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Multiple solutions for a class of elliptic equations with jumping nonlinearities ☆

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Dedicated to Luce Spennato, in memory of her kindness, goodness and generosity

Abstract

We consider a semilinear elliptic Dirichlet problem with jumping nonlinearity and, using variational methods, we show that the number of solutions tends to infinity as the number of jumped eigenvalues tends to infinity. In order to prove this fact, for every positive integer k we prove that, when a parameter is large enough, there exists a solution which presents k interior peaks. We also describe the asymptotic behaviour and the profile of this solution as the parameter tends to infinity.

Résumé

Nous considérons un problème de Dirichlet semi-linéaire avec le terme non linéaire qui interfère avec les valeurs propres de l'opérateur linéaire. Avec des méthodes variationnelles, nous montrons que le nombre de solutions est arbitrairement grand pourvu que le nombre de valeurs propres qui interfèrent avec le terme non linéaire soit suffisamment grand. Pour la démonstration nous prouvons que pour tout $k \in \mathbb{N}$ le problème a une solution qui présente k pics quand un paramètre est suffisamment grand. Nous décrivons aussi le comportement asymptotique et la forme de cette solution quand ce paramètre tend à l'infini.

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1. Introduction

Several papers have been devoted to study existence and multiplicity of solutions for semilinear elliptic problems of the following type

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$$\begin{cases} \Delta u + g(u) = \xi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded connected domain of $\mathbb{R}^n$, $\xi \in L^2(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{t\to-\infty} \frac{g(t)}{t} = \alpha \quad \text{and} \quad \lim_{t\to+\infty} \frac{g(t)}{t} = \beta \tag{1.2}$$

with α and β in $\mathbb{R}$. We assume, for example, $\alpha \leqslant \beta$ (the case $\alpha \geqslant \beta$ is similar).

We denote by λ_i (or also by $\lambda_i(\Omega)$) the eigenvalues of the Laplace operator $-\Delta$ in $H_0^1(\Omega)$. Since Ω is a connected domain, we have $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \dots$.

If there exists some eigenvalue λ_i such that $\lambda_i \in]\alpha, \beta[$, we say that g is a jumping nonlinearity and that λ_i is a jumped eigenvalue. It is well known that, if $g \in \mathcal{C}^1(\mathbb{R})$ and $g'(t) \neq \lambda_i$ $\forall t \in \mathbb{R}$, $\forall i \in \mathbb{N}$, then there exists a unique solution $u \in H_0^1(\Omega)$ for every $\xi \in L^2(\Omega)$. In fact, in this case one can apply for example a well-known result of Caccioppoli (see [9]).

The situation is very different if $g'(t)$ meets some eigenvalue λ_i, what happens, for example, if g is a jumping nonlinearity. The first paper concerning this case is due to Ambrosetti and Prodi (see [4]). They consider in [4] the problem

$$\begin{cases} \Delta u + g(u) = \xi_0 + te_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g \in \mathcal{C}^2(\mathbb{R})$, $\xi_0 \in L^2(\Omega)$, $t \in \mathbb{R}$ and e_1 is a positive eigenfunction corresponding to the first eigenvalue λ_1. Under the assumptions that $g''(t) > 0$ for all $t \in \mathbb{R}$ and

$$0 < \lim_{t\to-\infty} g'(t) < \lambda_1 < \lim_{t\to+\infty} g'(t) < \lambda_2, \tag{1.4}$$

they prove that there exists a function $\bar{t} : L^2(\Omega) \to \mathbb{R}$ such that problem (1.3) has exactly two solutions if $t > \bar{t}(\xi_0)$, exactly one solution if $t = \bar{t}(\xi_0)$ and no solution if $t < \bar{t}(\xi_0)$.

After the result of Ambrosetti and Prodi, several authors have studied semilinear problems where the nonlinear terms interfere with the spectrum of the linear operator and in particular (especially in the early 1980s) elliptic equations with jumping nonlinearities (see [1–3], [5–8], [10–18,20,21,23–32], [35–38], etc.). They apply in these papers analytical, topological and variational methods and exploit several tools (as topological degree, Morse index, Rybakowski index, etc.) in order to describe the right-hand side members ξ for which the problem has solution and to estimate the number of solutions. The literature on this subject is really very extensive and in recent years there has been a new growing interest in these problems (see [8,18]). Here we recall only the following results.

If no eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ belongs to the interval $[\alpha, \beta]$, then a well-known theorem of Rabinowitz (see [34]) applies and guarantees that problem (1.1) has at least one solution for every $\xi \in L^2(\Omega)$.

If $\alpha < \lambda_1 < \beta$, there exists a function $\bar{t} : L^2(\Omega) \to \mathbb{R}$ such that problem (1.3) has at least two solutions if $t > \bar{t}(\xi_0)$, at least one solution if $t = \bar{t}(\xi_0)$ and no solution if $t < \bar{t}(\xi_0)$ (see [4,7,3], etc.). If $\alpha < \lambda_1 < \lambda_2 < \beta$, there exist at least four solutions of problem (1.3) for $t > 0$ large enough (see [27,38], etc.).

If $n = 1$ (i.e. Ω is an interval) and $\alpha < \lambda_1 < \lambda_i < \beta$, then (see [29,12,36]) problem (1.3) has at least $2i$ distinct solutions for $t > 0$ large enough (indeed, exactly $2i$ solutions if suitable additional conditions are satisfied). Notice that this result does not hold in the case $n > 1$. In fact, in [14] Dancer considered problem (1.3) with $g(u) = -\alpha u^- + \beta u^+$ (where $u^+ = \max\{u, 0\}$ and $u^- = u^+ - u$) and showed that for every $i \geqslant 2$ there exists a smooth bounded domain Ω_i in $\mathbb{R}^n$, with $n > 1$, and a function $\xi_0 \in L^2(\Omega_i)$ such that problem (1.3), with $\Omega = \Omega_i$, has only four solutions for $t > 0$ large enough even if $\alpha < \lambda_1(\Omega_i) < \lambda_i(\Omega_i) < \beta$.

In the present paper our aim is to show that the number of solutions of a problem with jumping nonlinearity may be arbitrarily large, for any fixed domain Ω, provided the number of jumped eigenvalues is large enough. Therefore, we consider the following problem

$$\begin{cases} \Delta u - \alpha u^- + \beta u^+ = e_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where we fix $\alpha < \lambda_1$ and let $\beta \to +\infty$ (notice that, if in problem (1.3) we replace u by tu and let $t \to +\infty$, we obtain (1.5) as limit problem). We show that, for any fixed domain Ω, the number of distinct solutions tends to infinity as $\beta \to +\infty$. In fact, for every positive integer k we construct, for $\beta > 0$ large enough, a solution $u_{k,\beta}$ of problem (1.5), which presents k peaks and converges as $\beta \to +\infty$ to the solution $\frac{e_1}{\alpha-\lambda_1}$ (see also Remark 3.9).

The main result of this paper is presented in the following theorem.

Theorem 1.1. *Let Ω be a bounded connected domain of $\mathbb{R}^n$, with $n \geqslant 3$. Let us fix $\alpha < \lambda_1$. Then, for every positive integer k there exists $\bar\beta_k > 0$ such that for all $\beta > \bar\beta_k$ problem* (1.5) *has a solution $u_{k,\beta}$ satisfying the following properties*:

(I) *there exist a positive constant $\bar r$ and k points $x_1^\beta, \dots, x_k^\beta$ in Ω, with*

$$\operatorname{dist}(x_i^\beta, \partial\Omega) > \frac{\bar r}{\sqrt\beta} \quad \text{for } i = 1, \dots, k, \quad \text{and} \quad |x_i^\beta - x_j^\beta| \geqslant \frac{2\bar r}{\sqrt\beta} \quad \text{for } i \neq j, \tag{1.6}$$

such that, for every $\beta > \bar\beta_k$, $u_{k,\beta}(x) \leqslant 0$ $\forall x \in \Omega \setminus \bigcup_{i=1}^k B(x_i^\beta, \frac{\bar r}{\sqrt\beta})$ while $u_{k,\beta}^+ \not\equiv 0$ in $B(x_i^\beta, \frac{\bar r}{\sqrt\beta})$ for $i = 1, \dots, k$;

(II) *the points $x_1^\beta, \dots, x_k^\beta$ satisfy, in addition,*

$$\lim_{\beta\to+\infty} e_1(x_i^\beta) = \max_\Omega e_1 \quad \text{for } i = 1, \dots, k, \quad \text{and} \quad \lim_{\beta\to+\infty} \sqrt\beta\,|x_i^\beta - x_j^\beta| = \infty \quad \text{for } i \neq j;$$

(III) *for every $\beta > \bar\beta_k$, we have $u_{k,\beta}(x) > \frac{e_1(x)}{\alpha-\lambda_1}$ $\forall x \in \Omega$ and*

$$\lim_{\beta\to+\infty} \sup\left\{ u_{k,\beta}(x) - \frac{e_1(x)}{\alpha-\lambda_1} \colon x \in \Omega \setminus \bigcup_{i=1}^k B(x_i^\beta, \rho) \right\} = 0 \quad \forall \rho > 0; \tag{1.7}$$

(IV) *for every $i = 1, \dots, k$, the functions U_{k,β,x_i^β}, defined in $\sqrt\beta(\Omega - x_i^\beta)$ by*

$$U_{k,\beta,x_i^\beta}(x) = u_{k,\beta}\left(\frac{x}{\sqrt\beta} + x_i^\beta\right) \quad \forall x \in \sqrt\beta(\Omega - x_i^\beta),$$

converge as $\beta \to +\infty$ to the nonconstant radial solution U of the problem

$$\begin{cases} \Delta U + U^+ = 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} U(x) = \dfrac{1}{\alpha-\lambda_1} \max_\Omega e_1. \end{cases} \tag{1.8}$$

Moreover, the convergence is uniform on the compact subsets of $\mathbb{R}^n$.

Remark 1.2. It is clear that it follows from property (I) that the number of distinct solutions tends to infinity as $\beta \to +\infty$. From property (II) we infer that the peaks concentrate near the maximum points of e_1 and, if the distance between two peaks tends to zero as $\beta \to +\infty$, the approaching rate is less than the concentration rate. Property (III) shows that the peaks "are based" on the solution $\frac{e_1}{\alpha-\lambda_1}$, which is the minimal solution of the problem (as one can easily verify). Finally, property (IV) describes the asymptotic profile of the rescaled peaks. Let us point out that only if $n \geqslant 3$ the problem (1.8) has a nontrivial solution (since any bounded super-harmonic function in $\mathbb{R}^n$ with $n < 3$ is a constant function). In the cases $n = 1$ and $n = 2$ new, more refined, arguments have to be used in order to construct k-peak solutions and describe their asymptotic behaviour (see [33]).

The method we use for the proof of Theorem 1.1 is completely variational. The solutions are obtained as critical points of the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac12 \int_\Omega \left[|Du|^2 - \alpha(u^-)^2 - \beta(u^+)^2\right] dx + \int_\Omega e_1 u\, dx \quad \forall u \in H_0^1(\Omega). \tag{1.9}$$

Notice that similar phenomena occur also in other fields of Nonlinear Analysis. For example, k-peak solutions with similar properties as in Theorem 1.1 appear in many superlinear problems such as singular perturbation problems, nonlinear Schrödinger equations, nonlinear scalar field equations, elliptic equations involving critical or supercritical Sobolev exponents, etc. Solutions of this type are usually obtained by using a Lyapunov–Schmidt type finite-dimensional reduction method. In particular, we recall the paper [18] of Dancer and Yan, where they consider the problem

$$\begin{cases} \Delta u + |u|^p = te_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

with $p \in]1, \frac{n+2}{n-2}[$ (here the nonlinear term satisfies (1.2) with $\alpha = -\infty$ and $\beta = +\infty$). For every positive integer k, they prove the existence of a k-peak solution of (1.10) for $t > 0$ large enough (thus proving a well-known Lazer–McKenna conjecture). Concerning superlinear problems of this type, several results have been obtained in the last few years (see for example [19,22,40] and the references therein).

2. Notation and preliminary results

We look for solutions $u \in H_0^1(\Omega)$ of the following type. For every $\beta > 0$, set $r_\beta = \frac{3\bar{r}_1}{\sqrt{\beta}}$ where $\bar{r}_1$ is the radius of the ball in $\mathbb{R}^n$ for which the first eigenvalue of the Laplace operator is equal to 1, i.e.

$$\inf\left\{ \int_{B(0,\bar{r}_1)} |Du|^2\,dx\colon\ u \in H_0^1\big(B(0,\bar{r}_1)\big),\ \int_{B(0,\bar{r}_1)} u^2\,dx = 1 \right\} = 1. \tag{2.1}$$

For every positive integer k, consider the set

$$\Omega_{k,\beta} = \big\{(x_1,\ldots,x_k) \in \Omega^k\colon\ |x_i - x_j| \geqslant 2r_\beta \text{ if } i \neq j,\ \operatorname{dist}(x_i,\partial\Omega) \geqslant r_\beta \text{ for } i = 1,\ldots,k\big\}. \tag{2.2}$$

It is clear that, for every fixed $k \in \mathbb{N}$, $\Omega_{k,\beta} \neq \emptyset$ for β large enough and the balls $B(x_1,r_\beta),\ldots,B(x_k,r_\beta)$ are pairwise disjoint and included in Ω if $(x_1,\ldots,x_k) \in \Omega_{k,\beta}$.

We say that a function $u \in H_0^1(\Omega)$ is a k-peak function, with respect to the balls $B(x_1,r_\beta),\ldots,B(x_k,r_\beta)$, if $u^+ = \sum_{i=1}^k u_i^+$ where, for every $i = 1,\ldots,k$, u_i^+ is a nonnegative function in $H_0^1(\Omega)$ such that $u_i^+ \not\equiv 0$ and $u_i^+(x) = 0$ $\forall x \in \Omega \setminus B(x_i,r_\beta)$.

One can easily verify that, if a k-peak function u of this form is a solution of problem (1.5), then for every $i = 1,\ldots,k$ the function $t \mapsto f(u + tu_i^+)$ has for $t = 0$ the unique maximum point in the set $[-1,+\infty[$ and $f'(u)[u_i^+] = 0$, that is

$$\int_\Omega \big|Du_i^+\big|^2\,dx - \beta\int_\Omega \big(u_i^+\big)^2\,dx + \int_\Omega e_1 u_i^+\,dx = 0. \tag{2.3}$$

Therefore, it is natural to consider the subsets V_i of $H_0^1(\Omega)$ consisting of all the k-peak functions u, with respect to the balls $B(x_1,r_\beta),\ldots,B(x_k,r_\beta)$, such that $f'(u)[u_i^+] = 0$, and to look for critical points of the functional f constrained on the subsets V_i (even if they are not smooth manifolds and this fact gives some more problems when we have to prove that the constrained critical points actually give solutions of (1.5)).

For the k-peak functions u of the form described above we use also constraints of the following type (a barycenter type constraint)

$$\int_\Omega \big(u_i^+(x)\big)^2(x - x_i)\,dx = 0. \tag{2.4}$$

For every $i = 1,\ldots,k$, we denote by B_i the subset of $H_0^1(\Omega)$ consisting of all the k-peak functions u, with respect to the balls $B(x_1,r_\beta),\ldots,B(x_k,r_\beta)$, such that (2.4) holds.

Finally, let us denote by $S^\beta_{x_1,\ldots,x_k}$ the set of all the k-peak functions $u \in H_0^1(\Omega)$, with respect to the balls $B(x_1,r_\beta),\ldots,B(x_k,r_\beta)$, such that $u \in V_i \cap B_i$ for every $i = 1,\ldots,k$.

Proposition 2.1. *Let k be a positive integer, $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, $\alpha < \lambda_1$ and consider a point $(x_1, \dots, x_k) \in \Omega_{k,\beta}$.*

Then $S^\beta_{x_1,\dots,x_k} \neq \emptyset$ and the minimum of the functional f on the set $S^\beta_{x_1,\dots,x_k}$ is achieved.

Proof. First notice that $S^\beta_{x_1,\dots,x_k} \neq \emptyset$ because of the choice of the radius r_β. In fact, since $\sqrt{\beta} r_\beta > \bar{r}_1$, there exist nonnegative functions $v_i \in H^1_0(\Omega)$ such that $v_i = 0$ in $\Omega \setminus B(x_i, r_\beta)$, $\int_\Omega |Dv_i|^2 dx < \beta \int_\Omega v_i^2\, dx$ and $\int_\Omega (v_i(x))^2 (x - x_i)\, dx = 0$ (for example, we can choose as v_i a positive eigenfunction related to the first eigenvalue of the Laplace operator in $H^1_0(B(x_i, \tilde{r}_\beta))$, with $\frac{\bar{r}_1}{\sqrt{\beta}} < \tilde{r}_\beta < r_\beta$).

Therefore, $u = \sum_{i=1}^k t_i v_i \in S^\beta_{x_1,\dots,x_k}$ for suitable $t_i > 0$.

For every function $u \in S^\beta_{x_1,\dots,x_k}$, we have

$$f(u) = f\big(-u^-\big) + \sum_{i=1}^k f\big(u_i^+\big) \tag{2.5}$$

where $f(-u^-) \geqslant f(\frac{e_1}{\alpha - \lambda_1})$ (since $\alpha < \lambda_1$) and $f(u_i^+) > 0$ for $i = 1, \dots, k$ (because $u \in V_i$ implies $f(u_i^+) = \max\{f(tu_i^+)\colon t \geqslant 0\}$). It follows that $\inf_{S^\beta_{x_1,\dots,x_k}} f > -\infty$.

Let us consider a minimizing sequence $(u_j)_j$ for f on $S^\beta_{x_1,\dots,x_k}$. Taking into account that $u_j \in V_i$, namely

$$\int_\Omega \big|Du^+_{j,i}\big|^2 dx - \beta \int_\Omega \big(u^+_{j,i}\big)^2 dx + \int_\Omega e_1 u^+_{j,i}\, dx = 0, \tag{2.6}$$

and that $e_1 > 0$ in Ω, we have

$$\int_\Omega \big|Du^+_{j,i}\big|^2 dx < \beta \int_\Omega \big(u^+_{j,i}\big)^2 dx. \tag{2.7}$$

Therefore, if we set $v_{j,i} = \frac{u^+_{j,i}}{\|u^+_{j,i}\|_{L^2(\Omega)}}$ the sequence $(v_{j,i})_j$ is bounded in $H^1_0(\Omega)$. As a consequence, up to a subsequence, it converges as $j \to \infty$ to a function $v_i \in H^1_0(\Omega)$ in $L^2(\Omega)$, weakly in $H^1_0(\Omega)$ and almost everywhere in Ω (thus $v_i \geqslant 0$ in Ω and $\|v_i\|_{L^2(\Omega)} = 1$). Notice that $u_j \in V_i\ \forall j \in \mathbb{N}$ implies

$$\liminf_{j\to\infty} \big\|u^+_{j,i}\big\|_{L^2(\Omega)} > 0 \quad \text{for } i = 1, \dots, k. \tag{2.8}$$

In fact, arguing by contradiction, assume that for some $i \in \{1, \dots, k\}$, up to a subsequence, $\|u^+_{j,i}\|_{L^2(\Omega)} \to 0$ as $j \to \infty$. Then, from (2.6) we obtain

$$\big\|u^+_{j,i}\big\|_{L^2(\Omega)} \int_\Omega |Dv_{j,i}|^2 dx - \beta \big\|u^+_{j,i}\big\|_{L^2(\Omega)} + \int_\Omega e_1 v_{j,i}\, dx = 0 \tag{2.9}$$

and, as $j \to \infty$, $\int_\Omega e_1 v_i\, dx = 0$ which gives a contradiction because $e_1 > 0$, $v_i \geqslant 0$ in Ω and $v_i \not\equiv 0$.

Now, let us prove that

$$\limsup_{j\to\infty} \big\|u^+_{j,i}\big\|_{L^2(\Omega)} < +\infty \quad \text{for } i = 1, \dots, k. \tag{2.10}$$

Arguing again by contradiction, assume that for some $i \in \{1, \dots, k\}$, up to a subsequence, $\|u^+_{j,i}\|_{L^2(\Omega)} \to \infty$ as $j \to \infty$. Thus, from (2.6) we have

$$\int_\Omega |Dv_{j,i}|^2 dx - \beta + \big\|u^+_{j,i}\big\|^{-1}_{L^2(\Omega)} \int_\Omega e_1 v_{j,i}\, dx = 0 \tag{2.11}$$

which, as $j \to \infty$, implies

$$\lim_{j\to\infty} \int_\Omega |Dv_{j,i}|^2\,dx = \beta. \tag{2.12}$$

Taking into account that

$$\begin{aligned} f\left(u_{j,i}^+\right) &= \max\left\{ f\left(tu_{j,i}^+\right):\ t \geqslant 0\right\} \\ &= \max\left\{ \frac{t^2}{2}\left[\int_\Omega |Dv_{j,i}|^2\,dx - \beta\right] + t\int_\Omega e_1 v_{j,i}\,dx:\ t\geqslant 0\right\} \\ &= \frac{1}{2}\left[\beta - \int_\Omega |Dv_{j,i}|^2\,dx\right]^{-1}\left(\int_\Omega e_1 v_{j,i}\,dx\right)^2, \end{aligned} \tag{2.13}$$

from (2.12) we infer that $\lim_{j\to\infty} f(u_{j,i}^+) = +\infty$. As a consequence, we have also $\lim_{j\to\infty} f(u_j) = +\infty$, in contradiction with the fact that $(u_j)_j$ is a minimizing sequence. Therefore, the sequence $(u_j^+)_j$ is bounded in $L^2(\Omega)$. Notice that also the sequence $(u_j^-)_j$ is bounded in $L^2(\Omega)$ as one can easily verify taking into account that $\alpha < \lambda_1$. Thus, the sequence $(u_j)_j$ is bounded in $L^2(\Omega)$ and, as a consequence, also in $H_0^1(\Omega)$ (because $\sup_{j\in\mathbb{N}} f(u_j) < +\infty$). It follows that, up to a subsequence, u_j converges to a function $u \in H_0^1(\Omega)$ in $L^2(\Omega)$, weakly in $H_0^1(\Omega)$ and a.e. in Ω. The convergence in $L^2(\Omega)$ implies that u is a k-peak function with respect to the balls $B(x_1, r_\beta), \dots, B(x_k, r_\beta)$ (notice that $u_i^+ \not\equiv 0$ because of (2.8)) and that $u \in B_i$ for every $i = 1, \dots, k$. Now we prove that, as $j \to \infty$,

$$\int_\Omega \left|Du_j^-\right|^2 dx \to \int_\Omega \left|Du^-\right|^2 dx \quad \text{and} \quad \int_\Omega \left|Du_{j,i}^+\right|^2 dx \to \int_\Omega \left|Du_i^+\right|^2 dx \quad \text{for } i = 1, \dots, k \tag{2.14}$$

(namely, that $u_j \to u$ also in $H_0^1(\Omega)$) which allows us to conclude that $u \in V_i$ for $i = 1, \dots, k$ and that u is a minimizing function for f on $S^\beta_{x_1,\dots,x_k}$.

For the proof we argue by contradiction and assume that (up to a subsequence)

$$\begin{aligned} &\lim_{j\to\infty}\int_\Omega \left|Du_j^-\right|^2 dx > \int_\Omega \left|Du^-\right|^2 dx \quad \text{or} \\ &\lim_{j\to\infty}\int_\Omega \left|Du_{j,i}^+\right|^2 dx > \int_\Omega \left|Du_i^+\right|^2 dx \quad \text{for some } i \in \{1, \dots, k\}. \end{aligned} \tag{2.15}$$

Notice that, since

$$f(u_j) = f\left(-u_j^-\right) + \sum_{i=1}^k f\left(u_{j,i}^+\right), \tag{2.16}$$

we have (up to a subsequence)

$$\inf_{S^\beta_{x_1,\dots,x_k}} f = \lim_{j\to\infty} f(u_j) = \lim_{j\to\infty} f\left(-u_j^-\right) + \sum_{i=1}^k \lim_{j\to\infty} f\left(u_{j,i}^+\right) \tag{2.17}$$

where

$$\lim_{j\to\infty} f\left(-u_j^-\right) = f\left(-u^-\right) \quad \text{if} \quad \lim_{j\to\infty}\int_\Omega \left|Du_j^-\right|^2 dx = \int_\Omega \left|Du^-\right|^2 dx \tag{2.18}$$

while

$$\lim_{j\to\infty} f\left(-u_j^-\right) > f\left(-u^-\right) \quad \text{if} \quad \lim_{j\to\infty}\int_\Omega \left|Du_j^-\right|^2 dx > \int_\Omega \left|Du^-\right|^2 dx. \tag{2.19}$$

For $i = 1, \dots, k$ we have

$$\lim_{j\to\infty} f\big(u_{j,i}^+\big) = f\big(u_i^+\big) \text{ and } f'(u)\big[u_i^+\big] = 0 \quad \text{if} \quad \lim_{j\to\infty}\int_\Omega \big|Du_{j,i}^+\big|^2\,dx = \int_\Omega \big|Du_i^+\big|^2\,dx \tag{2.20}$$

(because $f'(u)[u_i^+] = \lim_{j\to\infty} f'(u_j)[u_{j,i}^+]$, in this case, and $f'(u_j)[u_{j,i}^+] = 0\ \forall j \in \mathbb{N}$ since $u_j \in V_i$).

On the contrary, if $\lim_{j\to\infty}\int_\Omega |Du_{j,i}^+|^2\,dx > \int_\Omega |Du_i^+|^2\,dx$, we have $f'(u)[u_i^+] < 0$. As a consequence, there exists $\bar t_i \in]0,1[$ such that $f'(\bar t_i u_i^+)[\bar t_i u_i^+] = 0$. It follows that

$$f\big(\bar t_i u_i^+\big) = \frac{1}{2}\bar t_i \int_\Omega e_1 u_i^+\,dx < \frac{1}{2}\int_\Omega e_1 u_i^+\,dx = \lim_{j\to\infty}\frac{1}{2}\int_\Omega e_1 u_{j,i}^+\,dx = \lim_{j\to\infty} f\big(u_{j,i}^+\big) \tag{2.21}$$

(where the first and the last equality hold because, if w is a nonnegative function in $H_0^1(\Omega)$, $f'(w)[w] = 0$ implies $f(w) = \frac{1}{2}\int_\Omega e_1 w\,dx$).

Therefore we infer that, if (2.15) occurs, there exists a function $\bar u \in S^\beta_{x_1,\dots,x_k}$ (of the form $\bar u = -u^- + \sum_{i=1}^k \bar t_i u_i^+$ with $\bar t_i \in]0,1]$ for $i = 1,\dots,k$) such that

$$f(\bar u) < \lim_{j\to\infty} f(u_j) = \inf_{S^\beta_{x_1,\dots,x_k}} f \tag{2.22}$$

which is a contradiction.

Thus $u_j \to u$ in $H_0^1(\Omega)$, $u \in S^\beta_{x_1,\dots,x_k}$ and $f(u) = \min_{S^\beta_{x_1,\dots,x_k}} f$. $\square$

Proposition 2.1 allows us to introduce the function $\varphi_\beta : \Omega_{k,\beta} \to \mathbb{R}$ defined by

$$\varphi_\beta(x_1,\dots,x_k) = \min_{S^\beta_{x_1,\dots,x_k}} f \quad \forall (x_1,\dots,x_k) \in \Omega_{k,\beta}. \tag{2.23}$$

Proposition 2.2. *For every positive integer k and for $\alpha < \lambda_1$, fix $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$. Then there exists $(x_1^\beta,\dots,x_k^\beta) \in \Omega_{k,\beta}$ such that $\varphi_\beta(x_1^\beta,\dots,x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$ (see* (2.23)*).*

Proof. Let us consider a sequence $(x_{1,j},\dots,x_{k,j})_j$ in $\Omega_{k,\beta}$ such that

$$\lim_{j\to\infty} \varphi_\beta(x_{1,j},\dots,x_{k,j}) = \sup_{\Omega_{k,\beta}} \varphi_\beta. \tag{2.24}$$

Since $\Omega_{k,\beta}$ is a compact set, there exists $(x_1^\beta,\dots,x_k^\beta) \in \Omega_{k,\beta}$ such that, up to a subsequence, $(x_{1,j},\dots,x_{k,j}) \to (x_1^\beta,\dots,x_k^\beta)$ as $j \to \infty$.

By Proposition 2.1, there exists $u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$. For every $j \in \mathbb{N}$, consider the function $\bar u_j \in S^\beta_{x_{1,j},\dots,x_{k,j}}$ such that $\bar u_j = -\bar u_j^- + \sum_{i=1}^k \bar u_{j,i}^+$, where $\bar u_{j,i}^+(x) = u_{\beta,i}^+(x + x_i^\beta - x_{i,j})$ and $-\bar u_j^-$ is the minimizing function for the minimum

$$\min\left\{ f(v):\ v \in H_0^1(\Omega),\ v \leqslant 0 \text{ in } \Omega,\ \int_\Omega v\bar u_{j,i}^+\,dx = 0 \text{ for } i = 1,\dots,k \right\} \tag{2.25}$$

(since $\alpha < \lambda_1$, there exists a unique minimizing function which depends continuously on the point $(x_{1,j},\dots,x_{k,j}) \in \Omega_{k,\beta}$).

Standard arguments show that $\bar u_j \to u_\beta$ and $f(\bar u_j) \to f(u_\beta)$ as $j \to \infty$. Thus, taking into account that $\min_{S^\beta_{x_{1,j},\dots,x_{k,j}}} f \leqslant f(\bar u_j)\ \forall j \in \mathbb{N}$ because $\bar u_j \in S^\beta_{x_{1,j},\dots,x_{k,j}}$, we obtain

$$\sup_{\Omega_{k,\beta}} \varphi_\beta = \lim_{j\to\infty} \varphi_\beta(x_{1,j},\dots,x_{k,j}) \leqslant \lim_{j\to\infty} f(\bar u_j) = f(u_\beta) = \varphi_\beta\big(x_1^\beta,\dots,x_k^\beta\big) \tag{2.26}$$

which implies $\varphi_\beta(x_1^\beta,\dots,x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$. $\square$

In order to describe the behaviour of the problem as $\beta \to +\infty$, we need also some preliminary results on the capacity.

For every smooth bounded domain A in $\mathbb{R}^n$ with $n \geqslant 3$, the capacity of A is defined by

$$\operatorname{cap}(A) = \min\left\{ \int_{\mathbb{R}^n} |Du|^2\,dx\colon\ u \in \mathcal{D}^{1,2}\big(\mathbb{R}^n\big),\ u \geqslant 1 \text{ a.e. in } A \right\}. \tag{2.27}$$

It is well known that there exists a unique minimizing function u_A. Moreover, we have $u_A = 1$ in A, $0 < u_A \leqslant 1$ in $\mathbb{R}^n \setminus A$, $\Delta u_A = 0$ in $\mathbb{R}^n \setminus \bar{A}$.

Notice that we have also

$$\operatorname{cap}(A) = -\int_{\partial A} (Du_A \cdot \nu)\,d\sigma \tag{2.28}$$

where ν denotes the outward normal on ∂A.

Lemma 2.3. *Let $A_1, A_2, \dots, A_s$, with $s > 1$, be s pairwise disjoint smooth bounded domains in $\mathbb{R}^n$ with $n \geqslant 3$. Then, we have*

$$\operatorname{cap}\left(\bigcup_{i=1}^{s} A_i\right) < \sum_{i=1}^{s} \operatorname{cap}(A_i). \tag{2.29}$$

Proof. For every $i = 1, \dots, s$, consider the minimizing function u_{A_i} for $\operatorname{cap}(A_i)$; then introduce the function $\mu \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ defined by

$$\mu(x) = \max\big\{u_{A_i}(x)\colon\ i = 1, \dots, s\big\} \quad \forall x \in \mathbb{R}^n. \tag{2.30}$$

Notice that μ is subharmonic (but not harmonic) in $\mathbb{R}^n \setminus \bigcup_{i=1}^{s} \bar{A}_i$. Therefore, if we set $A = \bigcup_{i=1}^{s} A_i$, we have $\mu \leqslant u_A$, $\mu \not\equiv u_A$ and

$$-\int_{\partial A} (Du_A \cdot \nu)\,d\sigma < -\sum_{i=1}^{s} \int_{\partial A_i} (Du_{A_i} \cdot \nu)\,d\sigma \tag{2.31}$$

because u_A (the minimizing function for $\operatorname{cap}(A)$) is harmonic in $\mathbb{R}^n \setminus \bar{A}$. Thus we obtain

$$\operatorname{cap}(A) = \int_{\mathbb{R}^n} |Du_A|^2\,dx = -\int_{\partial A} (Du_A \cdot \nu)\,d\sigma < -\sum_{i=1}^{s} \int_{\partial A} (Du_{A_i} \cdot \nu)\,d\sigma = \sum_{i=1}^{s} \int_{\mathbb{R}_n} |Du_{A_i}|^2\,dx = \sum_{i=1}^{s} \operatorname{cap}(A_i), \tag{2.32}$$

which completes the proof. □

3. Asymptotic behaviour and proof of the main results

Our next aim is to prove that, if $(x_1^\beta, \dots, x_k^\beta) \in \Omega_{k,\beta}$ and $u_\beta \in S_{x_1^\beta, \dots, x_k^\beta}^\beta$ is a function such that $f(u_\beta) = \varphi_\beta(x_1^\beta, \dots, x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$ (see Proposition 2.2), then u_β is a solution of problem (1.5) for β large enough. Therefore, we need to study the behaviour of u_β as $\beta \to +\infty$ and to describe the asymptotic profile of the function u_β (suitably rescaled).

Proposition 3.1. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, consider a point $(x_1^\beta, \dots, x_k^\beta) \in \Omega_{k,\beta}$ and a function $u_\beta \in S_{x_1^\beta, \dots, x_k^\beta}^\beta$ such that $f(u_\beta) = \varphi_\beta(x_1^\beta, \dots, x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$. Then we have*

(a) $u_\beta > \frac{e_1}{\alpha-\lambda_1}$ $\forall\beta$ *and* $u_\beta \to \frac{e_1}{\alpha-\lambda_1}$ *a.e. in* Ω *as* $\beta\to+\infty$;
(b)

$$\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]=\frac{k}{2(\alpha-\lambda_1)^2}\Big(\max_\Omega e_1\Big)^2\operatorname{cap}(\bar r_1) \tag{3.1}$$

where (for short) $\operatorname{cap}(\bar r_1)$ *denotes the capacity of the ball of radius* $\bar r_1$ *in* $\mathbb{R}^n$;
(c)

$$\lim_{\beta\to+\infty} e_1\big(x_i^\beta\big)=\max_\Omega e_1 \quad \textit{for } i=1,\dots,k, \tag{3.2}$$

$$\lim_{\beta\to+\infty}\sqrt{\beta}\,\big|x_i^\beta-x_j^\beta\big|=\infty \quad \textit{for } i\neq j. \tag{3.3}$$

Proof. Property (a) follows by standard arguments taking into account that, since $f(u_\beta)=\min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$, $-u_\beta^-$ is the unique minimizing function for the minimum

$$\min\left\{f(u)\colon u\in H_0^1(\Omega),\ u\leqslant 0 \text{ in } \Omega,\ \int_\Omega uu_\beta^+\,dx=0\right\} \tag{3.4}$$

and that $u_\beta^+=0$ in $\Omega\setminus\bigcup_{i=1}^k B(x_i^\beta,r_\beta)$ with $r_\beta\to 0$ as $\beta\to+\infty$.

Notice that, for every $u\in H_0^1(\Omega)$, $f(u)=f(-u^-)+f(u^+)$ and

$$f\big(-u^-\big)=\frac12\int_\Omega\big|Du^-\big|^2\,dx-\frac{\alpha}{2}\int_\Omega\big(u^-\big)^2\,dx-\int_\Omega e_1u^-\,dx.$$

If we set $v=-u^- - \frac{e_1}{\alpha-\lambda_1}$, taking into account that $f(\frac{e_1}{\alpha-\lambda_1})=\frac{1}{2(\alpha-\lambda_1)}$, a direct computation shows that

$$f\big(-u^-\big)=\frac{1}{2(\alpha-\lambda_1)}+\frac12\int_\Omega|Dv|^2\,dx-\frac{\alpha}{2}\int_\Omega v^2\,dx. \tag{3.5}$$

For the proof of property (b), we prove first that

$$\liminf_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\geqslant\frac{k}{2(\alpha-\lambda_1)^2}\Big(\max_\Omega e_1\Big)^2\operatorname{cap}(\bar r_1). \tag{3.6}$$

For every $\beta>0$ (β large enough) choose $(\bar x_1^\beta,\dots,\bar x_k^\beta)\in\Omega_{k,\beta}$ such that

$$\lim_{\beta\to+\infty}e_1\big(\bar x_i^\beta\big)=\max_\Omega e_1 \quad \text{for } i=1,\dots,k, \quad \text{and} \quad \lim_{\beta\to+\infty}\sqrt{\beta}\,\big|\bar x_i^\beta-\bar x_j^\beta\big|=\infty \quad \text{for } i\neq j. \tag{3.7}$$

Consider a function $\bar u_\beta\in S^\beta_{\bar x_1^\beta,\dots,\bar x_k^\beta}$ such that $f(\bar u_\beta)=\varphi_\beta(\bar x_1^\beta,\dots,\bar x_k^\beta)$. For every $r>0$, set $\Omega_r^\beta=\Omega\setminus\bigcup_{i=1}^k B(\bar x_i^\beta,\frac{r}{\sqrt{\beta}})$. Standard arguments show that

$$\lim_{\beta\to+\infty}\min\left\{\int_{\Omega_r^\beta}|Du|^2\,dx\colon u\in H_0^1(\Omega),\ \int_{\Omega_r^\beta}u^2\,dx=1\right\}=\lambda_1. \tag{3.8}$$

Therefore, since $\alpha<\lambda_1$, for every $r>3\bar r_1$ there exists $\beta_r>0$ such that

$$f(\bar u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\geqslant\sum_{i=1}^k f\big(\bar u_{\beta,i}^+\big)+\sum_{i=1}^k\frac12\int_{B(\bar x_i^\beta,r/\sqrt{\beta})}\big[|D\bar v_\beta|^2-\alpha\bar v_\beta^2\big]\,dx \quad \forall\beta>\beta_r \tag{3.9}$$

where $\bar v_\beta=-\bar u_\beta^- - \frac{e_1}{\alpha-\lambda_1}$. Taking into account that $f(\bar u_{\beta,i}^+)\geqslant 0$ for $i=1,\dots,k$, after rescaling we obtain

$$\beta^{\frac{n-2}{2}}\left[f(\bar u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\geqslant\sum_{i=1}^k\frac12\int\limits_{B(\sqrt\beta\bar x_i^\beta,r)}\left[|D\bar V_\beta|^2\,dx-\frac{\alpha}{\beta}\bar V_\beta^2\right]dx\quad\forall\beta>\beta_r,\tag{3.10}$$

where $\bar V_\beta(x)=\bar v_\beta(\frac{x}{\sqrt\beta})$.

Now, for $i=1,\dots,k$, consider the functions $\bar V_{\beta,\bar x_i}$, defined in $\sqrt\beta(\Omega-\bar x_i)$ by $\bar V_{\beta,\bar x_i}(x)=\bar V_\beta(x+\sqrt\beta\bar x_i)$. Since $\int_\Omega|D\bar u^+_{\beta,i}|^2\,dx\leqslant\beta\int_\Omega(\bar u^+_{\beta,i})^2\,dx$ (that is the subset of $B(\bar x_i,r_\beta)$ where $\bar u_\beta\geqslant0$ has the first eigenvalue which is not greater than β), standard arguments show that (up to a subsequence) $\bar V_{\beta,\bar x_i}$ converges as $\beta\to+\infty$ to a function $\bar V_{\infty,i}\in\mathcal D^{1,2}(\mathbb R^n)$ such that $0\leqslant\bar V_{\infty,i}(x)\leqslant\frac{1}{\lambda_1-\alpha}\max_\Omega e_1$ $\forall x\in\mathbb R^n$ and the set where $\bar V_{\infty,i}=\frac{1}{\lambda_1-\alpha}\max_\Omega e_1$ has first eigenvalue not greater than 1. Therefore, taking into account that the ball of radius $\bar r_1$ has the smallest capacity among the domains whose first eigenvalue is less than or equal to 1, we obtain

$$\int\limits_{\mathbb R^n}|D\bar V_{\infty,i}|^2\,dx\geqslant\frac{1}{(\lambda_1-\alpha)^2}\Big(\max_\Omega e_1\Big)^2\operatorname{cap}(\bar r_1)\quad\text{for } i=1,\dots,k.\tag{3.11}$$

Now, as $\beta\to+\infty$, we infer from (3.10) that

$$\liminf_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(\bar u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\geqslant\frac12\sum_{i=1}^k\int\limits_{B(0,r)}|D\bar V_{\infty,i}|^2\,dx\quad\forall r>3\bar r_1,\tag{3.12}$$

which, as $r\to\infty$, gives

$$\liminf_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(\bar u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\geqslant\frac12\sum_{i=1}^k\int\limits_{\mathbb R^n}|D\bar V_{\infty,i}|^2\,dx.\tag{3.13}$$

Taking into account that

$$f(u_\beta)=\max_{\Omega_{k,\beta}}\varphi_\beta\geqslant\varphi_\beta\big(\bar x_1^\beta,\dots,\bar x_k^\beta\big)=f(\bar u_\beta),\tag{3.14}$$

(3.6) follows easily from (3.11) and (3.13).

Let us prove that

$$\limsup_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\leqslant\frac{k}{2(\alpha-\lambda_1)^2}\Big(\max_\Omega e_1\Big)^2\operatorname{cap}(\bar r_1).\tag{3.15}$$

For every $\rho\in\,]\bar r_1,3\bar r_1[$, consider a positive radial function $w\in H^1_0(B(0,\rho))$ such that $\int_{B(0,\rho)}|Dw|^2\,dx<\int_{B(0,\rho)}w^2\,dx$. Then, denote by $\tilde u_\beta$ the function in $S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that

$$\tilde u^+_{\beta,i}(x)=t_i w\big(\sqrt\beta\big(x-x_i^\beta\big)\big)\quad\forall x\in B\left(x_i^\beta,\frac{\rho}{\sqrt\beta}\right),\tag{3.16}$$

with $t_i>0$ such that $f'(\tilde u^+_{\beta,i})[\tilde u^+_{\beta,i}]=0$ for $i=1,\dots,k$, and $\tilde u^-_\beta$ such that

$$\tilde u^-_\beta(x)=0\quad\forall x\in\bigcup_{i=1}^k B\left(x_i^\beta,\frac{\rho}{\sqrt\beta}\right),\tag{3.17}$$

$$f\big(-\tilde u^-_\beta\big)=\min\left\{f(v)\colon\ v\in H^1_0(\Omega),\ v\leqslant0\text{ in }\Omega,\ v=0\text{ in }\bigcup_{i=1}^k B\left(x_i^\beta,\frac{\rho}{\sqrt\beta}\right)\right\}\tag{3.18}$$

(notice that, since $\alpha<\lambda_1$ and $e_1>0$, this minimum is really achieved by a unique function which is negative in $\Omega\setminus\bigcup_{i=1}^k\bar B(x_i^\beta,\frac{\rho}{\sqrt\beta})$).

Since $\tilde u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$, we have

$$f(u_\beta)=\min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f\leqslant f(\tilde u_\beta)=\sum_{i=1}^k f\big(\tilde u^+_{\beta,i}\big)+f\big(-\tilde u^-_\beta\big). \tag{3.19}$$

Observe that, for $i=1,\dots,k$,

$$\begin{aligned}
f\big(\tilde u^+_{\beta,i}\big)&=\max_{t\geqslant 0} f\big(t\tilde u^+_{\beta,i}\big)\\
&=\max_{t\geqslant 0}\left\{\frac{t^2}{2}\left[\int_\Omega\big|D\tilde u^+_{\beta,i}\big|^2dx-\beta\int_\Omega\big(\tilde u^+_{\beta,i}\big)^2dx\right]+t\int_\Omega e_1\tilde u^+_{\beta,i}\,dx\right\}\\
&\leqslant\max_{t\geqslant 0}\left\{\frac{t^2}{2}\left[\int_\Omega\big|D\tilde u^+_{\beta,i}\big|^2dx-\beta\int_\Omega\big(\tilde u^+_{\beta,i}\big)^2dx\right]+t\max_\Omega e_1\int_\Omega \tilde u^+_{\beta,i}\,dx\right\}\\
&=\beta^{\frac{2-n}{2}}\max_{t\geqslant 0}\left\{\frac{t^2}{2}\int_{B(0,\rho)}\big[|Dw|^2-w^2\big]dx+\frac{1}{\beta}t\max_\Omega e_1\int_{B(0,\rho)} w\,dx\right\}.
\end{aligned} \tag{3.20}$$

Since the last maximum tends to zero as $\beta\to+\infty$ (as one can easily verify by a direct computation) it follows that

$$\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}f\big(\tilde u^+_{\beta,i}\big)=0\quad\text{for } i=1,\dots,k. \tag{3.21}$$

Now, set $\tilde v_\beta=-\tilde u^-_\beta-\frac{e_1}{\alpha-\lambda_1}$. By a direct computation we obtain

$$f\big(-\tilde u^-_\beta\big)=\frac{1}{2(\alpha-\lambda_1)}+\frac12\int_\Omega|D\tilde v_\beta|^2dx-\frac{\alpha}{2}\int_\Omega\tilde v_\beta^2\,dx, \tag{3.22}$$

which, combined with (3.19) and (3.21), gives

$$\limsup_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\leqslant\frac12\limsup_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\int_\Omega\big[|D\tilde v_\beta|^2-\alpha\tilde v_\beta^2\big]dx. \tag{3.23}$$

If we set $\tilde V_\beta(x)=\tilde v_\beta(\frac{x}{\sqrt\beta})$ $\forall x\in\sqrt\beta\,\Omega$, we have

$$\beta^{\frac{n-2}{2}}\int_\Omega\big[|D\tilde v_\beta|^2-\alpha\tilde v_\beta^2\big]dx=\int_{\sqrt\beta\,\Omega}\left[|D\tilde V_\beta|^2-\frac{\alpha}{\beta}\tilde V_\beta^2\right]dx. \tag{3.24}$$

Up to a subsequence, we can assume that there exist $x_1,\dots,x_k$ in $\bar\Omega$ such that $x_i^\beta\to x_i$, as $\beta\to+\infty$, for $i=1,\dots,k$. Moreover, for $i=2,\dots,k$, we can assume that $\sqrt\beta(x_i^\beta-x_1^\beta)$ converges or $\sqrt\beta|x_i^\beta-x_1^\beta|\to\infty$ as $\beta\to+\infty$; if we consider all the cases for which $\sqrt\beta(x_i^\beta-x_1^\beta)$ converges, we obtain a subset S_1 of $\{1,2,\dots,k\}$ such that $\sqrt\beta(x_i^\beta-x_j^\beta)$ converges as $\beta\to+\infty$ if i and j both belong to S_1, while $\sqrt\beta|x_i^\beta-x_j^\beta|\to\infty$ if, for example, $i\in S_1$ and $j\notin S_1$. Repeating this procedure, we can find a subset S_2 of $\{1,2,\dots,k\}\setminus S_1$ having similar properties, then a subset S_3 of $\{1,2,\dots,k\}\setminus(S_1\cup S_2)$, etc. Thus, we obtain h ($h\leqslant k$) pairwise disjoint subsets $S_1,\dots,S_h$ of $\{1,2,\dots,k\}$ such that $\bigcup_{i=1}^h S_i=\{1,2,\dots,k\}$ and, as $\beta\to+\infty$, $\sqrt\beta|x_i^\beta-x_j^\beta|\to\infty$ if i and j belong to different subsets while it remains bounded if i and j both belong to the same subset (moreover it is clear that, in this case, $x_i=x_j$). Now, standard arguments allow us to say that there exist h functions $\tilde W_1,\dots,\tilde W_h$ in $\mathcal D^{1,2}(\mathbb R^n)$ such that (up to a subsequence)

$$\lim_{\beta\to+\infty}\int_{\sqrt\beta\,\Omega}\left[|D\tilde V_\beta|^2-\frac{\alpha}{\beta}\tilde V_\beta^2\right]dx=\sum_{j=1}^h\int_{\mathbb R^n}|D\tilde W_j|^2\,dx \tag{3.25}$$

and, in addition, the function $\tilde{W}_j$ (for every $j = 1, \dots, h$) has the following property: if S_j consists of k_j elements, there exist k_j pairwise disjoint balls with radius ρ in $\mathbb{R}^n$, $B(c_1, \rho), \dots, B(c_{k_j}, \rho)$, such that

$$\int_{\mathbb{R}^n} |D\tilde{W}_j|^2\,dx = m_j^2 \operatorname{cap}\left(\bigcup_{i=1}^{k_j} B(c_i, \rho)\right) \tag{3.26}$$

where $m_j = \frac{e_1(x_i)}{\lambda_1 - \alpha}$ for $i \in S_j$ (different choices of i in S_j clearly give the same constant m_j).

Therefore, by (3.23)–(3.26), we have

$$\limsup_{\beta\to+\infty} \beta^{\frac{n-2}{2}}\left[f(u_\beta) - \frac{1}{2(\alpha-\lambda_1)}\right] \leqslant \frac{1}{2}\sum_{j=1}^{h} m_j^2 \operatorname{cap}\left(\bigcup_{i=1}^{k_j} B(c_i, \rho)\right) \quad \forall \rho \in]\bar{r}_1, 3\bar{r}_1[. \tag{3.27}$$

Now let $\rho \to \bar{r}_1$, then take into account that $\operatorname{cap}(\bigcup_{i=1}^{k_j} B(c_i, \bar{r}_1)) \leqslant k_j \operatorname{cap}(\bar{r}_1)$, because of Lemma 2.3. Thus, we obtain

$$\begin{aligned}\limsup_{\beta\to+\infty} \beta^{\frac{n-2}{2}}\left[f(u_\beta) - \frac{1}{2(\alpha-\lambda_1)}\right] &\leqslant \frac{1}{2}\sum_{j=1}^{h} m_j^2 \operatorname{cap}\left(\bigcup_{i=1}^{k_j} B(c_i, \bar{r}_1)\right)\\ &\leqslant \frac{1}{2}\sum_{j=1}^{h} m_j^2 k_j \operatorname{cap}(\bar{r}_1)\\ &\leqslant \frac{1}{2(\alpha-\lambda_1)^2}\left(\max_{\Omega} e_1\right)^2 k \operatorname{cap}(\bar{r}_1)\end{aligned} \tag{3.28}$$

(where the last inequality holds because $m_j^2 \leqslant \frac{1}{(\alpha-\lambda_1)^2}(\max_\Omega e_1)^2$ and $k_1 + k_2 + \dots + k_h = k$).

So (3.15) is proved and the proof of property (b) is complete.

In order to prove property (c), it suffices to observe that $m_j^2 < \frac{1}{(\alpha-\lambda_1)^2}(\max_\Omega e_1)^2$ if $e_1(x_i) < \max_\Omega e_1$ and that $\operatorname{cap}(\bigcup_{i=1}^{k_j} B(c_i, \bar{r}_1)) < k_j \operatorname{cap}(\bar{r}_1)$ if $k_j > 1$ (see Lemma 2.3). Therefore, if $k_j > 1$ for some $j \in \{1, 2, \dots, h\}$ or if $e_1(x_i) < \max_\Omega e_1$ for some $i \in \{1, \dots, k\}$, we have

$$\limsup_{\beta\to+\infty} \beta^{\frac{n-2}{2}}\left[f(u_\beta) - \frac{1}{2(\alpha-\lambda_1)}\right] < \frac{1}{2(\alpha-\lambda_1)^2}\left(\max_{\Omega} e_1\right)^2 k \operatorname{cap}(\bar{r}_1) \tag{3.29}$$

in contradiction with (3.6). Thus, we infer that $e_1(x_i) = \max_\Omega e_1$ for $i = 1, \dots, k$ (namely (3.2) holds) and, in addition, that $h = k$ and $k_j = 1$ for $j = 1, \dots, h$ (which gives (3.3)). □

Proposition 3.2. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, let $(x_1^\beta, \dots, x_k^\beta) \in \Omega_{k,\beta}$ and $u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \varphi_\beta(x_1^\beta, \dots, x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$. For every $i = 1, \dots, k$, let U_{β,x_i^β} be the function defined in $\sqrt{\beta}(\Omega - x_i^\beta)$ by $U_{\beta,x_i^\beta}(x) = u_\beta(\frac{x}{\sqrt{\beta}} + x_i^\beta)$ $\forall x \in \sqrt{\beta}(\Omega - x_i^\beta)$.*

Then, as $\beta \to +\infty$, U_{β,x_i^β} tends to a smooth radial function U such that

$$U^- = \frac{1}{\lambda_1 - \alpha}\max_{\Omega} e_1 (1 - u_{B(0,\bar{r}_1)}),$$

where $u_{B(0,\bar{r}_1)}$ is the minimizing function for the capacity of $B(0, \bar{r}_1)$, and U^+ is an eigenfunction corresponding to the first eigenvalue of $-\Delta$ in $H_0^1(B(0, \bar{r}_1))$ (that is $\int_{B(0,\bar{r}_1)} |DU^+|^2\,dx = \int_{B(0,\bar{r}_1)} (U^+)^2\,dx$). Moreover, the convergence is uniform on the compact subsets of $\mathbb{R}^n$.

Proof. Notice that, for every $i = 1, \dots, k$, $U^+_{\beta,x_i^\beta} 1_{B(0,3\bar{r}_1)} \in H_0^1(B(0, 3\bar{r}_1))$ (here we set $1_{B(0,3\bar{r}_1)}(x) = 1$ if $x \in B(0, 3\bar{r}_1)$, otherwise $1_{B(0,3\bar{r}_1)}(x) = 0$). Moreover, we have

$$\int_{B(0,3\bar{r}_1)} \left|DU^+_{\beta,x_i^\beta}\right|^2 dx \leqslant \int_{B(0,3\bar{r}_1)} \left(U^+_{\beta,x_i^\beta}\right)^2 dx \tag{3.30}$$

(because $f'(u_\beta)[u^+_{\beta,i}] = 0$). Therefore, there exists $E_i \in H^1_0(B(0,3\bar r_1))$ such that (up to a subsequence) $(\int_{B(0,3\bar r_1)}(U^+_{\beta,x_i^\beta})^2\,dx)^{-\frac12}U^+_{\beta,x_i^\beta} \to E_i$, as $\beta\to+\infty$, in $L^2(B(0,3\bar r_1))$, weakly in $H^1_0(B(0,3\bar r_1))$ and a.e. in $B(0,3\bar r_1)$. It follows that

$$\int_{B(0,3\bar r_1)} |DE_i|^2\,dx \leqslant 1 \tag{3.31}$$

and

$$\int_{B(0,3\bar r_1)} E_i^2(x)x\,dx = 0. \tag{3.32}$$

Arguing as in the proof of (3.6) and taking also into account property (c) of Proposition 3.1, we infer that there exist k domains $D_1,\dots,D_k$ in $\mathbb{R}^n$ such that

$$\liminf_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right] \geqslant \frac{1}{2(\alpha-\lambda_1)^2}\Big(\max_\Omega e_1\Big)^2\sum_{i=1}^k \operatorname{cap}(D_i) \tag{3.33}$$

and $U^-_{\beta,x_i^\beta}\to\frac{1}{\lambda_1-\alpha}\max_\Omega e_1(1-u_{D_i})$ as $\beta\to+\infty$ (here u_{D_i} is the minimizing function for $\operatorname{cap}(D_i)$). Moreover $E_i\in H^1_0(D_i)$, which implies that $\lambda_1(D_i)\leqslant 1$ ($\lambda_1(D_i)$ denotes the first eigenvalue of $-\Delta$ in $H^1_0(D_i)$). As a consequence, (3.33) is not in contradiction with (3.15) only if D_i is a ball of radius $\bar r_1$ for every $i=1,\dots,k$. In fact, only the balls of radius $\bar r_1$ have the smallest capacity among the domains D of $\mathbb{R}^n$ such that $\lambda_1(D)\leqslant 1$. Then, taking into account (3.31), we infer that E_i is an eigenfunction related to the first eigenvalue of the ball D_i (because $\lambda_1(D_i)=1$), hence (3.32) implies that $D_i=B(0,\bar r_1)$ for $i=1,\dots,k$.

Now, it remains to prove that U^+_{β,x_i^β} converges as $\beta\to+\infty$ to the positive eigenfunction $\bar E$ of $-\Delta$ in $H^1_0(B(0,\bar r_1))$ such that the function $U=\bar E+\frac{1}{\alpha-\lambda_1}\max_\Omega e_1(1-u_{B(0,\bar r_1)})$ is a smooth function.

Arguing by contradiction, assume that, as $\beta\to+\infty$, $U^+_{\beta,x_i^\beta}\to\bar t\bar E$ with $\bar t\neq 1$. Consider a radial function $\psi\in C^1_0(B(0,3\bar r_1))$ such that $\psi(x)=1$ if $|x|=\bar r_1$ and set $\psi_\beta(x)=\psi(\sqrt\beta(x-x_i^\beta))$. Then, a direct computation gives

$$\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}f'(u_\beta)[\psi_\beta]=(\bar t-1)\int_{\partial B(0,\bar r_1)}(DU\cdot\nu)\,d\sigma\neq 0 \tag{3.34}$$

(where ν denotes the outward normal on $\partial B(0,\bar r_1)$) and

$$\lim_{\beta\to+\infty}\beta^{\frac{n-1}{2}}\int_{B(x_i^\beta,r_\beta)}u^+_{\beta,i}(x)\psi_\beta(x)(x-x_i)\,dx=\int_{B(0,\bar r_1)}\big[U(x)+(\bar t-1)\bar E(x)\big]\psi(x)x\,dx=0. \tag{3.35}$$

Therefore, taking into account that u_β is the unique maximum point for f in the set $\{u_\beta+tu^+_{\beta,i}\colon t\geqslant -1\}$, by standard methods one can prove that there exists a continuous map $\theta:[-1,1]\to H^1_0(B(x_i^\beta,r_\beta))$ such that $\theta(t)=0$ for $|t|\geqslant\frac12$ and, for $|t|\leqslant\frac12$,

$$f\big(u_\beta+tu^+_{\beta,i}+\theta(t)\big)<f(u_\beta),\qquad \big[u_\beta+tu^+_{\beta,i}+\theta(t)\big]^+\not\equiv 0\quad\text{in } B\big(x_i^\beta,r_\beta\big),$$
$$\int_{B(x_i^\beta,r_\beta)}\big(\big[u_\beta+tu^+_{\beta,i}+\theta(t)\big]^+(x)\big)^2\big(x-x_i^\beta\big)\,dx=0.$$

Notice that

$$f'\big(u_\beta+tu^+_{\beta,i}+\theta(t)\big)\big[\big(u_\beta+tu^+_{\beta,i}+\theta(t)\big)^+\big]$$

depends continuously on t, it is positive for $t=-\frac12$ and negative for $t=\frac12$. Therefore, there exists $\tilde t\in\,]-\frac12,\frac12[$ such that

$$f'\big(u_\beta+\tilde tu^+_{\beta,i}+\theta(\tilde t)\big)\big[\big(u_\beta+\tilde tu^+_{\beta,i}+\theta(\tilde t)\big)^+\big]=0.$$

It follows that $u_\beta + \tilde{t}u^+_{\beta,i} + \theta(\tilde{t}) \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$, which is a contradiction because $f(u_\beta) = \min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$ and $f(u_\beta + \tilde{t}u^+_{\beta,i} + \theta(\tilde{t})) < f(u_\beta)$.

Thus we can conclude that $\bar{t} = 1$, which completes the proof. □

Lemma 3.3. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, let $(x_1^\beta,\dots,x_k^\beta) \in \Omega_{k,\beta}$ and $u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \varphi_\beta(x_1^\beta,\dots,x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$. Then, there exists $\tilde{\beta} > 0$ such that, for every $\beta > \tilde{\beta}$,*

$$\sup\left\{u_\beta(x):\ x \in \bigcup_{i=1}^k A_i^\beta\right\} < 0, \quad \textit{where } A_i^\beta = \left\{x \in \Omega:\ \frac{2}{3}r_\beta \leqslant \left|x - x_i^\beta\right| \leqslant r_\beta\right\}.$$

Proof. For $i = 1,\dots,k$, consider the function $U_{\beta,x_i^\beta} : \sqrt{\beta}(\Omega - x_i^\beta) \to \mathbb{R}$ introduced in Proposition 3.2. From Proposition 3.2 it follows that, when $\beta \to +\infty$, $U_{\beta,x_i^\beta} \to \frac{1}{\alpha-\lambda_1}\max_\Omega e_1(1 - u_{B(0,\bar{r}_1)})$ in $H^1(A_{2\bar{r}_1,3\bar{r}_1})$ where $A_{2\bar{r}_1,3\bar{r}_1}$ is the annulus $\{x \in \mathbb{R}^n:\ 2\bar{r}_1 \leqslant |x| \leqslant 3\bar{r}_1\}$. Taking into account the minimality property of u_β, it follows by standard methods that the convergence is uniform in $A_{2\bar{r}_1,3\bar{r}_1}$. Therefore, as $\beta \to +\infty$,

$$\sup_{A_i^\beta} u_\beta = \sup_{A_{2\bar{r}_1,3\bar{r}_1}} U_{\beta,x_i^\beta} \to \frac{1}{\alpha - \lambda_1}\max_\Omega e_1 \min_{A_{2\bar{r}_1,3\bar{r}_1}}(1 - u_{B(0,\bar{r}_1)}) < 0.$$

Since this fact holds for every $i = 1,\dots,k$, the proof is complete. □

Remark 3.4. As a consequence of Lemma 3.3, for $\beta > 0$ large enough, we have

$$u_\beta(x) < 0 \quad \forall x \in \Omega \setminus \bigcup_{i=1}^k B\left(x_i^\beta, \frac{2}{3}r_\beta\right). \tag{3.36}$$

In fact, the minimality property of u_β implies that

$$f(u_\beta) = \min\left\{f(u_\beta + \psi):\ \psi \in H_0^1(\Omega),\ \psi = 0 \text{ in } \bigcup_{i=1}^k B\big(x_i^\beta, r_\beta\big),\ u_\beta + \psi \leqslant 0 \text{ in } \Omega \setminus \bigcup_{i=1}^k B\big(x_i^\beta, r_\beta\big)\right\} \tag{3.37}$$

(since $\alpha < \lambda_1$, $\psi = 0$ is the unique minimizing function). Taking into account that $e_1 > 0$, $\sup\{u_\beta(x):\ x \in \partial B(x_i^\beta, r_\beta)\} < 0$ for $i = 1,\dots,k$ implies $u_\beta(x) < 0\ \forall x \in \Omega \setminus \bigcup_{i=1}^k B(x_i^\beta, r_\beta)$. This fact, combined with Lemma 3.3, proves our assertion.

Let us point out that this property is important because the condition $u_\beta \leqslant 0$ in $\Omega \setminus \bigcup_{i=1}^k B(x_i^\beta, r_\beta)$ is, indeed, a unilateral constraint (an obstacle type constraint) which would have given rise to a variational inequality if $u_\beta = 0$ somewhere in $\Omega \setminus \bigcup_{i=1}^k B(x_i^\beta, r_\beta)$. On the contrary, since (3.36) holds, u_β satisfies the equation $\Delta u_\beta + \alpha u_\beta = e_1$ in $\Omega \setminus \bigcup_{i=1}^k \bar{B}(x_i^\beta, \frac{2}{3}r_\beta)$.

Lemma 3.5. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, let $(x_1^\beta,\dots,x_k^\beta) \in \Omega_{k,\beta}$ and $u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$. Moreover, assume $\beta > \tilde{\beta}$ (see Lemma* 3.3).

Then, for every $i = 1,\dots,k$, there exists $\lambda_\beta^i \in \mathbb{R}^n$ such that

$$f'(u_\beta)[\psi] = \int_\Omega u^+_{\beta,i}(x)\psi(x)\big[\lambda_\beta^i \cdot \big(x - x_i^\beta\big)\big]\,dx \quad \forall \psi \in H_0^1\big(B\big(x_i^\beta, r_\beta\big)\big) \tag{3.38}$$

(*that is, u_β is a constrained critical point for the functional f on the constraint B_i*).

Proof. It suffices to apply a technique already used above, in order to deal with the constraint V_i which is not smooth.

Arguing by contradiction, assume that u_β is not a constrained critical point for f on B_i. Taking also into account that the map $t \mapsto f(u_\beta + tu^+_{\beta,i})$ has for $t=0$ the unique maximum point in the set $[-1,+\infty[$, it follows by standard methods that there exists a continuous map $\eta : [-1,1] \to H^1_0(B(x^\beta_i, r_\beta))$ such that $\eta(t) = 0$ for $|t| \geqslant \frac{1}{2}$,

$$f\big(u_\beta + tu^+_{\beta,i} + \eta(t)\big) < f(u_\beta) \quad \forall t \in [-1,1], \tag{3.39}$$

$[u_\beta + tu^+_{\beta,i} + \eta(t)]^+ \not\equiv 0$ in $B(x^\beta_i, r_\beta)$ and $u_\beta + tu^+_{\beta,i} + \eta(t)$ belongs to B_i $\forall t \in]-1,1]$, that is

$$\int\limits_{B(x_i,r_\beta)} \big(\big[u_\beta + tu^+_{\beta,i} + \eta(t)\big]^+(x)\big)^2 \big(x - x^\beta_i\big)\,dx = 0 \quad \forall t \in [-1,1]. \tag{3.40}$$

Notice that $\eta(-\frac{1}{2}) = \eta(\frac{1}{2}) = 0$ implies that

$$f'\big(u_\beta + tu^+_{\beta,i} + \eta(t)\big)\big[\big(u_\beta + tu^+_{\beta,i} + \eta(t)\big)^+\big] \tag{3.41}$$

is positive for $t = -\frac{1}{2}$ and negative for $t = \frac{1}{2}$. Therefore, since (3.41) depends continuously on t, there exists $\tilde{t} \in]-\frac{1}{2}, \frac{1}{2}[$ such that $u_\beta + \tilde{t}u^+_{\beta,i} + \eta(\tilde{t}) \in V_i$ (namely (3.41) is equal to zero for $t = \tilde{t}$). It follows that $u_\beta + \tilde{t}u^+_{\beta,i} + \eta(\tilde{t}) \in S^\beta_{x^\beta_1,\dots,x^\beta_k}$ and $f(u_\beta + \tilde{t}u^+_{\beta,i} + \eta(\tilde{t})) < f(u_\beta)$. Clearly, this fact gives a contradiction because $f(u_\beta) = \min_{S^\beta_{x^\beta_1,\dots,x^\beta_k}} f$.

Thus, u_β is a constrained critical point for f on B_i, namely there exists $\lambda^i_\beta \in \mathbb{R}^n$ such that (3.38) holds. □

Lemma 3.6. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, let $(x^\beta_1, \dots, x^\beta_k)$ and $(y^\beta_1, \dots, y^\beta_k)$ in $\Omega_{k,\beta}$, $u_\beta \in S^\beta_{x^\beta_1,\dots,x^\beta_k}$ and $v_\beta \in S^\beta_{y^\beta_1,\dots,y^\beta_k}$ such that $f(u_\beta) = \varphi_\beta(x^\beta_1,\dots,x^\beta_k) = \max_{\Omega_{k,\beta}} \varphi_\beta$ and $f(v_\beta) = \varphi_\beta(y^\beta_1,\dots,y^\beta_k)$. Moreover, assume that*

$$\lim_{\beta\to+\infty} \sqrt{\beta}\big(x^\beta_i - y^\beta_i\big) = 0 \quad \textit{for every } i = 1,\dots,k. \tag{3.42}$$

Then

$$\lim_{\beta\to+\infty} \sup_\Omega |u_\beta - v_\beta| = 0. \tag{3.43}$$

Let us consider the function V_{β,y^β_i} defined in $\sqrt{\beta}(\Omega - y^\beta_i)$ by

$$V_{\beta,y^\beta_i}(x) = v_\beta\left(\frac{x}{\sqrt{\beta}} + y^\beta_i\right) \quad \forall x \in \sqrt{\beta}\big(\Omega - y^\beta_i\big). \tag{3.44}$$

Then, also V_{β,y^β_i} (as U_{β,x^β_i}) converges to the function U (see Proposition 3.2*) and the convergence is uniform on the compact subsets of $\mathbb{R}^n$.*

Now, assume in addition that $u_\beta \not\equiv v_\beta$ for β large enough and set

$$(U_\beta - V_\beta)_{x^\beta_i}(x) = (u_\beta - v_\beta)\left(\frac{x}{\sqrt{\beta}} + x^\beta_i\right) \quad \forall x \in \sqrt{\beta}\big(\Omega - x^\beta_i\big). \tag{3.45}$$

Then, there exists $w_i \in \mathbb{R}^n$ such that, up to a subsequence, $(\sup_\Omega |u_\beta - v_\beta|)^{-1}(U_\beta - V_\beta)_{x^\beta_i}$ converges, as $\beta \to +\infty$, to $(DU \cdot w_i) \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ and the convergence is uniform on the compact subsets of $\mathbb{R}^n$. Moreover, $w_i \neq 0$ for some $i \in \{1,\dots,k\}$.

Proof. First notice that $\lim_{\beta\to+\infty} \sqrt{\beta}|y^\beta_i - y^\beta_j| = \infty$ for $i \neq j$ because of (3.3) and (3.42). Moreover, since $f(v_\beta) = \varphi_\beta(y^\beta_1,\dots,y^\beta_k) \leqslant \max_{\Omega_{k,\beta}} \varphi_\beta = f(u_\beta)$, from Proposition 3.1 (property (b)) we obtain

$$\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(v_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]\leqslant\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]=\frac{k}{2(\alpha-\lambda_1)^2}\Big(\max_\Omega e_1\Big)^2\operatorname{cap}(\bar r_1). \tag{3.46}$$

Therefore, we can say that $V_{\beta,y_i^\beta}\to U$ as $\beta\to+\infty$. In fact, all the properties of U_{β,x_i^β} we used to prove that $U_{\beta,x_i^\beta}\to U$ as $\beta\to+\infty$, are also satisfied by V_{β,y_i^β} as one can easily verify (see the proof of Proposition 3.2). Moreover, standard arguments allow us to say that the convergence is uniform on the compact subsets of $\mathbb{R}^n$.

In order to prove (3.43), notice that

$$\sup_{B(x_i^\beta,r_\beta)}|u_\beta-v_\beta|=\sup\left\{\left|U_{\beta,x_i^\beta}(x)-V_{\beta,y_i^\beta}\big(x+\sqrt{\beta}\big(x_i^\beta-y_i^\beta\big)\big)\right|:\ x\in B(0,3\bar r_1)\right\}. \tag{3.47}$$

Since both U_{β,x_i^β} and V_{β,y_i^β} converge to U as $\beta\to+\infty$ uniformly on the compact subsets of $\mathbb{R}^n$, taking also into account (3.42), we obtain $\lim_{\beta\to+\infty}\sup_{B(x_i^\beta,r_\beta)}|u_\beta-v_\beta|=0$ for $i=1,\dots,k$. Hence, since $\alpha<\lambda_1$, it follows by standard arguments that $\lim_{\beta\to+\infty}\sup_{\Omega\setminus\bigcup_{i=1}^k B(x_i^\beta,r_\beta)}|u_\beta-v_\beta|=0$, which completes the proof of property (3.43).

Now, we assume that $u_\beta\not\equiv v_\beta$ for β large enough and describe the behaviour of the function $(\sup_\Omega|u_\beta-v_\beta|)^{-1}(U_\beta-V_\beta)_{x_i^\beta}$ as $\beta\to+\infty$ (for $i=1,\dots,k$). Up to a subsequence, it converges to a function $Z_i\in\mathcal{D}^{1,2}(\mathbb{R}^n)$, with $\sup_{\mathbb{R}^n}|Z_i|\leqslant 1$, that satisfies the equations $\Delta Z+Z=0$ in $B(0,\bar r_1)$ and $\Delta Z=0$ in $\mathbb{R}^n\setminus\bar B(0,\bar r_1)$, as one can verify taking into account the behaviour of the functions U_{β,x_i^β} and V_{β,y_i^β}.

We prove that the interior and the exterior normal derivatives of Z_i on the boundary of $B(0,\bar r_1)$ coincide, so we can say that Z_i is a weak solution of the equation

$$\Delta Z+a(x)Z=0\quad\text{in }\mathbb{R}^n, \tag{3.48}$$

where $a(x)=1$ if $x\in B(0,\bar r_1)$ and $a(x)=0$ otherwise. In order to prove this fact, notice that Lemma 3.5 implies the existence of $\Lambda_{u,\beta}$ and $\Lambda_{v,\beta}$ in $\mathbb{R}^n$ such that

$$\begin{aligned}&\int_{B(0,3\bar r_1)}\left[DU_{\beta,x_i^\beta}D\Psi+\frac{\alpha}{\beta}U^-_{\beta,x_i^\beta}\Psi-U^+_{\beta,x_i^\beta}\Psi+\frac{1}{\beta}E_{1,\beta,x_i^\beta}\Psi\right]dx\\&\quad=\int_{B(0,3\bar r_1)}U^+_{\beta,x_i^\beta}(x)\Psi(x)(\Lambda_{u,\beta}\cdot x)\,dx\quad\forall\Psi\in H_0^1\big(B(0,3\bar r_1)\big)\end{aligned} \tag{3.49}$$

(where we set $E_{1,\beta,x_i^\beta}(x)=e_1(x/\sqrt{\beta}+x_i^\beta))$ and, for $\beta>0$ large enough so that $\sqrt{\beta}\,|x_i^\beta-y_i^\beta|<\bar r_1$ (see (3.42)),

$$\begin{aligned}&\int_{B(0,2\bar r_1)}\left[DV_{\beta,x_i^\beta}D\Psi+\frac{\alpha}{\beta}V^-_{\beta,x_i^\beta}\Psi-V^+_{\beta,x_i^\beta}\Psi+\frac{1}{\beta}E_{1,\beta,x_i^\beta}\Psi\right]dx\\&\quad=\int_{B(0,2\bar r_1)}V^+_{\beta,x_i^\beta}(x)\Psi(x)\big(\Lambda_{v,\beta}\cdot\big[x-\sqrt{\beta}\,\big(y_i^\beta-x_i^\beta\big)\big]\big)\,dx\quad\forall\Psi\in H_0^1\big(B(0,2\bar r_1)\big).\end{aligned} \tag{3.50}$$

We say that

$$\lim_{\beta\to+\infty}\Lambda_{u,\beta}=0\quad\text{and}\quad\lim_{\beta\to+\infty}\Lambda_{v,\beta}=0. \tag{3.51}$$

In fact, arguing by contradiction, assume for example that (up to a subsequence)

$$\lim_{\beta\to+\infty}|\Lambda_{u,\beta}|>0\quad\text{and}\quad\lim_{\beta\to+\infty}\frac{\Lambda_{u,\beta}}{|\Lambda_{u,\beta}|}=\bar\Lambda. \tag{3.52}$$

Now, set in (3.49) $\Psi(x)=\Psi_{\bar\Lambda}(x)=\zeta(x)(\bar\Lambda\cdot x)$ where ζ is a function in $C_0^1(B(0,3\bar r_1))$ such that $\zeta(x)=1$ $\forall x\in B(0,\bar r_1)$. Then, taking into account that

$$\int_{B(0,3\bar r_1)} \big[DUD\Psi_{\bar\Lambda} - U^+\Psi_{\bar\Lambda}\big]\,dx = 0 \quad \text{while} \quad \int_{B(0,\bar r_1)} U(x)(\bar\Lambda\cdot x)^2\,dx > 0, \tag{3.53}$$

as $\beta\to+\infty$ we obtain from (3.49)

$$0 = \bigg[\int_{B(0,\bar r_1)} U(x)(\bar\Lambda\cdot x)^2\,dx\bigg]^{-1} \int_{B(0,3\bar r_1)} \big[DUD\Psi_{\bar\Lambda} - U^+\Psi_{\bar\Lambda}\big]\,dx = \lim_{\beta\to+\infty}|\Lambda_{u,\beta}| > 0, \tag{3.54}$$

which is impossible. Therefore, we can conclude that $\lim_{\beta\to+\infty}\Lambda_{u,\beta}=0$.

In a similar way one can also prove that $\lim_{\beta\to+\infty}\Lambda_{v,\beta}=0$.

From (3.49) and (3.50), it follows that

$$\begin{aligned}
&\int_{B(0,2\bar r_1)} \bigg[D(U_\beta - V_\beta)_{x_i^\beta} D\Psi + \frac{\alpha}{\beta}\big(U_\beta^- - V_\beta^-\big)_{x_i^\beta}\Psi - \big(U_\beta^+ - V_\beta^+\big)_{x_i^\beta}\Psi\bigg]\,dx \\
&\quad = \int_{B(0,2\bar r_1)} \big(U_\beta^+(x) - V_\beta^+(x)\big)_{x_i^\beta}\Psi(x)(\Lambda_{u,\beta}\cdot x)\,dx + \int_{B(0,2\bar r_1)} V^+_{\beta,x_i^\beta}(x)\Psi(x)\big((\Lambda_{u,\beta}-\Lambda_{v,\beta})\cdot x\big)\,dx \\
&\qquad + \big(\Lambda_{v,\beta}\cdot\sqrt{\beta}\big(y_i^\beta - x_i^\beta\big)\big)\int_{B(0,2\bar r_1)} V^+_{\beta,x_i^\beta}(x)\Psi(x)\,dx \quad \forall\Psi\in H_0^1\big(B(0,2\bar r_1)\big).
\end{aligned} \tag{3.55}$$

We say that

$$\limsup_{\beta\to+\infty}\frac{|(\Lambda_{v,\beta}\cdot\sqrt{\beta}(y_i^\beta - x_i^\beta))|}{\sup_\Omega|u_\beta - v_\beta|} < +\infty \tag{3.56}$$

and

$$\limsup_{\beta\to+\infty}\frac{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|}{\sup_\Omega|u_\beta - v_\beta|} < +\infty. \tag{3.57}$$

First, let us prove (3.56). Arguing by contradiction, assume for example that (up to a subsequence)

$$\lim_{\beta\to+\infty}\frac{(\Lambda_{v,\beta}\cdot\sqrt{\beta}(y_i^\beta - x_i^\beta))}{\sup_\Omega|u_\beta - v_\beta|} = +\infty. \tag{3.58}$$

Now, if (3.57) holds, we choose in (3.55) a function Ψ such that $\Psi(x)>0\ \forall x\in B(0,\bar r_1)$. Thus, letting $\beta\to+\infty$, we obtain

$$\begin{aligned}
&\bigg[\int_{B(0,\bar r_1)} U\Psi\,dx\bigg]^{-1}\bigg[\int_{B(0,2\bar r_1)} DZ_iD\Psi\,dx - \int_{B(0,\bar r_1)} Z_i\Psi\,dx\bigg] + \bar r_1\limsup_{\beta\to+\infty}\frac{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|}{\sup_\Omega|u_\beta - v_\beta|} \\
&\quad \geqslant \lim_{\beta\to+\infty}\frac{(\Lambda_{v,\beta}\cdot\sqrt{\beta}(y_i^\beta - x_i^\beta))}{\sup_\Omega|u_\beta - v_\beta|} = +\infty
\end{aligned} \tag{3.59}$$

which is a contradiction (because of (3.57)).

In a similar way one can argue when (3.57) still holds while in (3.58) $+\infty$ is replaced by $-\infty$. So we can say that (3.56) holds when (3.57) holds. In the other case, there exists $\tilde\Lambda$ such that, up to a subsequence,

$$\lim_{\beta\to+\infty}\frac{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|}{\sup_\Omega|u_\beta - v_\beta|} = +\infty \quad \text{and} \quad \lim_{\beta\to+\infty}\frac{\Lambda_{u,\beta}-\Lambda_{v,\beta}}{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|} = \tilde\Lambda. \tag{3.60}$$

So, we can choose in (3.55) a function $\tilde\Psi$ in $H_0^1(B(0,2\bar r_1))$, which is strictly positive in $B(\frac{\bar r_1}{2}\tilde\Lambda,\frac{\bar r_1}{4})$ and vanishes elsewhere. Thus, we have $\int_{B(0,\bar r_1)}U(x)\tilde\Psi(x)\,dx>0$ and $\int_{B(0,\bar r_1)}U(x)\tilde\Psi(x)(\tilde\Lambda\cdot x)\,dx>0$ (because $(\tilde\Lambda\cdot x)>0\ \forall x\in B(\frac{\bar r_1}{2}\tilde\Lambda,\frac{\bar r_1}{4})$).

Therefore, letting $\beta \to +\infty$ in (3.55), from (3.58) and (3.60) we obtain

$$\int_{B(0,2\bar r_1)} DZ_i D\tilde\Psi\,dx - \int_{B(0,\bar r_1)} Z_i\tilde\Psi\,dx = \lim_{\beta\to+\infty}\frac{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|}{\sup_\Omega|u_\beta-v_\beta|}\int_{B(0,\bar r_1)} U\tilde\Psi(\tilde\Lambda\cdot x)\,dx + \lim_{\beta\to+\infty}\frac{(\Lambda_{v,\beta}\cdot\sqrt{\beta}(y_i^\beta-x_i^\beta))}{\sup_\Omega|u_\beta-v_\beta|}\int_{B(0,\bar r_1)} U\tilde\Psi\,dx = +\infty, \tag{3.61}$$

which is a contradiction. In a similar way one can argue when (3.57) does not hold and (3.58) holds with $-\infty$ in place of $+\infty$ (it suffices to replace $\tilde\Lambda$ by $-\tilde\Lambda$ when we choose $\tilde\Psi$). Thus, (3.56) is proved in any case.

Now we can prove (3.57). Arguing again by contradiction, assume that, up to a subsequence, (3.60) holds. Then, choosing $\tilde\Psi$ as before and letting $\beta\to+\infty$ in (3.55), we obtain

$$\int_{B(0,2\bar r_1)} DZ_i D\tilde\Psi\,dx - \int_{B(0,\bar r_1)} Z_i\tilde\Psi\,dx \geqslant \lim_{\beta\to+\infty}\frac{|\Lambda_{u,\beta}-\Lambda_{v,\beta}|}{\sup_\Omega|u_\beta-v_\beta|}\int_{B(0,\bar r_1)} U\tilde\Psi(\tilde\Lambda\cdot x)\,dx + \liminf_{\beta\to+\infty}\frac{(\Lambda_{v,\beta}\cdot\sqrt{\beta}(y_i^\beta-x_i^\beta))}{\sup_\Omega|u_\beta-v_\beta|}\int_{B(0,\bar r_1)} U\tilde\Psi\,dx = +\infty \tag{3.62}$$

(where the last equality holds because of (3.56)). This gives a contradiction, so (3.57) is proved too.

Now, for every smooth function $\bar\Psi : \partial B(0,\bar r_1)\to\mathbb{R}$, we can choose a smooth function Ψ in $H_0^1(B(0,2\bar r_1))$ such that $\Psi(x)=\bar\Psi(x)\ \forall x\in\partial B(0,\bar r_1)$ and, moreover,

$$\int_{B(0,\bar r_1)} U(x)\Psi(x)x\,dx = 0 \quad\text{and}\quad \int_{B(0,\bar r_1)} U(x)\Psi(x)\,dx = 0. \tag{3.63}$$

Therefore, letting $\beta\to+\infty$ in (3.55), we obtain

$$\int_{B(0,\bar r_1)} DZ_i D\Psi\,dx - \int_{B(0,\bar r_1)} Z_i\Psi\,dx + \int_{A(\bar r_1,2\bar r_1)} DZ_i D\Psi\,dx = 0 \tag{3.64}$$

where $A(\bar r_1,2\bar r_1)=B(0,2\bar r_1)\setminus\bar B(0,\bar r_1)$. It follows that

$$\int_{\partial B(0,\bar r_1)} \Psi(DZ_i\cdot\nu)\,d\sigma + \int_{\partial A(\bar r_1,2\bar r_1)} \Psi(DZ_i\cdot\nu)\,d\sigma = 0 \tag{3.65}$$

where ν denotes the outward normal. Taking into account that $\Psi=\bar\Psi$ on $\partial B(0,\bar r_1)$ and $\Psi=0$ on $\partial B(0,2\bar r_1)$, since $\bar\Psi$ is an arbitrary function, we infer that the interior and the exterior normal derivatives of Z_i on $\partial B(0,\bar r_1)$ indeed coincide, so that Z_i is a weak solution of Eq. (3.48).

Let us point out that every solution Z of Eq. (3.48) must satisfy the condition $\int_{\partial B(0,\bar r_1)} Z\,d\sigma=0$. In fact, (3.48) implies

$$0=\int_{B(0,\bar r_1)}(\Delta Z+Z)U\,dx = -\int_{\partial B(0,\bar r_1)} Z(DU\cdot\nu)\,d\sigma + \int_{B(0,\bar r_1)} Z(\Delta U+U)\,dx = -\int_{\partial B(0,\bar r_1)} Z(DU\cdot\nu)\,d\sigma, \tag{3.66}$$

where $(DU\cdot\nu)$ is a nonzero constant on the boundary of $B(0,\bar r_1)$. Moreover, Z_i is the unique solution Z in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ satisfying the condition $Z(x)=Z_i(x)\ \forall x\in\partial B(0,\bar r_1)$. In fact, the Dirichlet problem in $B(0,\bar r_1)$

$$\begin{cases} \Delta Z+Z=0 & \text{in } B(0,\bar r_1),\\ Z=Z_i & \text{on } \partial B(0,\bar r_1),\end{cases} \tag{3.67}$$

has only solutions of the form $Z=Z_i+tU$, with $t\in\mathbb{R}$, and $Z_i+tU\in\mathcal{D}^{1,2}(\mathbb{R}^n)$ if and only if $t=0$ (because $\lim_{|x|\to\infty}U(x)=\frac{1}{\alpha-\lambda_1}\max_\Omega e_1\neq 0$).

Also, notice that the functions of the form $(DU\cdot w)$ belong to $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and are solutions of Eq. (3.48) for every $w\in\mathbb{R}^n$; moreover, if $w\neq 0$, $(DU\cdot w)$ is an eigenfunction for the Laplace–Beltrami operator on the sphere $\partial B(0,\bar r_1)$. Thus, in order to prove that Z_i is a function of this form, it suffices to prove that $\int_{\partial B(0,\bar r_1)} Z_i\Phi\,d\sigma=0$ for any other eigenfunction of the Laplace–Beltrami operator on $\partial B(0,\bar r_1)$. We have already proved this fact in the case where Φ is constant on $\partial B(0,\bar r_1)$ (that is Φ is an eigenfunction corresponding to the first eigenvalue 0). In order to deal with the other cases, let us consider the unit sphere S in $\mathbb{R}^n$. It is well known (see [39]) that the first nonzero eigenvalue for the Laplace–Beltrami operator on S is $(n-1)$, that its multiplicity is n and that the coordinate functions x_j $(j=1,\dots,n)$, and the linear combinations of these, are the corresponding eigenfunctions. By rescaling, it follows that the functions of the form $(DU\cdot w)$, with $w\in\mathbb{R}^n\setminus\{0\}$, are the eigenfunctions corresponding to the second eigenvalue of the Laplace–Beltrami operator on the sphere $\partial B(0,\bar r_1)$.

Now, for every eigenfunction Φ of the Laplace–Beltrami operator on the unit sphere S, we consider the function $h(r)=\int_S Z_i(rx)\Phi(x)\,d\sigma$ (defined for $r\geqslant 0$). Notice that $\Phi(x)$ is an eigenfunction on S if and only if $\Phi(\frac{x}{\bar r_1})$ is an eigenfunction on $\partial B(0,\bar r_1)$. Therefore, our aim is now to show that $h(\bar r_1)=0$ for the eigenfunctions Φ on S, corresponding to eigenvalues greater than $(n-1)$.

Here we argue as in [15]. Taking into account Eq. (3.48), a direct computation shows that the function h is a weak solution of the equation

$$-\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}h(r)\right)+\frac{1}{r^2}\alpha h(r)=a(r)h(r)\quad\text{in }]0,+\infty[,\tag{3.68}$$

where α denotes the eigenvalue corresponding to the eigenfunction Φ (here we set $a(r)=1$ for $r\in[0,\bar r_1]$, $a(r)=0$ for $r>\bar r_1$).

We prove that, if $\alpha>n-1$, then $h(r)=0$ $\forall r\geqslant 0$ (not only for $r=\bar r_1$). Notice that $h(0)=0$ because $\int_S\Phi(x)\,d\sigma=0$ if $\alpha\geqslant n-1$. Arguing by contradiction, assume that $h\not\equiv 0$ in $[0,+\infty[$. We say that, in this case, h cannot have more than one zero in $]0,+\infty[$. In fact, if z_1 and z_2, $z_1<z_2$, are two positive zeroes of h, we can say that $h'(z_1)\neq 0$ (otherwise $h\equiv 0$ because it is solution of a Cauchy problem in z_1 for Eq. (3.68)). Clearly, we may assume $h'(z_1)>0$ (otherwise we replace h by $-h$) so that, if we set

$$\tilde z_2=\sup\bigl\{r\in\,]z_1,z_2[\colon\ h(t)>0\ \forall t\in\,]z_1,r[\bigr\},\tag{3.69}$$

we have $\tilde z_2\in\,]z_1,z_2]$. Now, set $\bar h(r)=-\frac{d}{dr}U(r,0,\dots,0)$. A simple calculation shows that $\bar h(r)$ is a positive solution of Eq. (3.68) for $\alpha=n-1$ (it suffices to choose, for example, $\Phi(x)=x_1$ in the definition of h and take into account that $\frac{\partial U}{\partial x_1}$ is a weak solution of (3.48)). Let us consider the function $\chi(r):=\bar h'(r)h(r)-\bar h(r)h'(r)$. A direct computation shows that

$$\chi'(r)+\frac{n-1}{r}\chi(r)=[n-1-\alpha]\frac{\bar h(r)h(r)}{r^2}\quad\forall r>0.\tag{3.70}$$

It follows that

$$\chi'(r)+\frac{n-1}{r}\chi(r)<0\quad\forall r\in\,]z_1,\tilde z_2[,\tag{3.71}$$

because $\alpha>n-1$, $\bar h(r)>0$ $\forall r>0$ and $h(r)>0$ $\forall r\in\,]z_1,\tilde z_2[$. As a consequence, the function

$$\Theta(r):=r^{n-1}\chi(r)=r^{n-1}\bigl(\bar h'(r)h(r)-\bar h(r)h'(r)\bigr)\tag{3.72}$$

is strictly decreasing on $]z_1,\tilde z_2[$. Therefore, since $\Theta(z_1)=-z_1^{n-1}\bar h(z_1)h'(z_1)<0$, we have also $\Theta(\tilde z_2)<0$ which is impossible because $\Theta(\tilde z_2)=-\tilde z_2^{n-1}\bar h(\tilde z_2)h'(\tilde z_2)$, with $h'(\tilde z_2)\leqslant 0$.

Thus, h has at the most one zero in $]0,+\infty[$. Therefore, if we set

$$\tilde r=\sup\bigl\{r>0\colon\ h(t)\neq 0\ \forall t\in\,]0,r[\bigr\},\tag{3.73}$$

we have $\tilde r\in\,]0,+\infty]$. It is clear that we may assume $h(r)>0$ $\forall r\in\,]0,\tilde r[$ (otherwise we replace h by $-h$).

If $\tilde r<+\infty$, we have $h(\tilde r)=0$ and $h'(\tilde r)\leqslant 0$. Moreover, we have $\Theta(\tilde r)<0$ because $\lim_{r\to 0}\Theta(r)=0$ and $\Theta(r)$ is strictly decreasing in $]0,\tilde r[$ (as we infer from (3.70), since $h(r)>0$ $\forall r\in\,]0,\tilde r[$). Therefore, we have $\Theta(\tilde r)=-\tilde r^{n-1}\bar h(\tilde r)h'(\tilde r)<0$, which is impossible because $\bar h(\tilde r)>0$ and $h'(\tilde r)\leqslant 0$.

Thus $\tilde{r} = +\infty$ and

$$\Theta(r) \leqslant \Theta(\bar{r}_1) < 0 \quad \forall r \geqslant \bar{r}_1. \tag{3.74}$$

Notice that, for $r \geqslant \bar{r}_1$, $\bar{h}(r) = \frac{\bar{c}_1}{r^{n-1}}$ for a suitable constant $\bar{c}_1 > 0$. Therefore, for $c = -\frac{\Theta(\bar{r}_1)}{\bar{c}_1} > 0$, we have

$$h'(r) + \frac{n-1}{r}h(r) \geqslant c > 0 \quad \forall r \geqslant \bar{r}_1. \tag{3.75}$$

Now, set $H(r) = r^{n-1}h(r)$. By a simple computation, it follows from (3.75) that $H'(r) \geqslant cr^{n-1}$ $\forall r \geqslant \bar{r}_1$. Therefore, we have $\liminf_{r\to\infty} \frac{H(r)}{r^n} \geqslant \frac{c}{n}$, that is $\liminf_{r\to\infty} \frac{h(r)}{r} \geqslant \frac{c}{n} > 0$, which implies $\lim_{r\to\infty} h(r) = +\infty$.

Clearly, this fact gives a contradiction because $\sup_{\mathbb{R}^n} |Z_i| \leqslant 1$ and, as a consequence, $\sup\{|h(r)|\colon r \geqslant 0\} < +\infty$.

Thus, we can conclude that, for every $i = 1, \dots, k$, $Z_i = (DU \cdot w_i)$ for a suitable $w_i \in \mathbb{R}^n$.

It remains to show that $w_i \neq 0$ for some $i \in \{1, \dots, k\}$. Arguing by contradiction, assume that $w_i = 0$ for every $i = 1, \dots, k$. Then, standard arguments show that, for every $i = 1, \dots, k$,

$$\Bigl(\sup_{\Omega}|u_\beta - v_\beta|\Bigr)^{-1}(U_\beta - V_\beta)_{x_i^\beta} \to 0, \tag{3.76}$$

as $\beta \to +\infty$, uniformly on the compact subsets of $\mathbb{R}^n$; it follows that

$$\lim_{\beta\to+\infty}\Bigl(\sup_{\Omega}|u_\beta - v_\beta|\Bigr)^{-1} \sup_{\bigcup_{i=1}^k B(x_i^\beta, r_\beta)} |u_\beta - v_\beta| = 0 \tag{3.77}$$

and, as a consequence, that $(\sup_\Omega |u_\beta - v_\beta|)^{-1}(u_\beta - v_\beta) \to 0$ as $\beta \to +\infty$ uniformly in Ω, which is obviously a contradiction. So the proof is complete. □

Remark 3.7. As a first consequence of Lemma 3.6, we can say that there exists $\tilde{\beta}_k > 0$ such that, for $\beta > \tilde{\beta}_k$, u_β is the unique function in $S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$. In fact, arguing by contradiction, assume that (up to a subsequence) there exists another minimizing function $v_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$, $v_\beta \not\equiv u_\beta$, such that $f(v_\beta) = \min_{S^\beta_{x_1^\beta,\dots,x_k^\beta}} f$.

A direct computation shows that

$$\int_{B(0,3\bar{r}_1)} \bigl(V^+_{\beta,x_i^\beta}(x)\bigr)^2 x\,dx = \int_{B(0,3\bar{r}_1)} \bigl(U^+_{\beta,x_i^\beta}(x)\bigr)^2 x\,dx + 2\int_{B(0,3\bar{r}_1)} U^+_{\beta,x_i^\beta}(x)\bigl(V^+_\beta - U^+_\beta\bigr)_{x_i^\beta}(x)x\,dx + o\Bigl(\sup_{\Omega}|v_\beta - u_\beta|\Bigr) \quad \text{for } i = 1, \dots, k. \tag{3.78}$$

Since u_β and v_β both belong to $S^\beta_{x_1^\beta,\dots,x_k^\beta}$, we have (for β large enough)

$$\int_{B(0,3\bar{r}_1)} \bigl(V^+_{\beta,x_i^\beta}(x)\bigr)^2 x\,dx = \int_{B(0,3\bar{r}_1)} \bigl(U^+_{\beta,x_i^\beta}(x)\bigr)^2 x\,dx = 0.$$

Therefore, as $\beta \to +\infty$, from (3.78) we obtain

$$\int_{B(0,\bar{r}_1)} U(x)\bigl(DU(x)\cdot w_i\bigr)x\,dx = 0. \tag{3.79}$$

It follows that $w_i = 0$. In fact, if $w_i \neq 0$, $(DU(x)\cdot w_i)(x \cdot w_i) < 0$ for $(x \cdot w_i) \neq 0$.

Therefore, taking also into account that U has radial symmetry, we infer that

$$\int_{B(0,\bar{r}_1)} U(x)\bigl(DU(x)\cdot w_i\bigr)x\,dx = -\bar{c}w_i, \tag{3.80}$$

for a suitable positive constant $\bar{c}$, in contradiction with (3.79). Thus $w_i = 0$ for every $i = 1, \dots, k$, which is impossible because of Lemma 3.6 (since $w_i \neq 0$ for some $i \in \{1, \dots, k\}$).

Hence we can conclude that, for $\beta > 0$ large enough, u_β is the unique minimizing function for f on $S^\beta_{x_1^\beta, \dots, x_k^\beta}$.

Proposition 3.8. *For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, let $(x_1^\beta, \dots, x_k^\beta) \in \Omega_{k,\beta}$ and $u_\beta \in S^\beta_{x_1^\beta, \dots, x_k^\beta}$ such that $f(u_\beta) = \varphi_\beta(x_1^\beta, \dots, x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$. Then, there exists $\bar{\beta}_k > 0$ such that u_β is a solution of problem* (1.5) *for every $\beta > \bar{\beta}_k$.*

Proof. We have to prove that the Lagrange multipliers (see Lemma 3.5) are equal to zero for β large enough, that is there exists $\bar{\beta}_k > 0$ such that, for every $i = 1, \dots, k$, $\lambda^i_\beta = 0$ $\forall \beta > \bar{\beta}_k$. Arguing by contradiction, assume that, for some $i \in \{1, \dots, k\}$, there exists a sequence $(\beta_j)_{j \in \mathbb{N}}$ such that $\lim_{j\to\infty} \beta_j = +\infty$ and $\lambda^i_{\beta_j} \neq 0$ $\forall j \in \mathbb{N}$. Clearly, we can assume that

$$\limsup_{j\to\infty} \frac{|\lambda^m_{\beta_j}|}{|\lambda^i_{\beta_j}|} \leqslant 1 \quad \text{for } m = 1, \dots, k \tag{3.81}$$

(otherwise we replace i by some $m \neq i$ and the sequence $(\beta_j)_j$ by a suitable subsequence). Since $|\lambda^i_{\beta_j}| \neq 0$, there exists $\lambda^i \in \mathbb{R}^n$, with $|\lambda^i| = 1$, such that (up to a subsequence) $\frac{\lambda^i_{\beta_j}}{|\lambda^i_{\beta_j}|} \to \lambda^i$ as $j \to \infty$. Then, let us choose $(y_1^{\beta_j}, \dots, y_k^{\beta_j}) \in \Omega_{k,\beta_j}$ and $v_{\beta_j} \in S^{\beta_j}_{y_1^{\beta_j}, \dots, y_k^{\beta_j}}$, with $f(v_{\beta_j}) = \varphi_{\beta_j}(y_1^{\beta_j}, \dots, y_k^{\beta_j})$, in such a way that $y_m^{\beta_j} = x_m^{\beta_j}$ for $m \neq i$, $y_i^{\beta_j} = x_i^{\beta_j} + \frac{\varepsilon_j}{\sqrt{\beta_j}} \lambda^i$, with $\varepsilon_j > 0$ $\forall j \in \mathbb{N}$, $\lim_{j\to\infty} \varepsilon_j = 0$ and

$$\lim_{j\to\infty} \frac{\varepsilon_j \beta_j^{\frac{3}{2}}}{|\lambda^i_{\beta_j}|} = 0 \tag{3.82}$$

(this choice of $y_1^{\beta_j}, \dots, y_k^{\beta_j}$ is indeed possible because $\lim_{j\to\infty} \sqrt{\beta_j} |x_i^{\beta_j} - x_m^{\beta_j}| = \infty$ for $m \neq i$).

A direct computation shows that

$$f(v_{\beta_j}) = f(u_{\beta_j}) + f'(u_{\beta_j})[v_{\beta_j} - u_{\beta_j}] + R_j \tag{3.83}$$

where, for j large enough,

$$R_j \geqslant \frac{1}{2} \int_{\Omega_j} \left[\left| D(v_{\beta_j} - u_{\beta_j}) \right|^2 - \alpha (v_{\beta_j} - u_{\beta_j})^2 \right] dx - \frac{\beta_j}{2} \sum_{m=1}^k \int_{B(x_m^{\beta_j}, r_{\beta_j})} (v_{\beta_j} - u_{\beta_j})^2 \, dx \tag{3.84}$$

with $\Omega_j = \Omega \setminus \bigcup_{m=1}^k B(x_m^{\beta_j}, r_{\beta_j})$. Notice that

$$\lim_{j\to\infty} \min \left\{ \int_{\Omega_j} |Du|^2 \, dx \colon u \in H_0^1(\Omega), \int_{\Omega_j} u^2 \, dx = 1 \right\} = \lambda_1 \tag{3.85}$$

which, since $\alpha < \lambda_1$, implies

$$\int_{\Omega_j} \left[\left| D(v_{\beta_j} - u_{\beta_j}) \right|^2 - \alpha (v_{\beta_j} - u_{\beta_j})^2 \right] dx \geqslant 0 \tag{3.86}$$

for j large enough.

Now, fix a function $\eta \in C_0^1(B(0, 3\bar{r}_1))$ such that $\eta(x) = 1$ $\forall x \in B(0, 2\bar{r}_1)$ and, for $m = 1, \dots, k$, set $\eta_j^m(x) = \eta[\sqrt{\beta_j}(x - x_m^{\beta_j})]$ $\forall x \in B(x_m^{\beta_j}, r_{\beta_j})$ ($\eta_j^m(x) = 0$ elsewhere). Then, for j large enough so that $u_{\beta_j} < 0$ in

$\Omega \setminus \bigcup_{m=1}^{k} B(x_m^{\beta_j}, \frac{2}{3} r_{\beta_j})$, we have

$$f'(u_{\beta_j})\left[\left(1-\sum_{m=1}^{k}\eta_j^m\right)(v_{\beta_j}-u_{\beta_j})\right]=0. \tag{3.87}$$

Thus, from (3.83), (3.84), (3.86) and (3.87), we obtain

$$f(v_{\beta_j})-f(u_{\beta_j})\geqslant\sum_{m=1}^{k}f'(u_{\beta_j})\big[\eta_j^m(v_{\beta_j}-u_{\beta_j})\big]-\frac{\beta_j}{2}\sum_{m=1}^{k}\int\limits_{B(x_m^{\beta_j},r_{\beta_j})}(v_{\beta_j}-u_{\beta_j})^2\,dx, \tag{3.88}$$

where

$$f'(u_{\beta_j})\big[\eta_j^m(v_{\beta_j}-u_{\beta_j})\big]=\int\limits_{\Omega}u^+_{\beta_j,m}(x)\eta_j^m(x)(v_{\beta_j}-u_{\beta_j})(x)\big[\lambda^m_{\beta_j}\cdot\big(x-x_m^{\beta_j}\big)\big]\,dx\quad\text{for } m=1,\dots,k, \tag{3.89}$$

because $\eta_j^m(v_{\beta_j}-u_{\beta_j})\in H_0^1(B(x_m^{\beta_j}, r_{\beta_j}))$ (see Lemma 3.5).

After rescaling, we have (for j large enough)

$$\begin{aligned}\frac{\beta_j^{\frac{n+1}{2}}}{|\lambda^i_{\beta_j}|s_j}\big[f(v_{\beta_j})-f(u_{\beta_j})\big]\geqslant{}&\sum_{m=1}^{k}\int\limits_{B(0,3\bar r_1)}U^+_{\beta_j,x_m^{\beta_j}}(x)\eta(x)\frac{1}{s_j}(V_{\beta_j}-U_{\beta_j})_{x_m^{\beta_j}}(x)\frac{1}{|\lambda^i_{\beta_j}|}\big(\lambda^m_{\beta_j}\cdot x\big)\,dx\\&-\frac{s_j\beta_j^{\frac{3}{2}}}{2|\lambda^i_{\beta_j}|}\sum_{m=1}^{k}\int\limits_{B(0,3\bar r_1)}\frac{1}{s_j^2}(V_{\beta_j}-U_{\beta_j})^2_{x_m^{\beta_j}}\,dx,\end{aligned} \tag{3.90}$$

where, for short, we set $s_j=\sup_\Omega|v_{\beta_j}-u_{\beta_j}|$ (notice that $s_j\neq0\ \forall j\in\mathbb{N}$ because $v_{\beta_j}\neq u_{\beta_j}$, since $\varepsilon_j>0$). From Lemma 3.6 we infer that there exist $w_1,\dots,w_k$ in $\mathbb{R}^n$ such that (up to a subsequence) $\frac{1}{s_j}(V_{\beta_j}-U_{\beta_j})_{x_m^{\beta_j}}\to(DU\cdot w_m)$, as $j\to\infty$, uniformly in $B(0,3\bar r_1)$ for every $m=1,\dots,k$.

We say that $w_m=0$ for $m\neq i$. In fact,

$$\int\limits_{B(0,3\bar r_1)}\big(V^+_{\beta_j,x_m^{\beta_j}}(x)\big)^2x\,dx=\int\limits_{B(0,3\bar r_1)}\big(U^+_{\beta_j,x_m^{\beta_j}}(x)\big)^2x\,dx+2\int\limits_{B(0,3\bar r_1)}U^+_{\beta_j,x_m^{\beta_j}}(x)\big(V^+_{\beta_j}-U^+_{\beta_j}\big)_{x_m^{\beta_j}}(x)x\,dx+o(s_j). \tag{3.91}$$

Because of the choice of $(y_1^{\beta_j},\dots,y_k^{\beta_j})$, if $m\neq i$, we have

$$\int\limits_{B(0,3\bar r_1)}\big(V^+_{\beta_j,x_m^{\beta_j}}(x)\big)^2x\,dx=\int\limits_{B(0,3\bar r_1)}\big(U^+_{\beta_j,x_m^{\beta_j}}(x)\big)^2x\,dx=0 \tag{3.92}$$

for j large enough.

Therefore, as $j\to\infty$, we obtain

$$\int\limits_{B(0,\bar r_1)}U(x)\big(DU(x)\cdot w_m\big)x\,dx=0, \tag{3.93}$$

which implies $w_m=0$ (in fact, (3.93) holds if and only if $w_m=0$, as one can easily verify).

On the contrary, for $m=i$ we have (for j large enough)

$$\int\limits_{B(0,3\bar r_1)}\big(U^+_{\beta_j,x_i^{\beta_j}}(x)\big)^2x\,dx=0 \tag{3.94}$$

while

$$\int_{B(0,3\bar r_1)} \big(V^+_{\beta_j,x_i^{\beta_j}}(x)\big)^2 x\,dx = \varepsilon_j \lambda^i \int_{B(0,3\bar r_1)} \big(V^+_{\beta_j,x_i^{\beta_j}}(x)\big)^2\,dx. \tag{3.95}$$

As $j \to \infty$, from (3.91) and (3.95) we obtain

$$\lim_{j\to\infty} \frac{\varepsilon_j}{s_j} \lambda^i \int_{B(0,\bar r_1)} U^2\,dx = 2 \int_{B(0,\bar r_1)} U(x)\big(DU(x)\cdot w_i\big)x\,dx, \tag{3.96}$$

where $w_i \neq 0$ because $w_m = 0$ for $m \neq i$ (see Lemma 3.6). It follows that $\lim_{j\to\infty} \frac{\varepsilon_j}{s_j} \in\ ]0, +\infty[$ and $w_i = -c_i\lambda^i$ for a suitable positive constant c_i.

Now we can pass to the limit as $j \to \infty$ in (3.90). So (taking also into account (3.81) and (3.82)) we obtain

$$\liminf_{j\to\infty} \frac{\beta_j^{\frac{n+1}{2}}}{|\lambda^i_{\beta_j}|s_j}\big[f(v_{\beta_j}) - f(u_{\beta_j})\big] \geqslant -c_i \int_{B(0,\bar r_1)} U(x)\big(DU(x)\cdot\lambda^i\big)\big(x\cdot\lambda^i\big)\,dx > 0, \tag{3.97}$$

where the last inequality holds because $c_i > 0$ and $U(x)(DU(x)\cdot\lambda^i)(x\cdot\lambda^i) < 0$ almost everywhere in $B(0,\bar r_1)$ (for every x such that $(x\cdot\lambda^i) \neq 0$).

It is clear that (3.97) gives a contradiction because $f(v_{\beta_j}) \leqslant f(u_{\beta_j})$ $\forall j \in \mathbb{N}$, since $f(v_{\beta_j}) = \varphi_{\beta_j}(y_1^{\beta_j}, \dots, y_k^{\beta_j})$ while $f(u_{\beta_j}) = \max_{\Omega_{k,\beta}} \varphi_{\beta_j}$.

Thus, we can say that u_β is a solution of problem (1.5) for $\beta > 0$ large enough, which completes the proof. □

Proof of Theorem 1.1. For every positive integer k, for $\alpha < \lambda_1$ and for $\beta > 0$ large enough so that $\Omega_{k,\beta} \neq \emptyset$, consider a point $(x_1^\beta, \dots, x_k^\beta) \in \Omega_{k,\beta}$ and a function $u_\beta \in S^\beta_{x_1^\beta,\dots,x_k^\beta}$ such that $f(u_\beta) = \varphi_\beta(x_1^\beta, \dots, x_k^\beta) = \max_{\Omega_{k,\beta}} \varphi_\beta$ (see Propositions 2.1 and 2.2).

If we set $u_{k,\beta} = u_\beta$, then all the assertions in Theorem 1.1 follow directly from Propositions 3.1, 3.2 and 3.8 as one can easily verify. □

Remark 3.9. The method used to prove Theorem 1.1 may be easily adapted to prove that, for all $k \in \mathbb{N}$ and $\xi_0 \in L^2(\Omega)$, the problem (1.3) with $g(u) = -\alpha u^- + \beta u^+$ has, for β and t positive and large enough, a solution $u_{k,\beta,t}$ such that, as $t \to +\infty$, $\frac{1}{t}u_{k,\beta,t}$ tends to the solution $u_{k,\beta}$ obtained in Theorem 1.1 for problem (1.5). Thus, our method applies also in the case of the problem considered by Dancer in [14]. Indeed, the same result holds for more general functions g satisfying (1.2) (see [33]).

Remark 3.10. One can easily verify that (since $\alpha < \lambda_1$) the trivial solution $\frac{e_1}{\alpha-\lambda_1}$ is a strict local minimum point for the functional f and that

$$\lim_{t\to\infty} f\left(\frac{e_1}{\alpha-\lambda_1} + te_1\right) = -\infty \quad \forall \beta > \lambda_1. \tag{3.98}$$

Moreover, f satisfies the well-known Palais–Smale compactness condition. Therefore, there exists a mountain pass type solution. Let us point out that this solution differs from the ones given by Theorem 1.1 because they correspond to higher critical values (even in the case $k = 1$).

In fact, let us fix $\bar x \in \Omega$ such that $e_1(\bar x) < \max_\Omega e_1$ and choose a function $\bar u_\beta \in S^\beta_{\bar x}$, such that $f(\bar u_\beta) = \min_{S^\beta_{\bar x}} f$. Then, consider the continuous path $p : [0, +\infty[\ \to H^1_0(\Omega)$ defined as follows:

$$p(t) = \begin{cases} (1-t)\frac{e_1}{\alpha-\lambda_1} + t(\bar u_\beta - \bar u_\beta^+) & \forall t \in [0,1], \\ \bar u_\beta - 2\bar u_\beta^+ + t\bar u_\beta^+ & \forall t \geqslant 1. \end{cases} \tag{3.99}$$

One can easily verify that

$$p(0)=\frac{e_1}{\alpha-\lambda_1},\qquad \lim_{t\to+\infty}\left\|\frac{e_1}{\alpha-\lambda_1}-p(t)\right\|_{H_0^1(\Omega)}=+\infty,$$
$$\lim_{t\to+\infty} f\big(p(t)\big)=-\infty \quad\text{and}\quad \max\big\{f\big(p(t)\big):\ t\geqslant 0\big\}=f(\bar u_\beta). \tag{3.100}$$

Therefore, the mountain pass level is less than or equal to $f(\bar u_\beta)$. The solution u_β obtained in this paper satisfies property (b) of Proposition 3.1. Arguing as in the proof of Proposition 3.1 one can verify that

$$\lim_{\beta\to+\infty}\beta^{\frac{n-2}{2}}\left[f(\bar u_\beta)-\frac{1}{2(\alpha-\lambda_1)}\right]=\frac{1}{2(\alpha-\lambda_1)^2}\big(e_1(\bar x)\big)^2\operatorname{cap}(\bar r_1). \tag{3.101}$$

Since $e_1(\bar x)<\max_\Omega e_1$, it follows that $f(\bar u_\beta)<f(u_\beta)$ for β large enough, which proves our assertion. Indeed (see [33]) the mountain pass solution has one peak localized near the boundary of Ω; moreover, the number of solutions of this type, with peaks near $\partial\Omega$, may be related to the geometrical properties of $\partial\Omega$ (on the other hand, also the number of solutions with a prescribed number of interior peaks may be related to the shape of Ω).

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