omega) \circ W \). We have \( W = \tilde{W} \circ W_1 \) with

\[
W_1 u = \lim_{t \to +\infty} e^{itH_q} e^{-itH_0} u \quad \text{for any } u \in L^2(\mathbb{R}^2),
\]

\[
\tilde{W} = \lim_{t \to +\infty} e^{itH_\omega} e^{-itH_q} \quad \text{for any } u \in L^2.
\]

By standard theory \( W_1 \) is an isometric isomorphism of \( L^2(\mathbb{R}^2) \) into itself with inverse \( Z_1 u = \lim_{t \to +\infty} e^{itH_0} e^{-itH_q} u \) and by Lemma 5.7 \( \tilde{W} \) is an isomorphism \( L^2(\mathbb{R}^2) \to L^2_c(H_\omega) \) with inverse \( \tilde{Z} \). Then by product rule the limit in (2) exists and we have \( Z = Z_1 \circ \tilde{Z} \) with \( Z \) the inverse of \( W \).

**Lemma 5.10.** For \( u \in L^{2,s}(\mathbb{R}^2) \) with \( s > 1/2 \) we have

\[
W u = u - \frac{1}{2\pi i} \int_{|\lambda| \geq \omega} R_{H_\omega}^-(\lambda) V_\omega \left[ R_{H_0}^+(\lambda) - R_{H_0}^-(\lambda) \right] u d\lambda.
\]

**Proof.** \( W u \in L^2(\mathbb{R}^2) \), but the above formula is meaningful in the larger space \( L^{2,-s}(\mathbb{R}^2) \). For \( u \in L^{2,-s}(\mathbb{R}^2) \cap L^2_c(H_\omega) \) and for \( \langle u, v \rangle_2 = \int_{\mathbb{R}^2} u \cdot \bar{v} dx \) the standard \( L^2 \) pairing, we have by Plancherel

\[
\langle W u, v \rangle_2 = \langle u, v \rangle_2 + \lim_{\epsilon \to 0^+} \int_0^{+\infty} \left[ V_\omega e^{-iH_0t} - e^{-iH_qt} \right]_2 dt
\]

\[
= \langle u, v \rangle_2 + \lim_{\epsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ A R_{H_0}(\lambda + i\epsilon) u, B R_{H_0}^+(\lambda + i\epsilon) v \right]_2 d\lambda.
\]

By the orthogonality in \( L^2(\mathbb{R}) \) of boundary values of Hardy functions in \( H^2(\mathbb{C}_+) \) and in \( H^2(\mathbb{C}_-) \) we have for \( \epsilon > 0 \)

\[
\int_{-\infty}^{+\infty} \left[ A R_{H_0}(\lambda + i\epsilon) u, B R_{H_0}^+(\lambda + i\epsilon) v \right]_2 d\lambda = \int_{-\infty}^{+\infty} \left[ A \left[ R_{H_0}(\lambda + i\epsilon) - R_{H_0}(\lambda - i\epsilon) \right] u, B R_{H_0}^+(\lambda + i\epsilon) v \right]_2 d\lambda.
\]

By \( u \in L^{2,s}(\mathbb{R}^2) \) and \( v \in L^{2,s}(\mathbb{R}^2) \cap L^2_c(H_\omega) \) the limit in the right-hand side for \( \epsilon \searrow 0 \) exists and we have

\[
\langle W u, v \rangle_2 = \langle u, v \rangle_2 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ A \left[ R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0) \right] u, B R_{H_0}^+(\lambda + i0) v \right]_2 d\lambda
\]

\[
= \langle u, v \rangle_2 + \frac{1}{2\pi} \int_{|\lambda| \geq \omega} \left[ A \left[ R_{H_0}(\lambda + i0) - R_{H_0}(\lambda - i0) \right] u, B R_{H_0}^+(\lambda + i0) v \right]_2 d\lambda.
\]

This yields Lemma 5.10.

The crucial part of our linear theory is the proof of the following analogue of [40]:

**Lemma 5.11.** For any \( p \in (1, \infty) \) the restrictions of \( W \) and \( Z \) to \( L^2 \cap L^p \) extend into operators such that for \( C(\omega) < \infty \) semicontinuous in \( \omega \)

\[
\| W \|_{L_p(\mathbb{R}^2), L^p_c(H_\omega)} + \| Z \|_{L^p_c(H_\omega), L^p(\mathbb{R}^2)} < C(\omega).
\]
In the next two sections we will consider $W$ only, since the proof for $Z$ is similar. The argument in the following two sections is a transposition of [40]. We consider diagonal matrices

$$E_+ = \text{diag}(1, 0) \quad \text{and} \quad E_- = \text{diag}(0, 1).$$

Keeping in mind Lemma 5.10, $\sigma_1 R(z) = - R(-z) \sigma_1$ for $R(z)$ equal to $R_{H_0}(z)$ or to $R_{H_0}(z)$ and $\sigma_1 L^2(H_0) = L^2(H_0)$, it is easy to conclude that the $L^p$ boundness of $W$ is equivalent to $L^p$ boundness of

$$U u := \int_{\lambda \geq \omega} R^-_{H_0}(\lambda) V_\omega [R^\tau_{H_0}(\lambda) - R^-_{H_0}(\lambda)] u d\lambda,$$

$$= \int_{\lambda \geq \omega} R^-_{H_0}(\lambda) V_\omega [R_0^+(\lambda) - R_0^-(\lambda)] E_+ u d\lambda.$$

As in [40] we deal separately with high, treated in Section 6, and low energies, treated in Section 7. We introduce cut-off functions $\psi_1(x) \in C_0^\infty(\mathbb{R})$, and $\psi_2(x) \in C_0^\infty(\mathbb{R})$, with $\psi_1(x) + \psi_2(x) = 1$, $\psi_1(-x) = \psi_1(x)$, $\psi_1(x) = 1$ for $|x| \leq C$ and $\psi_1(x) = 0$ or $|x| > 2C$ for some $C > \omega$.

6. $L^p$ boundness of $U$: high energies

This part is almost the same of the corresponding part in [40]. For $\psi_1(x)$ the cutoff function introduced after Lemma 5.11, $\psi_1(H_0)$ is a convolution operator with symbol $\psi_1(|\xi|^2 + \omega)$. Both $\psi_1(H_0)$ and $\psi_2(H_0)$ are bounded operators in $L^p(\mathbb{R}^2)$ for any $p \in [1, \infty]$. In order to estimate the high frequency part (the so called high energy) $U \psi_2(H_0)$, we expand $R^-_{H_0}(\lambda)$ into the sum of few terms of Born series

$$R^-_{H_0}(\lambda) = R^-_{H_0}(\lambda) - R^-_{H_0}(\lambda) V_\omega R^-_{H_0}(\lambda) + R^-_{H_0}(\lambda) V_\omega R^-_{H_0}(\lambda) V_\omega R^-_{H_0}(\lambda),$$

getting by Lemma 5.10 the decomposition $U = U_1 + U_2 + U_3$ with

$$U_1 u = -\frac{1}{2\pi i} \int R^-_{H_0}(\lambda) V_\omega R_0^+(\lambda - \omega) E_+ u d\lambda,$$

$$U_2 u = \frac{1}{2\pi i} \int R^-_{H_0}(\lambda) V_\omega R^\tau_{H_0}(\lambda) V_\omega R_0^+(\lambda - \omega) E_+ u d\lambda,$$

$$U_3 u = -\frac{1}{2\pi i} \int R^-_{H_0}(\lambda) V_\omega R^-_{H_0}(\lambda) V_\omega R_0^+(\lambda - \omega) E_+ u d\lambda.$$

Lemma 6.1. The operator $U_1 \psi_2(H_0)$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. Specifically for any $s > 1$ there exists a constant $C_s > 0$ so that for $T = U_1 \psi_2(H_0)$

$$\|Tu\|_{L^p} \leq C_s \|x^s V_\omega\|_{L^2} \|u\|_{L^p} \quad \text{for all } u \in L^p(\mathbb{R}^2). \tag{1}$$

Proof. Recall $R_0(z) = (-\Delta - z)^{-1}$ and $R^\pm_{H_0}(z) = \text{diag}(R_0^\pm(z - \omega), -R_0^\pm(z + \omega))$. For $u = (u_1, u_2)$, and for $\mathcal{F}$ the Fourier transform, we are reduced to operators of schematic form

$$\mathcal{F}(E_{\pm} U_1 u)(\xi) = \int_{\lambda \geq \omega} d\lambda \int_{\mathbb{R}^2} \frac{1}{|\xi|^2 + \omega + \lambda + i0} \hat{u}_1(\xi - \eta) \delta(\lambda - (|\xi - \eta|^2 + \omega)) \hat{V}(\eta) d\eta,$$

with $\hat{V}$ the Fourier transform of the generic component of $V_\omega$. Then

$$E_{\pm} U_1 u = \int_{\mathbb{R}^2} d\eta \hat{V}(\eta) T_{\eta}^\pm u_1 \eta$$
where \( u_{1\eta}(x) = e^{ix\cdot\eta}u_1(x), \) \( T_\eta^{-1}u_{1\eta} = \frac{1}{i\pi} K_0(\sqrt{\eta^2/4 + \omega |\cdot|}) * u_{1\eta} \) and by [39]

\[
T_\eta^+ u_{1\eta}(x) = \frac{i}{2|\eta|} \int_0^\infty e^{i|\eta|}u_{1\eta}(x + t\eta/|\eta|) dt.
\]

By [40] we have that \( T = E_+U_1 \) satisfies inequality (1) while for \( T = E_-U_1 \) we use

\[
\|T_\eta^\pm u\|_{L^p} \leq \frac{1}{4\pi} \|K_0(\sqrt{\eta^2/4 + \omega |x|})\|_{L^1} \|u_1\|_{L^p} \leq C(\eta)^{-1} \|u_1\|_{L^p}
\]

and so \( \|E_-U_1u\|_{L^p} \lesssim \|\hat{V}(\eta)/\langle \eta \rangle\|_{L^1} \|u_1\|_{L^p}. \)

\[ \square \]

**Lemma 6.2.** The operator \( U_2\psi_2(H_0) \) is bounded in \( L^p(\mathbb{R}^2) \) for all \( 1 < p < \infty \), moreover, there exists a constant \( C_s > 0 \) so that for \( T = U_2\psi_2(H_0) \)

\[
\|Tu\|_{L^p} \leq C_s \|\langle x \rangle^s V_o\|_{L^2}^2 \|u\|_{L^p} \quad \text{for all } u \in L^p(\mathbb{R}^2)
\]

is valid, provided \( s > 1 \).

**Proof.** By [39] and with the notation of Lemma 6.1 we are reduced to a combination of operators

\[
L_{\pm,\pm}u = \int_{\mathbb{R}^2} d\eta_1 T_{\eta_1}^\pm \int_{\mathbb{R}^2} d\eta_2 \hat{V}(\eta_1)\hat{V}(\eta_2 - \eta_1)T_{\eta_2}^\pm u_{1\eta_2}.
\]

\( T_f = L_{-,-}u \) satisfies inequality (1) by [40, Proposition 2.2]. The other cases follow from Lemma 6.1. For example, for \( K(\eta_1, \eta_2) = \hat{V}(\eta_1)\hat{V}(\eta_2 - \eta_1) \) and \( \hat{K}(x, \eta_2) = \int d\eta e^{i\eta \cdot x}K(\eta, \eta_2), \)

\[
\|L_{\pm,\pm}u\|_{L^p} = \left\| \int_{\mathbb{R}^2} d\eta_2 \int_{\mathbb{R}^2} d\eta_1 K(\eta_1, \eta_2)T_{\eta_1}^\pm T_{\eta_2}^\pm u_{1\eta_2} \right\|_{L^p}
\leq \tilde{C}_s \int_{\mathbb{R}^2} d\eta_2 \|\langle x \rangle^s \hat{K}(x, \eta_2)\|_{L^2}^2 \|T_{\eta_2}^\pm u_{1\eta_2}\|_{L^p}
\leq \tilde{C}_s \int_{\mathbb{R}^2} d\eta_2 \|\langle x \rangle^s \hat{K}(x, \eta_2)\|_{L^2}^2 \|\hat{K}(\eta_2)\|_{L^2}^{-1} \|u_{1\eta_2}\|_{L^p} \|u_1\|_{L^p} \|\hat{V}(\eta)/\langle \eta \rangle\|_{L^1} \|u_1\|_{L^p}. \)

\[ \square \]

**Lemma 6.3.** Set \( T = U_3\psi_2(H_0) \). Then \( T \) is bounded in \( L^p(\mathbb{R}^2) \) for all \( 1 \leq p \leq \infty \).

**Proof.** Schematically

\[
E_+U_3\psi_2(H_0)u = \int_{k \geq 0} R_{H_0}^{-}(k^2) V F(k^2 + \omega)V[R_{H_0}^{+(k^2)} - R_{H_0}^{-(k^2)}] \psi_2(\lambda + \omega)u k dk,
\]

with \( F(k^2 + \omega) = R_{H_0}^{-(k^2)} V R^{-(k^2)} \) and \( V \) the generic component of \( V_o \). By (3) Lemma 5.4 for \( G_{k,y}^{\pm}(x) = e^{\pm ik|y|}G^{\pm}(x - y, k) \) with \( G^{\pm}(x, k) = \pm \frac{i}{2} H_0^{\pm}(k|x|) \) we have the following analogue of inequality (3.5) [40]

\[
\left| \frac{d}{dk} \left[ F(k^2 + \omega)V G_{k,y}^{\pm}(x), VG_{k,x}^{\pm}(y) \right] \right| \leq C_j \|\langle x \rangle^s V_o\|_{L^\infty}^3 \left\| \frac{k}{3} \sqrt{(x)} \langle y \rangle \right\|
\]

(1)

and by [40, Proposition 3.1] this yields the desired result for \( T = E_+U_3\psi_2(H_0) \). Since (1) continues to hold if we replace \( G_{k,x} \) with \( e^{-ik|x|} G_{k,x} \) with \( G_{k,x}(y) = G(x - y, k) \), where \( G(x, k) = K_0(\sqrt{k^2 + \omega|x|}) \), we get also the desired result for \( T = E_-U_3\psi_2(H_0) \). \( \square \)
7. \(L^p\) boundness of \(U\): low energies

Set
\[
Tu := \int_{\lambda \geq \omega} R_{H_\omega}^- (\lambda) V_\omega \left[ R_0^-(\lambda - \omega) - R_0^-(\lambda - \omega) \right] \psi_1(\lambda) E_+ u \, d\lambda.
\]

We want to prove:

**Lemma 7.1.** For any \(p \in (1, \infty)\) the restriction of \(T\) on \(L^2 \cap L^p\) extends into an operator such that \(\|T\|_{L^p(\mathbb{R}^2), L^p(\mathbb{R}^2)} < C(\omega)\) for \(C(\omega) < \infty\) semicontinuous in \(\omega\).

Let \(V_\omega = V = \{V_\ell: \ell, j = 1, 2\}, W = \{W_\ell: \ell, j = 1, 2\}\) with \(W_{12} = W_{21} = 0\), \(W_{22} = 1 \in \mathbb{R}\) and \(W_{11}(x) = 1\) for \(V_{11}(x) \geq 0\) and \(W_{11}(x) = -1\) for \(V_{11}(x) < 0\). Set \(B^* = (x)^{-N}\) for some large \(N > 0\), and \(A = \{A_\ell: \ell, j = 1, 2\}\) with \(A_{11}(x) = |V_{11}(x)|\), \(A_{12}(x) = W_{11}(x)V_{12}(x)\) and \(A_{2j}(x) = V_{2j}(x)\). Then \(W^2 = 1\), \(B^* WA = V\). Let \(k > 0\) be such that \(k^2 = \lambda - \omega\) and set \(M(k) = W + AR_{H_0}^-(\lambda)B^*\). Then
\[
R_{H_\omega}^-(\lambda) = R_{H_0}^-(\lambda) - R_{H_0}^-(\lambda)B^* M^{-1}(k) A R_{H_0}^-(\lambda).
\]

We have \(M(k) = W + c^-(k)P + A\tilde{G}_0 B^* + O(k^2 \log k)\) where: \(c^-(k) = a^- + b^- \log k\); \(P\) is a projection in \(L^2\) defined by
\[
P = \begin{bmatrix} A_{11} & \{ B^*_{11} \} \\ A_{21} \|V_{11}\|_{L^1} \end{bmatrix},
\]
\[
\tilde{G}_0 = \text{diag} \left( -\frac{1}{2\pi} \log |x|^*, -R_0(-2\omega) \right);
\]
\[
\|d^j/dk^j O(k^2 \log k)\|_{L^2, L^2} \leq Ck^{2-j}(\log k), \quad j = 0, 1, 2, \ 0 < k < c.
\]

Let \(Q = 1 - P\) and let \(M_0 = W + A\tilde{G}_0 B^*\). Then \(QM_0Q\) is invertible in \(QL^2\) if and only if \(\omega\) is not a resonance or an eigenvalue for \(H_\omega\) and in that case
\[
M^{-1}(k) = g^*(k) (P - PM_0 QD_0 - QD_0 QM_0 PM_0 QD_0 Q + QD_0 Q + O(k^2 \log k))
\]
with \(g(k) = c^- \log k + d^-\) for \(c^- \neq 0\) and \(D_0 = (QM_0Q)^{-1}\) by [17]. We claim now that \(QD_0 Q - QWQ\) is a Hilbert–Schmidt operator. In fact, following the argument in Lemma 3 [18], we get that the operator \(L = P + QM_0 Q\) is invertible in \(QL^2\), and \(D_0 = QL^{-1} Q\). We have
\[
L = W + \left[ A\tilde{G}_0 B^* + P + PM_0 P - PM_0 Q - QM_0 P \right].
\]

Set \(L := W(1 + \tilde{S})\), the operators \(P\), \(PM_0P\), \(PM_0Q\), \(QM_0P\) are of rank one while \(A\tilde{G}_0 B^*\) is a Hilbert–Schmidt operator. From the fact that \(W\) is invertible, we get that also \((1 + \tilde{S})\) is invertible. Moreover the identity \((1 + \tilde{S})^{-1} = 1 - \tilde{S}(1 + \tilde{S})^{-1}\) yields
\[
L^{-1} = W + \left[ A\tilde{G}_0 B^* + P + PM_0 P - PM_0 Q - QM_0 P \right],
\]
that is the product of an Hilbert–Schmidt operator with one in \(B(L^2(\mathbb{R}^2), L^2(\mathbb{R}^2))\). Finally, an application of the Theorem VI.22, Chapter VI, in [25], shows that \(L^{-1} - W\) is of Hilbert–Schmidt type.

So we are reduced to the following list of operators:
\[
T^+_0 u := \int_{0}^{\infty} R_0^+(k^2) E_+ V_\omega E_+ \left[ R^+_0(k^2) - R_0^-(k^2) \right] \psi_1(\lambda) uk \, dk,
\]
and \(T^-_0\) defined as above but with \(R_0^+(k^2) E_+\) replaced by \(R_0(-k^2 - 2\omega) E_-\) which are bounded in \(L^p\) for \(1 < p < \infty\) by Lemma 6.1;
\[
T^+_1 u := \int_{0}^{\infty} R_0^-(k^2) E_+ N(k) \left[ R^+_0(k^2) - R_0^-(k^2) \right] \psi_1(\lambda) E_+ uk \, dk
\]
with
\[ \left\| d^j / dk^j N(k^2 \log k) \right\|_{L^2 \rightarrow L^2} \leq C k^{2-j} (\log k), \quad j = 0, 1, 2, \ 0 < k < c \]
which is bounded in \( L^p \) for \( 1 \leq p \leq \infty \) by Proposition 4.1 [40];

\[ T_2^\pm u \coloneqq \int_0^\infty R_0(k^2) E B^* (d(k) F + L + W) A \left[ R_0^+(k^2) - R_0^-(k^2) \right] \psi_1(\lambda) E u \, dk \]

with \( F \) a rank 3 operator, \( L \) a Hilbert–Schmidt operator in \( L^2 \), and \( d(k) = g^{-1}(k) \). There are also operators \( T_j \), for \( j = 0, 1, 2 \), defined as above but with \( R_0^-(k^2) E \) replaced by \( R_0(-k^2 + 2\omega) E \) and bounded in \( L^p \). So \( T_2^\pm = T_{2,1}^\pm d(\sqrt{-\Delta}) + T_{2,2}^\pm + T_{2,3}^\pm \) with \( T_{2,j}^\pm \) for \( j = 1, 2, 3 \) operators bounded in \( L^p \) for \( 1 < p < \infty \) because of the following statement proved in [40] (the + case is exactly that in [40], and the − case can be proved following the same argument):

if \( K \) is an operator with integral kernel \( K(x,y) \) such that for some \( s > 1 \)
\[ \| K \|_s \coloneqq \left( \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy (\int_{\mathbb{R}^2} \langle x \rangle^2 |K(x, x-y)|^2)^{1/2} < \infty \]
then the operators
\[ Z^\pm u \coloneqq \int_0^\infty R_0^+(k^2) K \left[ R_0^+(k^2) - R_0^-(k^2) \right] uk \, dk, \]

\[ Z^- u \coloneqq \int_0^\infty R_0(-k^2 + 2\omega) K \left[ R_0^+(k^2) - R_0^-(k^2) \right] uk \, dk \]

are bounded in \( L^p \) for \( 1 < p < \infty \) with \( \| Z^\pm \|_{L^p,L^p} < C_{s,p} \| K \|_s \).

8. Proofs of Lemmas 3.2, 3.3 and 3.4

We mimic Mizumachi [22]. By the limiting absorption principle we have
\[ P_{c}(\omega) e^{-itH_\omega} f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-it\lambda} (\lambda) P_{c}(\omega) \left[ R_{H_\omega}^+(\lambda) - R_{H_\omega}^-(\lambda) \right] f \, d\lambda. \]

We consider a smooth function \( \chi(x) \) satisfying \( 0 \leq \chi(x) \leq 1 \) for \( x \in \mathbb{R} \), \( \chi(x) = 1 \) if \( x \geq 2 \) and \( \chi(x) = 0 \) if \( x \leq 1 \). \( \chi_M(x) \) is an even function satisfying \( \chi_M(x) = \chi(x - M) \) for \( x \geq 0 \). Let \( \bar{\chi}_M(x) = 1 - \chi_M(x) \). We have:

**Lemma 8.1.** For any fixed \( s > 1 \) there exists a positive \( C(\omega) \) upper semicontinuous in \( \omega \), such that for any \( u \in \mathcal{S}(\mathbb{R}^2) \) we have
\[ \left\| R_{H_\omega}^\pm(\lambda) f \right\|_{L^2(\sigma_{c}(H_\omega);L^2)} \leq C \| f \|_{L^2}. \]

First, we prove Lemma 3.2 assuming Lemma 8.1.

**Proof of Lemma 3.2.** We split
\[ P_{c}(\omega) e^{-itH_\omega} f = P_{c}(\omega) e^{-itH_\omega} \chi_M(H_\omega) f + P_{c}(\omega) e^{-itH_\omega} \bar{\chi}_M(H_\omega) f \]
with
Proof of Lemma 3.3.

By Plancherel’s identity and Hölder inequalities we have

\[ \left\| \int_0^t e^{-i(t-s)H_\omega} P_c(\omega) g(s, \cdot) \, ds \right\|_{L^{2/3} L^{2/3}'} \leq \| R^+_{H_\omega}(\cdot) P_c(\omega) \tilde{\chi}_{[0, +\infty)} \ast \hat{g}(\lambda, \cdot) \|_{L^{1/3} L^{1/3}'} \]

Integrating by parts, in \( S'_c(\mathbb{R}^2) \) for any \( t \neq 0 \) and \( f \in S_1(\mathbb{R}^2) \)

\[ P_c(\omega) e^{-itH_\omega} f = \frac{(it)^{-j}}{2\pi i} \int_\mathbb{R} d\lambda e^{-it\lambda} \partial^j_\lambda P_c(\omega) \left\{ (R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)) \chi_M(\lambda) \right\} f. \]

Since by (3) Lemma 5.4 for high energies we have

\[ \left\| \partial^j_\lambda P_c(\omega) R^\pm_{H_\omega}(\lambda) \right\|_2 \sim (\lambda)^{-j+1/2} \]

the above integral absolutely converges in \( (x)^{-(j+1)/2} \) \( L^2 \) for \( j \geq 2 \). Let \( g(t, x) \in S(\mathbb{R} \times \mathbb{R}^2) \). By Fubini and integration by parts, \( j \geq 2 \),

\[ \left\| \chi_M(H_\omega) e^{-itH_\omega} P_c(\omega) f, g \right\|_{L^{2/3} L^{2/3}'} \leq (2\pi)^{-1/2} \| \chi_M(\lambda) (R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)) f \|_{L^2(\sigma_c(H_\omega); L^{1-j}_t)} \| \hat{g}(\lambda, \cdot) \|_{L^2 L^{2-j}_x} \]

In a similar way we have

\[ \left\| e^{-itH_\omega} \tilde{\chi}_M(H_\omega) f, g \right\|_{L^{2/3} L^{2/3}'} \leq (2\pi)^{-1/2} \left( \| \tilde{\chi}_M(H_\omega) (R^+_{H_\omega}(\lambda) - R^-_{H_\omega}(\lambda)) f \|_{L^2(\sigma_c(H_\omega); L^{1-j}_t)} \| g \|_{L^2 L^{2-j}_x} \right) \]

therefore we achieve

\[ \left\| e^{-itH_\omega} P_c(\omega) f, g \right\|_{L^{2/3} L^{2/3}'} \leq (2\pi)^{-1/2} \left( \| \chi_M(\lambda) (R_{H_\omega}(\lambda + i0) - R_{H_\omega}(\lambda - i0)) f \|_{L^2(\sigma_c(H_\omega); L^{1-\epsilon}_t)} \right. \]

and by Lemma 8.1 this estimate yields Lemma 3.2. \( \square \)
By Lemma 5.4 \( \sup_{\lambda \geq 0} \| R_{H_\omega}^\pm(\lambda) P_c(\omega) \|_{B(L^{2,s},L^{2,-r})} \lesssim \langle \lambda \rangle^{-1/2} \), and so
\[
\sup_{\lambda \in \mathbb{R}} \| R_{H_\omega}^+(\lambda) P_c(\omega) \|_{B(L^{2,s},L^{2,-r})} \| g \|_{L^{2,s} L^2_t} \leq C \| g \|_{L^{2,s} L^2_t}.
\]
The above inequalities yields Lemma 3.3. \( \square \)

**Proof of Lemma 3.4.** Let \((q, r)\) be admissible and let \( T \) be an operator defined by
\[
Tg(t) = \int_{\mathbb{R}} ds e^{-i(t-s)H_\omega} P_c(\omega) g(s).
\]
Using Lemmas 3.2 and 3.3 we get \( f := \int_{\mathbb{R}} ds e^{isH_\omega} P_c(\omega) g(s) \in L^2(\mathbb{R}) \) and that there exists a \( C > 0 \) such that
\[
\| Tg(t) \|_{L^2_t L^2_s} \leq C \| g \|_{L^2_t L^2_s}
\]
for every \( g \in S(\mathbb{R} \times \mathbb{R}^2) \). Since \( q > 2 \), it follows from Lemma 3.1 in [28] (see also [2]) and (1) that
\[
\left\| \int_{s<t} ds e^{-i(t-s)H_\omega} P_c(\omega) g(s) \right\|_{L^q_t L^p_s} \lesssim \| g \|_{L^2_t L^2_s}.
\]
This yields Lemma 3.4. \( \square \)

To prove Lemma 8.1 observe that it is not restrictive to prove
\[
\| R_{H_\omega}^\pm(\lambda) f \|_{L^2_{\lambda}(\mathbb{R}^+; L^{2,-r}\mathbb{R}^+)} \leq C \| f \|_{L^2_t}.
\]
Following the argument in [22, Section 4] we need the following:

**Lemma 8.2.** There exists a positive constant \( C \) such that for \( s > 1 \)
\[
\| R_{H_\omega}^\pm(\lambda) f \|_{L^2_{\lambda}(\mathbb{R}^+; L^{2,-s}\mathbb{R}^+)} \leq C \| f \|_{L^2_t}.
\]

**Proof.** \( E_+ R_{H_0}^\pm(\lambda, -\omega) E_+ f \) and by Lemma 4.2 [22] we get
\[
\| R_{H_0}^\pm(\lambda) E_+ f \|_{L^{2,-s}(0,\infty)} \leq C \sup_x \| R_{H_0}^\pm(\lambda) E_+ f \|_{L^{2,-s}(0,\infty)} \leq C \| E_+ f \|_{L^2_t}.
\]
We have \( E_- R_{H_0}^\pm(\lambda) f = - R_0(-\omega - \lambda) E_- f = - \frac{-\Delta + \omega - \lambda}{-\Delta + 2\omega + \lambda} R_0^+(\lambda - \omega) E_- f \). So by (1)
\[
\| E_- R_{H_0}^\pm(\lambda) f \|_{L^{2,-s}(\mathbb{R}^+,\infty)} \leq \frac{-\Delta + \omega - \lambda}{-\Delta + 2\omega + \lambda} \| R_0^+(\lambda - \omega) E_- f \|_{L^{2,-s}(0,\infty)} \leq C_1 \| R_0^+(\lambda) E_- f \|_{L^{2,-s}(0,\infty)} \leq C_1 C \| E_- f \|_{L^2_t} \quad \square
\]

**Proof of inequality (8.1).** We consider the operator \( h_q = -\Delta + q(x) \) introduced in Section 5 and \( H_q = \sigma_3(h_q + \omega) \). We claim that
\[
\| R_{H_q}^\pm(\lambda) f \|_{L^2_{\lambda}(\mathbb{R}^+; L^{2,-s}\mathbb{R}^+)} \leq C \| f \|_{L^2_t}.
\]
Indeed \( E_+ R_{H_q}^\pm(\lambda, -\omega) E_+ f \) and \( \| R_{H_q}^\pm(\lambda) E_+ f \|_{L^{2,-s}(0,\infty)} \| \| \leq C \| f \|_{L^2_t} \) by [22, Lemma 4.1]. On the other hand
\[
E_- R_{H_q}^\pm(\lambda) f = - R_{h_q}(-\lambda - \omega) E_- f = - R_0(-\lambda - \omega) E_- f + R_0(-\lambda - \omega) q R_{h_q}(-\lambda - \omega) E_- f.
\]
The bound for the first term comes from Lemma 8.2 and
\[
\| R_0(-\lambda - \omega) q R_{h_q}(-\lambda - \omega) E_- f \|_{L^{2,-s} L^2_t} \lesssim \| R_0(-\lambda - \omega) q R_{h_q}(-\lambda - \omega) E_- f \|_{L^{\infty} L^2_t} \lesssim \| q R_{h_q}(-\lambda - \omega) E_- f \|_{L^{\infty} L^2_t} \leq C \| E_- f \|_{L^2_t}.
\]
Armed with inequality (1) we consider the identity
\[
R_{H_0}^\pm (\lambda) = (1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda)
\]
\[
= R_{H_q}^\pm (\lambda) - R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda).
\]
(8.2)

By (1) it is enough to bound the last term in the last sum. This is bounded by
\[
\left\| R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda)f \right\|_{L_3^2 L_4^{-\delta}}
\]
\[
\leq \left\| R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q)(1 + R_{H_q}^\pm (\lambda)(V_\omega - \sigma_3 q))^{-1} R_{H_q}^\pm (\lambda)f \right\|_{L_3^2 L_4^{-\delta}}
\]
\[
\lesssim \| f \|_{L_3^2}
\]
by (1) and by the fact that the above \(L^\infty_\lambda(\omega, \infty)\) norms are bounded by Lemmas 5.1 and 5.4. 

9. Proof of Lemma 4.7

The proof is standard and analogous to [9, Lemma 5.8]. Recall:

**Lemma 4.7.** We have for \(\varphi(x)\) and \(\varphi(t, x)\) Schwarz functions, for \(t \in [0, \infty)\) and for fixed \(s > 1\) sufficiently large
\[
\| e^{-iH_0t} R_{H_q}^+ (\Lambda) P_c(\omega)\varphi \|_{L_3^2 L_4^{-\delta}} < C(\Lambda, \omega) \| \varphi(\cdot) \|_{L_3^2},
\]
\[
\left\| \int_0^t e^{-iH_0(t-\tau)} R_{H_q}^+ (\Lambda) P_c(\omega)\varphi(\cdot) d\tau \right\|_{L_3^2 L_4^{-\delta}} < C(\Lambda, \omega) \| \varphi(t, \cdot) \|_{L_3^2 L_4^{-\delta}}
\]
with \(C(\Lambda, \omega)\) upper semicontinuous in \(\omega\) and in \(\Lambda \geq \omega\).

**Proof.** We consider \(\omega < a/ < a < \Lambda < b < \infty\) and the partition of unity \(1 = g + \tilde{g}\) with \(g \in C^\infty_0(\mathbb{R})\) with \(g = 1\) in \([a, b]\) and \(g = 0\) in \([a/2, 2b]\). By Lemma 3.2 we get
\[
\| e^{-iH_0t} R_{H_q}^+ (\Lambda) P_c(\omega)\tilde{g}(H_0)\varphi \|_{L_3^2 L_4^{-\delta}} \lesssim C(\omega) \| R_{H_q}^+ (\Lambda) P_c(\omega)\tilde{g}(H_0)\varphi \|_{L_3^2}
\]
\[
\lesssim C(\omega)c_0(a, b, \omega) \| \varphi \|_{L_3^2}.
\]
Similarly by the proof of Lemma 3.3, for any \(s > 1\)
\[
\left\| \int_0^t e^{-i(t-s)H_0} R_{H_q}^+ (\Lambda) P_c(\omega)\tilde{g}(H_0)\varphi(s, \cdot) d\tau \right\|_{L_3^2 L_4^{-\delta}}
\]
\[
\lesssim \| R_{H_q}^+ (\Lambda) P_c(\omega)\tilde{g}(H_0)\varphi(s, \cdot) \|_{L_3^2 L_4^{-\delta}}
\]
\[
\lesssim \| R_{H_q}^+ (\Lambda) P_c(\omega)\tilde{g}(H_0)\varphi(s, \cdot) \|_{L_3^2 L_4^{-\delta}}
\]
\[
\lesssim C(s, a, b, \omega) \| \varphi \|_{L_3^2 L_4^{-\delta}}
\]
by \((\lambda - \Lambda) R_{H_q}^+ (\Lambda) R_{H_q}^+ (\Lambda) = R_{H_q}^+ (\Lambda) - R_{H_q}^+ (\Lambda)\), Lemma 5.4 and \(|\lambda - \Lambda| \geq a \wedge b\). We consider now
\[
\langle x \rangle^{-\gamma} g(H_0) e^{-iH_0t} R_{H_q}(\Lambda + i\epsilon) P_c(H_0)(y)^{-\gamma}
\]
\[
= e^{-i\Lambda t} \langle x \rangle^{-\gamma} \int_0^s e^{-i(H_0-t-\epsilon)s} g(H_0) P_c(H_0) d\tau \langle y \rangle^{-\gamma}.
\]
(9.1,e)

We claim the following:
Lemma 9.1. There are functions \(u(x, \xi)\) defined for \(x \in \mathbb{R}^2\) and for \(|\xi| \in [a/2, 2b]\) with values in \(\mathbb{C}^2\) such that for any \(\chi \in C_0^\infty(a/2, 2b)\) we have (for \( i \alpha \sigma_f \) the product row column and \( i u \) the transpose of a column vector)
\[
\chi(H_\omega) f(x) = (2\pi)^{-2} \int \mathbb{R}^4 u(x, \xi) \chi(\xi)|\xi|^2 + \omega) d\xi dy.
\] (9.2)

There are constants \(c_{a\beta}\) such that
\[
|a_x^\alpha a_\xi^\beta u(x, \xi)| \leq c_{a\beta}(x)|\beta| \quad \text{for all } x \in \mathbb{R}^2 \text{ and } |\xi| \in [a/2, 2b].
\] (9.3)

Let us assume Lemma 9.1. Then we can write the kernel of operator (9.1) as
\[
\langle x \rangle^{-\gamma} g(H_\omega) e^{-i H_\omega t} R_{H_\omega}(A + i \epsilon)(y)^{—\gamma} = \text{constant} \langle x \rangle^{-\gamma} \int \mathbb{R}^3 u(x, \xi) e^{-i(\alpha(\xi^2 + \omega) - A - i \epsilon)c} g(\xi^2 + \omega)^{\gamma} u(y, \xi) d\xi (y)^{—\gamma}.
\] (9.4)

Estimates (9.3) and elementary integration by parts yields
\[
|\langle x \rangle^{—\gamma + \tau}(y)^{—\gamma + \tau} e^{-\epsilon t} \rangle \leq C(x)^{—\gamma + \tau} (y)^{—\gamma + \tau} e^{-\epsilon t}.
\]

For \(\gamma > \tau > 1\) and \(r \geq 3\), we obtain
\[
\left\| e^{-i H_\omega t} R_{H_\omega}^+(A) g(H_\omega) P_c(\omega) \varphi \right\|_{L^2_t(0, \infty), L^2_x} \leq C \| \varphi(x) \|_{L^2_x}.
\]

Similarly
\[
\left\| \int_0^t e^{-i(t-s) H_\omega} R_{H_\omega}^+(A) P_c(\omega) g(H_\omega) \varphi(s, \cdot) ds \right\|_{L^2_t L^2_x} \leq \left\| \int_0^t (t-s)^{—\gamma} \| \varphi(s, \cdot) \|_{L^2_x} \right\|_{L^2_t} \leq C \| \varphi \|_{L^2_t L^2_x}.
\]

We need now to prove Lemma 9.1. \(\Box\)

10. Proof of Lemma 9.1

First of all we explain how to define the \(u(x, \xi)\). We set \(V_\omega = B^*A\) with \(A(x)\) and \(B^*(x)\) rapidly decreasing and continuous. Then we have

Lemma 10.1. For any \(\lambda > \omega\) and any \(\xi \in \mathbb{R}^2\) with \(\lambda = \omega + |\xi|^2\), in \(L^2(\mathbb{R}^2)\) the system
\[
(1 + AR_{H_\omega}^+(\lambda) B^*) \tilde{u} = Ae^{-i\xi \cdot x} e_1
\] (1)

admits exactly one solution \(\tilde{u}(x, \xi) \in H^2\) such that for any \([a, b] \subset (1, \infty) \setminus \sigma_p(H)\) there is a fixed \(C < \infty\) such that for any \(\lambda \in [a, b]\) and any \(\xi\) as above we have
\[
\| \tilde{u}(\cdot, \xi) \|_{H^2} \leq C.
\] (2)

Proof. \(AR_{H_\omega}^+(\lambda) B^*\) is compact and \(\ker(1 + AR_{H_\omega}^+(\lambda) B^*) = \{0\}\) for \(\lambda > \omega\) by [11], since in that case \(\lambda \notin \sigma_p(H_\omega)\). By Fredholm alternative we get existence and uniqueness of \(\tilde{u}(x, \xi)\). Regularity theory and continuity of the coefficients of system (1) with respect to \(\xi\) yield (2). \(\Box\)

Let now \(i e_1 = (1, 0)\) and \(G_0(|x|, k) = \text{diag}(\frac{1}{4} H_0^+(k|x|), -\frac{1}{2a} K_0(\sqrt{k^2 + 2\omega|x|}))\) for \(k > 0\). We have \(G_0(r, k) = \frac{i}{2\sqrt{k} \sqrt{4k^2 + 2\omega r}} e^{ikr} e_1 + O(r^{—\frac{3}{2}})\) and \(\partial_k G_0(r, k) = -k \frac{2k}{4\sqrt{k} \sqrt{4k^2 + 2\omega r}} e^{ikr} e_1 + O(r^{—\frac{3}{2}})\). We set
\[
u(x, \xi) = e^{-i\xi \cdot x} e_1 + v(x, \xi) = e^{-i\xi \cdot x} e_1 - R_{H_\omega}^+(\lambda) B^* \tilde{u}(\cdot, \xi).
\]
Then $(H_\omega - \lambda)u(x, \xi) = B^*(Ae^{-i\xi \cdot x}e_1 - \tilde{u} - AR^{-1}_H(\lambda)B^*\tilde{u}) = 0$. Notice $B^*\tilde{u} = V_0u$ so $v(x, \xi) = e^{-ix \cdot \xi}w(x, \xi)$ where $w(x, \xi)$ is the unique solution in $L^2_{\omega, s}$, $s > 1$, of the integral equation
\begin{equation}
  w(x, \xi) = -F(x, \xi) - \int G_0(|x-z|, |\xi|)e^{i(x-z) \cdot \xi}V_0(z)w(z, \xi)\,dz.
\end{equation}
with
\begin{equation}
  F(x, \xi) = \int G_0((x-z), |\xi|)V_0(z)e^{i(x-z) \cdot \xi}e_1\,dz.
\end{equation}

It is elementary to show that, for $|\xi| \in [a, b]$, then $|\partial^\alpha_\xi \partial^\beta_\xi F(x, \xi)| \leq \tilde{c}_{\alpha\beta}(x)|\beta|^{-1/2}$. By standard arguments and Lemmas 5.3 and 5.4 we have $|\partial^\alpha_\xi \partial^\beta_\xi w(x, \xi)| \leq \tilde{c}_{\alpha\beta}(x)|\beta|$. This yields (9.3). To get (9.2) we follow the presentation in Chapter 9 [32]. We denote by $R^\pm_{H_0}(x, y, k)$ the kernel of $R^\pm_{H_0}(k^2 + \omega)$. We set
\begin{equation}
  R^+_{H_0}(x, y, k) = G_0(|x-y|, k) + h(x, y, k)
\end{equation}
with $h(x, y, k) = -R^+_{H_0}(k^2 + \omega)V_0G_0(| \cdot - y|, k)$. Let $(r, \Sigma)$ be polar coordinates on the sphere $S^1$, then we claim:

**Lemma 10.2.** Let $k > 0$. For $r \to \infty$ we have uniform convergence on compact sets of, with $u \cdot (1,0)$ the raw column product between column $u$ and raw $(1,0)$,
\begin{align}
  R^+_{H_0}(x, r\Sigma, k) &= \frac{i\sqrt{2}}{4\sqrt{1\pi kr}}e^{ikr}u(x, k\Sigma) \cdot (1,0) + O(r^{-2}), \\
  \frac{\partial}{\partial r} R^+_{H_0}(x, r\Sigma, k) &= -\frac{i\sqrt{2}}{4\sqrt{1\pi kr}}ke^{ikr}u(x, k\Sigma) \cdot (1,0) + O(r^{-2}), \\
  R^+_{H_0}(r\Sigma, y, k) &= \frac{i\sqrt{2}}{4\sqrt{1\pi kr}}e^{ikr}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(y, k\Sigma)\sigma_3 + O(r^{-2}), \\
  \frac{\partial}{\partial r} R^+_{H_0}(r\Sigma, y, k) &= -\frac{i\sqrt{2}}{4\sqrt{1\pi kr}}ke^{ikr}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u(y, k\Sigma)\sigma_3 + O(r^{-2}).
\end{align}

For $R^-_{H_0}(x, y, k)$ the asymptotic expansion follows from $R^-_{H_0}(x, y, k) = \overline{R^+_{H_0}(x, y, k)}$.

We write $R^+_{H_0}(x, r\Sigma, k) = G_0(|x-r\Sigma|, k) + h(x, r\Sigma, k)$ with
\begin{align}
  h(x, r\Sigma, k) &= -R^+_{H_0}(k^2 + \omega)V_0G_0(| \cdot - r\Sigma|, k) \\
  &= -R^+_{H_0}(k^2 + \omega)\begin{bmatrix} V_0(x) & \frac{i\sqrt{2}}{4\sqrt{1\pi kr}}e^{ikr}e^{-ik\Sigma \cdot x} \text{diag}(1,0) \end{bmatrix} + O(r^{-3/2}).
\end{align}

We have
\begin{align}
  \left\| V_0(x)G_0(|x-r\Sigma|, k) - V_0(x)\frac{i\sqrt{2}}{4\sqrt{1\pi kr}}e^{ikr}e^{-ik\Sigma \cdot x} \text{diag}(1,0) \right\|_{L^2_{\omega, s}} = O(r^{-3/2}).
\end{align}
From $v(x, \xi) = -R^+_{H_0}(k^2 + \omega)V_0(x)e^{-ik\Sigma \cdot x}e_1$, with $'e_1 = (1,0)$ we get
\begin{align}
  v(x, \xi)'e_1 &= -R^+_{H_0}(k^2 + \omega)V_0(x)e^{-ik\Sigma \cdot x} \text{diag}(1,0).
\end{align}

Then we conclude for any $s > 1$
\begin{align}
  \left\| h(x, r\Sigma, k) - \frac{i\sqrt{2}}{4\sqrt{1\pi kr}}v(x, k\Sigma)'e_1 \right\|_{L^2_{\omega, s}} = O(r^{-3/2})
\end{align}
and
\[ \left\| R^+_{H_0}(x, r, \Sigma, k) - \frac{i}{4\sqrt{r}} u(x, k \Sigma)\gamma_1 e_1 \right\|_{L^2} = O(r^{-3/2}). \]

Then point wise
\[ h(x, r, \Sigma, k + i0) - \frac{i}{4\sqrt{r} \pi kr} u(x, k \Sigma)\gamma_1 e_1 = O(r^{-3/2}) \]
and
\[ R^+_{H_0}(x, r, \Sigma, k) - \frac{i}{4\sqrt{r} \pi kr} u(x, k \Sigma)\gamma_1 e_1 = O(r^{-3/2}). \]

This yields (1) in Lemma 10.2. (2) can be obtained with a similar argument. (3) and (4) follow from (1) and (2) by
\[ \sigma_3 R^\pm_{H_0}(x, y, k)\sigma_3 = R^\pm_{H_0}(x, y, k) = i R^\mp_{H_0}(y, x, k). \]

By Lemma 3.5 for \( v \in L^2(H_0) \cap C^\infty \) and for \( \varphi \in C^\infty_0(\mathbb{R}) \) supported in \((\omega, \infty)\) we have
\[ \varphi(H_0)v(x) = \frac{2}{\pi} \int_0^\infty k \, dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) G(x, y, k)v(y) \, dy. \]

We prove (here \( u^\dagger \bar{u} \) is a raw column product between column \( u \) and raw \( \bar{u} \))
\[ \Im R^+_{H_0}(x, y, k) = \frac{1}{8\pi} \int_{S^1} u(x, k \Sigma)\bar{u}(y, k \Sigma)\sigma_3 \, d\Sigma, \]
where \( d\Sigma \) is the standard measure on \( S^1 \). By the Green theorem for \( S_R = \{ z \in \mathbb{R}^2 : |z| = R \} \), \( |x| < R \), \( |y| < R \) and \( r = |z| \).

By Green theorem for \( S_R = \{ z \in \mathbb{R}^2 : |z| = R \} \), \( |x| < R \) and \( |y| < R \),
\[ \Im R^+_{H_0}(x, y, k) = \frac{1}{2\pi} \int_{S_R} I(x, y, z, k) \, d\ell(z), \]
where
\[ I(x, y, z, k) := R^+_{H_0}(x, z, k)\gamma_3 \partial_{z_1} R^+_{H_0}(z, y, k) - (\partial_{z_1} R^+_{H_0}(x, z, k))\gamma_3 R^+_{H_0}(z, y, k). \]

By Lemma 10.2
\[ \left| \Im R^+_{H_0}(x, y, k) - \frac{1}{8\pi} \int_{S^1} u(x, k \Sigma)\bar{u}(y, k \Sigma)\sigma_3 \, d\Sigma \right| \]
\[ = \frac{R}{2\pi} \int_{S^1} I(x, y, r, \Sigma, k) \, d\Sigma - \frac{1}{8\pi} \int_{S^1} u(x, k \Sigma)\bar{u}(y, k \Sigma)\sigma_3 \, d\Sigma \lessapprox O(R^{-2}). \]

Therefore, taking \( R \to +\infty \), we arrive at (3). Moreover, we obtain
\[ \varphi(H_0)v(x) = \frac{2}{\pi} \int_0^\infty k \, dk \int_{\mathbb{R}^2} \varphi(k^2 + \omega) G(x, y, k)v(y) \, dy \]
\[ = \frac{1}{4\pi^2} \int_0^\infty k \, dk \int_{\mathbb{R}^2} \int_{S^1} u(x, k \Sigma)\bar{u}(y, k \Sigma)\sigma_3 v(y) \varphi(k^2 + \omega) \, d\Sigma \, dy \]
\[ = (2\pi)^{-2} \int_{\mathbb{R}^4} u(x, \xi)\bar{u}(y, \xi)\sigma_3 v(y) \varphi(|\xi|^2 + \omega) \, d\xi \, dy, \]
that is the integral representation (9.2). This completes the proof of Lemma 9.1. \( \Box \)
References