Global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical Hartree equation in $\mathbb{R}^d$

Changxing Miao $^a$,*, Guixiang Xu $^a$, Lifeng Zhao $^b$

$^a$ Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China
$^b$ Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

Received 27 May 2008; received in revised form 8 January 2009; accepted 14 January 2009
Available online 7 February 2009

Abstract

We prove the global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation with low regularity data in $\mathbb{R}^d$, $d \geq 3$. Precisely, we show that a unique and global solution exists for initial data in the Sobolev space $H^s(\mathbb{R}^d)$ with $s > \frac{4(\gamma - 2)}{(3\gamma - 4)}$, which also scatters in both time directions. This improves the result in [M. Chae, S. Hong, J. Kim, C.W. Yang, Scattering theory below energy for a class of Hartree type equations, Comm. Partial Differential Equations 33 (2008) 321–348], where the global well-posedness was established for any $s > \max(1/2, 4(\gamma - 2)/(3\gamma - 4))$. The new ingredients in our proof are that we make use of an interaction Morawetz estimate for the smoothed out solution $Iu$, instead of an interaction Morawetz estimate for the solution $u$, and that we make careful analysis of the monotonicity property of the multiplier $m(\xi) \cdot |\xi|^p$.

As a byproduct of our proof, we obtain that the $H^s$ norm of the solution obeys the uniform-in-time bounds.

© 2009 Elsevier Masson SAS. All rights reserved.

MSC: 35Q40; 35Q55; 47J35

Keywords: Almost interaction Morawetz estimate; Well-posedness; Hartree equation; I-method; Uniform bound

1. Introduction

In this paper, we study the global well-posedness of the following initial value problem (IVP) for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation.

\[
\begin{aligned}
  iu_t + \Delta u &= (|x|^{-\gamma} * |u|^2)u, \\
  u(0) &= u_0(x) \in H^s(\mathbb{R}^d),
\end{aligned}
\]

where $H^s$ denotes the usual inhomogeneous Sobolev space of order $s$. It is a classical model introduced in [26].

We adopt the following standard notion of local well-posedness, that is, we say that the IVP (1.1) is locally well-posed in $H^s$ if for any $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_s)$ depending only on the norm of the initial data, such that a solution to the IVP exists on the time interval $[0, T)$, is unique in a certain Banach space of functional

* Corresponding author.

E-mail addresses: miao_changxing@iapcm.ac.cn (C. Miao), xu_guixiang@iapcm.ac.cn (G. Xu), zhaolifengustc@yahoo.cn (L. Zhao).
X ⊂ C([0, T], H^s), and the solution map from H_x^s to C([0, T], H^s) depends continuously. If T can be taken arbitrarily large, we say that the IVP (1.1) is globally well-posed.

Local well-posedness for the IVP (1.1) in H^s for any s > γ/2 − 1 was established in [18]. A local solution also exists for H^{γ/2−1} initial data, but the time of existence depends not only on the H^{γ/2−1} norm of u_0, but also on the profile of u_0. For more details on local well-posedness see [18].

L^2 solutions of (1.1) enjoy mass conservation

\[ \|u(t, \cdot)\|_{L^2(\mathbb{R}^d)} = \|u_0(\cdot)\|_{L^2(\mathbb{R}^d)}. \]

Moreover, H^1 solutions enjoy energy conservation

\[ E(u)(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{4} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^{\gamma}} |u(t, x)|^2 |u(t, y)|^2 \, dx \, dy = E(u)(0), \]

which together with mass conservation and the local theory immediately yields global well-posedness for (1.1) with initial data in H^1. A large amount of work has been devoted to global well-posedness and scattering for the Hartree equation, see [7–11,13,15,17–23].

Existence of global solutions in \( \mathbb{R}^3 \) to (1.1) corresponding to initial data below the energy threshold was recently obtained in [5] by using the method of “almost conservation laws” or “I-method” (for a detailed description of this method, see [25] or Section 3 below) and the interaction Morawetz estimate for the solution \( u \), where global well-posedness was obtained in \( H^s(\mathbb{R}^3) \) with \( s > \max(1/2, 4(\gamma - 2)/(3\gamma - 4)) \) (see Fig. 1). Since authors in [5] used the interaction Morawetz estimate, which involves \( H^{1/2} \) norm of the solution, the restriction condition \( s \geq 1/2 \) is prerequisite. In order to resolve IVP (1.1) in \( H^s \), \( s < \frac{1}{2} \) by still using the interaction Morawetz estimate, we need return to the interaction Morawetz estimate for the smoothed out version \( Iu \) of the solution, which is used simultaneously in [2] and [6].

In this paper, we consider the case \( d \geq 3 \) and we prove the following result:

**Theorem 1.1.** Let \( 2 < \gamma < 3 \leq d \), the initial value problem (1.1) is globally well-posed in \( H^s(\mathbb{R}^d) \) for any \( s > \frac{4(\gamma - 2)}{3\gamma - 4} \). Moreover the solution satisfies

\[ \sup_{t \in [0, \infty)} \|u(t)\|_{H^s(\mathbb{R}^d)} \leq C(\|u_0\|_{H^s}), \]

and there is scattering for these solutions, that is, the wave operators exist and there is asymptotic completeness on all of \( H^s(\mathbb{R}^d) \).
Remark 1.1. As for the case $3 \leq \gamma < 4 \leq d$, local well-posedness for the IVP (1.1) in $H^s$ holds for any $s > \frac{\gamma}{2} - 1$. Note that in this case, we have

$$\frac{\gamma}{2} - 1 \geq \frac{1}{2},$$

which satisfies the need of the regularity of the interaction Morawetz estimate. Hence for the case $3 \leq \gamma < 4 \leq d$, we only need to combine "I-method" with the interaction Morawetz estimate for the solution, instead of the interaction Morawetz estimate for the smoothed out version $Iu$ of the solution, to obtain the low regularity global solution of the IVP (1.1), just as in [3].

Our method follows closely the recent developments in [2,4] and [6], where the main two ingredients are the "I-method" and an almost Morawetz type estimate following the work of Lin and Strauss [16]. In order to obtain the low regularity global solution of the IVP (1.1), we combine I-method with an interaction Morawetz estimate for the smoothed out version $Iu$ of the solution. By comparison with the interaction Morawetz estimate for the solution in [5], such a Morawetz estimate for an almost solution is the main novelty of this paper, which helps us to lower the need on the regularity of the initial data. In addition, we do not use the monotonicity property of the multiplier $m(\xi) \cdot \langle \xi \rangle^p$ in the proof of the almost conservation law.

Last, we organize this paper as following: In Section 2, we introduce some notation and state some important propositions that we will use throughout this paper. In Section 3, we review the I-method, prove the local well-posedness theory for $Iu$ and obtain an upper bound on the increment of the modified energy. In Section 4, we prove the "almost interaction Morawetz estimate" for the smoothed out version $Iu$ of the solution. Finally in Section 5, we give the details of the proof of the global well-posedness stated in Theorem 1.1.

2. Notation and preliminaries

2.1. Notation

In what follows, we use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant $C$. If $A \lesssim B$ and $B \lesssim A$, we say that $A \approx B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition $(a) := 1 + |a|$ and $a \pm : = a \pm \epsilon$ with $0 < \epsilon \ll 1$. The reader also has to be alert that we sometimes do not explicitly write down constants that depend on the $L^2$ norm of the solution. This is justified by the conservation of the $L^2$ norm.

2.2. Definition of spaces

We use $L^r_x(\mathbb{R}^d)$ to denote the Lebesgue space of functions $f : \mathbb{R}^d \to \mathbb{C}$ whose norm

$$\|f\|_{L^r_x} := \left( \int_{\mathbb{R}^d} |f(x)|^r \, dx \right)^{\frac{1}{r}}$$

is finite, with the usual modification in the case $r = \infty$. We also use the space–time Lebesgue spaces $L^q_t L^r_x$ which are equipped with the norm

$$\|u\|_{L^q_t L^r_x} := \left( \int_J \|u(t,x)\|_{L^r_x}^q \, dt \right)^{\frac{1}{q}}$$

for any space–time slab $J \times \mathbb{R}$, with the usual modification when either $q$ or $r$ are infinity. When $q = r$, we abbreviate $L^q_t L^r_x$ by $L^q_{t,x}$.

As usual, we define the Fourier transform of $f(x) \in L^1_x$ by

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) \, dx.$$
We define the fractional differentiation operator $|\nabla|^\alpha x$ for any real $\alpha$ by

$$|
abla|^\alpha u(\xi) := |\xi|^\alpha \hat{u}(\xi),$$

and analogously

$$\langle \nabla \rangle^\alpha u(\xi) := \langle \xi \rangle^\alpha \hat{u}(\xi).$$

The (in)homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ ($H^s(\mathbb{R}^d)$) is given via

$$\|u\|_{\dot{H}^s} := \|\nabla|^su\|_{L^2(\mathbb{R}^d)},$$

$$\|u\|_{H^s} := \|\langle \nabla \rangle^su\|_{L^2(\mathbb{R}^d)},$$

and more general (in)homogeneous Sobolev spaces are given via

$$\|u\|_{\dot{H}^{s,p}} := \|\nabla|^su\|_{L^p(\mathbb{R}^d)},$$

$$\|u\|_{H^{s,p}} := \|\langle \nabla \rangle^su\|_{L^p(\mathbb{R}^d)}.$$
Proposition 2.2. Let $d \geq 3$. For $r < \infty$, $0 \leq \frac{2}{q} \leq \min(\delta(r), 1)$, we have
\[
\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{\delta(r)}_{\frac{d}{q} + \frac{1}{2} +}}.
\]
While for $2 \leq q \leq \infty$, $r = \infty$, we have
\[
\|u\|_{L^q_t L^\infty_x} \lesssim \|u\|_{X^{d}_{\frac{d}{q} + \frac{1}{2} +}}.
\]

We will also need the Littlewood–Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leq 2$ which equals 1 on the ball $|\xi| \leq 1$. For each dyadic number $N \in 2\mathbb{Z}$, we define the Littlewood–Paley operators
\[
\hat{P}_{\leq N} f(\xi) := \varphi \left( \frac{\xi}{N} \right) \hat{f}(\xi), \quad \hat{P}_{\geq N} f(\xi) := \left( 1 - \varphi \left( \frac{\xi}{N} \right) \right) \hat{f}(\xi),
\]
\[
\hat{P}_N f(\xi) := \left( \varphi \left( \frac{\xi}{N} \right) - \varphi \left( \frac{2\xi}{N} \right) \right) \hat{f}(\xi).
\]
Similarly we can define $P_{< N}$, $P_{\geq N}$, and $P_{M < N} = P_{\leq N} - P_{\leq M}$, whenever $M$ and $N$ are dyadic numbers. We will frequently write $f_{\leq N}$ for $P_{\leq N} f$ and similarly for the other operators.

The Littlewood–Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every $L^p_x$ and $\dot{H}^s_x$ space for $1 \leq p \leq \infty$ and $s \geq 0$. They also obey the following Sobolev and Bernstein estimates
\[
\|P_{\geq N} f\|_{L^p} \lesssim N^{-s} \|\nabla^s P_{\geq N} f\|_{L^p},
\]
\[
\|\nabla^s P_{\leq N} f\|_{L^q} \lesssim N^{s + \frac{d}{q} - \frac{d}{p}} \|P_{\leq N} f\|_{L^p},
\]
\[
\|\nabla^{\pm s} P_N f\|_{L^q} \lesssim N^{\pm s + \frac{d}{q} - \frac{d}{p}} \|P_N f\|_{L^p},
\]
whenever $s \geq 0$ and $1 \leq p \leq q \leq \infty$.

3. The I-method and the modified local well-posedness

3.1. The I-operator and the hierarchy of energies

Let us define the operator $I$. For $s < 1$ and a parameter $N \gg 1$, let $m(\xi)$ be the following smooth monotone multiplier:
\[
m(\xi) := \begin{cases} 
1, & \text{if } |\xi| < N, \\
\left( \frac{N}{|\xi|} \right)^{1-s}, & \text{if } |\xi| > 2N.
\end{cases}
\]
We define the multiplier operator $I : H^s \to H^1$ by
\[
\hat{I} u(\xi) = m(\xi) \hat{u}(\xi).
\]
The operator $I$ is smoothing of order $1 - s$ and we have from Bernstein’s estimates that
\[
\|u\|_{H^{s_0}} \lesssim \|I u\|_{H^{s_0+1-s}} \lesssim N^{1-s} \|u\|_{H^{s_0}},
\]
\[
\|u\|_{X^{s_0}_{\delta_00}} \lesssim \|I u\|_{X^{s_0+1-s}_{\delta_00}} \lesssim N^{1-s} \|u\|_{X^{s_0}_{\delta_00}}
\]
for any $s_0, \delta_00 \in \mathbb{R}$.

We set
\[
\tilde{E}(u) = E(I u),
\]
where
\[
E(u)(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4} \iint \frac{1}{|x - y|^\nu} |u(t, x)|^2 |u(t, y)|^2 \, dx \, dy.
\]
We call $\tilde{E}(u)$ the modified energy. Since we will focus on the analysis of the modified energy, we collect some facts concerning the calculus of multilinear forms used to define the modified energy.

If $k \geq 2$ is an even integer, we define a spatial multiplier of order $k$ to be the function $M_k(\xi_1, \xi_2, \ldots, \xi_k)$ on

$$\Gamma_k = \{(\xi_1, \xi_2, \ldots, \xi_k) \in (\mathbb{R}^d)^k : \sum_{j=1}^{k} \xi_j = 0\},$$

which we endow with the standard measure $\delta(\xi_1 + \xi_2 + \cdots + \xi_k)$. If $M_k$ is a multiplier of order $k$, $1 \leq j \leq k$ is an index and $l \geq 1$ is an even integer, the elongation $X_j^l(M_k)$ of $M_k$ is defined to be the multiplier of order $k + l$ given by

$$X_j^l(M_k)(\xi_1, \xi_2, \ldots, \xi_{k+1}) = M_k(\xi_1, \ldots, \xi_{j-1}, \xi_j + 1 + \xi_{j+l}, \xi_{j+l+1}, \ldots, \xi_{k+l}).$$

Also if $M_k$ is a multiplier of order $k$ and $u_1, u_2, \ldots, u_k$ are functions on $\mathbb{R}^d$, we define the $k$-linear functional

$$A_k(M_k; u_1, u_2, \ldots, u_k) = \text{Re} \int_{\Gamma_k} M_k(\xi_1, \xi_2, \ldots, \xi_k) \prod_{j=1}^{k} \tilde{u}_j(\xi_j)$$

and we adopt the notation $A_k(M_k; u) = A_k(M_k; u, \tilde{u}, \ldots, \tilde{u})$. We observe that the quantity $A_k(M_k; u)$ is invariant

1. if one permutes the even arguments $\xi_2, \xi_4, \ldots, \xi_k$ of $M_k$;
2. if one permutes the odd arguments $\xi_1, \xi_3, \ldots, \xi_{k-1}$ of $M_k$;
3. if one makes the change of $M_k(\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_k) \mapsto M_k(-\xi_2, -\xi_1, \ldots, -\xi_k, -\xi_{k-1})$.

If $u$ is a solution of (1.1), the following differentiation law holds for the multiplier forms $A_k(M_k; u)$

$$\partial_t A_k(M_k; u) = A_k \left( i M_k \sum_{j=1}^{k} (-1)^j |\xi_j|^2 \hat{u}_j + \hat{u} \right) + A_{k+2} \left( i \sum_{j=1}^{k} (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X^2_j(M_k; u) \right)$$

(3.2)

where we used the notational convention $\xi_{a,b} = \xi_a + \xi_b$, $\xi_{a,b,c} = \xi_a + \xi_b + \xi_c$, etc. Indeed, note that

$$i \partial_t u_j + \Delta u_j = (|x|^{-\gamma} * |u_j|^2) u_j, \quad \text{for } 0 \leq j \leq k,$$

we take Fourier transformation in spatial variable and deduce

$$\partial_t \hat{u}_j(\xi_j) = -i|\xi_j|^2 \hat{u}_j(\xi_j) - i \int_{\xi_j = \xi_{j+1,j+2}} \hat{u}_{j+1,j+2}^{-(d-\gamma)} \hat{u}_j(\xi_j) \hat{u}_j(\xi_{j+1,j+2}) \, d\xi_{j+1,j+2},$$

$$\partial_t \hat{u}_j(\xi_j) = +i|\xi_j|^2 \hat{u}_j(\xi_j) + i \int_{\xi_j = \xi_{j+1,j+2}} \hat{u}_{j+1,j+2}^{-(d-\gamma)} \hat{u}_j(\xi_j) \hat{u}_j(\xi_{j+1,j+2}) \, d\xi_{j+1,j+2}.$$ 

From the above identities, we can obtain (3.2).

Using the above notation, the modified energy (3.1) can be written as follows:

$$\tilde{E}(u) = A_2 \left(-\frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u\right) + A_4 \left(\frac{1}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u\right)$$

where we abbreviate $m(\xi_j)$ as $m_j$.

Together with the differentiation rules (3.2) and the symmetry properties of $k$-linear functional $A_k(M_k; u)$, we obtain

$$\partial_t A_2 \left(-\frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u\right) = A_2 \left(-\frac{i}{2} \xi_1 m_1 \cdot \xi_2 m_2 \sum_{j=1}^{2} (-1)^j |\xi_j|^2; u\right)$$
Then for any

where we used the fact that \( \sum_{j=1}^{2} (-1)^j |\xi_j|^2 = 0 \) on the hyperplane \( \Gamma_2 \). At the same time, we have

\[
\partial_t A_4 \left( \frac{1}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u \right) \\
= A_4 \left( \frac{1}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4 \sum_{j=1}^{4} (-1)^j |\xi_j|^2; u \right) \\
+ A_6 \left( \frac{1}{4} \sum_{j=1}^{4} (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X_j^2 (|\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4); u \right) \\
= -A_4 \left( i |\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 m_2 m_3 m_4; u \right) - A_6 \left( i |\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_1, 2, 3 m_4 m_5 m_6; u \right) \\
= -A_4 \left( i |\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 m_2 m_3 m_4; u \right) + A_6 \left( i |\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_1, 2, 3 (m_1, 2, 3 - m_4 m_5 m_6); u \right).
\]

The fundamental theorem of calculus together with these estimates implies the following proposition, which will be used to prove that \( \tilde{E} \) is almost conserved.

**Proposition 3.1.** Let \( u \) be an \( H^1 \) solution to (1.1). Then for any \( T \in \mathbb{R} \) and \( \delta > 0 \), we have

\[
\tilde{E}(u)(T + \delta) - \tilde{E}(u)(T) = \int_T^{T+\delta} A_4(M_4; u) \, dt + \int_T^{T+\delta} A_6(M_6; u) \, dt
\]

with

\[
M_4 = i |\xi_{2,3}|^{-(d-\gamma)} |\xi_1|^2 m_1 (m_1 - m_2 m_3 m_4); \\
M_6 = i |\xi_{2,3}|^{-(d-\gamma)} |\xi_{4,5}|^{-(d-\gamma)} m_1, 2, 3 (m_1, 2, 3 - m_4 m_5 m_6).
\]

Furthermore if \( |\xi_j| \ll N \) for all \( j \), then the multipliers \( M_4 \) and \( M_6 \) vanish on \( \Gamma_4 \) and \( \Gamma_6 \), respectively.

### 3.2. Modified local well-posedness

In this subsection, we shall prove a local well-posedness result for the modified solution \( Iu \) and some a priori estimates for it.

Let \( J = [t_0, t_1] \) be an interval of time. We denote by \( Z_I(J) \) the following space:

\[
Z_I(J) = S_I(J) \cap X^{1, \frac{d}{2} +} (J)
\]

where

\[
S_I(J) = \left\{ u; \sup_{(q,r) \text{ admissible}} \| (\nabla) Iu \|_{L_q^q L_r^r (\mathbb{R}^d)} < \infty \right\},
\]

\[
X^{1, \frac{d}{2} +} (J) = \left\{ u; \| Iu \|_{X^{1, \frac{d}{2} +} (J \times \mathbb{R}^d)} < \infty \right\}.
\]

**Proposition 3.2.** Let \( 2 < \gamma < 3 \leq d, s > \frac{\gamma}{2} - 1 \), and consider the IVP

\[
i Iu_t + \Delta_I u = I (|x|^{-\gamma} \ast |u|^2 u), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R},
\]

\[
Iu(t_0, x) = Iu_0(x) \in H^s(\mathbb{R}^d).
\]

Then for any \( u_0 \in H^s \), there exists a time interval \( J = [t_0, t_0 + \delta] \), \( \delta = \delta(\| Iu_0 \|_{H^s}) \) and there exists a unique \( u \in Z_I(J) \) solution to (3.3). Moreover it is continuity with respect to the initial data.
Proof. The proof of this proposition proceeds by the usual fixed point method on the space $Z_I(J)$. Since the estimates are very similar to the ones we provide in the proof of Proposition 3.3 below, in particular (3.9) and (3.10), we omit the details. □

**Proposition 3.3.** Let $2 < \gamma < 3 < d$ and $s > \frac{\gamma}{2} - 1$. If $u$ is a solution to the IVP (3.3) on the interval $J = [t_0, t_1]$, which satisfies the following a priori bound

$$\|Iu\|_{L_t^4 H_x^s}^{d-2} (J \times \mathbb{R}^d) < \mu,$$

where $\mu$ is a small universal constant, then

$$\|u\|_{Z_I(J)} \lesssim \|u_0\|_{H^s}.$$

**Proof.** We start by obtaining a control of the Strichartz norms. Applying $(q, r)$ in Proposition 2.1. For any pair of admissible exponents $(q, r)$ following lines. We note that a similar approach was used in [3]. We write

$$\langle\nabla\rangle u = \int_{\mathbb{R}^d} \hat{u}(\xi) e^{i \xi \cdot x} \langle|\xi|\rangle d\xi,$$

where we used Hölder’s inequality and Hardy–Littlewood–Sobolev’s inequality.

In order to obtain an upper bound on $\|u\|_{L_t^q L_x^r}$, we perform a Littlewood–Paley decomposition along the following lines. We note that a similar approach was used in [3]. We write

$$u = u_{N_0} + \sum_{j=1}^{\infty} u_{N_j},$$

where $u_{N_0}$ has spatial frequency support for $|\xi| \leq N$, while $u_{N_j}$ is such that its spatial Fourier support transform is supported for $|\xi| \approx N_j = 2^j h$ with $h_j \gtrsim \log N$ and $j = 1, 2, \ldots$. By the triangle inequality and Hölder’s inequality, we have

$$\|u\|_{L_t^q L_x^r} \lesssim \|u_{N_0}\|_{L_t^q L_x^r} + \sum_{j=1}^{\infty} \|u_{N_j}\|_{L_t^q L_x^r} \lesssim \|u_{N_0}\|_{L_t^q L_x^r} + \sum_{j=1}^{\infty} \|u_{N_j}\|_{L_t^q L_x^r}.$$
\[ \| \langle \nabla \rangle I u_{N_j} \|_{L_t^6 L_x^{\frac{6d}{3d-s}}} \approx N_j \left( \frac{N}{N_j} \right)^{1-s} \| u_{N_j} \|_{L_t^6 L_x^{\frac{6d}{3d-s}}} \], \quad j = 1, 2, \ldots,
\]
\[ \| I u_{N_j} \|_{L_t^6 L_x^{\frac{6d}{3d-s}}} \approx \left( \frac{N}{N_j} \right)^{1-s} \| u_{N_j} \|_{L_t^6 L_x^{\frac{6d}{3d-s}}} \], \quad j = 1, 2, \ldots.

Now we use these estimates to obtain the following upper bound on (3.7)
\[ \| u \|_{L_t^6 L_x^{\frac{6d}{3d+3\delta}}} \lesssim \| I u_{N_0} \|_{L_t^6 H^\delta_x} \| u_{N_0} \|_{Z^1(J)} \]
\[ + \sum_{j=1}^\infty \left( \frac{1}{N_j} \left( \frac{N_j}{N} \right)^{1-s} \| u_{N_j} \|_{Z^1(J)} \right)^{2-\frac{2}{\gamma}} \left( \left( \frac{N_j}{N} \right)^{1-s} \| u_{N_j} \|_{Z^1(J)} \right)^{\frac{\gamma}{2}} \]
\[ \lesssim \mu^\delta \| u \|_{Z^1(J)} + N^{-2-\frac{2}{\gamma}} \| u \|_{Z^1(J)} \], \quad (3.8)

which together with (3.5) implies that
\[ \| \langle \nabla \rangle I u \|_{L_t^6 L_x^4} \lesssim \| I u_0 \|_{H^1} + \mu^\delta \| u \|_{Z^1(J)} + N^{-(4-\gamma)} \| u \|_{Z^1(J)}^3 \], \quad (3.9)

Now we shall obtain a control of the \( X^{s,b} \) norm. We use Duhamel’s formula and the theory of \( X^{s,b} \) spaces \([12,25]\) to obtain
\[ \| I u \|_{X^{s,b}} \lesssim \| I u_0 \|_{H^1} + \| \langle \nabla \rangle I (|x|^{-\gamma} |u|^2 u) \|_{X^{s,b}} \]
\[ \lesssim \| I u_0 \|_{H^1} + \| \langle \nabla \rangle I (|x|^{-\gamma} |u|^2 u) \|_{L_t^2 L_x^{\frac{6d}{3d+3\delta}}} \]
\[ \lesssim \| I u_0 \|_{H^1} + \| \langle \nabla \rangle I u \|_{L_t^{\frac{6d}{3d+\delta}} L_x^{\frac{6d}{3d+3\delta}} \}} + \| \langle \nabla \rangle I u \|_{L_t^{\frac{6d}{3d+\delta}} L_x^{\frac{6d}{3d+3\delta}} \}} \], \quad (3.10)

An upper bound on \( \| u \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} \) is given by (3.8). In order to obtain an upper bound on \( \| u \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} \), we proceed as follows. First we perform a dyadic decomposition and write \( u \) as in (3.6). The triangle inequality applied on (3.6) gives for any \( 0 < \delta < \frac{\gamma}{2} - 1 \)
\[ \| u \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} \lesssim \| I u_{N_0} \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} + \sum_{j=1}^\infty \| u_{N_j} \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} \]
\[ = \| I u_{N_0} \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} + \sum_{j=1}^\infty N_j^{\frac{3d-s}{3}} N^{s-1} \| \langle \nabla \rangle^{1-\delta} I u_{N_j} \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} + \]
\[ \lesssim \| I u \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} + \| \langle \nabla \rangle^{1-\delta} I u \|_{L_t^{6d} L_x^{\frac{6d}{3d+3\delta}}} \lesssim \| I u \|_{X^{s,b}} \], \quad (3.11)

where we used Proposition 2.2. By applying the inequalities (3.8) and (3.11) to bound the right-hand side of (3.10), we obtain
\[ \| I u \|_{X^{s,b}} \lesssim \| I u_0 \|_{H^1} + \mu^\delta \| u \|_{Z^1(J)}^3 + N^{-(2-\frac{2}{\gamma})} \| u \|_{Z^1(J)}^3 \], \quad (3.12)

The desired bound follows from (3.9) and (3.12) by choosing \( N \) sufficiently large. \( \Box \)

3.3. An upper bound on the increment of \( \tilde{E}(u) \)

**Decomposition remark.** Our approach to prove a decay for the increment of the modified energy is based on obtaining certain multilinear estimates in appropriate functional spaces which are \( L^2 \)-based (more details, see [24]). Hence, whenever we perform a Littlewood–Paley decomposition of a function we shall assume that the Fourier transforms of the Littlewood–Paley pieces are positive. Moreover, we will ignore the presence of conjugates. At the end we will always keep a decay factor \( C(N_1, N_2, \ldots) \) in order to perform the summations.
Now we proceed to prove the almost conservation law of the modified energy. In Proposition 3.1, we prove that an increment of the modified energy can be expressed as
\[ \tilde{E}(u)(T + \delta) - \tilde{E}(u)(T) = T + \delta \int_T^T \Lambda_4(M_4; u) \, dt + T + \delta \int_T^T \Lambda_6(M_6; u) \, dt \]
with
\[ M_4 = i|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1(m_1 - 2m_3m_4); \]
\[ M_6 = i|\xi_{2,3}|^{-(d-\gamma)}|\xi_{4,5}|^{-(d-\gamma)}m_{1,2,3}(m_{1,2,3} - 4m_5m_6). \]
Hence in order to control the increment of the modified energy, we shall find an upper bound on the \( \Lambda_4(M_4; u) \) and \( \Lambda_6(M_6; u) \) forms, which we do in the following propositions. First we give the quadrilinear estimate.

**Proposition 3.4.** For any Schwartz function \( u \), and any \( \delta \approx 1 \) just as in Proposition 3.2, we have that
\[ \left| T + \delta \int_T^T \Lambda_4(M_4; u) \, dt \right| \lesssim N^{1-} \| Iu \|^4_{X^{1,\frac{1}{2}+}}, \quad (3.13) \]
for \( s > \frac{\gamma}{2} - 1 \).

**Proof.** By Plancherel’s theorem, we aim to prove that
\[ \left| \int_T^T \int_T^T |\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1(m_1 - 2m_3m_4) \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4) \right| \]
\[ \lesssim N^{1-} C(N_1, N_2, N_3, N_4) \prod_{j=1}^4 \| Iu_j \|^4_{X^{1,\frac{1}{2}+}}, \quad (3.14) \]
where \( C(N_1, N_2, N_3, N_4) \) is a decay just as the remark above, and it allows us to sum over all dyadic shells. The analysis which follows will not rely on the complex conjugate structure in \( \Lambda_4(M_4; u) \). Thus, by symmetry, we may assume that \( N_2 \geq N_3 \geq N_4 \).

**Case 1.** \( N \gg N_2 \). According to the definition of \( m(\xi) \), the multiplier
\[ |\xi_{2,3}|^{-(d-\gamma)}m_1(m_1 - 2m_3m_4) \]
is identically 0, the bound (3.13) holds trivially.

**Case 2.** \( N_2 \gg N \gg N_3 \gg N_4 \). Since \( \sum_{j=1}^4 \xi_j = 0 \), we have \( N_1 \approx N_2 \). We aim for (3.14) with a decay factor
\[ C(N_1, N_2, N_3, N_4) = N_2^{0-}. \]

By the mean value theorem, we have the following pointwise bound
\[ |m_1(m_1 - 2m_3m_4)| = |m_1(m_{2,3,4} - 2m_3m_4)| \]
\[ \lesssim m_1 |\nabla m(\xi) \cdot (\xi_3 + \xi_4)| \quad \text{where } |\xi| \approx |\xi_2| \]
\[ \lesssim m_1 m_2 N_3 \frac{N_2}{N_2} \]
Hence by Hölder’s inequality and Hardy–Littlewood–Sobolev’s inequality, Proposition 2.2 and Bernstein’s estimate, we obtain
\[ \text{LHS of (3.14)} \lesssim N_1^2 m_1 m_2 \left| \int_{T}^{T+\delta} \int_{\Gamma_4} \left| \xi_{2,3}^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4) \right| \right| \]
\[
\lesssim N_1^2 m_1 m_2 \frac{N_3}{N_2} \| u_1 \| L^\gamma_1 L^\delta_4 \| u_2 \| L^\gamma_1 L^\delta_4 \| u_3 \| L^\gamma_1 L^\delta_4 \| u_4 \| L^\gamma_1 L^\delta_4 \]
\[
\lesssim N_1^2 m_1 m_2 \frac{N_3}{N_2} N_4^{-\gamma-2} \prod_{j=1}^{4} \| u_j \|_{\mathcal{X}^{0,\frac{1}{2}+}}.
\]

where we used the fact that
\[
\begin{align*}
\frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} &= 1, \\
\delta \left( \frac{6d}{3d-4} \right) - \frac{2}{3} &= 0, \\
d \left( \frac{6d}{3d-2} \right) - \frac{2}{6} &= 0, \\
\delta \left( \frac{6d}{3d+10-6\gamma} \right) - \frac{2}{6} &= \gamma - 2.
\end{align*}
\]

It suffices to show that
\[
N_1^2 m_1 m_2 \frac{N_3}{N_2} N_4^{-\gamma-2} \lesssim N^{-1+} N_2^{-1} m_1 N_1 m_2 N_2 (N_3) (N_4).
\]

We reduce to show that
\[
N_1^{-1} N_2^{0+} \lesssim N_2 (N_3) N_3^{-1} (N_4) N_4^{2-\gamma}.
\]

This is true since
\[
N_2 \gtrsim N_1^{-1+} N_2^{0+}; \quad (N_3) N_3^{-1} \gtrsim 1; \quad (N_4) N_4^{2-\gamma} \gtrsim 1.
\]

**Case 3.** \( N_2 \gtrsim N_3 \gtrsim N \). In this case, we use the trivial pointwise bound
\[
|m_1 (m_1 - m_2 m_3 m_4)| \lesssim m_1^2.
\]
The frequency interactions fall into two subcategories, depending on which frequency is comparable to \( N_2 \).

**Case 3a.** \( N_1 \approx N_2 \gtrsim N_3 \gtrsim N \). In this case, we prove the decay factor
\[
C(N_1, N_2, N_3, N_4) = N_3^0-
\]
in \( (3.14) \). This allows us to directly sum in \( N_3 \) and \( N_4 \), and sum in \( N_1 \) and \( N_2 \) after applying Cauchy–Schwarz to those factors.

By Hölder’s inequality, Hardy–Littlewood–Sobolev’s inequality, Proposition 2.2 and Bernstein’s estimate, we obtain
\[
\text{LHS of (3.14)} \lesssim N_1^2 m_1^2 \left| \int_{T}^{T+\delta} \int_{\Gamma_4} \left| \xi_{2,3}^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4) \right| \right| \]
\[
\lesssim N_1^2 m_1^2 \| u_1 \| L^\gamma_1 L^\delta_4 \| u_2 \| L^\gamma_1 L^\delta_4 \| u_3 \| L^\gamma_1 L^\delta_4 \| u_4 \| L^\gamma_1 L^\delta_4 \]
\[
\lesssim N_1^2 m_1^2 N_4^{-\gamma-2} \prod_{j=1}^{4} \| u_j \|_{\mathcal{X}^{0,\frac{1}{2}+}}.
\]

It suffices to show that
\[
N_1^2 m_1^2 N_4^{-\gamma-2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_2 m_3 N_3 (N_4).
\]

We reduce to show that
\[ N^{1-\gamma} N_3^{\frac{10}{3}} \lesssim m_3 N_3 m_4 (N_4) N_4^{2-\gamma}. \]

This is true since for \( s \geq \gamma - 2 \), we have
\[ m_3 N_3 m_4 (N_4) N_4^{2-\gamma} \geq m_3 N_3 m_4 (N_4)^{3-\gamma} \gtrsim m_3 N_3 \gtrsim N^{1-\gamma} N_3^{\frac{10}{3}}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-decreasing if \( s + p \geq 1 \). While for \( \frac{\gamma}{2} - 1 < s < \gamma - 2 \), we have
\[ m_3 N_3 m_4 (N_4) N_4^{2-\gamma} \gtrsim m_3 N_3 m_3 N_3^{3-\gamma} \gtrsim N^{1-\gamma} N_3^{\frac{10}{3}} \gtrsim N^{1-\gamma} N_3^{\frac{10}{3}}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-increasing if \( s + p < 1 \).

**Case 3b.** \( N_2 \approx N_3 \gtrsim N, N_2 \gtrsim N_1 \). In this case, we prove the decay factor
\[ C(N_1, N_2, N_3, N_4) = N_2^{-\frac{1}{2}} \]
in (3.14). This will allow us to directly sum in all the \( N_j \).

By Hölder’s inequality, Hardy–Littlewood–Sobolev’s inequality, Proposition 2.2 and Bernstein’s estimate once again, we obtain
\[
\text{LHS of (3.14)} \lesssim N_1^2 \int_0^{T+\delta} \int \int |\xi_{2,3}^{2}|^{-(d-\gamma)} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \hat{u}_4(t, \xi_4) \left| \frac{\partial}{\partial t} \right|_{\xi_{2,3}^{2}} \nonumber \]
\[
\lesssim N_1^2 \int_0^{T+\delta} \int \int \frac{\partial}{\partial t} \| u_1 \|_{L^2_t L^\infty_x} \| u_2 \|_{L^2_t L^\infty_x} \| u_3 \|_{L^2_t L^\infty_x} \| u_4 \|_{L^2_t L^\infty_x} \nonumber \]
\[
\lesssim N_1^2 \int_0^{T+\delta} \int \int \| u_j \|_{H^{0,\frac{1}{2}}} \nonumber \]

It suffices to show that
\[ N_1^2 m_1^2 N_4^{\gamma - 2} \lesssim N^{1-\gamma} N_2^{-\frac{1}{2}} m_1 N_1 m_2 N_2 m_3 N_3 (N_4). \]

Note that
\[ N_1^2 m_1^2 \lesssim m_1 N_1 m_2 N_2. \]
We reduce to show that
\[ N^{1-\gamma} N_2^{\frac{10}{3}} \lesssim m_3 N_3 m_4 (N_4) N_4^{2-\gamma}. \]

This is true since for \( s \geq \gamma - 2 \), we have
\[ m_3 N_3 m_4 (N_4) N_4^{2-\gamma} \geq m_3 N_3 m_4 (N_4)^{3-\gamma} \gtrsim m_3 N_3 \approx m_2 N_2 \gtrsim N^{1-\gamma} N_2^{\frac{10}{3}}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-decreasing if \( s + p \geq 1 \). While for \( \frac{\gamma}{2} - 1 < s < \gamma - 2 \), we have
\[ m_3 N_3 m_4 (N_4) N_4^{2-\gamma} \gtrsim m_3 N_3 m_3 N_3^{3-\gamma} \approx m_2 N_2^{4-\gamma} \gtrsim N^{1-\gamma} N_3^{\frac{10}{3}} \gtrsim N^{1-\gamma} N_3^{\frac{10}{3}}, \]
where we used the fact that \( m(\xi) \langle \xi \rangle^p \) is monotone non-increasing if \( s + p < 1 \). This completes the proof. \( \square \)

In order to make use of quadrilinear estimate (Proposition 3.4) to obtain sextilinear estimate, we first give a lemma

**Lemma 3.1.** Assume \( u, \delta \) are as in Proposition 3.2, and \( P_{N_{1,2,3}} \) the Littlewood–Paley projection onto the \( N_{1,2,3} \) frequency shell. Then
\[
\| P_{N_{1,2,3}} (I (u|\nabla|^{-(d-\gamma)}|u|^2)) \|_{L^2_t L^\infty_x} \lesssim N_{1,2,3} \| u \|_{H^{0,\frac{1}{2}}}^{3}. \]
Proof. We write $u = u_L + u_H$ where
\[
\text{supp } \hat{u}_I(t, \xi) \subseteq \{ |\xi| < 2 \},
\]
\[
\text{supp } \hat{u}_H(t, \xi) \subseteq \{ |\xi| > 1 \}.
\]
Hence,
\[
\left\| P_{N,2,3} \left( I \left( |\nabla|^{-\left( -d - \gamma \right)} |u|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| P_{N,2,3} \left( I \left( |u_L|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
+ \left\| P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\]

Consider the first term. By Hölder’s inequality, Hardy–Littlewood–Sobolev’s inequality and Proposition 2.2, we have
\[
\left\| P_{N,2,3} \left( I \left( |u_L|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |u_L|^3 \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |u|^3 \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |u|^3 \right\|_{X^{\frac{3}{2} - \frac{1}{2}, \frac{1}{2}}}
\leq N_{1,2,3} \left\| |u|^3 \right\|_{X^{\frac{3}{2} - \frac{1}{2}, \frac{1}{2}}},
\]

since
\[
N_{1,2,3} \geq 1, \quad \text{and} \quad 0 \leq d \times \left( 2 - \frac{9d - 6\gamma - 4}{18d} \right) - \frac{2}{9} = \frac{\gamma}{3} \leq 1.
\]

We estimate the second term. By Sobolev’s inequality and using the Leibniz rule for the operator $|\nabla|^{2-\gamma I}$ (Principle A.5 in [25]) and Proposition 2.2, we have
\[
\left\| \frac{1}{N_{1,2,3}} P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{-1} P_{N,2,3} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{2-\gamma I} \left( I \left( |u_H|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{2-\gamma I} |u_H|^2 \right\|_{L_t^\frac{6d}{18-6\gamma+12} L_x^{\frac{18d}{18-6\gamma+12}}}
\lesssim \left\| |\nabla|^{2-\gamma I} |u_H|^3 \right\|_{X^{3,\frac{1}{2}, \frac{1}{2}}},
\]

As for the third term. By Sobolev’s inequality and using the Leibniz rule for the operator $|\nabla|^{2-\gamma I}$ and Proposition 2.2 again, we have
\[
\left\| \frac{1}{N_{1,2,3}} P_{N,2,3} \left( I \left( |u_L|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{-1} P_{N,2,3} \left( I \left( |u_L|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{2-\gamma I} \left( I \left( |u_L|^2 \right) \right) \right\|_{L_t^1 L_x^{\frac{6d}{5}} L^{\frac{18}{9}}}
\lesssim \left\| |\nabla|^{2-\gamma I} |u_L|^2 \right\|_{L_t^\frac{6d}{18-6\gamma+12} L_x^{\frac{18d}{18-6\gamma+12}}}
\lesssim \left\| |\nabla|^{2-\gamma I} |u_L|^3 \right\|_{X^{3,\frac{1}{2}, \frac{1}{2}}},
\]
Almost interaction Morawetz estimate

Theorem 4.1. Let \( u \) be a Schwartz solution to

\[
\begin{align*}
  iu_{t} + \Delta u &= \widetilde{N}(u), & (x, t) \in \mathbb{R}^d \times [0, T],
  \\
  iu_{t} + \Delta u &= I(\widetilde{N}(u)), & (x, t) \in \mathbb{R}^d \times [0, T].
\end{align*}
\]

where \( \widetilde{N}(u) = (|x|^{-\gamma} * |u|^2)u \). Let \( Iu \) be a solution to

\[
   iu_{t} + \Delta u = I(\widetilde{N}(u)),
\]

Then

\[
\begin{align*}
\| \nabla |^{-2} Iu \|_{L^4_t L^4_x} &\lesssim \| Iu \|_{L^\infty_t H^1_x} \| Iu \|_{L^2_t L^2_x}^3 + \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla a \cdot \{ \widetilde{N}_{\text{bad}}, Iu(t, x_1)Iu(t, x_2) \} dx_1 dx_2 dt
\end{align*}
\]

with \( \{ \cdot, \cdot \}_p \) is the momentum bracket defined by

\[
\{ f, g \}_p = \text{Re}(f \nabla \bar{g} - g \nabla \bar{f}),
\]
and

\[ \widetilde{N}_{\text{bad}} = \sum_{i=1}^{2} (I\widetilde{N}_i(u_i) - \widetilde{N}_i(Iu_i)) \prod_{j=1, j\neq i}^{2} Iu_j, \]

where \( u_i \) is a solution to

\[ iu_t + \Delta u = \widetilde{N}(u), \quad (x_i, t) \in \mathbb{R}^d \times \mathbb{R}, \quad d \geq 3, \]

(4.3)

here \( x_i \in \mathbb{R}^d \), not a coordinate. In particular, on a time interval \( J_k \) where the local well-posedness Proposition 3.2 holds, we have that

\[
\int_{J_k} \int_{\mathbb{R}^d} \nabla a \cdot \left\{ \widetilde{N}_{\text{bad}}, Iu(t,x_1)Iu(t,x_2) \right\} \rho dx_1 dx_2 dt \lesssim \frac{1}{N^1-} \|u\|_{Z^1(J_k)}^6.
\]

Toward this goal, we recall the idea of the proof of the interaction Morawetz estimate for the defocusing non-linear cubic Schrödinger equation in three space dimensions [3]. We present the result using a tensor of Schrödinger solutions that emerged in [2,6]. We first recall the generalized Virial identity [2,16].

**Proposition 4.1.** Let \( a : \mathbb{R}^d \to \mathbb{R} \) be convex and \( u \) be a smooth solution to the solution

\[ iu_t + \Delta u = \widetilde{N}(u), \quad (t,x) \in [0, T] \times \mathbb{R}^d. \]

(4.4)

Then the following inequality holds

\[
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{0}^{T} (-\Delta a) |u(t,x)|^2 dx dt + 2 \int_{\mathbb{R}^d} \int_{0}^{T} \nabla a \cdot \{\widetilde{N}, u\} \rho dx dt \\
&\lesssim |M_a(T) - M_a(0)|
\end{aligned}
\]

where \( M_a(t) \) is the Morawetz action corresponding to \( u \) and is given by

\[ M_a(t) = 2 \int_{\mathbb{R}^d} \nabla a(x) \cdot \text{Im}(\overline{u}(x) \nabla u(x)) dx. \]

**Proof of Theorem 4.1.** Now we rewrite Eq. (4.1) as

\[ iIu_t + \Delta Iu = \widetilde{N}(Iu) + \left( I(\widetilde{N}(u)) - \widetilde{N}(Iu) \right). \]

By Proposition 4.1, we have

\[
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{0}^{T} (-\Delta a) |Iu(t,x)|^2 dx dt - \int_{\mathbb{R}^d} \int_{0}^{T} \left| Iu(t,y) \right|^2 \frac{x - y}{|x - y|} (\nabla |\nabla|^{-\frac{(d-4)}{2}} |Iu|^2 |Iu|^2)(t,x) dx dy dt \\
&\lesssim \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \nabla a(x) \cdot \text{Im}(\overline{Iu}(x) \nabla Iu(x)) dx \right| + \int_{\mathbb{R}^d} \int_{0}^{T} \nabla a \cdot \{ I\widetilde{N}(u) - \widetilde{N}(Iu), Iu \} \rho dx dt,
\end{aligned}
\]

where the second term on the left-hand side

\[
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{0}^{T} \left| Iu(t,y) \right|^2 \frac{x - y}{|x - y|} (\nabla |\nabla|^{-\frac{(d-4)}{2}} |Iu|^2 |Iu|^2)(t,x) dx dy dt \\
&\lesssim \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \nabla a(x) \cdot \text{Im}(\overline{Iu}(x) \nabla Iu(x)) dx \right| + \int_{\mathbb{R}^d} \int_{0}^{T} \nabla a \cdot \{ I\widetilde{N}(u) - \widetilde{N}(Iu), Iu \} \rho dx dt,
\end{aligned}
\]

(4.5)

is created by the term \( \widetilde{N}(Iu) \) and the commutator \( I(\widetilde{N}(u)) - \widetilde{N}(Iu) \) creates the second term on the right-hand side

\[
\begin{aligned}
&\int_{\mathbb{R}^d} \int_{0}^{T} \nabla a \cdot \{ I\widetilde{N}(u) - \widetilde{N}(Iu), Iu \} \rho dx dt,
\end{aligned}
\]

(4.6)
By symmetry, we know that the term (4.5) is a positive term that we can ignore, which is analogue to the case in [19,20]. Next we turn to the estimate of the error term (4.6). In addition, the conjugation will play no crucial role in the forthcoming argument.

Now define the tensor product \( u := (u_1 \otimes u_2)(t, x) \) for \( x \) in
\[
\mathbb{R}^{d+d} = \{ x = (x_1, x_2): \ x_1 \in \mathbb{R}^d, \ x_2 \in \mathbb{R}^d \}
\]
by the formula
\[
(u_1 \otimes u_2)(t, x) = u_1(t, x_1)u_2(t, x_2),
\]
let us set
\[
IU(t, x) = \sum_{j=1}^2 Iu(t, x_j).
\]

If \( u \) solves (4.4) for \( d \) dimensions, then \( IU \) solves (4.4) for \( 2d \) dimensions, with right-hand side \( \tilde{N}_I \) given by
\[
\tilde{N}_I = \sum_{i=1}^2 \left( I(\tilde{N}_i(u_i)) \prod_{j=1, j\neq i}^2 Iu_j \right).
\]

Now let us decompose
\[
\tilde{N}_I = \tilde{N}_{\text{good}} + \tilde{N}_{\text{bad}}
\]
\[
\triangleq \sum_{i=1}^2 \left( \tilde{N}_i(Iu_i) \prod_{j=1, j\neq i}^2 Iu_j \right) + \sum_{i=1}^2 \left( I(\tilde{N}_i(u_i)) - I(\tilde{N}_i(Iu_i)) \right) \prod_{j=1, j\neq i}^2 Iu_j.
\]

The first term summand creates a positive term that we can ignore again. The term we call \( \tilde{N}_{\text{bad}} \) produces the error term. Now we pick \( a(x) = a(x_1, x_2) = |x_1 - x_2| \) where \( (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d \). Hence we have
\[
\left\| \nabla \left| \frac{d-3}{2} Iu \right| L_{\infty}^{\frac{4}{3}} L_{\infty}^{\frac{4}{3}} \leq \left\| Iu \right\|_{L_{\infty}^{\frac{4}{3}} H_{\infty}^{\frac{4}{3}}}^3 + \left| \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{\text{bad}}, Iu(t, x_1)Iu(t, x_2) \} \right| dx_1 dx_2 dt,
\]

Note that the second term of the right-hand side comes from the momentum bracket term in the proof of Proposition 4.1. Following with the same calculations in [2], we deduce that
\[
\mathcal{E} := \left| \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{\text{bad}}, Iu(t, x_1)Iu(t, x_2) \} \right| dx_1 dx_2 dt 
\leq \left\| I(\tilde{N}(u)) - \tilde{N}(Iu) \right\|_{L_{\infty}^{\frac{4}{3}} L_{\infty}^{\frac{4}{3}}} + \left\| \nabla a \cdot \{ \tilde{N}(u), Iu(t, x) \} \right\|_{L_{\infty}^{\frac{4}{3}} L_{\infty}^{\frac{4}{3}}} \left\| u \right\|_{Z_{1}(J)}^5. \tag{4.7}
\]

Now we proceed to estimate \( \left\| \nabla a \cdot \{ I(\tilde{N}(u)) - \tilde{N}(Iu) \} \right\|_{L_{\infty}^{\frac{4}{3}} L_{\infty}^{\frac{4}{3}}} \), which is the harder term. The term \( \left\| I(\tilde{N}(u)) - \tilde{N}(Iu) \right\|_{L_{\infty}^{\frac{4}{3}} L_{\infty}^{\frac{4}{3}}} \) can be estimated in the same way. Note that
\[
\tilde{N}(u) = (|x|^{-\gamma} * |u|^2)u,
\]
we have
\[
\mathcal{F}_x (\nabla_x (I(\tilde{N}(u)) - \tilde{N}(Iu)))(\xi) = \int_{\xi = \sum_{j=1}^3 \xi_j} i\xi \xi_2 \xi_3^{-\gamma} (m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3))\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3.
\]

We decompose \( u \) into a sum of dyadic pieces \( u_j \) localized around \( N_j \), then
Hence, by Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we have
\[
\begin{align*}
\|\nabla_x(I(\tilde{N}(u)) - \tilde{N}(Iu))\|_{L^1_t L^2_x} \\
= \|\mathcal{F}_x(\nabla_x(I(\tilde{N}(u)) - \tilde{N}(Iu)))(\xi)\|_{L^1_t L^2_x} \\
\lesssim \sum_{N_1, N_2, N_3} \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi||\xi_{2,3}|^{-(d-\gamma)} m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3) |\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x}.
\end{align*}
\]
Since the conjugation plays no crucial role here, without loss of generality, we assume that
\[N_1 \gtrsim N_2 \gtrsim N_3.\]
Set
\[\sigma(\xi_1, \xi_2, \xi_3) = |\xi_1 + \xi_2 + \xi_3| m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3),\]
then
\[\sigma(\xi_1, \xi_2, \xi_3) = \sum_{j=1}^4 \chi_j(\xi_1, \xi_2, \xi_3) \sigma(\xi_1, \xi_2, \xi_3) := \sum_{j=1}^4 \sigma_j(\xi_1, \xi_2, \xi_3),\]
where \(\chi_j(\xi_1, \xi_2, \xi_3)\) is a smooth characteristic function of the set \(\Omega_j\) defined as follows:
\begin{itemize}
  \item \(\Omega_1 = \{||\xi_i| \approx N_1, i = 1, 2, 3; N \gg N_1\}\);
  \item \(\Omega_2 = \{||\xi_i| \approx N_1, i = 1, 2, 3; N \gtrsim N \gg N_2\}\);
  \item \(\Omega_3 = \{||\xi_i| \approx N_1, i = 1, 2, 3; N_1 \gg N_2 \gtrsim N \gg N_3\}\);
  \item \(\Omega_4 = \{||\xi_i| \approx N_1, i = 1, 2, 3; N_1 \gg N_2 \gtrsim N_3 \gtrsim N\}\).
\end{itemize}
Hence, we have
\[
\begin{align*}
\|\nabla_x(I(\tilde{N}(u)) - \tilde{N}(Iu))\|_{L^1_t L^2_x} \\
\lesssim \sum_{N_1, N_2, N_3} \sum_{j=1}^4 \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_j(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x} \\
:= \sum_{N_1, N_2, N_3} \sum_{j=1}^4 L_j.
\end{align*}
\]
**Contribution of \(L_1\).** Since \(\sigma_1\) is identically zero when \(N \geq 4N_1\), \(L_1\) gives no contribution to the sum above.

**Contribution of \(L_2\).** By the mean value theorem, we have the pointwise bound
\[\sigma_2(\xi_1, \xi_2, \xi_3) \lesssim N_1 \cdot m_1 N_2 \frac{N_2}{N_1} = m_1 N_2.\]
Hence, by Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we obtain
\[
\begin{align*}
L_2 = \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \sigma_2(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x} \\
\lesssim m_1 N_2 \left\| \int_{|\xi_j| \approx N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x}.
\end{align*}
\]
\[ \lesssim m_1 N_2 \| u_1 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \| u_2 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \| u_3 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \lesssim m_1 N_2 N_3^{\gamma - 2} \prod_{j=1}^3 \| u_j \|_{X^{\frac{1}{2}+}}, \]

where we used the fact that
\[ \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1, \quad \frac{3d - 4}{6d} + \frac{3d - 4}{6d} + \frac{3d - 6\gamma + 8}{6d} + \frac{1}{2} + 1, \]
\[ \delta \left( \frac{6d}{3d - 4} \right) \frac{2}{3} = 0, \quad \delta \left( \frac{6d}{3d - 6\gamma + 8} \right) \frac{2}{3} = \gamma - 2. \]

It suffices to show that
\[ m_1 N_2 N_3^{\gamma - 2} \lesssim N^{-1+} N_1^{0+} m_1 N_1 \langle N_2 \rangle \langle N_3 \rangle. \]

We reduce to show that
\[ N_1^{1-} N_1^{0+} \lesssim N_1 \langle N_2 \rangle N_2^{-1} \langle N_3 \rangle N_3^{2-\gamma}. \]

This is true since
\[ N_1 \gtrsim N_1^{1-}; \quad \langle N_2 \rangle N_2^{-1} \gtrsim 1; \quad \langle N_3 \rangle N_3^{2-\gamma} \gtrsim 1. \]

**Contribution of \( L_3 \).** Note that
\[ \sigma_3(\xi_1, \xi_2, \xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1. \]

Hence, by Hölder’s inequality, Hardy–Littlewood–Sobolev’s inequality, Proposition 2.2 and Bernstein’s estimate, we have
\[ L_3 = \left\| \int_{|\xi_j| = N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_2, \xi_3|^{-(d-\gamma)} \sigma_3(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x} \]
\[ \lesssim m_1 N_1 \left\| \int_{|\xi_j| = N_j, \xi = \sum_{j=1}^3 \xi_j} |\xi_2, \xi_3|^{-(d-\gamma)} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L^1_t L^2_x} \]
\[ \lesssim m_1 N_1 \| u_1 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \| u_2 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \| u_3 \|_{L^1_t L^{\frac{6d}{3d-4}}_x} \]
\[ \lesssim m_1 N_1 N_3^{\gamma - 2} \prod_{j=1}^3 \| u_j \|_{X^{\frac{1}{2}+}}. \]

It suffices to show that
\[ m_1 N_1 N_3^{\gamma - 2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_2 \langle N_3 \rangle. \]

We reduce to show that
\[ N^{1-} N_2^{0+} \lesssim m_2 N_2 \langle N_3 \rangle N_3^{2-\gamma}. \]

This is true since
\[ m_2 N_2 \gtrsim N^{1-} N_2^{0+}; \quad \langle N_3 \rangle N_3^{2-\gamma} \gtrsim 1. \]
Contribution of $L_4$. Note that

$$\sigma_4(\xi_1, \xi_2, \xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1.$$ 

Hence, by Hölder’s inequality, Hardy–Littlewood–Sobolev’s inequality, Proposition 2.2 and Bernstein’s estimate, we obtain

$$L_4 = \left\| \int_{|\xi_j| \leq N_j} |\xi_2, 3|^{-3} \sigma_4(\xi_1, \xi_2, \xi_3) \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_t^1 L_x^6} \lesssim m_1 N_1 \left\| \int_{|\xi_j| \leq N_j} |\xi_2, 3|^{-3} \hat{u}(\xi_1) \hat{u}(\xi_2) \hat{u}(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right\|_{L_t^1 L_x^6} \lesssim m_1 N_1 \|u_1\|_{L_t^3 L_x^6} \|u_2\|_{L_t^3 L_x^6} \|u_3\|_{L_t^3 L_x^6} \lesssim m_1 N_1 N_3^{2-\gamma} \prod_{j=1}^{3} \|u_j\|_{X^{0,1/2}_x}.

It suffices to show that

$$m_1 N_1 N_3^{2-\gamma} \lesssim N_1^{1+} N_2^{0-} m_1 N_1 m_2 N_3 N_3.$$ 

We reduce to show that

$$N_1^{1-} m_2 N_3 N_3^{2-\gamma}.$$ 

This is true since for $s \geq \gamma - 2$, we have

$$m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim m_2 N_2 \gtrsim N_1^{1-} N_2^{0+}$$

where we used the fact that $m(\xi)\langle \xi \rangle$ is monotone non-decreasing if $s + p \geq 1$. While for $\frac{\gamma}{2} - 1 < \gamma - 2$, we have

$$m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim N_1^{1-} N_2^{0+}$$

where we used the fact that $m(\xi)\langle \xi \rangle$ is monotone non-increasing if $s + p < 1$. 

5. Proof of Theorem 1.1

We first scale the solution. Suppose that $u(t, x)$ is a global in time solution to (1.1) with initial data $u_0 \in C_0^\infty(\mathbb{R}^d)$. Setting

$$u^\lambda(t, x) = \lambda^{-\frac{n+2-\gamma}{2}} u \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right),$$

we choose a parameter $\lambda$ so that $\|I u_0^\lambda\|_{H^1} = O(1)$, that is

$$\lambda \approx N_1^{\frac{-1}{2\gamma-2}}.$$ 

(5.1)

Next, let us define

$$S := \{0 \leq t < \infty: \|I u^\lambda\|_{L_t^4 \dot{H}^{\frac{d-3}{2}-\frac{1}{2}} ([0, t] \times \mathbb{R}^d)} \lesssim K \lambda^\frac{1}{2} (t^{-1}) \},$$

with $K$ a constant to be chosen later. We claim that $S$ is the whole interval $[0, \infty)$. Indeed, assume by contradiction that it is not so, then since

$$\|I u^\lambda\|_{L_t^4 \dot{H}^{\frac{d-3}{2}-\frac{1}{2}} ([0, t] \times \mathbb{R}^d)}$$
is a continuous function of time, there exists a time $T \in [0, \infty)$ such that

$$
\| I u^\lambda \|_{L^4 H^{\frac{d-3}{4}}(0,T) \times \mathbb{R}^d} > K \lambda^{3\left(\frac{q}{2} - 1\right)}, \quad (5.2)
$$

$$
\| I u^\lambda \|_{L^4 H^{\frac{d-3}{4}}(0,T) \times \mathbb{R}^d} \leq 2K \lambda^{3\left(\frac{q}{2} - 1\right)}. \quad (5.3)
$$

We now split the interval $[0, T]$ into subintervals $J_k, k = 1, \ldots, L$, in such a way that

$$
\| I u^\lambda \|_{L^4 H^{\frac{d-3}{4}}(J_k \times \mathbb{R}^d)} \leq \mu, \quad \text{with } \mu \text{ as in Proposition 3.3.}
$$

This is possible because of (5.3). Then, the number $L$ of possible subintervals must satisfy

$$
L \approx \frac{(2K \lambda)^{\frac{3}{4}\left(\frac{q}{2} - 1\right)}}{\mu} \approx \frac{(2K)^{\frac{3}{4}\left(\frac{q}{2} - 1\right)}}{\mu}. \quad (5.4)
$$

From Propositions 3.2 and 3.3, we know that

$$
\sup_{t \in [0,T]} E(I u^\lambda(t)) \lesssim E(I u^\lambda_0) + \frac{L}{N^{1-}}
$$

and by our choice (5.1) of $\lambda$, $E(I u^\lambda_0) \lesssim 1$. Hence, in order to guarantee that

$$
E(I u^\lambda(t)) \lesssim 1
$$

holds for all $t \in [0, T]$, we need to require that

$$
L \lesssim N^{1-}.
$$

According to (5.4), this is fulfilled as long as

$$
\frac{(2K)^{\frac{3}{4}\left(\frac{q}{2} - 1\right)}}{\mu} \lesssim N^{1-}. \quad (5.5)
$$

From our choice of $\lambda$, the expression (5.5) implies that

$$
\frac{(2K)^{4\left(\frac{q}{2} - 1\right)}}{\mu} \lesssim N^{1- - \frac{1}{\gamma} - \frac{3\left(\frac{q}{2} - 1\right)}{\gamma}}.
$$

Thus this is possible for $s > \frac{4(\gamma - 2)}{3\gamma - 4}$ and a large number $N$.

Now recall the a priori estimate (4.2)

$$
\| \nabla |^{-\frac{d-3}{4}} I u^\lambda \|_{L^4_{t,x} L^4_{t,x}} \lesssim \| I u^\lambda \|_{L^\infty_{t,x} H^1_{t,x}} \| I u^\lambda \|_{L^3_{t,x} L^6_{t,x}}^3 + \int_0^T \int_{\mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{bad}, I u^\lambda(t, x_1) I u^\lambda(t, x_2) \} \rho \, dx_1 \, dx_2 \, dt.
$$

Set

$$
\text{Error}(t) := \int_{\mathbb{R}^d} \nabla a \cdot \{ \tilde{N}_{bad}, I u^\lambda(t, x_1) I u^\lambda(t, x_2) \} \rho \, dx_1 \, dx_2.
$$

By Theorem 4.1 and Proposition 3.3 on each interval $J_k$, we have that

$$
\left| \int_{J_k} \text{Error}(t) \, dt \right| \lesssim \frac{1}{N^{1-}} \| u^\lambda \|_{Z_{J_k}^1} \lesssim \frac{1}{N^{1-}}.
$$

Summing all the $J_k$’s, we have that

$$
\left| \int_0^T \text{Error}(t) \, dt \right| \leq \frac{L}{N^{1-}} \lesssim N^{0+}.
$$
Therefore, by our choice (5.1) of \( \lambda \), we obtain
\[
\| |\nabla|^{\frac{d-3}{2}} Iu^\lambda \|_{L^2_x L^4_t}^4 \lesssim \| Iu^\lambda \|_{L^\infty_t L^4_x}^3 \| Iu^\lambda \|_{L^\infty_t L^2_x}^3 + N^{0+} \lesssim C \lambda^{3(\frac{d}{2}-1)}.
\]
This estimate contradicts (5.2) for an appropriate choice of \( K \). Hence \( S = [0, \infty) \). In addition, let \( T_0 \) be chosen arbitrarily, we have also proved that for \( s > \frac{4(y-2)}{3y-4} \),
\[
\| Iu^\lambda (\lambda^2 T_0) \|_{H^s_x} = O(1).
\]
Then
\[
\| u(T_0) \|_{H^s} = \| u(T_0) \|_{L^2} + \| u(T_0) \|_{H^s} \\
= \| u_0 \|_{L^2} + \lambda^{s-\frac{d}{2}+1} \| u^\lambda (\lambda^2 T_0) \|_{H^s} \\
\lesssim \lambda^{s-\frac{d}{2}+1} \| Iu^\lambda (\lambda^2 T_0) \|_{H^s} \lesssim \lambda^{s-\frac{d}{2}+1} \approx N^{1-s}.
\]
Since \( T_0 \) is arbitrarily large, the a priori bound on the \( H^s \) norm concludes the global well-posedness of the Cauchy problem (1.1).

Note that we have obtained that
\[
\| Iu^\lambda \|_{L^4_t L^d_x} \lesssim C(\| u_0 \|_{H^s}),
\]
this together with Propositions 2.2, 3.3 and the property of the operator \( I \) imply that
\[
\sup_{(q,r) \text{ admissible}} \| (\nabla)^{\frac{d}{2}} u \|_{L^q_t L^r_x([0, +\infty) \times \mathbb{R}^d)} \lesssim \| u \|_{Z_1([0, +\infty))} \lesssim C(\| u_0 \|_{H^s}) \lesssim C(\| u_0 \|_{H^s}).
\]
The scattering result can be obtained from the well-known standard argument [1,3]. This completes the proof.

Acknowledgements

The authors thank the referees and the associated editor for their invaluable comments and suggestions which helped improve the paper greatly. C. Miao and G. Xu were partly supported by the NSF of China (No. 10725102, No. 10801015), and L. Zhao was supported by China postdoctoral science foundation project. The second author would like to thank Nikolaos Tzirakis for helpful communications.

References