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Point-condensation phenomena and saturation effect for the one-dimensional Gierer–Meinhardt system

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Abstract

In this paper, we are concerned with peak solutions to the following one-dimensional Gierer–Meinhardt system with saturation:

$$\begin{cases} 0=\varepsilon^2 A''-A+\dfrac{A^2}{H(1+\kappa A^2)}+\sigma, & A>0,\ x\in(-1,1),\\ 0=DH''-H+A^2, & H>0,\ x\in(-1,1),\\ A'(\pm 1)=H'(\pm 1)=0, \end{cases}$$

where ε, $D>0$, $\kappa\geqslant 0$, $\sigma\geqslant 0$. The saturation effect of the activator is given by the parameter κ. We will give a sufficient condition of κ for which point-condensation phenomena emerge. More precisely, for fixed $D>0$, we will show that the Gierer–Meinhardt system admits a peak solution when ε is sufficiently small under the assumption: κ depends on ε, namely, $\kappa=\kappa(\varepsilon)$, and there exists a limit $\lim_{\varepsilon\to 0}\kappa\varepsilon^{-2}=\kappa_0$ for certain $\kappa_0\in[0,\infty)$.

MSC: 35K57; 35Q80; 92C15

Keywords: Gierer–Meinhardt system; Saturation effect; Pattern formation; Nonlinear elliptic system

1. Introduction

In this paper, we are concerned with the following system of ordinary differential equations:

$$\begin{cases} 0=\varepsilon^2 A''-A+\dfrac{A^2}{H(1+\kappa A^2)}+\sigma, & A>0,\ x\in(-1,1),\\ 0=DH''-H+A^2, & H>0,\ x\in(-1,1),\\ A'(\pm 1)=H'(\pm 1)=0, \end{cases} \tag{1}$$

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where unknowns are $A = A(x)$ and $H = H(x)$. $\varepsilon > 0$, $D > 0$, $\kappa \geqslant 0$ and $\sigma \geqslant 0$ are constants. This system arises as a steady-state problem of the 1-dimensional Gierer–Meinhardt system with saturation which was proposed by A. Gierer and H. Meinhardt [4]. The general Gierer–Meinhardt system is written by

$$\begin{cases} A_t = \varepsilon^2 \Delta A - A + \dfrac{A^p}{H^q(1+\kappa A^p)} + \sigma, & A > 0,\ x \in \Omega,\ t > 0, \\ \tau H_t = D\Delta H - H + \dfrac{A^r}{H^s}, & H > 0,\ x \in \Omega,\ t > 0, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ A(x,0) = A_0(x), \qquad H(x,0) = H_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where $A = A(x,t)$ and $H = H(x,t)$, $\tau > 0$, Δ is the Laplace operator in $\mathbb{R}^N$, Ω is a bounded smooth domain in $\mathbb{R}^N$, ν is the unit outer normal to $\partial\Omega$. The exponents satisfy the conditions $p > 1$, $q, r > 0$, $s \geqslant 0$, and $0 < (p-1)/q < r/(s+1)$. The unknowns $A(x,t)$ and $H(x,t)$ represent the concentrations of an activator and an inhibitor, respectively, at $x \in \Omega$ and time $t > 0$. A_0 and H_0 are their initial data. One of the parameters of (2), κ stands for the degree of a saturation effect to the reaction term of the activator. The term $\sigma \geqslant 0$ is the source term. σ represent the source rate of the activator. This system expresses some models of biological pattern formation. It is known that (2) has various kinds of striking solutions when ε is small and D is large. In particular, we are mainly interested in a solution such that the activator A is concentrated at a finite number of points in $\overline{\Omega}$. Such a solution is called a "peak solution". Peak solutions represent point-condensation phenomena of the activator. When $\kappa = 0$ (no saturation case), a lot of methods to construct peak solutions were established by many mathematicians. However, when $\kappa > 0$, it is not trivial whether a peak solution also exists or not. When $\kappa > 0$ is fixed independently of ε, due to the bistable nonlinearity, solutions with transition layers may exist. Indeed, M. del Pino [3] showed the existence of solutions with multiple layers when the domain Ω is a ball. See also [1,16,7].

We introduce the shadow system of (2). Dividing the second equation in (2) by D and taking the limit $D \to \infty$ formally, we have $\Delta H = 0$ in Ω and $\frac{\partial H}{\partial \nu} = 0$ on $\partial\Omega$. This means that $H(x,t)$ does not depend on x, and hence we can regard $H(x,t) = \xi(t)$. Thus we have the following system which is called the shadow system of (2):

$$\begin{cases} A_t = \varepsilon^2 \Delta A - A + \dfrac{A^p}{\xi^q(1+\kappa A^p)} + \sigma, & A > 0,\ x \in \Omega,\ t > 0, \\ \tau \xi_t = \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \left(-\xi + \dfrac{A^r}{\xi^s}\right) dx, & \xi > 0,\ t > 0, \\ \dfrac{\partial A}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ A(x,0) = A_0(x), \qquad \xi(0) = \xi_0, & x \in \Omega. \end{cases} \tag{3}$$

For this shadow system, in the case $(p,q,r,s) = (2,1,2,0)$, $\sigma = 0$, $\kappa > 0$, J. Wei and M. Winter [22] showed that the shadow system (3) admits a stationary solution concentrating at one point of the boundary for sufficiently small ε, and the stability was studied. In [8], multi-boundary peak stationary solutions to (3) has been constructed for sufficiently small ε in the case where $\Omega \subset \mathbb{R}^N$ is axially symmetric with respect to x_N-axis, $(p,q,r,s) = (2,1,2,0)$, $\sigma = 0$, $\kappa > 0$, $N \leqslant 5$. Moreover, multi-boundary peak stationary solutions to the original Gierer–Meinhardt system (2) was constructed near the solution to the shadow system (3) for sufficiently large D by using the implicit function theorem. The result was extended to the case $\sigma > 0$ in [10]. In [22,8,10], it was supposed that $\kappa \geqslant 0$ depends on ε, namely $\kappa = \kappa(\varepsilon)$, and there exists a limit $\kappa\varepsilon^{-2N} \to \kappa_0 \in [0,\infty)$ as $\varepsilon \to 0$ for certain κ_0. This condition is called a "weak saturation" condition. This condition gives one of the sufficient conditions for which peak solutions appear.

The method, namely, to find a stationary solution to (2) near the stationary solution to the shadow system (3) by the implicit function theorem, is one of the methods to construct a solution to the Gierer–Meinhardt system (2), which was developed from the work by W.-M. Ni and I. Takagi [14]. In general, the number D must be large enough in the method. However, the following question arises, "for $D > 0$ given arbitrarily, does the Gierer–Meinhardt system (2) possess a peak solution under the weak saturation condition?". The purpose of this paper is to construct a 1-peak solution concentrating at $x = 0$ to the 1-dimensional Gierer–Meinhardt system (1) for any fixed finite D (which is called the strong coupling case) under the weak saturation condition.

We give remarks on other related results. For fixed $\kappa > 0$, M. Mimura, M. Tabata and Y. Hosono [1] showed the existence of interior transition layers by using the singular perturbation method in the case $N = 1$. Y. Nishiura [15] showed that, for some 1-dimensional reaction–diffusion systems including the Gierer–Meinhardt system (2), the bifurcating branch emanating from a uniform state continues to exist until it is connected to the singularly perturbed solutions when one of the diffusion constants is sufficiently large. Multi-peak stationary solutions to (2) were first constructed by I. Takagi [17] in the case $\kappa = 0$, $N = 1$. Moreover, its stability was discussed in [5]. In the case $\kappa = 0$ and $N = 1$, J. Wei and M. Winter [23] studied the existence and stability of symmetric and asymmetric multi-peak stationary solutions to (2), and they showed that multi-peak stationary solutions are generated by exactly two types of peaks if the peaks are separated. In the case $\kappa = 0$ and $N = 2$, multi-interior peak stationary solutions were constructed and the stability was discussed in [19–21]. With respect to the stability analysis for the Gierer–Meinhardt system and its shadow system, see [12,7,9], and the references therein. Some a priori estimate for a stationary solutions to (2) were given in [6,2,13]. For other results related to the Gierer–Meinhardt system, see [11,18,24] and the references therein.

Finally, we state remarks on our notation. For a domain $\Omega \subset \mathbb{R}$, we use standard Lebesgue spaces and Sobolev spaces $L^2(\Omega)$, $L^\infty(\Omega)$, $H^2(\Omega)$, and so on, with the usual norm. Throughout this paper, unless otherwise stated, we use the symbols C, C', C'', c, c', c'' as positive constants, but they need not have the same value in each situation.

This paper is composed as follows. In Section 2, we will state our main results, Theorem 1 and Theorem 2. In Section 3, we will prepare some lemmas and state an outline of our construction of a solution. In Section 4, we will give some estimates in order to prove the theorems. In Sections 5 and 6, we will give the proofs of Theorems 1 and 2.

2. Main results

We need some preliminaries to state our main results. We introduce a solution denoted by w_δ to the following problem:

$$\begin{cases} w'' - w + f_\delta(w) = 0, & w > 0, \text{ in } \mathbb{R}, \\ w(0) = \max\limits_{y \in \mathbb{R}} w(y), & w(y) \to 0 \text{ as } |y| \to \infty, \end{cases} \tag{4}$$

$$f_\delta(w) := \frac{w^2}{1 + \delta w^2}. \tag{5}$$

It is known that, there exists a constant $\delta_* > 0$, the problem (4) has a unique solution w_δ for each $\delta \in [0, \delta_*)$, and w_δ is radially symmetric, namely, $w_\delta(y) = w_\delta(-y)$, $y \in \mathbb{R}$. This fact was established in [22]. The number δ_* is given by

$$\delta_* = \sup\left\{\delta > 0\colon \text{ there exists } a > 0 \text{ such that } \int_0^a \bigl(-t + f_\delta(t)\bigr)\,dt = 0\right\}.$$

For fixed $D > 0$, let $G_D(x, z)$ be Green's function to

$$\begin{cases} DG_{xx}(x, z) - G(x, z) = -\delta_z(x) & \text{in } (-1, 1), \\ G_x(\pm 1, z) = 0. \end{cases} \tag{6}$$

$G_D(x, z)$ can be written explicitly

$$G_D(x, z) = \begin{cases} \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1 + x)] \cosh[\theta(1 - z)], & -1 < x < z, \\ \frac{\theta}{\sinh(2\theta)} \cosh[\theta(1 - x)] \cosh[\theta(1 + z)], & z < x < 1, \end{cases} \tag{7}$$

where $\theta := D^{-1/2}$. We put

$$\alpha_D := \frac{1}{G_D(0, 0)}. \tag{8}$$

Moreover, the non-smooth part of $G_D(x, z)$ is given by

$$K_D\bigl(|x - z|\bigr) = \frac{1}{2\sqrt{D}} e^{-\frac{1}{\sqrt{D}}|x - z|}. \tag{9}$$

Let $H_D(x,z)$ be the regular part of $G_D(x,z)$,

$$G_D(x,z) = K_D\big(|x-z|\big) - H_D(x,z).$$

$H_D(x,z)$ is C^∞ in both x and z.

Next, we prepare a cut-off function. Let $\chi \in C_0^\infty(\mathbb{R})$ be a function such that, $0 \leqslant \chi \leqslant 1$, $\chi(x) = 0$ for $|x| < 1$, $\chi(x) = 1$ for $|x| > 2$. Let r_0 be a fixed constant such that $0 < r_0 < 1/2$, for example, $r_0 = 1/10$. We will use a cut-off function in the form $\chi(\frac{x}{r_0})$. Note that $\chi(\frac{x}{r_0}) = 0$ for $|x| > 2r_0$.

We suppose the following assumption on the constant κ in (1).

(A) $\kappa \geqslant 0$ depends on ε, and there exists a limit

$$\lim_{\varepsilon\to 0} \kappa\varepsilon^{-2} = \kappa_0 \tag{10}$$

for some $\kappa_0 \in [0,\infty)$.

Let us state our main results. We first state a result in the case $\sigma = 0$.

Theorem 1. *Let $\sigma = 0$. Fix $D > 0$ arbitrarily. We suppose* (A)*, and let the value $\kappa_0\alpha_D^2$ be sufficiently small. Then, for sufficiently small $\varepsilon > 0$,* (1) *admits a* 1*-peak radially symmetric solution $(A_\varepsilon(x), H_\varepsilon(x))$ such that $A_\varepsilon(x)$ concentrates at $x = 0$. More precisely, there exists $\delta_\varepsilon \in [0,\delta_*)$ for each ε sufficiently small such that $\delta_\varepsilon \to \delta_0$ as $\varepsilon \to 0$ for some $\delta_0 \in [0,\delta_*)$ which is decided by κ_0 and D and satisfies*

$$\delta_0\bigg(\int_{\mathbb{R}} w_{\delta_0}^2(y)\,dy\bigg)^2 = \kappa_0\alpha_D^2, \tag{11}$$

and A_ε takes the form:

$$A_\varepsilon(x) = \frac{1}{\varepsilon\int_{\mathbb{R}} w_{\delta_\varepsilon}^2}\bigg\{\alpha_D w_{\delta_\varepsilon}\bigg(\frac{x}{\varepsilon}\bigg)\chi\bigg(\frac{x}{r_0}\bigg) + \varepsilon\phi_\varepsilon\bigg(\frac{x}{\varepsilon}\bigg)\bigg\}, \quad x \in (-1,1), \tag{12}$$

where α_D is defined by (8)*, w_δ is the unique solution to* (4)*, and $\phi_\varepsilon(y)$ is a radially symmetric function on $\Omega_\varepsilon := (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ such that*

$$\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leqslant C \tag{13}$$

holds for some constant $C > 0$ independent of ε. H_ε has the following property:

$$H_\varepsilon(0) = \frac{1}{\varepsilon\int_{\mathbb{R}} w_{\delta_\varepsilon}^2}\big(\alpha_D + O(\varepsilon)\big) \quad \text{as } \varepsilon \to 0. \tag{14}$$

Next, we state a result in the case $\sigma \neq 0$.

Theorem 2. *Let $\sigma > 0$. We assume the same assumption on κ as in Theorem* 1*. Then,* (1) *admits a radially symmetric solution provided ε is sufficiently small. More precisely, if we fix $\overline{\sigma} > 0$ and $\gamma \in (0,1/2)$, there exists $\hat{\varepsilon}_1 > 0$ such that, for all $\varepsilon \in (0,\hat{\varepsilon}_1)$ and $\sigma \in (0,\overline{\sigma})$,* (1) *admits a radially symmetric solution $(A_{\varepsilon,\sigma}(x), H_{\varepsilon,\sigma}(x))$, and $A_{\varepsilon,\sigma}$ takes the form*:

$$A_{\varepsilon,\sigma}(x) = \frac{1}{\varepsilon\int_{\mathbb{R}} w_{\delta_\varepsilon}}\bigg\{\alpha_D w_{\delta_\varepsilon}\bigg(\frac{x}{\varepsilon}\bigg)\chi\bigg(\frac{x}{r_0}\bigg) + \varepsilon\phi_\varepsilon\bigg(\frac{x}{\varepsilon}\bigg) + \varepsilon^\gamma\phi_{\varepsilon,\sigma}\bigg(\frac{x}{\varepsilon}\bigg)\bigg\} + \sigma, \quad x \in (-1,1), \tag{15}$$

where δ_ε and ϕ_ε are given in Theorem 1*, and $\phi_{\varepsilon,\sigma}(y)$ is a radially symmetric function on Ω_ε such that*

$$\|\phi_{\varepsilon,\sigma}\|_{H^2(\Omega_\varepsilon)} \leqslant \overline{\sigma} \tag{16}$$

holds, and $H_{\varepsilon,\sigma}$ satisfies

$$H_{\varepsilon,\sigma}(0) = \frac{1}{\varepsilon \int_{\mathbb{R}} w_{\delta_\varepsilon}^2}\big(\alpha_D + O(\varepsilon) + \big(\overline{\sigma} + \overline{\sigma}^2\big)O\big(\varepsilon^\gamma\big)\big), \tag{17}$$

as $\varepsilon \to 0$, where $O(\varepsilon)$ and $O(\varepsilon^\gamma)$ are independent of σ.

Remark 3. The setting of the domain $(-1, 1)$ is not essential. For given $k \in \mathbb{N}$, if we construct a 1-peak solution to (1) on smaller domain in advance, then we can obtain a k-peak symmetric solution to (1) by reflections.

Remark 4. The assumption "$\kappa_0\alpha_D^2$ is sufficiently small" in Theorem 1 is due to some technical reason. See Remark 14 stated later.

3. Basic analysis and preliminaries

In this section, we prepare some lemmas to prove Theorem 1 and state the outline of our construction. We first define some function spaces as follows:

$$L_r^2(\mathbb{R}) := \big\{u \in L^2(\mathbb{R}):\ u(x) = u(-x),\ x \in \mathbb{R}\big\}, \tag{18}$$

$$H_r^2(\mathbb{R}) := H^2(\mathbb{R}) \cap L_r^2(\mathbb{R}), \tag{19}$$

and for a domain $(-a, a)$, $a \in (0, \infty)$,

$$L_r^2(-a, a) := \big\{u \in L^2(-a, a):\ u(x) = u(-x),\ x \in (-a, a)\big\}, \tag{20}$$

$$H_r^2(-a, a) := H^2(-a, a) \cap L_r^2(-a, a), \tag{21}$$

$$H_{r,\nu}^2(-a, a) := \big\{u \in H_r^2(-a, a):\ u'(\pm a) = 0\big\}. \tag{22}$$

Because we will frequently use rescaling, we introduce the following notations.

Definition 5. Put $\Omega_\varepsilon := (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$.
For a function $u : (-1, 1) \to \mathbb{R}$, let $\overline{u}(y) := u(\varepsilon y)$, $y \in \Omega_\varepsilon$.
Inversely, for a function $v : \Omega_\varepsilon \to \mathbb{R}$, let $\underline{v}(x) := v(\frac{x}{\varepsilon})$, $x \in (-1, 1)$.

3.1. Basic analysis

For the unique solution w_δ to (4), let us state some known facts. After that, we state some new lemmas.

Lemma 6. *For each $\delta \in [0, \delta_*)$, the unique radially symmetric solution w_δ has the following properties*:

(i) $w_\delta \in C^\infty(\mathbb{R})$.
(ii) *Let*

$$L_\delta := \frac{d^2}{dx^2} - 1 + f_\delta'(w_\delta) : H^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

where $f_\delta'(w_\delta) = 2w_\delta/(1 + \delta w_\delta^2)$. Then, $\mathrm{Ker}(L_\delta) = \mathrm{span}\{w_\delta'\}$.
(iii) *If we restrict the domain to* $\mathrm{Dom}(L_\delta) = H_r^2(\mathbb{R})$, *then L_δ has a bounded inverse $L_\delta^{-1} : L_r^2(\mathbb{R}) \to H_r^2(\mathbb{R})$.*
(iv) *If we fix $\overline{\delta} \in (0, \delta_*)$, then there exist constants $C, c > 0$ such that*

$$w_\delta(y),\ \left|\frac{d^n w_\delta}{dy^n}(y)\right| \leqslant Ce^{-c|y|}, \quad y \in \mathbb{R},\ n = 1, 2, \tag{23}$$

holds for any $\delta \in [0, \overline{\delta}]$.

Proof. (i)–(iii) have been proven in Lemma 2.2 of [22]. (iv) have been proven in Lemma 2.4 of [8]. □

We state continuity and differentiability of w_δ on δ.

Lemma 7. *As a $C^1(\mathbb{R})$-valued function of δ, w_δ satisfies the following*:

(i) *w_δ is continuous in $\delta \in [0, \delta_*)$ with respect to the $C^1(\mathbb{R})$-norm.*
(ii) *w_δ is of class $C^1((0, \delta_*), C^1(\mathbb{R}))$.*

Proof. This fact was proven in Lemma 2.3 of [22] (see also Lemma 2.3 of [8]). □

Let us denote the derivatives of w_δ in x and in δ by $w'_\delta(x)$ and $\frac{dw_\delta}{d\delta}$, respectively. Next, we state some useful formulae.

Lemma 8. *The following identities hold*:

$$L_\delta w_\delta = f'_\delta(w_\delta)w_\delta - f_\delta(w_\delta), \tag{24}$$

$$L_\delta \frac{dw_\delta}{d\delta} = f_\delta^2(w_\delta), \tag{25}$$

$$L_\delta \left(w_\delta + 2\delta \frac{dw_\delta}{d\delta} + \frac{1}{2} y \cdot w'_\delta \right) = w_\delta, \tag{26}$$

$$L_\delta \left(w_\delta + 2\delta \frac{dw_\delta}{d\delta} \right) = f_\delta(w_\delta). \tag{27}$$

Proof. These facts were proven in Lemma 2.3 of [22]. □

Lemma 9. *$w_\delta \to b$ in $C^2_{\mathrm{loc}}(\mathbb{R})$ holds as $\delta \to \delta_*$, where $b > 0$ is the second positive root of $-t + f_{\delta_*}(t) = 0$, $t \in \mathbb{R}$.*

Proof. This fact was proven in Lemma 2.3 of [22]. □

Lemma 10. *For any $\delta \in (0, \delta_*)$, it holds that*

$$\frac{d}{d\delta} \left(\int_{-\infty}^{\infty} w_\delta^2(y)\, dy \right) > 0. \tag{28}$$

Proof. This fact was proven in Lemma 2.6 of [22]. □

Lemma 11. *For fixed $\bar{\delta} \in (0, \delta_*)$, there exists constant $C > 0$ such that*

$$\left\| \frac{dw_\delta}{d\delta} \right\|_{H^2(\mathbb{R})} \leqslant C \tag{29}$$

holds for any $\delta \in (0, \bar{\delta})$.

Proof. It is easy to see that L_δ^{-1} is bounded uniformly in $\delta \in [0, \bar{\delta}]$. By using (25) and Lemma 6(iv), we can estimate by some constants $C, C' > 0$ independent of $\delta \in [0, \bar{\delta}]$ as follows:

$$\left\| \frac{dw_\delta}{d\delta} \right\|_{H^2(\mathbb{R})} = \left\| L_\delta^{-1} f_\delta^2(w_\delta) \right\|_{H^2(\mathbb{R})} \leqslant C \left\| f_\delta^2(w_\delta) \right\|_{L^2(\mathbb{R})} \leqslant C'. \tag{30}$$

Hence we complete the proof. □

Lemma 12.

(i) *For each $\delta \in [0, \delta_*)$, if $\phi \in H_r^2(\mathbb{R})$ satisfies the following*:

$$\phi'' - \phi + f_\delta'(w_\delta)\phi - \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} f_\delta(w_\delta) = 0 \quad \textit{in } \mathbb{R}, \tag{31}$$

$$\gamma \neq \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{dw_\delta}{d\delta}}, \tag{32}$$

then $\phi = 0$.

(ii) *There exists* $\delta_1 \in (0, \delta_*)$ *such that, for* $\delta \in [0, \delta_1)$, *if* $\phi \in H_r^2(\mathbb{R})$ *satisfies the following*:

$$\phi'' - \phi + f_\delta'(w_\delta)\phi - \gamma \frac{\int_{\mathbb{R}} f_\delta(w_\delta)\phi}{\int_{\mathbb{R}} w_\delta^2} w_\delta = 0 \quad \textit{in } \mathbb{R}, \tag{33}$$

$$\gamma \neq \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} L_\delta^{-1}(w_\delta) f_\delta(w_\delta)}, \tag{34}$$

then $\phi = 0$.

Before the proof, we state some remarks. Lemma 10 implies that $\int_{\mathbb{R}} w_\delta \frac{dw_\delta}{d\delta} > 0$ for any $\delta \in (0, \delta_*)$. Hence, we first notice that

$$0 < \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{dw_\delta}{d\delta}} \leqslant 1, \quad \delta \in [0, \delta_*). \tag{35}$$

Secondly, we consider the value of $\int_{\mathbb{R}} L_\delta^{-1}(w_\delta) f_\delta(w_\delta)$. By using (26) and integration by parts, we have

$$\begin{aligned}
\lim_{\delta \to 0} \int_{\mathbb{R}} L_\delta^{-1}(w_\delta) f_\delta(w_\delta) &= \lim_{\delta \to 0} \int_{\mathbb{R}} \left(w_\delta(y) + 2\delta \frac{dw_\delta}{d\delta}(y) + \frac{1}{2} y \cdot w_\delta'(y) \right) f_\delta\big(w_\delta(y)\big)\, dy \\
&= \int_{\mathbb{R}} \left(w_0^3(y) + \frac{1}{2} y \cdot w_0'(y) w_0^2(y) \right) dy \\
&= \int_{\mathbb{R}} \left(w_0^3(y) - \frac{1}{6} w_0^3(y) \right) dy \\
&= \frac{5}{6} \int_{\mathbb{R}} w_0^3(y)\, dy > 0.
\end{aligned} \tag{36}$$

Here, we note that $\delta \int_{\mathbb{R}} \frac{dw_\delta}{d\delta} f_\delta(w_\delta)\, dy \to 0$ as $\delta \to 0$ by Lemma 11. Moreover, we see that

$$w_0'' - w_0 + w_0^2 = 0, \qquad \int_{\mathbb{R}} w_0'' w_0 - \int_{\mathbb{R}} w_0^2 + \int_{\mathbb{R}} w_0^3 = 0, \qquad \int_{\mathbb{R}} (w_0')^2 + \int_{\mathbb{R}} w_0^2 = \int_{\mathbb{R}} w_0^3. \tag{37}$$

Therefore, $\int_{\mathbb{R}} w_0^3 > \int_{\mathbb{R}} w_0^2$. Thus we have

$$\left. \frac{\int_{\mathbb{R}} w_\delta^2}{\int_{\mathbb{R}} L_\delta^{-1}(w_\delta) f_\delta(w_\delta)} \right|_{\delta = 0} = \frac{\int_{\mathbb{R}} w_0^2}{\frac{5}{6} \int_{\mathbb{R}} w_0^3} < \frac{6}{5}. \tag{38}$$

Proof. (i) By using (27), the equation (31) can be written as follows:

$$\begin{aligned}
&L_\delta \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} f_\delta(w_\delta), \qquad \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} L_\delta^{-1}\big(f_\delta(w_\delta)\big), \\
&\int_{\mathbb{R}} w_\delta \phi = \gamma \frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} \left(\int_{\mathbb{R}} w_\delta^2 + 2\delta \int_{\mathbb{R}} w_\delta \frac{dw_\delta}{d\delta} \right).
\end{aligned}$$

Hence, $\int_{\mathbb{R}} w_\delta \phi = 0$ must be hold by (32). Thus we have $L_\delta \phi = 0$, $\phi \in H_r^2(\mathbb{R})$, and hence $\phi = 0$ by Lemma 6(iii).

(ii) Define δ_1 by

$$\delta_1 := \sup\left\{\delta \in (0, \delta_*)\colon \int_{\mathbb{R}} L_{\delta'}^{-1}(w_{\delta'}) f'_{\delta'}(w_{\delta'}) > 0 \text{ for } \delta' \in (0, \delta)\right\}. \tag{39}$$

This δ_1 is well defined by (36). Then we can prove by the same argument as in the proof of (i). □

Now, we define an operator $\mathcal{L}_\delta$ on $L^2(\mathbb{R})$ with $\mathrm{Dom}(\mathcal{L}_\delta) = H^2(\mathbb{R})$ by

$$\mathcal{L}_\delta \phi = \phi'' - \phi + f'_\delta(w_\delta)\phi - 2\frac{\int_{\mathbb{R}} w_\delta \phi}{\int_{\mathbb{R}} w_\delta^2} f_\delta(w_\delta). \tag{40}$$

Its conjugate operator is given by

$$\mathcal{L}_\delta^* \psi = \psi'' - \psi + f'_\delta(w_\delta)\psi - 2\frac{\int_{\mathbb{R}} f_\delta(w_\delta)\psi}{\int_{\mathbb{R}} w_\delta^2} w_\delta, \quad \psi \in H^2(\mathbb{R}). \tag{41}$$

Let us define δ_2 by

$$\delta_2 = \sup\left\{\delta \in (0, \delta_1)\colon \frac{\int_{\mathbb{R}} w_{\delta'}^2}{\int_{\mathbb{R}} L_{\delta'}^{-1}(w_{\delta'}) f_{\delta'}(w_{\delta'})} < 2 \text{ for } \delta' \in (0, \delta)\right\}, \tag{42}$$

where δ_1 is defined by (39). This δ_2 is well defined by (38).

Lemma 13. *For the operators $\mathcal{L}_\delta$ and $\mathcal{L}_\delta^*$, and δ_2 defined above, there hold that*

(i) $\mathrm{Ker}(\mathcal{L}_\delta) \cap H_r^2(\mathbb{R}) = \{0\}$ *for any* $\delta \in [0, \delta_*)$,
(ii) $\mathrm{Ker}(\mathcal{L}_\delta^*) \cap H_r^2(\mathbb{R}) = \{0\}$ *for any* $\delta \in [0, \delta_2)$.

Proof. This lemma is a consequence of Lemma 12. □

Remark 14. We do not know whether $\mathrm{Ker}(\mathcal{L}_\delta^*) \cap H_r^2(\mathbb{R})$ is trivial or not for δ near δ_*. If $\mathrm{Ker}(\mathcal{L}_\delta^*) \cap H_r^2(\mathbb{R}) = \{0\}$ holds for all $\delta \in [0, \delta_*)$, then we can remove the assumption "$\kappa_0\alpha_D^2$ is sufficiently small" in Theorem 1. However, it seems to be a difficult problem.

3.2. Outline of our construction

We state an outline of our construction. We see by Lemmas 9–11 that there exists unique $\delta_\varepsilon \in [0, \delta_*)$ such that

$$\delta_\varepsilon \left(\int_{\mathbb{R}} w_{\delta_\varepsilon}^2\right)^2 = \kappa\varepsilon^{-2}\alpha_D^2 \tag{43}$$

holds for each $\varepsilon > 0$. By the assumption (A), in the limit $\varepsilon \to 0$, there hold that

$$\delta_\varepsilon \to \delta_0, \quad \delta_0 \left(\int_{\mathbb{R}} w_{\delta_0}^2\right)^2 = \kappa_0\alpha_D^2, \tag{44}$$

as $\varepsilon \to 0$, for some $\delta_0 \in [0, \delta_*)$. We assume henceforth that $\kappa_0\alpha_D^2 \geqslant 0$ is small enough so that $\delta_0 \in [0, \delta_2)$, where δ_2 is given by (42). Then we note that there exists $\bar{\delta} \in (0, \delta_2)$ such that $\delta_\varepsilon \in [0, \bar{\delta}]$ holds for all $\varepsilon > 0$ sufficiently small. Hence, we may assume that $c < \int_{\mathbb{R}} w_{\delta_\varepsilon}^2(y)\,dy < C$ holds for all ε sufficiently small, the constants $c, C > 0$ are independent of ε.

Put

$$c_\varepsilon := \frac{1}{\varepsilon \int_{\mathbb{R}} w_{\delta_\varepsilon}^2}. \tag{45}$$

We consider the following problem for a and h:

$$\begin{cases} \varepsilon^2 a'' - a + \dfrac{a^2}{h(1+\delta_\varepsilon \alpha_D^{-2} a^2)} + \sigma_\varepsilon = 0, & a > 0,\ x \in (-1,1), \\ Dh'' - h + c_\varepsilon a^2 = 0, & h > 0,\ x \in (-1,1), \\ a'(\pm 1) = h'(\pm 1) = 0, \end{cases} \tag{46}$$

where

$$\sigma_\varepsilon := \frac{\sigma}{c_\varepsilon}. \tag{47}$$

If we obtain a solution to (46), then we obtain a solution to (1) by putting $A(x) = c_\varepsilon a(x)$ and $H(x) = c_\varepsilon h(x)$. For $U \in H^2_{r,\nu}(\Omega_\varepsilon)$, let $T[\underline{U}]$ be a unique solution to the following problem for v:

$$\begin{cases} Dv'' - v + c_\varepsilon \underline{U}^2 = 0, & x \in (-1,1), \\ v'(\pm 1) = 0. \end{cases} \tag{48}$$

Here, the under-bar and over-bar notation is due to Definition 5. Moreover, we put

$$S[U](y) := U''(y) - U(y) + \frac{U^2(y)}{\overline{T[\underline{U}]}(y)(1+\delta_\varepsilon \alpha_D^{-2} U^2(y))}, \quad y \in \Omega_\varepsilon. \tag{49}$$

If we can find $U \in H^2_{r,\nu}(\Omega_\varepsilon)$ such that, $S[U] + \sigma_\varepsilon = 0$, $U > 0$ in Ω_ε, then we obtain a solution to (46) by putting $a(x) = \underline{U}(x)$ and $h(x) = T[\underline{U}](x)$.

Here, we note that $T[\underline{U}]$ is written by using Green's function as follows:

$$T[\underline{U}](x) = c_\varepsilon \int_{-1}^{1} G_D(x,z) \underline{U}^2(z)\, dz, \quad x \in (-1,1), \tag{50}$$

for $U \in L^2(\Omega_\varepsilon)$. In particular, $T[\underline{U}]$ is radially symmetric provided U is radially symmetric. Now, let us define an approximate function w_ε as follows:

$$w_\varepsilon(x) := \alpha_D w_{\delta_\varepsilon}\left(\frac{x}{\varepsilon}\right) \chi\left(\frac{x}{r_0}\right), \tag{51}$$

where $\alpha_D = G_D(0,0)^{-1}$, w_{δ_ε} is the unique solution to (4) for $\delta = \delta_\varepsilon$, χ is the cut-off function defined in the previous section. We will first consider the case $\sigma = 0$ and prove Theorem 1 in Section 5. For the purpose, we will seek $U \in H^2_{r,\nu}(\Omega_\varepsilon)$ such that $S[U] = 0$, $U > 0$ in Ω_ε in the form $U(y) = \overline{w}_\varepsilon(y) + \varepsilon \phi(y)$ for some $\phi \in H^2_{r,\nu}(\Omega_\varepsilon)$. Next, we will consider the case $\sigma \neq 0$ in Section 6. Note that $(\sigma_\varepsilon =)\sigma / c_\varepsilon \leqslant C\sigma\varepsilon$ holds for some constant $C > 0$ independent of ε sufficiently small. Therefore, we can prove Theorem 2 by a perturbation argument.

4. Basic estimates

In this section, we show some basic estimates.

Lemma 15. *There exists $c_1 > 0$ such that $T[w_\varepsilon](x) \geqslant c_1$, $x \in (-1,1)$, for all ε sufficiently small.*

Proof.

$$\begin{aligned} T[w_\varepsilon](x) &= c_\varepsilon \int_{-1}^{1} G_D(x,z) w_\varepsilon^2(z)\, dx \\ &= \alpha_D^2 c_\varepsilon \int_{-1}^{1} G_D(x,z) w_{\delta_\varepsilon}^2\left(\frac{z}{\varepsilon}\right) \chi^2\left(\frac{z}{r_0}\right) dz \end{aligned}$$

$$= \alpha_D^2 \varepsilon c_\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} G_D(x, \varepsilon z) w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz$$

$$\geqslant \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \frac{\theta}{\sinh(2\theta)} \int_{-1/\varepsilon}^{1/\varepsilon} w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz$$

$$= \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_0}^2} \frac{\theta}{\sinh(2\theta)} \int_{-\infty}^{\infty} w_{\delta_0}^2(z)\, dz + o(1),$$

as $\varepsilon \to 0$, where $o(1)$ is uniform in $x \in (-1, 1)$. This estimate completes the proof. □

Next, we show the following elementary inequality.

Lemma 16. *For the non-smooth part $K_D(|x - z|)$ of $G_D(x, z)$, the following estimate holds*:

$$\left|K_D(|x|) - K_D(|y|)\right| \leqslant \frac{1}{2\sqrt{D}} \left\{ \frac{1}{\sqrt{D}} \left(\left||x| - |y|\right|\right) + \frac{1}{2} \left(\frac{1}{\sqrt{D}}\right)^2 \left(|x|^2 + |y|^2\right) \right\}. \tag{52}$$

Proof. This lemma is easily verified by (9) and the following elementary inequality:

$$1 - |x| \leqslant e^{-|x|} \leqslant 1 - |x| + \frac{1}{2}|x|^2.$$

Thus, we omit the details. □

Lemma 17. *For w_ε defined by* (51)*, it holds that*

$$T[w_\varepsilon](0) = \alpha_D + O(\varepsilon), \tag{53}$$

as $\varepsilon \to 0$.

Proof. Note that

$$T[w_\varepsilon](0) = c_\varepsilon \int_{-1}^{1} G_D(0, z) w_\varepsilon^2(z)\, dz = \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \int_{-1/\varepsilon}^{1/\varepsilon} G_D(0, \varepsilon z) w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz,$$

and the following inequality holds:

$$\int_{|z| < \frac{r_0}{\varepsilon}} G_D(0, \varepsilon z) w_{\delta_\varepsilon}^2(z)\, dz \leqslant \int_{-1/\varepsilon}^{1/\varepsilon} G_D(0, \varepsilon z) w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz \leqslant \int_{|z| < \frac{2r_0}{\varepsilon}} G_D(0, \varepsilon z) w_{\delta_\varepsilon}^2(z)\, dz. \tag{54}$$

The left-hand side of (54) is written as follows:

$$(\text{l.h.s.}) = G_D(0, 0) \int_{|z| < \frac{r_0}{\varepsilon}} w_{\delta_\varepsilon}^2(z)\, dz + \int_{|z| < \frac{r_0}{\varepsilon}} \left\{G_D(0, \varepsilon z) - G_D(0, 0)\right\} w_{\delta_\varepsilon}^2(z)\, dz \equiv I + II.$$

Moreover, noting $\alpha_D^{-1} = G_D(0, 0)$, we can estimate by Lemma 6(iv) so that

$$I = \alpha_D^{-1} \left\{ \int_{\mathbb{R}} w_{\delta_\varepsilon}^2(z)\, dz - \int_{|z| > \frac{r_0}{\varepsilon}} w_{\delta_\varepsilon}^2(z)\, dz \right\} = \alpha_D^{-1} \int_{\mathbb{R}} w_{\delta_\varepsilon}^2(z)\, dz + \text{e.s.t.}, \tag{55}$$

where "e.s.t." means "exponentially small term". Next, we can estimate by Lemma 16 and the mean value theorem as follows:

$$|II| \leqslant \int\limits_{|z|<\frac{r_0}{\varepsilon}} \big|K_D(\varepsilon|z|) - K_D(0)\big| w_{\delta_\varepsilon}^2(z)\,dz + \int\limits_{|z|<\frac{r_0}{\varepsilon}} \big|H_D(0,\varepsilon z) - H_D(0,0)\big| w_{\delta_\varepsilon}^2(z)\,dz$$
$$\leqslant C \int\limits_{|z|<\frac{r_0}{\varepsilon}} \varepsilon|z| w_{\delta_\varepsilon}^2(z)\,dz$$
$$\leqslant C'\varepsilon.$$

Here, we note that, $\varepsilon^2|z|^2 < \varepsilon|z|r_0$ for $|z| < r_0/\varepsilon$, $\int_{\mathbb{R}} |z| w_{\delta_\varepsilon}^2(z)\,dz$ is bounded uniformly in ε sufficiently small since we may assume $\delta_\varepsilon \in [0, \bar{\delta}]$ and we can apply Lemma 6(iv). Hence

$$\big(\text{l.h.s. of (54)}\big) = \alpha_D^{-1} \int\limits_{\mathbb{R}} w_{\delta_\varepsilon} + O(\varepsilon). \tag{56}$$

We can see that the right-hand side of (54) have the same behavior as (56). Thus we have

$$T[w_\varepsilon](0) = \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \left(\alpha_D^{-1} \int\limits_{\mathbb{R}} w_{\delta_\varepsilon}^2(z)\,dz + O(\varepsilon) \right) = \alpha_D + O(\varepsilon).$$

Thus we complete the proof. □

Lemma 18. *For some constant $C > 0$, the following estimate holds*:

$$\big|T[w_\varepsilon](\varepsilon y) - T[w_\varepsilon](0)\big| \leqslant C\big(\varepsilon|y| + \varepsilon\big),\ \ y \in \Omega_\varepsilon, \tag{57}$$

for all ε sufficiently small.

Proof.

$$T[w_\varepsilon](\varepsilon y) - T[w_\varepsilon](0) = c_\varepsilon \int\limits_{-1}^{1} \big\{G_D(\varepsilon y, z) - G_D(0,z)\big\} w_\varepsilon^2(z)\,dz$$
$$= \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \int\limits_{-1/\varepsilon}^{1/\varepsilon} \big\{G_D(\varepsilon y, \varepsilon z) - G_D(0, \varepsilon z)\big\} w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz$$
$$= \frac{\alpha_D^2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \Bigg[\int\limits_{-1/\varepsilon}^{1/\varepsilon} \big\{K_D(\varepsilon|y-z|) - K_D(\varepsilon|z|)\big\} w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz$$
$$- \int\limits_{-1/\varepsilon}^{1/\varepsilon} \big\{H_D(\varepsilon y, \varepsilon z) - H_D(0, \varepsilon z)\big\} w_{\delta_\varepsilon}^2(z) \chi^2\left(\frac{\varepsilon}{r_0} z\right) dz \Bigg].$$

Now, by Lemma 16, and noting $\varepsilon|z| \leqslant 1$ for $|z| \leqslant 1/\varepsilon$, the following estimate holds:

$$\big|K_D(\varepsilon|y-z|) - K_D(\varepsilon|z|)\big| \leqslant C\big(\varepsilon\big||y| - |y-z|\big| + \varepsilon^2\big(|y-z|^2 + |z|^2\big)\big) \leqslant C'\varepsilon\big(|y| + |z|\big), \quad y, z \in \Omega_\varepsilon,$$

for some $C, C' > 0$ independent of ε, y and z. Moreover, we can estimate by the Maclaurin expansion as follows:

$$\big|H_D(\varepsilon y, \varepsilon z) - H_D(0, \varepsilon z)\big| \leqslant C''\varepsilon\big(|y| + |z|\big), \quad y, z \in \Omega_\varepsilon,$$

for some $C'' > 0$ independent of ε, y and z. Thus we have

$$\left|T[w_\varepsilon](\varepsilon y)-T[w_\varepsilon](0)\right|\leqslant \varepsilon\frac{\alpha_D^2}{\int_{\mathbb{R}}w_{\delta_\varepsilon}^2}\left(C'+C''\right)\int_{-1/\varepsilon}^{1/\varepsilon}\left(|y|+|z|\right)w_{\delta_\varepsilon}^2(z)\,dz\leqslant C'''\left(\varepsilon|y|+\varepsilon\right),\quad y\in\Omega_\varepsilon,$$

for some $C'''>0$ independent of ε and y. Thus we complete the proof. □

Lemma 19. *There exists $C_1>0$ such that*

$$\left\|S[\overline{w}_\varepsilon]\right\|_{L^2(\Omega_\varepsilon)}\leqslant C_1\varepsilon, \tag{58}$$

for all ε sufficiently small.

Proof. It is easily to see that $\Delta\overline{w}_\varepsilon-\overline{w}_\varepsilon=-f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D+\text{e.s.t.}$ in $L^2(\Omega_\varepsilon)$ as $\varepsilon\to 0$. Hence,

$$\begin{aligned}S[\overline{w}_\varepsilon](y)&=-f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D+\frac{1}{\overline{T[w_\varepsilon]}(y)}\frac{\overline{w}_\varepsilon^2(y)}{1+\delta_\varepsilon\alpha_D^{-2}\overline{w}_\varepsilon^2(y)}+\text{e.s.t.}\\&=-f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D+\frac{1}{\overline{T[w_\varepsilon]}(y)}\frac{\alpha_D^2w_{\delta_\varepsilon}^2(y)\chi^2(\frac{\varepsilon}{r_0}y)}{1+\delta_\varepsilon w_{\delta_\varepsilon}^2(y)\chi^2(\frac{\varepsilon}{r_0}y)}+\text{e.s.t.}\\&=-f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D+\frac{\alpha_D^2}{\overline{T[w_\varepsilon]}}f_{\delta_\varepsilon}(w_{\delta_\varepsilon})+\text{e.s.t.}\quad\text{in }L^2(\Omega_\varepsilon).\end{aligned}$$

By Lemma 17, we have

$$\begin{aligned}-f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D+\frac{\alpha_D^2}{\overline{T[w_\varepsilon]}}f_{\delta_\varepsilon}(w_{\delta_\varepsilon})&=f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D\left\{-1+\frac{\alpha_D}{T[w_\varepsilon](0)}+\frac{\alpha_D}{\overline{T[w_\varepsilon]}(y)}-\frac{\alpha_D}{T[w_\varepsilon](0)}\right\}\\&=f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\alpha_D\left\{O(\varepsilon)+\frac{\alpha_D}{T[w_\varepsilon](\varepsilon y)T[w_\varepsilon](0)}\left(T[w_\varepsilon](0)-T[w_\varepsilon](\varepsilon y)\right)\right\}.\end{aligned}$$

Moreover, by Lemma 18, the following estimate holds:

$$\begin{aligned}\left\|f_{\delta_\varepsilon}(w_{\delta_\varepsilon})\left(T[w_\varepsilon](0)-T[w_\varepsilon](\varepsilon y)\right)\right\|_{L^2(\Omega_\varepsilon)}^2&=\int_{\Omega_\varepsilon}\frac{w_{\delta_\varepsilon}^4}{(1+\delta_\varepsilon w_{\delta_\varepsilon}^2)^2}\left(T[w_\varepsilon](0)-T[w_\varepsilon](\varepsilon y)\right)^2dy\\&\leqslant C\int_{\Omega_\varepsilon}w_{\delta_\varepsilon}^4(y)\left(\varepsilon|y|+\varepsilon\right)^2dy\leqslant C'\varepsilon^2,\end{aligned}$$

for some constants $C, C'>0$ independent of ε sufficiently small. From these estimates and by Lemma 15, we have a conclusion. □

Next, we give the derivatives of T and S. The proofs of Lemmas 20, 21 below are uninteresting calculation. So we give their proofs in Appendix A.

Lemma 20. *If we regard T as a mapping form $L^2(-1,1)$ into $L^\infty(-1,1)$, then T is Fréchet differentiable on $L^2(-1,1)$, and its derivative at $u\in L^2(-1,1)$ is given by*

$$T'[u]\phi=2c_\varepsilon\int_{-1}^{1}G_D(x,z)u(z)\phi(z)\,dz,\quad \phi\in L^2(-1,1). \tag{59}$$

Moreover, for some constant $C>0$ independent of ε sufficiently small, the following estimates hold:

$$\left\|\overline{T[\underline{u}+\underline{h}]}-\overline{T[\underline{u}]}-\overline{T'[\underline{u}]\underline{h}}\right\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\|h\|_{L^2(\Omega_\varepsilon)}^2, \tag{60}$$

$$\left\|\overline{T'[\underline{u}]\underline{h}}\right\|_{L^\infty(\Omega_\varepsilon)}\leqslant C\|u\|_{L^2(\Omega_\varepsilon)}\|h\|_{L^2(\Omega_\varepsilon)}, \tag{61}$$

for any $u, h\in L^2(\Omega_\varepsilon)$.

For $\tau > 0$, we define a ball in $H^2(\Omega_\varepsilon)$ as follows:

$$B_\tau(\overline{w}_\varepsilon) := \{u \in H^2(\Omega_\varepsilon)\colon \|\overline{w}_\varepsilon - u\|_{H^2(\Omega_\varepsilon)} < \tau\}. \tag{62}$$

Let us fix $\tau > 0$ so that

$$T[\underline{u}](x) \geqslant \frac{1}{2}c_1, \quad x \in (-1, 1), \tag{63}$$

holds for all $u \in B_\tau(\overline{w}_\varepsilon)$ and ε sufficiently small, where c_1 is a constant given in Lemma 15.

Lemma 21. *For all ε sufficiently small, $S : H^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ is Fréchet differentiable on $B_\tau(\overline{w}_\varepsilon)$, and its derivative at $u \in B_\tau(\overline{w}_\varepsilon)$ is given by*

$$S'[u]\phi = \phi'' - \phi + \frac{2u\phi}{\overline{T[\underline{u}]}(1+\delta_\varepsilon\alpha_D^{-2}u^2)^2} - \frac{u^2(\overline{T'[\underline{u}]\underline{\phi}})}{\overline{T[\underline{u}]}^2(1+\delta_\varepsilon\alpha_D^{-2}u^2)}, \quad \phi \in H^2(\Omega_\varepsilon). \tag{64}$$

Moreover, the following estimates hold: for $u \in B_\tau(\overline{w}_\varepsilon)$, $\phi \in H^2(\Omega_\varepsilon)$ and $h \in H^2(\Omega_\varepsilon)$, $\|h\|_{H^2(\Omega_\varepsilon)} \ll 1$,

$$\big\|S[u+h] - S[u] - S'[u]h\big\|_{L^2(\Omega_\varepsilon)} \leqslant C\big(\|h\|^2_{L^2(\Omega_\varepsilon)} + \|h\|_{L^\infty(\Omega_\varepsilon)}\|h\|_{L^2(\Omega_\varepsilon)}\big), \tag{65}$$

$$\big\|S'[u+h]\phi - S'[u]\phi\big\|_{L^2(\Omega_\varepsilon)} \leqslant C\big(\|h\|_{L^2(\Omega_\varepsilon)} + \|h\|_{L^\infty(\Omega_\varepsilon)}\big)\|\phi\|_{L^2(\Omega_\varepsilon)}, \tag{66}$$

where $C > 0$ is independent of u, ϕ, h and ε sufficiently small.

Remark 22. For the estimate (66), we note that the term ϕ'' vanishes in $S'[u+h]\phi - S'[u]\phi$. Actually, (66) also holds for $\phi \in L^2(\Omega_\varepsilon)$.

5. Construction of a solution for $\sigma = 0$

In this section, we construct the 1-peak solution to (1) in the case $\sigma = 0$ and prove Theorem 1. Therefore, we always assume $\sigma = 0$ throughout this section. Our construction is based on the argument due to the contraction mapping principle, which was used in [14,23,8], and so on.

Now we define an operator $\tilde{L}_\varepsilon$ on $L^2(\Omega_\varepsilon)$ with $\mathrm{Dom}(\tilde{L}_\varepsilon) = H^2_{r,\nu}(\Omega_\varepsilon)$ by

$$\tilde{L}_\varepsilon\phi := S'[\overline{w}_\varepsilon]\phi = \phi'' - \phi + \frac{2\overline{w}_\varepsilon\phi}{\overline{T[w_\varepsilon]}(1+\delta_\varepsilon\alpha_D^{-2}\overline{w}_\varepsilon^2)^2} - \frac{\overline{w}_\varepsilon^2(\overline{T'[w_\varepsilon]\underline{\phi}})}{\overline{T[w_\varepsilon]}^2(1+\delta_\varepsilon\alpha_D^{-2}\overline{w}_\varepsilon^2)}. \tag{67}$$

Then its conjugate operator $\tilde{L}_\varepsilon^*$ is given by $\mathrm{Dom}(\tilde{L}_\varepsilon^*) = H^2_{r,\nu}(\Omega_\varepsilon)$ and

$$\tilde{L}_\varepsilon^*\psi = \psi'' - \psi + \frac{2\overline{w}_\varepsilon\psi}{\overline{T[w_\varepsilon]}(1+\delta_\varepsilon\alpha_D^{-2}\overline{w}_\varepsilon^2)^2} - \left(\overline{T'[w_\varepsilon]\left[\frac{w_\varepsilon\underline{\psi}}{T[w_\varepsilon]^2(1+\delta_\varepsilon\alpha_D^{-2}w_\varepsilon^2)}\right]}\right)\overline{w}_\varepsilon. \tag{68}$$

The most important thing for our construction is the invertibility of $\tilde{L}_\varepsilon$. We will notice that the limits of $\tilde{L}_\varepsilon$ and $\tilde{L}_\varepsilon^*$ as $\varepsilon \to 0$ are $\mathcal{L}_{\delta_0}$ and $\mathcal{L}_{\delta_0}^*$ in some sense.

Proposition 23. *There exist $\varepsilon_0 > 0$ and $\lambda > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following inequality holds:*

$$\|\tilde{L}_\varepsilon\phi\|_{L^2(\Omega_\varepsilon)} \geqslant \lambda\|\phi\|_{H^2(\Omega_\varepsilon)}, \quad \phi \in H^2_{r,\nu}(\Omega_\varepsilon). \tag{69}$$

In particular, if δ_0 given in (44) is small so that $\delta_0 \in [0, \delta_2)$, then

$$\mathrm{Ran}(\tilde{L}_\varepsilon) = L^2_r(\Omega_\varepsilon), \tag{70}$$

holds for $\varepsilon \in (0, \varepsilon_0)$, and hence, $\tilde{L}_\varepsilon : H^2_{r,\nu}(\Omega_\varepsilon) \to L^2_r(\Omega_\varepsilon)$ has a bounded inverse $\tilde{L}_\varepsilon^{-1}$.

Before the proof, we make sure of the following extension and embedding lemmas on Ω_ε and a priori elliptic estimate. Although they are elementary and well-known facts, we need to state their ε-dependence clearly because our domain Ω_ε depends on ε. So, we give their proofs in Appendix A for the completeness.

Lemma 24 *(Extension lemma). For fixed $\bar{\varepsilon} > 0$, there exists an extension operator E from $H^2_r(\Omega_\varepsilon)$ into $H^2_r(\mathbb{R})$, and there exists $C > 0$ depending only on $\bar{\varepsilon}$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$,*

$$\|Eu\|_{H^2(\mathbb{R})} \leqslant C\|u\|_{H^2(\Omega_\varepsilon)}, \quad u \in H^2_r(\Omega_\varepsilon). \tag{71}$$

Lemma 25 *(Embedding lemma). For fixed $\bar{\varepsilon} > 0$, there exists $C > 0$ depending only on $\bar{\varepsilon}$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$,*

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\|u\|_{H^2(\Omega_\varepsilon)}, \quad u \in H^2(\Omega_\varepsilon). \tag{72}$$

Lemma 26 *(A priori elliptic estimate). For fixed $\bar{\varepsilon} > 0$ and $f \in L^2(\Omega_\varepsilon)$, let $\varepsilon \in (0, \bar{\varepsilon})$ and $u \in H^2_\nu(\Omega_\varepsilon)$ satisfy the following equation*:

$$-u'' + u = f \quad in\ \Omega_\varepsilon. \tag{73}$$

Then, the following estimate holds:

$$\|u\|_{H^2(\Omega_\varepsilon)} \leqslant C\|f\|_{L^2(\Omega_\varepsilon)}, \tag{74}$$

the constant $C > 0$ is independent of u, f and $\varepsilon \in (0, \bar{\varepsilon})$.

Proof of Proposition 23. We first prove (69). Let the contrary be true. Then there exist $\{\varepsilon_n\}_{n=1}^\infty$ and $\phi_n \in H^2_{r,\nu}(\Omega_{\varepsilon_n})$ such that

$$\begin{cases} \varepsilon_n \to 0, \qquad \|\tilde{L}_{\varepsilon_n}\phi_n\|_{L^2(\Omega_{\varepsilon_n})} \to 0, & \text{as } n \to \infty, \\ \|\phi_n\|_{H^2(\Omega_{\varepsilon_n})} = 1, & n = 1, 2, \ldots. \end{cases} \tag{75}$$

Then, each ϕ_n can be extended to an element of $H^2_r(\mathbb{R})$ by the extension lemma. For simplicity, let us denote the extended function $E\phi_n$ by ϕ_n again. Note that $\|\phi_n\|_{H^2(\mathbb{R})} \leqslant M$ holds for some constant $M > 0$ independent of n. Hence, we can pick up a subsequence (we denote the subsequence by $\{\phi_n\}$ simply), such that,

$$\phi_n \rightharpoonup \phi \quad \text{in } H^2(\mathbb{R}), \tag{76}$$

$$\phi_n \to \phi \quad \text{in } L^2_{\text{loc}}(\mathbb{R}) \text{ and } L^\infty_{\text{loc}}(\mathbb{R}), \tag{77}$$

as $n \to \infty$, for some $\phi \in H^2_r(\mathbb{R})$, where "$\rightharpoonup$" means the weak-limit. Let us denote δ_{ε_n} and Ω_{ε_n} corresponding to ε_n by δ_n and Ω_n, respectively. Recall that $\delta_n \to \delta_0$ as $n \to \infty$. We claim that:

Claim. *For any $\varphi \in C_0^\infty(\mathbb{R})$, it holds that*

$$(\tilde{L}_{\varepsilon_n}\phi_n, \varphi)_{L^2(\Omega_n)} \to (\mathcal{L}_{\delta_0}\phi, \varphi)_{L^2(\mathbb{R})} \quad (n \to \infty). \tag{78}$$

Indeed, let $K := \operatorname{supp}(\varphi)$ for $\varphi \in C_0^\infty(\mathbb{R})$. We may assume $\Omega_n \supset K$ considering n is large enough. Then,

$$(\tilde{L}_{\varepsilon_n}\phi_n, \varphi)_{L^2(\Omega_n)} = \int_K \phi_n''\varphi - \int_K \phi_n\varphi + \int_K \frac{2\overline{w}_{\varepsilon_n}\phi_n\varphi}{\overline{T[w_{\varepsilon_n}]}(1 + \delta_n\alpha_D^{-2}\overline{w}^2_{\varepsilon_n})^2} - \int_K \frac{\overline{w}^2_{\varepsilon_n}(\overline{T'[w_{\varepsilon_n}]\underline{\phi_n}})\varphi}{\overline{T[w_{\varepsilon_n}]}^2(1 + \delta_n\alpha_D^{-2}\overline{w}^2_{\varepsilon_n})}.$$

Let us consider each term. We first notice that

$$\int_K \phi_n''\varphi - \int_K \phi_n\varphi \to \int_K (\phi'' - \phi)\varphi \quad (n \to \infty).$$

Recall $\overline{w}_{\varepsilon_n}(y) = \alpha_D w_{\delta_n}(y)\chi(\frac{\varepsilon_n}{r_0}y)$. For each $y \in K$, we have

$$\begin{aligned} \frac{2\overline{w}_{\varepsilon_n}(y)\phi_n(y)\varphi(y)}{\overline{T[w_{\varepsilon_n}]}(y)(1 + \delta_n\alpha_D^{-2}\overline{w}^2_{\varepsilon_n}(y))^2} &= \frac{2\alpha_D w_{\delta_n}(y)\chi(\frac{\varepsilon_n}{r_0}y)\phi_n(y)\varphi(y)}{T[w_{\varepsilon_n}](\varepsilon_n y)(1 + \delta_n w^2_{\delta_n}(y)\chi^2(\frac{\varepsilon_n}{r_0}y))^2} \\ &\to \frac{2w_{\delta_0}(y)\phi(y)\varphi(y)}{(1 + \delta_0 w^2_{\delta_0}(y))^2} = f'_{\delta_0}(w_{\delta_0})\phi(y)\varphi(y), \end{aligned}$$

as $n \to \infty$. By applying Lebesgue's convergence theorem, we can see that

$$\int_K \frac{2\overline{w}_{\varepsilon_n}\phi_n\varphi}{\overline{T[w_{\varepsilon_n}]}(1+\delta_n\alpha_D^{-2}\overline{w}_{\varepsilon_n}^2)^2} \to \int_K f'_{\delta_0}(w_{\delta_0})\phi\varphi \quad (n\to\infty).$$

Next, for each $y \in K$, let

$$\begin{aligned}
\big(\overline{T'[w_{\varepsilon_n}]\underline{\phi_n}}\big)(y) &= 2c_{\varepsilon_n}\int_{-1}^{1} G_D(\varepsilon_n y, z)w_{\varepsilon_n}(z)\phi_n\left(\frac{z}{\varepsilon_n}\right)dz \\
&= \frac{2\alpha_D}{\int_{\mathbb{R}} w_{\delta_n}^2}\int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, \varepsilon_n z)w_{\delta_n}(z)\chi\left(\frac{\varepsilon_n}{r_0}z\right)\phi_n(z)\,dz \\
&= \frac{2\alpha_D}{\int_{\mathbb{R}} w_{\delta_n}^2}\Bigg\{\int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, 0)w_{\delta_n}(z)\chi\left(\frac{\varepsilon_n}{r_0}z\right)\phi_n(z)\,dz \\
&\quad + \int_{-1/\varepsilon_n}^{1/\varepsilon_n}\big[G_D(\varepsilon_n y, \varepsilon_n z) - G_D(\varepsilon_n y, 0)\big]w_{\delta_n}(z)\chi\left(\frac{\varepsilon_n}{r_0}z\right)\phi_n(z)\,dz\Bigg\}.
\end{aligned}$$

We notice that

$$\frac{2\alpha_D}{\int_{\mathbb{R}} w_{\delta_n}^2}\int_{-1/\varepsilon_n}^{1/\varepsilon_n} G_D(\varepsilon_n y, 0)w_{\delta_n}(z)\chi\left(\frac{\varepsilon_n}{r_0}z\right)\phi_n(z)\,dz \to \frac{2}{\int_{\mathbb{R}} w_{\delta_0}^2}\int_{\mathbb{R}} w_{\delta_0}(z)\phi(z)\,dz$$

as $n \to \infty$ for each $y \in K$. By the same estimate as was used in the proof of Lemma 18, the following estimate holds:

$$\begin{aligned}
&\left|\frac{2\alpha_D}{\int_{\mathbb{R}} w_{\delta_n}^2}\int_{-1/\varepsilon_n}^{1/\varepsilon_n}\big[G_D(\varepsilon_n y, \varepsilon_n z) - G_D(\varepsilon_n y, 0)\big]w_{\delta_n}(z)\chi\left(\frac{\varepsilon_n}{r_0}z\right)\phi_n(z)\,dz\right| \\
&\quad\leqslant C\varepsilon_n\int_{\Omega_n}\big(|y|+|z|\big)w_{\delta_n}(z)\big|\phi_n(z)\big|\,dz \\
&\quad\leqslant C\varepsilon_n\big(|y|\cdot\|w_{\delta_n}\|_{L^2(\Omega_n)} + \|zw_{\delta_n}\|_{L^2(\Omega_n)}\big)\|\phi_n\|_{L^2(\Omega_n)} \\
&\quad\leqslant C'\varepsilon_n\big(1+|y|\big),
\end{aligned}$$

for some constants $C, C' > 0$ independent of n. Hence, for each $y \in K$, it holds that

$$\big(\overline{T'[w_{\varepsilon_n}]\underline{\phi_n}}\big)(y) \to 2\frac{\int_{\mathbb{R}} w_{\delta_0}\phi}{\int_{\mathbb{R}} w_{\delta_0}^2} \quad (n\to\infty). \tag{79}$$

Noting (79), we can see by Lebesgue's convergence theorem that

$$\int_K \frac{\overline{w}_{\varepsilon_n}^2(\overline{T'[w_{\varepsilon_n}]\underline{\phi_n}})\varphi}{\overline{T[w_{\varepsilon_n}]}^2(1+\delta_n\alpha_D^{-2}\overline{w}_{\varepsilon_n}^2)} \to 2\frac{\int_{\mathbb{R}} w_{\delta_0}\phi}{\int_{\mathbb{R}} w_{\delta_0}^2}\int_K f_{\delta_0}(w_{\delta_0})\varphi \quad (n\to\infty).$$

By these observations, the claim is verified.

On the other hand, we notice that

$$\big|(\tilde{L}_{\varepsilon_n}\phi_n, \varphi)_{L^2(\Omega_n)}\big| \leqslant \|\tilde{L}_{\varepsilon_n}\phi_n\|_{L^2(\Omega_n)}\|\varphi\|_{L^2(\Omega_n)} \to 0 \tag{80}$$

as $n \to \infty$ for any $\varphi \in C_0^\infty(\mathbb{R})$. Combining (78) and (80), we have

$$(\mathcal{L}_{\delta_0}\phi, \varphi)_{L^2(\mathbb{R})} = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}). \tag{81}$$

Therefore, $\mathcal{L}_{\delta_0}\phi = 0$, $\phi \in H_r^2(\mathbb{R})$. Thus we conclude $\phi = 0$ by Lemma 13.

Next, we claim that:

Claim.

$$\|\phi_n\|_{H^2(\Omega_n)} \to 0 \quad \text{as } n \to \infty. \tag{82}$$

Indeed, by Lemma 26, we have

$$\begin{aligned}\|\phi_n\|_{H^2(\Omega_n)} &\leqslant C\left\{\|\tilde{L}_{\varepsilon_n}\phi_n\|_{L^2(\Omega_n)} + \left\|\frac{2\overline{w}_{\varepsilon_n}\phi_n}{\overline{T[w_{\varepsilon_n}]}(1+\delta_n\alpha_D^{-2}\overline{w}_{\varepsilon_n}^2)^2}\right\|_{L^2(\Omega_n)} + \left\|\frac{\overline{w}_{\varepsilon_n}^2(\overline{T'[w_{\varepsilon_n}]\underline{\phi_n}})}{\overline{T[w_{\varepsilon_n}]}^2(1+\delta_n\alpha_D^{-2}\overline{w}_{\varepsilon_n}^2)}\right\|_{L^2(\Omega_n)}\right\}\\ &\equiv C(I+II+III).\end{aligned} \tag{83}$$

By (75), $I \to 0$ as $n \to \infty$. Moreover, by the exponentially decay estimate of Lemma 6(iv) and the fact $\phi_n \to \phi = 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$ and $L^2_{\mathrm{loc}}(\mathbb{R})$, we can see that $II, III \to 0$ as $n \to \infty$.

However, (82) contradicts $\|\phi_n\|_{H^2(\Omega_n)} = 1$. Thus (69) is verified.

Next, we show (70). We note that (69) implies the range of $\tilde{L}_\varepsilon$ is closed. Hence, by a general theory of the functional analysis, $\mathrm{Ran}(\tilde{L}_\varepsilon) = L_r^2(\Omega_\varepsilon)$ if and only if $\tilde{L}_\varepsilon^*$ is one to one. However, by the same argument as was used in the proof of (69), we can show that $\tilde{L}_\varepsilon^*$ is one to one for sufficiently small ε under the assumption where $\delta_\varepsilon \to \delta_0 \in [0, \delta_2)$ as $\varepsilon \to 0$. Therefore, we omit the details. □

At last, we construct a solution to (1) and complete the proof of Theorem 1. Let us find $\phi \in H_{r,\nu}^2(\Omega_\varepsilon)$ such that $S[\overline{w}_\varepsilon + \varepsilon\phi] = 0$ for sufficiently small ε. Note that it is equivalent to the following: for $\phi \in H_{r,\nu}^2(\Omega_\varepsilon)$,

$$\begin{aligned}&S[\overline{w}_\varepsilon + \varepsilon\phi] = 0,\\ &\big(\tilde{L}_\varepsilon(\varepsilon\phi)\big) = S'[\overline{w}_\varepsilon](\varepsilon\phi) = -S[\overline{w}_\varepsilon + \varepsilon\phi] + S'[\overline{w}_\varepsilon](\varepsilon\phi),\\ &\varepsilon\tilde{L}_\varepsilon\phi = -S[\overline{w}_\varepsilon] - \big(S[\overline{w}_\varepsilon + \varepsilon\phi] - S[\overline{w}_\varepsilon] - S'[\overline{w}_\varepsilon](\varepsilon\phi)\big),\\ &\phi = \frac{1}{\varepsilon}\big\{-\tilde{L}_\varepsilon^{-1}\big[S[\overline{w}_\varepsilon]\big] - \tilde{L}_\varepsilon^{-1}\big[S[\overline{w}_\varepsilon + \varepsilon\phi] - S[\overline{w}_\varepsilon] - S'[\overline{w}_\varepsilon](\varepsilon\phi)\big]\big\} =: M_\varepsilon(\phi).\end{aligned}$$

Hence, we only need to find a fixed point ϕ of M_ε. Define

$$B := \left\{\phi \in H_{r,\nu}^2(\Omega_\varepsilon):\ \|\phi\|_{H^2(\Omega_\varepsilon)} < \frac{2C_1}{\lambda}\right\}, \tag{84}$$

where C_1 and λ are constants given in Lemma 19 and Proposition 23, respectively. Let us show that M_ε is a contraction mapping on B when ε is sufficiently small.

Proposition 27. *There exists $\varepsilon_1 > 0$ such that, for $\varepsilon \in (0, \varepsilon_1)$, M_ε is a contraction mapping on B.*

Proof. For $\phi \in B$, note that $M_\varepsilon(\phi) \in H_{r,\nu}^2(\Omega_\varepsilon)$. Moreover, by Lemma 19, (65) and (72), we can estimate as follows:

$$\begin{aligned}\big\|M_\varepsilon(\phi)\big\|_{H^2(\Omega_\varepsilon)} &\leqslant \frac{1}{\varepsilon\lambda}\big\{\big\|S[\overline{w}_\varepsilon]\big\|_{L^2(\Omega_\varepsilon)} + \big\|S[\overline{w}_\varepsilon + \varepsilon\phi] - S[\overline{w}_\varepsilon] - S'[\overline{w}_\varepsilon](\varepsilon\phi)\big\|_{L^2(\Omega_\varepsilon)}\big\}\\ &\leqslant \frac{1}{\varepsilon\lambda}\big\{C_1\varepsilon + C\varepsilon^2\big(\|\phi\|^2_{L^2(\Omega_\varepsilon)} + \|\phi\|_{L^\infty(\Omega_\varepsilon)}\|\phi\|_{L^2(\Omega_\varepsilon)}\big)\big\}\\ &\leqslant \frac{1}{\lambda}\big\{C_1 + C'\varepsilon\|\phi\|^2_{H^2(\Omega_\varepsilon)}\big\}\\ &\leqslant \frac{1}{\lambda}\left\{C_1 + C'\varepsilon\frac{4C_1^2}{\lambda^2}\right\},\end{aligned}$$

where $C, C' > 0$ are independent of ε sufficiently small. Hence, if ε is small so that $\varepsilon < \lambda^2/(8C_1C')$, then $\|M_\varepsilon(\phi)\|_{H^2(\Omega_\varepsilon)} < \frac{2C_1}{\lambda}$ for $\phi \in B$. Therefore, M_ε is a mapping form B into itself for sufficiently small ε.

For $\phi_1, \phi_2 \in B$, by (65), (66) and (72), we can estimate as follows:

$$\begin{aligned}
&\|M_\varepsilon(\phi_1) - M_\varepsilon(\phi_2)\|_{H^2(\Omega_\varepsilon)} \\
&\quad\leqslant \frac{1}{\varepsilon\lambda}\|S[\overline{w}_\varepsilon + \varepsilon\phi_1] - S[\overline{w}_\varepsilon + \varepsilon\phi_2] - S'[\overline{w}_\varepsilon](\varepsilon\phi_1) + S'[\overline{w}_\varepsilon](\varepsilon\phi_2)\|_{L^2(\Omega_\varepsilon)} \\
&\quad\leqslant \frac{1}{\varepsilon\lambda}\big\{\big\|S\big[\overline{w}_\varepsilon + \varepsilon\phi_2 + \varepsilon(\phi_1 - \phi_2)\big] - S[\overline{w}_\varepsilon + \varepsilon\phi_2] - S'[\overline{w}_\varepsilon + \varepsilon\phi_2]\big(\varepsilon(\phi_1 - \phi_2)\big)\big\|_{L^2(\Omega_\varepsilon)} \\
&\qquad + \varepsilon\|S'[\overline{w}_\varepsilon + \varepsilon\phi_2](\phi_1 - \phi_2) - S'[\overline{w}_\varepsilon](\phi_1 - \phi_2)\|_{L^2(\Omega_\varepsilon)}\big\} \\
&\quad\leqslant \frac{C}{\varepsilon\lambda}\big\{\varepsilon^2\|\phi_1 - \phi_2\|^2_{H^2(\Omega_\varepsilon)} + \varepsilon^2\|\phi_2\|_{H^2(\Omega_\varepsilon)}\|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)}\big\} \\
&\quad\leqslant C'\varepsilon\|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)},
\end{aligned}$$

where $C, C' > 0$ are independent of ε sufficiently small. Therefore, M_ε is a contraction mapping on B provided ε is small enough. □

Proof of Theorem 1. By Proposition 27, M_ε has a unique fixed point in B if ε is sufficiently small. Let $\phi_\varepsilon \in B$ be the fixed point. Then ϕ_ε satisfies $S[\overline{w}_\varepsilon + \varepsilon\phi_\varepsilon] = 0$. As we stated in Section 3.2, by putting $A_\varepsilon(x) := c_\varepsilon(w_\varepsilon(x) + \varepsilon\underline{\phi}_\varepsilon(x))$ and $H_\varepsilon(x) := c_\varepsilon T[w_\varepsilon + \varepsilon\underline{\phi}_\varepsilon](x)$, we obtain a solution to (1). We can see that this $(A_\varepsilon, H_\varepsilon)$ satisfies (12)–(14). Thus we complete the proof. □

6. Construction of a solution for $\sigma > 0$

In this section, we construct a solution to (1) in the case $\sigma > 0$ and prove Theorem 2. Let us treat σ as a parameter. To lead precise estimates, we fix $\overline{\sigma} > 0$ arbitrarily, and we will consider $\sigma \in (0, \overline{\sigma})$. Let $\phi_\varepsilon \in B$ be a unique fixed point of M_ε given in the proof of Theorem 1. Put

$$U_\varepsilon(y) := \overline{w}_\varepsilon(y) + \varepsilon\phi_\varepsilon, \quad y \in \Omega_\varepsilon, \tag{85}$$

and we define an operator $\hat{L}_\varepsilon$ on $L^2(\Omega_\varepsilon)$ with $\mathrm{Dom}(\hat{L}_\varepsilon) = H^2_{r,\nu}(\Omega_\varepsilon)$ by

$$\hat{L}_\varepsilon\phi := S'[U_\varepsilon + \sigma_\varepsilon]\phi, \quad \phi \in \mathrm{Dom}(\hat{L}_\varepsilon). \tag{86}$$

We note that

$$\|\sigma_\varepsilon\|_{H^2(\Omega_\varepsilon)} = \sigma_\varepsilon\|1\|_{H^2(\Omega_\varepsilon)} = \frac{\sigma_\varepsilon}{\sqrt{2\varepsilon}} = \frac{\varepsilon\sigma\int_{\mathbb{R}} w^2_{\delta_\varepsilon}(y)\,dy}{\sqrt{2\varepsilon}} < C\sqrt{\varepsilon}\sigma$$

holds for some constant $C > 0$ independent of ε sufficiently small. Thus, we may assume $U_\varepsilon + \sigma_\varepsilon \in B_\tau(\overline{w}_\varepsilon)$ for sufficiently small ε and $\sigma \in (0, \overline{\sigma})$. Then we have the following proposition.

Proposition 28. *There exists $\hat{\varepsilon}_0 > 0$ depending on $\overline{\sigma}$ such that, for $\varepsilon \in (0, \hat{\varepsilon}_0)$ and $\sigma \in (0, \overline{\sigma})$, $\hat{L}_\varepsilon$ has a bounded inverse $\hat{L}_\varepsilon^{-1} : L^2_r(\Omega_\varepsilon) \to H^2_{r,\nu}(\Omega_\varepsilon)$, and the following estimate holds*:

$$\|\hat{L}_\varepsilon^{-1}\phi\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{2}{\lambda}\|\phi\|_{L^2(\Omega_\varepsilon)}, \quad \phi \in L^2_r(\Omega_\varepsilon), \tag{87}$$

where λ is a constant given in Proposition 23.

Proof. Let $\phi \in L^2_r(\Omega_\varepsilon)$ be given. For $\psi \in H^2_{r,\nu}(\Omega_\varepsilon)$, the following equations are equivalent:

$$\begin{aligned}
&\hat{L}_\varepsilon\psi = \phi, \\
&\tilde{L}_\varepsilon\psi - (\tilde{L}_\varepsilon\psi - \hat{L}_\varepsilon\psi) = \phi, \\
&(I - K_\varepsilon)\psi := \psi - \tilde{L}_\varepsilon^{-1}(\tilde{L}_\varepsilon\psi - \hat{L}_\varepsilon\psi) = \tilde{L}_\varepsilon^{-1}\phi.
\end{aligned} \tag{88}$$

Noting $\tilde{L}_\varepsilon\psi - \hat{L}_\varepsilon\psi = S'[\overline{w}_\varepsilon]\psi - S'[U_\varepsilon + \sigma_\varepsilon]\psi$, we can regard K_ε is a mapping from $L^2_r(\Omega_\varepsilon)$ into itself. We can estimate so that

$$\begin{aligned}\|K_\varepsilon\psi\|_{L^2(\Omega_\varepsilon)} \leqslant \|K_\varepsilon\psi\|_{H^2(\Omega_\varepsilon)} &\leqslant \frac{1}{\lambda}\|\tilde{L}_\varepsilon\psi - \hat{L}_\varepsilon\psi\|_{L^2(\Omega_\varepsilon)}\\ &= \frac{1}{\lambda}\big\|S'[\overline{w}_\varepsilon + \varepsilon\phi_\varepsilon + \sigma_\varepsilon]\psi - S'[\overline{w}_\varepsilon]\psi\big\|_{L^2(\Omega_\varepsilon)}\\ &\leqslant \frac{C}{\lambda}\big(\|\varepsilon\phi_\varepsilon + \sigma_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\varepsilon\phi_\varepsilon + \sigma_\varepsilon\|_{L^\infty(\Omega_\varepsilon)}\big)\|\psi\|_{L^2(\Omega_\varepsilon)}\\ &\leqslant C'(\varepsilon + \sqrt{\varepsilon}\overline{\sigma})\|\psi\|_{L^2(\Omega_\varepsilon)},\end{aligned}$$

by Lemma 21, where $C' > 0$ is independent of $\sigma \in (0, \overline{\sigma})$ and ε. Therefore, $\|K_\varepsilon\|_{L^2_r(\Omega_\varepsilon)\to L^2_r(\Omega_\varepsilon)} \leqslant 1/2$ holds provided ε is small enough. Hence, by the Neumann series theory, $(I - K_\varepsilon)^{-1} : L^2_r(\Omega_\varepsilon) \to L^2_r(\Omega_\varepsilon)$ exists. Thus, we have $\psi = (I - K_\varepsilon)^{-1}\phi \equiv \hat{L}_\varepsilon\phi$. Moreover, from (88), we have $\psi \in H^2_{r,\nu}(\Omega_\varepsilon)$ and the estimate:

$$\|\psi\|_{H^2(\Omega_\varepsilon)} \leqslant \|K_\varepsilon\psi\|_{H^2(\Omega_\varepsilon)} + \big\|\tilde{L}_\varepsilon^{-1}\psi\big\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{1}{2}\|\psi\|_{H^2(\Omega_\varepsilon)} + \frac{1}{\lambda}\|\phi\|_{L^2(\Omega_\varepsilon)}.$$

Hence, $\|\psi\|_{H^2(\Omega_\varepsilon)} \leqslant \frac{2}{\lambda}\|\phi\|_{L^2(\Omega_\varepsilon)}$ follows. □

We put

$$\hat{B} := \big\{\phi \in H^2_{r,\nu}(\Omega_\varepsilon):\ \|\phi\|_{H^2(\Omega_\varepsilon)} < \overline{\sigma}\big\},$$

and fix $\gamma \in (0, 1/2)$ arbitrarily. Let us find $\phi \in \hat{B}$ such that $S[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma\phi] + \sigma_\varepsilon = 0$. We note that this is equivalent to the following:

$$\begin{aligned}&-\varepsilon^\gamma\hat{L}_\varepsilon\phi = S\big[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma\phi\big] - S[U_\varepsilon + \sigma_\varepsilon] - S'[U_\varepsilon + \sigma_\varepsilon]\big(\varepsilon^\gamma\phi\big) + S[U_\varepsilon + \sigma_\varepsilon] + \sigma_\varepsilon,\\ &\phi = -\frac{1}{\varepsilon^\gamma}\hat{L}_\varepsilon^{-1}\big(S\big[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma\phi\big] - S[U_\varepsilon + \sigma_\varepsilon] - S'[U_\varepsilon + \sigma_\varepsilon]\big(\varepsilon^\gamma\phi\big) + S[U_\varepsilon + \sigma_\varepsilon] + \sigma_\varepsilon\big) =: M_{\varepsilon,\sigma}(\phi).\end{aligned}$$

Proposition 29. *For $\overline{\sigma}$ and γ, there exists $\hat{\varepsilon}_1 > 0$ such that, for $\varepsilon \in (0, \hat{\varepsilon}_1)$ and $\sigma \in (0, \overline{\sigma})$, $M_{\varepsilon,\sigma}$ is a contraction mapping from $\hat{B}$ into itself.*

Proof. By Proposition 28, we have

$$\begin{aligned}\big\|M_{\varepsilon,\sigma}(\phi)\big\|_{H^2(\Omega_\varepsilon)} &\leqslant \frac{2}{\varepsilon^\gamma\lambda}\big\{\big\|S\big[U_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma\phi\big] - S[U_\varepsilon + \sigma_\varepsilon] - S'[U_\varepsilon + \sigma_\varepsilon]\big(\varepsilon^\gamma\phi\big)\big\|_{L^2(\Omega_\varepsilon)}\\ &\quad + \big\|S[U_\varepsilon + \sigma_\varepsilon]\big\|_{L^2(\Omega_\varepsilon)} + \|\sigma_\varepsilon\|_{L^2(\Omega_\varepsilon)}\big\}\\ &\equiv \frac{2}{\varepsilon^\gamma\lambda}(I + II + III).\end{aligned}$$

Moreover, by Lemmas 21, 25, we can see that the following estimates hold: for $\phi \in \hat{B}$,

$$\begin{aligned}&I \leqslant 2C\varepsilon^{2\gamma}\|\phi\|^2_{H^2(\Omega_\varepsilon)} \leqslant 2C\varepsilon^{2\gamma}\overline{\sigma}^2,\\ &II \leqslant \big\|S[U_\varepsilon + \sigma_\varepsilon] - S[U_\varepsilon] - S'[U_\varepsilon]\sigma_\varepsilon\big\|_{L^2(\Omega_\varepsilon)} + \big\|S'[U_\varepsilon]\sigma_\varepsilon\big\|_{L^2(\Omega_\varepsilon)} \leqslant C'\sqrt{\varepsilon}\big(\overline{\sigma}^2 + \overline{\sigma}\big),\\ &III \leqslant C''\sqrt{\varepsilon}\overline{\sigma},\end{aligned}$$

the constants $C, C', C'' > 0$ are independent of $\sigma \in (0, \overline{\sigma})$ and ε sufficiently small. Thus we have

$$\big\|M_{\varepsilon,\sigma}(\phi)\big\|_{H^2(\Omega_\varepsilon)} \leqslant C'''\big(\overline{\sigma}^2 + \overline{\sigma}\big)\max\big\{\varepsilon^{1/2-\gamma}, \varepsilon^\gamma\big\}$$

for some constant $C''' > 0$ independent of σ and ε sufficiently small. Hence, $\|M_{\varepsilon,\sigma}(\phi)\|_{H^2(\Omega_\varepsilon)} < \overline{\sigma}$ holds for all $\phi \in \hat{B}$ and $\sigma \in (0, \overline{\sigma})$ provided ε is small enough.

Let $\phi_1, \phi_2 \in \hat{B}$. Then, by the same argument as was used in the proof of Proposition 27, we can see that

$$\left\| M_{\varepsilon,\sigma}(\phi_1) - M_{\varepsilon,\sigma}(\phi_2) \right\|_{H^2(\Omega_\varepsilon)} \leqslant C\overline{\sigma}\varepsilon^\gamma \|\phi_1 - \phi_2\|_{H^2(\Omega_\varepsilon)}$$

holds for some constant $C > 0$ independent of σ and ε sufficiently small. Hence, there exists $\hat{\varepsilon}_1 > 0$, $M_{\varepsilon,\sigma}$ is a contraction mapping on $\hat{B}$ for $\varepsilon \in (0, \hat{\varepsilon}_1)$ and $\sigma \in (0, \overline{\sigma})$. Thus we complete the proof. □

Proof of Theorem 2. By Proposition 29, $M_{\varepsilon,\sigma}$ has a unique fixed point in $\hat{B}$ if ε is sufficiently small. Let $\phi_{\varepsilon,\sigma} \in \hat{B}$ be the fixed point. Then $\phi_{\varepsilon,\sigma}$ satisfies $S[\overline{w}_\varepsilon + \varepsilon\phi_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \phi_{\varepsilon,\sigma}] = 0$. As we stated in Section 3.2, by putting $A_{\varepsilon,\sigma}(x) := c_\varepsilon(\overline{w_\varepsilon}(x) + \varepsilon\underline{\phi}_\varepsilon(x) + \sigma_\varepsilon + \varepsilon^\gamma \underline{\phi}_{\varepsilon,\sigma})$ and $H_{\varepsilon,\sigma}(x) := c_\varepsilon T[w_\varepsilon + \varepsilon\underline{\phi}_\varepsilon + \sigma_\varepsilon + \varepsilon^\gamma \underline{\phi}_{\varepsilon,\sigma}](x)$, we obtain a solution to (1). We can easily see that this $(A_{\varepsilon,\sigma}, H_{\varepsilon,\sigma})$ satisfies (15)–(17). Thus we complete the proof. □

Appendix A

In this section, we give the proofs of Lemmas 20, 21, 24, 25 26. We first prove Lemmas 24, 25, 26.

Proof of Lemma 24. In general, $H^2(\mathbb{R})$-extensions denoted by $E_1 u$ and $E_2 v$ of $u \in H^2(0, \infty)$ and $v \in H^2(-\infty, 0)$ are given by

$$E_1 u(x) = \begin{cases} u(x), & x > 0, \\ 3u(-x) - 2u(-2x), & x < 0, \end{cases} \qquad E_2 v(x) = \begin{cases} 3v(-x) - 2v(-2x), & x > 0, \\ u(x), & x < 0, \end{cases}$$

respectively, and

$$\|E_1 u\|_{H^2(\mathbb{R})} \leqslant C\|u\|_{H^2(0,\infty)}, \qquad \|E_2 v\|_{H^2(\mathbb{R})} \leqslant C\|v\|_{H^2(-\infty,0)}, \tag{89}$$

hold for some $C > 0$ independent of u and v. By translation, we see that there exists $H^2(\mathbb{R})$-extensions denoted by $\tilde{E}_1 u$ and $\tilde{E}_2 v$ of $u \in H^2(-\frac{1}{\varepsilon}, \infty)$ and $v \in H^2(-\infty, \frac{1}{\varepsilon})$. Because translation does not change the H^2-norm,

$$\|\tilde{E}_1 u\|_{H^2(\mathbb{R})} \leqslant C\|u\|_{H^2(-\frac{1}{\varepsilon},\infty)}, \qquad \|\tilde{E}_2 v\|_{H^2(\mathbb{R})} \leqslant C\|v\|_{H^2(-\infty,\frac{1}{\varepsilon})}, \tag{90}$$

hold for the same constant C as that in (89). Now, let $\varphi \in C^\infty(\mathbb{R})$ be a function such that, $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ for $x \leqslant -\frac{1}{3}$, $\varphi(x) = 0$ for $x > \frac{1}{3}$. Moreover, we take φ so that

$$1 - \varphi(x) = \varphi(-x), \quad x \in \mathbb{R}, \tag{91}$$

holds. We define $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$. Then, for fixed $\overline{\varepsilon} > 0$, we note that the estimates

$$\sup_{x\in\mathbb{R}} |\varphi'_\varepsilon(x)|, \ \sup_{x\in\mathbb{R}} |\varphi''_\varepsilon(x)| \leqslant M, \quad \varepsilon \in (0, \overline{\varepsilon}), \tag{92}$$

hold for some constant $M > 0$ depending only on φ and independent of $\varepsilon \in (0, \overline{\varepsilon})$. For $u \in H_r^2(\Omega_\varepsilon)$, if we regard $(\varphi_\varepsilon u)(x) = 0$ for $x \in [-\frac{1}{\varepsilon}, \infty)$, then $\varphi_\varepsilon u \in H^2(-\frac{1}{\varepsilon}, \infty)$. Note that $\|\varphi_\varepsilon u\|_{H^2(-\frac{1}{\varepsilon},\infty)} \leqslant C'\|u\|_{H^2(\Omega_\varepsilon)}$ holds for all $\varepsilon \in (0, \overline{\varepsilon})$ by (92). We extend $\varphi_\varepsilon u$ by $\tilde{E}_1$, $\tilde{E}_1(\varphi_\varepsilon u) \in H^2(\mathbb{R})$. Similarly, we can regard $(1 - \varphi_\varepsilon)u \in H^2(-\infty, \frac{1}{\varepsilon})$, and $\tilde{E}_2((1 - \varphi_\varepsilon)u) \in H^2(\mathbb{R})$. Define $Eu := \tilde{E}_1(\varphi_\varepsilon u) + \tilde{E}_2((1 - \varphi_\varepsilon)u)$. Then, this E is a desired extension operator from $H_r^2(\Omega_\varepsilon)$ into $H_r^2(\mathbb{R})$. Indeed, we can easily see that Eu gives the $H^2(\mathbb{R})$-extension of u. Moreover, Eu is radially symmetric by our construction. Note the estimate

$$\begin{aligned} \|Eu\|_{H^2(\mathbb{R})} &\leqslant \left\|\tilde{E}_1(\varphi_\varepsilon u)\right\|_{H^2(\mathbb{R})} + \left\|\tilde{E}_2\big((1 - \varphi_\varepsilon)u\big)\right\|_{H^2(\mathbb{R})} \\ &\leqslant C\left\{\|\varphi_\varepsilon u\|_{H^2(-\frac{1}{\varepsilon},\infty)} + \left\|(1 - \varphi_\varepsilon)u\right\|_{H^2(-\infty,\frac{1}{\varepsilon})}\right\} \leqslant 2CC'\|u\|_{H^2(\Omega_\varepsilon)}. \end{aligned}$$

Thus we complete the proof. □

Proof of Lemma 25. Let E be the extension operator given by Lemma 24. By Morrey's inequality, we have

$$\|u\|_{L^\infty(\Omega_\varepsilon)} = \|Eu\|_{L^\infty(\Omega_\varepsilon)} \leqslant \|Eu\|_{L^\infty(\mathbb{R})} \leqslant C''\|Eu\|_{H^2(\mathbb{R})} \leqslant CC''\|u\|_{H^2(\Omega_\varepsilon)}.$$

The constants $C, C'' > 0$ are independent of $\varepsilon \in (0, \overline{\varepsilon})$. □

Proof of Lemma 26. We first note that

$$\|u\|_{H^1(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)} \tag{93}$$

holds. This can be easily confirmed by multiplying (73) by u and integrating over Ω_ε. For fixed $c \in (0, \frac{1}{2})$, let $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$ be functions such that

$$\varphi_1(x) = \begin{cases} 1, & |x+1| < c, \\ 0, & |x+1| > 2c, \end{cases} \qquad \varphi_2(x) = \begin{cases} 1, & |x-1| < c, \\ 0, & |x-1| > 2c. \end{cases}$$

Define $\varphi_0(x) := 1 - (\varphi_1(x) + \varphi_2(x))$ for $x \in (-1, 1)$. Let

$$\varphi_j^\varepsilon(x) := \varphi_j(\varepsilon x), \quad j = 0, 1, 2.$$

Then, note that $|\frac{d^n \varphi_j^\varepsilon}{dx^n}(x)|$, $j = 0, 1, 2$, $n = 1, 2$, are bounded uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Now, $\varphi_0^\varepsilon u$ solves the following equation:

$$-\big(\varphi_0^\varepsilon u\big)'' + \big(\varphi_0^\varepsilon u\big) = -\big(\varphi_0^\varepsilon\big)'' u - 2\big(\varphi_0^\varepsilon\big)' u' + \varphi_0^\varepsilon f =: g_0^\varepsilon \quad \text{in } \Omega_\varepsilon. \tag{94}$$

Note that $\|g_0^\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant C\|f\|_{L^2(\Omega_\varepsilon)}$ holds for some constant $C > 0$ independent of $\varepsilon \in (0, \bar{\varepsilon})$ by (93). We extend $\varphi_0^\varepsilon u$ and g_0^ε as 0 in $\mathbb{R} \setminus \Omega_\varepsilon$. Then $\varphi_0^\varepsilon u \in H^2(\mathbb{R})$ and $g_0^\varepsilon \in L^2(\mathbb{R})$ satisfy the same equation as that in (94) over $\mathbb{R}$. Thus, by using a priori estimate of solutions for elliptic equations in whole space, we have

$$\big\|\varphi_0^\varepsilon u\big\|_{H^2(\Omega_\varepsilon)} = \big\|\varphi_0^\varepsilon u\big\|_{H^2(\mathbb{R})} \leqslant C'\big\|g_0^\varepsilon\big\|_{L^2(\mathbb{R})} \leqslant CC'\|f\|_{L^2(\Omega_\varepsilon)}, \tag{95}$$

for some constant $C' > 0$. Next, we consider $\varphi_1^\varepsilon u$. We extend $\varphi_1^\varepsilon u = 0$ for $x \geqslant \frac{1}{\varepsilon}$. Then $\varphi_1^\varepsilon u \in H^2(-\frac{1}{\varepsilon}, \infty)$. Moreover, we extend it to $H^2(\mathbb{R})$-function by reflection (this extension is possible since u satisfies $u'(-\frac{1}{\varepsilon}) = 0$). Then we notice that φ_1^ε satisfies

$$-\big(\varphi_1^\varepsilon u\big)'' - \big(\varphi_1^\varepsilon u\big) = -\big(\varphi_1^\varepsilon\big)'' u - 2\big(\varphi_1^\varepsilon\big)' u' + \varphi_1^\varepsilon f \quad \text{in } \mathbb{R}. \tag{96}$$

Hence, by the same argument as was used for $\varphi_0^\varepsilon u$, we have $\|\varphi_1^\varepsilon u\|_{H^2(\Omega_\varepsilon)} \leqslant C''\|f\|_{L^2(\Omega_\varepsilon)}$ for some constant $C'' > 0$ independent of $\varepsilon \in (0, \bar{\varepsilon})$. We can estimate for $\varphi_2^\varepsilon u$ in the same way. Thus we have

$$\|u\|_{H^2(\Omega_\varepsilon)} \leqslant \big\|\varphi_0^\varepsilon u\big\|_{H^2(\Omega_\varepsilon)} + \big\|\varphi_1^\varepsilon u\big\|_{H^2(\Omega_\varepsilon)} + \big\|\varphi_2^\varepsilon u\big\|_{H^2(\Omega_\varepsilon)} \leqslant C'''\|f\|_{L^2(\Omega_\varepsilon)} \tag{97}$$

for some constant $C''' > 0$ independent of u, f and $\varepsilon \in (0, \bar{\varepsilon})$. □

Proof of Lemma 20. It is easily to see that the Fréchet derivative of T at $u \in L^2(-1, 1)$ is given by $T'[u]$ of (59). Hence, we only show the inequalities (60) and (61). Noting that $c \leqslant G_D(x, z) \leqslant C$, $x, z \in (-1, 1)$, holds for some $C, c > 0$, we can estimate as follows:

$$\begin{aligned} \big|\overline{T[\underline{u} + \underline{h}]}(y) - \overline{T[\underline{u}]}(y) - \big(\overline{T'[\underline{u}]\underline{h}}\big)(y)\big| &= c_\varepsilon \left| \int_{-1}^{1} G_D(\varepsilon y, z)\underline{h}^2(z)\,dz \right| = \frac{1}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \left| \int_{-1/\varepsilon}^{1/\varepsilon} G_D(\varepsilon y, \varepsilon z) h^2(z)\,dz \right| \\ &\leqslant C'\|h\|^2_{L^2(\Omega_\varepsilon)}, \quad y \in \Omega_\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \big|\big(\overline{T'[\underline{u}]\underline{h}}\big)(y)\big| &= 2c_\varepsilon \left| \int_{-1}^{1} G_D(\varepsilon y, z)\underline{u}(z)\underline{h}(z)\,dz \right| = \frac{2}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \left| \int_{-1/\varepsilon}^{1/\varepsilon} G_D(\varepsilon y, \varepsilon z) u(z) h(z)\,dz \right| \\ &\leqslant C'\|u\|_{L^2(\Omega_\varepsilon)}\|h\|_{L^2(\Omega_\varepsilon)}, \quad y \in \Omega_\varepsilon, \end{aligned}$$

for any $u, h \in L^2(\Omega_\varepsilon)$, where $C' > 0$ is independent of ε sufficiently small. Thus we complete the proof. □

Proof of Lemma 21. Let us show the inequalities (65) and (66). We first note that $\|u\|_{L^\infty(\Omega_\varepsilon)}, \|u\|_{L^2(\Omega_\varepsilon)} \leqslant C_\tau$ holds for any $u \in B_\tau(\overline{w}_\varepsilon)$, where $C_\tau > 0$ is some constant independent of u and ε sufficiently small. For simplicity of notation, we put

$$g_\varepsilon(t) := \frac{t^2}{1+\delta_\varepsilon \alpha_D^{-2} t^2}. \tag{98}$$

Let $u \in B_\tau(\overline{w}_\varepsilon)$, $h \in H^2(\Omega_\varepsilon)$, $\|h\|_{H^2(\Omega_\varepsilon)} \ll 1$, and

$$\begin{aligned} S[u+h]-S[u]-S'[u]h &= \frac{g_\varepsilon(u+h)}{\overline{T[\underline{u}+\underline{h}]}} - \frac{g_\varepsilon(u)}{\overline{T[\underline{u}]}} - \frac{g'_\varepsilon(u)h}{\overline{T[\underline{u}]}} + \frac{(\overline{T'[\underline{u}]\underline{h}})g_\varepsilon(u)}{\overline{T[\underline{u}]}^2} \\ &= \frac{1}{\overline{T[\underline{u}+\underline{h}]}}\left\{g_\varepsilon(u+h)-g_\varepsilon(u)-g'(u)h\right\} \\ &\quad + g_\varepsilon(u)\left\{\frac{1}{\overline{T[\underline{u}+\underline{h}]}} - \frac{1}{\overline{T[\underline{u}]}} - \frac{(\overline{T'[\underline{u}]\underline{h}})}{\overline{T[\underline{u}]}^2}\right\} \\ &\quad + g'_\varepsilon(u)h\left\{\frac{1}{\overline{T[\underline{u}+\underline{h}]}} - \frac{1}{\overline{T[\underline{u}]}}\right\} \\ &\equiv I + II + III. \end{aligned}$$

Note that

$$\left|g_\varepsilon(u+h)-g_\varepsilon(u)-g'_\varepsilon(u)h\right| = \left|\int_0^1 \left\{g'_\varepsilon(u+th)-g'_\varepsilon(u)\right\}dt\cdot h\right| \leqslant M|h|^2$$

holds for some constant $M>0$ independent of ε sufficiently small. By this,

$$\|I\|_{L^2(\Omega_\varepsilon)} \leqslant C\left(\int_{\Omega_\varepsilon} h^4\right)^{\frac{1}{2}} \leqslant C\|h\|_{L^\infty(\Omega_\varepsilon)}\|h\|_{L^2(\Omega_\varepsilon)}$$

holds for some constant $C>0$ independent of ε sufficiently small. Next, let

$$\begin{aligned} II &= \frac{g_\varepsilon(u)}{\overline{T[\underline{u}+\underline{h}]}\,\overline{T[\underline{u}]}}\left\{\overline{T[\underline{u}]}^2 - \overline{T[\underline{u}+\underline{h}]}\,\overline{T[\underline{u}]} + \overline{T[\underline{u}+\underline{h}]}\big(\overline{T'[\underline{u}]\underline{h}}\big)\right\} \\ &= \frac{g_\varepsilon(u)}{\overline{T[\underline{u}+\underline{h}]}\,\overline{T[\underline{u}]}}\Big[\overline{T[\underline{u}]}\left\{\overline{T[\underline{u}]} - \overline{T[\underline{u}+\underline{h}]} + \big(\overline{T'[\underline{u}]\underline{h}}\big)\right\} - \big(\overline{T'[\underline{u}]\underline{h}}\big)\left\{\overline{T[\underline{u}+\underline{h}]} - \overline{T[\underline{u}]}\right\}\Big]. \end{aligned}$$

Note that

$$\left\|\frac{g_\varepsilon(u)}{\overline{T[\underline{u}+\underline{h}]}\,\overline{T[\underline{u}]}}\right\|_{L^2(\Omega_\varepsilon)}, \qquad \left\|\overline{T[\underline{u}]}\right\|_{L^\infty(\Omega_\varepsilon)},$$

are bounded independently of u and ε sufficiently small. Hence, by applying (60) and (61), we can estimate so that

$$\|II\|_{L^2(\Omega_\varepsilon)} \leqslant C'\|h\|^2_{L^2(\Omega_\varepsilon)}$$

for some constant $C'>0$ independent of ε sufficiently small. By the same estimate, we have $\|III\|_{L^2(\Omega_\varepsilon)} \leqslant C''\|h\|^2_{L^2(\Omega_\varepsilon)}$ for some constant $C''>0$ independent of ε sufficiently small. By these estimates, we obtain (65).

Let $u \in B_\tau(\overline{w}_\varepsilon)$, $h \in H^2(\Omega_\varepsilon)$, $\|h\|_{H^2(\Omega_\varepsilon)} \ll 1$, $\phi \in H^2(\Omega_\varepsilon)$, and

$$\begin{aligned} &\left\|S'[u+h]\phi - S'[u]\phi\right\|_{L^2(\Omega_\varepsilon)} \\ &\quad = \left\|\left(\frac{g'_\varepsilon(u+h)}{\overline{T[\underline{u}+\underline{h}]}} - \frac{g'_\varepsilon(u)}{\overline{T[\underline{u}]}}\right)\phi + \frac{(\overline{T'[\underline{u}+\underline{h}]\underline{\phi}})}{\overline{T[\underline{u}+\underline{h}]}^2}g_\varepsilon(u+h) - \frac{(\overline{T'[\underline{u}]\underline{\phi}})}{\overline{T[\underline{u}]}}g_\varepsilon(u)\right\|_{L^2(\Omega_\varepsilon)} \end{aligned}$$

$$\leqslant \left\| \left(\frac{g_\varepsilon'(u+h)}{\overline{T[\underline{u}+\underline{h}]}} - \frac{g_\varepsilon'(u)}{\overline{T[\underline{u}]}} \right) \phi \right\|_{L^2(\Omega_\varepsilon)} + \left\| \frac{(\overline{T'[\underline{u}+\underline{h}]\underline{\phi}})}{\overline{T[\underline{u}+\underline{h}]}^2} \big(g_\varepsilon(u+h) - g_\varepsilon(u) \big) \right\|_{L^2(\Omega_\varepsilon)}$$
$$+ \left\| g_\varepsilon(u) \left(\frac{(\overline{T'[\underline{u}+\underline{h}]\underline{\phi}})}{\overline{T[\underline{u}+\underline{h}]}^2} - \frac{(\overline{T'[\underline{u}]\underline{\phi}})}{\overline{T[\underline{u}]}^2} \right) \right\|_{L^2(\Omega_\varepsilon)}$$
$$\equiv IV + V + VI.$$

Let us estimate each term. By applying Lemma 20 and the mean value theorem,

$$IV \leqslant \|\phi\|_{L^2(\Omega_\varepsilon)} \left\{ \left\| g_\varepsilon'(u+h) \left\{ \frac{1}{\overline{T[\underline{u}+\underline{h}]}} - \frac{1}{\overline{T[\underline{u}]}} \right\} \right\|_{L^\infty(\Omega_\varepsilon)} + \left\| \frac{1}{\overline{T[\underline{u}]}} \left\{ g_\varepsilon'(u+h) - g_\varepsilon'(u) \right\} \right\|_{L^\infty(\Omega_\varepsilon)} \right\}$$
$$\leqslant C \|\phi\|_{L^2(\Omega_\varepsilon)} \left\{ \left\| \overline{T[\underline{u}+\underline{h}]} - \overline{T[\underline{u}]} \right\|_{L^\infty(\Omega_\varepsilon)} + \left\| g_\varepsilon'(u+h) - g_\varepsilon'(u) \right\|_{L^\infty(\Omega_\varepsilon)} \right\}$$
$$\leqslant C' \|\phi\|_{L^2(\Omega_\varepsilon)} \big(\|h\|_{L^2(\Omega_\varepsilon)} + \|h\|_{L^\infty(\Omega_\varepsilon)} \big),$$

$$V \leqslant C \left\| \overline{T'[\underline{u}+\underline{h}]\underline{\phi}} \right\|_{L^\infty(\Omega_\varepsilon)} \left\| g_\varepsilon(u+h) - g_\varepsilon(u) \right\|_{L^2(\Omega_\varepsilon)} \leqslant C' \|\phi\|_{L^2(\Omega_\varepsilon)} \|h\|_{L^2(\Omega_\varepsilon)},$$

$$VI \leqslant C \left\| \big(\overline{T'[\underline{u}+\underline{h}]\underline{\phi}} \big) \overline{T[\underline{u}]}^2 - \big(\overline{T'[\underline{u}]\underline{\phi}} \big) \overline{T[\underline{u}+\underline{h}]}^2 \right\|_{L^\infty(\Omega_\varepsilon)}$$
$$\leqslant C \left\{ \left\| \overline{T'[\underline{u}+\underline{h}]\underline{\phi}} \right\|_{L^\infty(\Omega)} \left\| \overline{T[\underline{u}]}^2 - \overline{T[\underline{u}+\underline{h}]}^2 \right\|_{L^\infty(\Omega_\varepsilon)} \right.$$
$$\left. + \left\| \overline{T[\underline{u}+\underline{h}]}^2 \right\|_{L^\infty(\Omega_\varepsilon)} \left\| \overline{T'[\underline{u}+\underline{h}]\underline{\phi}} - \overline{T'[\underline{u}]\underline{\phi}} \right\|_{L^\infty(\Omega_\varepsilon)} \right\}$$
$$\leqslant C' \|\phi\|_{L^2(\Omega_\varepsilon)} \|h\|_{L^2(\Omega_\varepsilon)}$$

hold for some constants $C, C' > 0$ independent of ε sufficiently small. Here, we used the fact that we may assume there exists a constant $M' > 0$ independent of ε such that

$$\left\| \overline{T[\underline{u}]} \right\|_{L^\infty(\Omega_\varepsilon)}, \left\| \overline{T[\underline{u}+\underline{h}]} \right\|_{L^\infty(\Omega_\varepsilon)} \leqslant M'$$

holds as long as $u \in B_\tau(\overline{w}_\varepsilon)$ and $\|h\|_{H^2(\Omega_\varepsilon)} \ll 1$. Indeed, for example,

$$\big| T[\underline{u}](x) \big| = c_\varepsilon \left| \int_{-1}^{1} G_D(x,z) \underline{u}^2(z)\, dz \right| = \frac{1}{\int_{\mathbb{R}} w_{\delta_\varepsilon}^2} \left| \int_{\Omega_\varepsilon} G_D(x, \varepsilon z) u^2(z)\, dz \right| \leqslant C C_\tau^2.$$

By these estimates, we complete the proof. □

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