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Exact reconstruction of damaged color images using a total variation model

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Abstract

In this paper the reconstruction of damaged piecewice constant color images is studied using an RGB total variation based model for colorization/inpainting. In particular, it is shown that when color is known in a uniformly distributed region, then reconstruction is possible with maximal fidelity.

MSC: 49J99; 26B30; 68U10

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1. Introduction

In this work we address the following colorization problem: How can a color image be recovered when the underlying gray level function is known everywhere but only small patches of color are available? Among the various approaches used in the study of this problem (e.g., [7–10,20,23,24]), we highlight the Red–Green–Blue (RGB) total variation model proposed by Fornasier in [14,15], and subsequently studied in [16]. A main motivation for that work was the restoration of damaged frescoes during WWII. The numerical implementation of this model usually provides very good results (see [16]); see also the work of Kang and March [17], where the Chromaticity/Brightness representation of colors is used in place of the RGB one.

In the RGB model, a color is identified with a vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, whose components ξ_1, ξ_2, and ξ_3 correspond to the different channels red, green, and blue. The color image to be reconstructed is represented by a function of bounded variation $u_0 \in BV(R; \mathbb{R}^3)$, where the open rectangle $R := (0, a) \times (0, b) \subset \mathbb{R}^2$ is the domain of the image.

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The goal is to reconstruct the original color image u_0 through a variational model starting from the knowledge of the gray level of u_0 on a given open subset D of R (the *damaged region*) together with the exact information of u_0 on $R \setminus D$ (the *undamaged region*). In [16] (see also [14,15]), the authors propose to minimize a functional of the form

$$\Phi(Du)(R) + \lambda \int_{R\setminus D} (u - u_0)^p \, dx + \mu \int_D \big(L(u\cdot e) - L(u_0 \cdot e)\big)^p \, dx \tag{1.1}$$

among all functions $u \in BV(R;\mathbb{R}^3)$, where Φ is a convex function on $\mathbb{R}^{3\times 3}$, Du is the gradient measure of u, so that $\Phi(Du)(R)$ is the corresponding (possibly anisotropic) total variation, $\lambda, \mu \in (0,\infty)$, $1 \leqslant p < \infty$, $L:\mathbb{R} \to [0,\infty)$ is an increasing nonlinear function, and $e \in \mathbb{R}^3$ is a unit vector. The map $u \mapsto L(u\cdot e)$ represents a nonlinear projection, which associates to each color u the corresponding gray level $L(u \cdot e)$.

The purpose of this paper is to study the faithfulness of the reconstruction provided by the model (1.1), with particular emphasis on the possible creation of new, spurious contours in the restored image. For this reason, we will consider only images that exhibit "perfect" fidelity, in the sense that we require $u = u_0$ $\mathcal{L}^2$-a.e. on $R \setminus D$ and $u \cdot e = u_0 \cdot e$ $\mathcal{L}^2$-a.e. in D, equivalently, $\lambda = \mu = \infty$. For simplicity, we take Φ to be the Euclidean norm. Therefore, we are led to the total variation-type minimization problem (see [22])

$$\inf\big\{|Du|(R)\colon\ u \in \mathrm{Ad}(u_0, D)\big\}, \tag{1.2}$$

where the class of admissible color images $\mathrm{Ad}(u_0, D)$ is defined as

$$\mathrm{Ad}(u_0, D) := \big\{u \in BV\big(R;\mathbb{R}^3\big),\ u = u_0\ \mathcal{L}^2\text{-a.e. on } R\setminus D,\ u\cdot e = u_0 \cdot e\ \mathcal{L}^2\text{-a.e. in } D\big\}.$$

Simple explicit examples (see Example 6.1) show that in general the solution to (1.2) may present spurious contours. However, some numerical experiments performed in [16] seem to indicate that the model (1.1) provides good reconstruction results when the exact information of the color is known over a small but uniformly distributed area (see [16, Figs. 8.1 and 8.2]), and, intuitively, we expect that the two models describe similar phenomena provided λ and μ are large enough. This is speculative, since noise and blurring may prevent exact reconstruction to be attainable. As it turns out, the study of the simplified model (1.2) is analytically very challenging even when considering very special color images u_0 (which will exclude images with textures) and very special geometries for the damaged region D. The analysis requires new ideas, which do not rely on classical mathematical tools.

Indeed, the exact reconstructibility of u_0 reduces to proving that u_0 is the unique solution of the 1-Laplacian Neumann problem (2.3). This question is far from trivial since standard PDE methods for elliptic equations with an underlying strictly convex integrand, e.g., the p-Laplacian with $p > 1$, fail to apply in our case. Since the 1-Laplacian is not defined at points in which Du vanishes, we need to introduce a very weak notion of solution, inspired by what is commonly done in the scalar setting (see, e.g., [6,11,18]). Precisely, in the first main result of the paper, Theorem 1.8, we prove the existence of a suitable tensor-valued calibration, i.e., a divergence-free tensor field with norm not exceeding one and with prescribed normal traces on the discontinuity set of u_0. The construction of such a calibration is new, and exploits in a rather subtle way the non-simply connected geometry of the damaged region (see Lemma 5.1). Although calibration methods have been previously used to deal with total variation based functionals, this seems to be the first paper where related techniques are applied in the corresponding vectorial setting.

Moreover, in Theorems 1.8 and 1.10 we show that the reconstructibility of u_0 over a neighborhood of the discontinuity set is closely related to the problem of finding the extension of a vector-valued Lipschitz function defined on an unbounded domain with minimal norm of its gradient. Note that for scalar-valued functions, the corresponding extension problem (1.17) has been studied by several authors (see, e.g., the review paper [4]) starting from the pioneering work of Kirszbraun [19], McShane [21], and Whitney [25], see also the seminal paper of Aronsson [3]. Very little is known in the vectorial setting.

Finally, in Theorem 1.13 we validate in our simplified model (1.2) the numerical experiments performed in [16] for (1.1), namely, we show that a special class of piecewise constant images u_0 are exactly reconstructible, provided that the exact information on the colors is known over a (possibly) small but sufficiently well distributed region. The result is nontrivial, and it amounts to exhibiting a large family of explicit solutions to problem (1.2). We remark that only very few explicit solutions are known in the scalar case and, to our knowledge, no results prior to ours are available in the vectorial setting.

In view of these considerations, we regard the analysis in this paper as a first necessary step toward the understanding of the more realistic model (1.1).

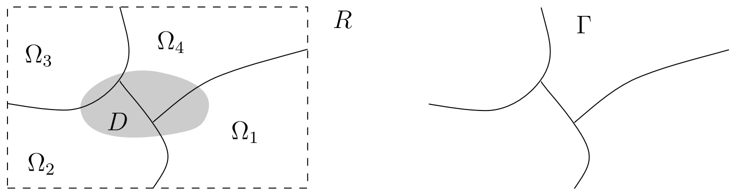

Fig. 1. An admissible image u_0, with its contour Γ and a damaged region D.

We now describe the results of the paper more in details.

As mentioned above, we will restrict the study to the case in which the original image belongs to a special class $\mathrm{Im}(R)$ of piecewise constant functions, precisely, u_0 belongs to $\mathrm{Im}(R)$ if there exist $\{\xi_k\}_{k=1}^N \subset \mathbb{R}^3$ with $\xi_k \neq \xi_h$ for $k \neq h$ and a family $\{\Omega_k\}_{k=1}^N$ of mutually disjoint open Lipschitz subsets of R such that

$$R = \Gamma \cup \bigcup_{k=1}^N \Omega_k, \qquad u_0 = \sum_{k=1}^N \xi_k 1_{\Omega_k}, \tag{1.3}$$

where $\Gamma := \bigcup_{k=1}^N \Sigma_k$, with $\Sigma_k := \partial\Omega_k \cap R$. Note that since Ω_k is Lipschitz, it has only finitely many connected components. We refer to Γ as the *contour* of u_0 and Σ_k is called the kth *contour* (cf. Fig. 1).

Definition 1.1. A color image $u_0 \in \mathrm{Im}(R)$ is said to be *reconstructible* over an open subset D of R if it is a minimizer of (1.2).

Given $u_0 \in \mathrm{Im}(R)$ and an arbitrary open set $D \subset R$, in general u_0 need not be reconstructible over D. Some examples may be found in Section 6. In particular, when *neighboring colors* ξ_h and ξ_k (i.e., $\mathcal{H}^1(\partial\Omega_k \cap \partial\Omega_h) > 0$) of u_0 have the same gray level, i.e., $\xi_h \cdot e = \xi_k \cdot e$, and the damaged region D contains part of $\partial\Omega_k \cap \partial\Omega_h$, then u_0 may fail to be reconstructible over D. This leads us to the notion of compatibility of neighboring colors and to restricting our analysis to specific geometries of the damaged region D. Precisely, for $k = 1, \dots, N$ decompose

$$\Sigma_k = \bigcup_{h \neq k} \Sigma_{k,h},$$

where $\Sigma_{k,h} := \partial\Omega_k \cap \partial\Omega_h$. We define the *compatibility vector field* $z_k \in L^\infty(\Sigma_k; \langle e\rangle^\perp)$ of u_0 as follows:

$$z_k(x) := P\left(\frac{\xi_k - \xi_h}{|\xi_k - \xi_h|}\right) \quad \text{if } x \in \Sigma_{k,h},\ h \neq k,\ \text{with } \mathcal{H}^1(\Sigma_{k,h}) > 0,$$

where $P : \mathbb{R}^3 \to \langle e\rangle^\perp$ denotes the projection of $\mathbb{R}^3$ on the orthogonal space $\langle e\rangle^\perp$ to e, i.e.,

$$P(\xi) := \xi - (\xi \cdot e)e.$$

Thus z_k is a piecewise constant vector field on Σ_k, and it is constant on each $\Sigma_{k,h}$. By construction, $\|z_k\|_{L^\infty(\Sigma_k;\langle e\rangle^\perp)} \leqslant 1$. In addition, $|z_k| = 1$ over some $\Sigma_{k,h}$ if and only if the two neighboring colors ξ_h and ξ_k have the same gray level. Thus, the condition

$$\sup_{1 \leqslant k \leqslant N} \|z_k\|_{L^\infty(\Sigma_k;\langle e\rangle^\perp)} < 1, \tag{1.4}$$

is equivalent to saying that u_0 does not have neighboring colors with the same gray level.

Concerning the restriction imposed on the damaged region, we will often assume that the intersection of the contour of u_0 with the boundary of the damaged region D has zero length, i.e.,

$$\mathcal{H}^1(\Gamma \cap \partial D) = 0. \tag{1.5}$$

In the statement of the first main theorem (see Theorem 1.2) we use a particular class $\mathcal{F}(D)$ of divergence-free vectors fields. To fix the notation, let $\{\varepsilon_1, \varepsilon_2\}$ be an orthonormal basis of $\langle e\rangle^\perp$, and decompose the generic vector $z \in \langle e\rangle^\perp$ as $z = z^{(1)}\varepsilon_1 + z^{(2)}\varepsilon_2$. In $\mathbb{R}^2$ we consider the canonical basis $\{e_1, e_2\}$, where $e_1 = (1,0)$, $e_2 = (0,1)$, and for every $v=(v_1,v_2)\in\mathbb{R}^2$ we define $v^\perp:=(v_2,-v_1)$. Given a tensor $M\in\langle e\rangle^\perp\otimes\mathbb{R}^2$, write $M=\varepsilon_1\otimes M^{(1)}+\varepsilon_2\otimes M^{(2)}$ with $M^{(1)},M^{(2)}\in\mathbb{R}^2$. Hence, if $\xi\in\mathbb{R}^2$,

$$M[\xi]=\big(M^{(1)}\cdot\xi\big)\varepsilon_1+\big(M^{(2)}\cdot\xi\big)\varepsilon_2,$$

and the Euclidean norm in $\langle e\rangle^\perp\otimes\mathbb{R}^2$ is

$$|M|=\sqrt{\big|M^{(1)}\big|^2+\big|M^{(2)}\big|^2},$$

induced by the scalar product

$$M:M_*=\big(M^{(1)}\cdot M_*^{(1)}\big)+\big(M^{(2)}\cdot M_*^{(2)}\big).$$

If $M:D\to\langle e\rangle^\perp\otimes\mathbb{R}^2$ is a smooth tensor field, then $\operatorname{div}M:D\to\langle e\rangle^\perp$ is defined by

$$\operatorname{div}M:=\operatorname{div}M^{(1)}\varepsilon_1+\operatorname{div}M^{(2)}\varepsilon_2,$$

where $\operatorname{div}M^{(i)}$ is the divergence of the vector field $M^{(i)}:D\to\mathbb{R}^2$, $i=1,2$. Hence, the divergence theorem becomes

$$\int_{\partial D}M[\nu_D]\cdot\varphi\,d\mathcal{H}^1=\int_D(\nabla\varphi:M+\varphi\cdot\operatorname{div}M)\,dx,$$

whenever $D\subset\mathbb{R}^2$ is an open bounded Lipschitz domain and $\varphi:D\to\langle e\rangle^\perp$ is sufficiently regular. Here ν_D is the outer unit normal to D.

We now introduce $\mathcal{F}(D)$. Given an open set $D\subset R$ with Lipschitz boundary, we define $\mathcal{F}(D)$ as the space of all tensor fields $M\in L^\infty(D;\langle e\rangle^\perp\otimes\mathbb{R}^2)$ with

$$\|M\|_{L^\infty(D;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant 1,$$

and zero distributional divergence, i.e.,

$$\langle\operatorname{div}M,\varphi\rangle=-\int_D\nabla\varphi:M\,dx=0\quad\text{for all }\varphi\in C_c^\infty\big(D;\langle e\rangle^\perp\big).$$

By [2, Theorem 1.2] every $M\in\mathcal{F}(D)$ admits a normal trace, that is, there exists a linear operator

$$\Psi:\big\{M\in L^\infty\big(D;\langle e\rangle^\perp\otimes\mathbb{R}^2\big):\ \operatorname{div}M=0\big\}\to L^\infty\big(\partial D;\langle e\rangle^\perp\big)$$

such that

$$\big\|\Psi(M)\big\|_{L^\infty(\partial D;\langle e\rangle^\perp)}\leqslant\|M\|_{L^\infty(D;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant 1\tag{1.6}$$

and

$$\int_{\partial D}\Psi(M)\cdot\varphi\,d\mathcal{H}^1=\int_D\nabla\varphi:M\,dx\tag{1.7}$$

for every $\varphi\in C^\infty(\overline{D};\langle e\rangle^\perp)$, and

$$\Psi(M)(x)=M(x)\big[\nu_D(x)\big]\quad\text{for }\mathcal{H}^1\text{-a.e. }x\in\partial D,$$

whenever $M\in C^\infty(\overline{D};\langle e\rangle^\perp\otimes\mathbb{R}^2)\cap\mathcal{F}(D)$. As usual we define $M[\nu_D]:=\Psi(M)$ for every $M\in\mathcal{F}(D)$.

The next theorem states that the reconstructibility of u_0 is equivalent to the existence of a suitable tensor-valued calibration.

Theorem 1.2. *Let $u_0\in\operatorname{Im}(R)$ and let $D\subset R$ be an open set with Lipschitz boundary satisfying* (1.5). *Then the following three conditions are equivalent*:

(i) *u_0 is reconstructible over D*;

(ii) *for every $1 \leqslant k \leqslant N$ and $\varphi \in BV(D \cap \Omega_k; \langle e\rangle^{\perp})$,*

$$|D\varphi|(D \cap \Omega_k) + \int_{\partial D \cap \Omega_k} |\varphi| \, d\mathcal{H}^1 + \int_{D \cap \Sigma_k} z_k \cdot \varphi \, d\mathcal{H}^1 \geqslant 0;$$

(iii) *for every $1 \leqslant k \leqslant N$ there exists a tensor field $M_k \in \mathcal{F}(D \cap \Omega_k)$ such that*

$$M_k[\nu_{D\cap\Omega_k}] = -z_k \quad \text{on } D \cap \Sigma_k, \tag{1.8}$$

where z_k is the compatibility vector field of u_0.

Moreover, if (1.4) *holds, then any of the conditions* (i)–(iii) *is equivalent to*

(iv) *u_0 is the unique minimizer for the model on D.*

Remark 1.3. In view of (1.8) and of Lemma 4.1, the tensor field $M : D \to \langle e\rangle^{\perp} \otimes \mathbb{R}^2$ defined by $M := M_k$ in $D \cap \Omega_k$, $1 \leqslant k \leqslant N$, is divergence-free in D. The conditions on M can be considered as a weak formulation of a 1-Laplacian Neumann problem (see Remark 2.2 for more details).

Since $D \subset \mathbb{R}^2$, any divergence-free field rotated of $\frac{\pi}{2}$ is locally the gradient of a Lipschitz function. Based on this observation, part (i) of the following proposition provides a method to construct a tensor field M satisfying part (iii) of Theorem 1.2.

To give the precise statement, we introduce the following notation, which will be used throughout the paper: Given a vector $v = (v_1, v_2) \in \mathbb{R}^2$, we define $v^{\perp} := (v_2, -v_1)$.

Proposition 1.4. *Let $A \subset \mathbb{R}^2$ be an open set with Lipschitz boundary, let $\Sigma \subset \partial A$, and let $g \in L^{\infty}(\Sigma; \langle e\rangle^{\perp})$.*

(i) *If there exists a Lipschitz function $f : \overline{A} \to \langle e\rangle^{\perp}$ such that $\partial_{\tau_A} f = g$ on Σ, where $\partial_{\tau_A} f$ is the tangential derivative of f with respect to the orientation induced by $\tau_A := (\nu_A)^{\perp}$, then the tensor field $M : A \to \langle e\rangle^{\perp} \otimes \mathbb{R}^2$ defined by*

$$M^{(j)} := -\big(\nabla f^{(j)}\big)^{\perp}, \quad j = 1, 2,$$

is divergence free in D and satisfies

$$M[\nu_A] = g \quad \text{on } \Sigma. \tag{1.9}$$

Moreover, if $\|\nabla f\|_{L^{\infty}(A; \langle e\rangle^{\perp} \otimes \mathbb{R}^2)} \leqslant 1$, then $M \in \mathcal{F}(A)$.

(ii) *Conversely, if A is simply connected, given $M \in \mathcal{F}(A)$ such that* (1.9) *holds, there exists a Lipschitz function $f : \overline{A} \to \langle e\rangle^{\perp}$ such that $\|\nabla f\|_{L^{\infty}(A; \langle e\rangle^{\perp} \otimes \mathbb{R}^2)} \leqslant 1$ and $\partial_{\tau_A} f = g$ on Σ.*

Remark 1.5. In view of Theorem 1.2(i) and (iii) and Proposition 1.4, we remark that the reconstructibility of u_0 can be reduced to a Lipschitz extension problem.

The proof of the previous proposition is standard and we omit it.

Next we focus our attention on special classes of damaged regions. As already mentioned, we are particularly interested in undamaged regions $R \setminus D$ having small area but that are uniformly distributed in R. Since in this case the damaged area is very large, it is reasonable to assume that it contains a small neighborhood of Γ (see Fig. 3). Note that if $D_1 \subset D_2 \subset R$, then

$$\mathrm{Ad}(u_0, D_1) \subset \mathrm{Ad}(u_0, D_2),$$

therefore if u_0 is reconstructible over D_2, then it is also reconstructible over D_1. Hence, as a starting point, we begin to study the case in which u_0 is reconstructible over a δ-neighborhood of the contour Γ, i.e., $D = \Gamma(\delta)$ for $\delta > 0$ sufficiently small, where

$$\Gamma(\delta) := \big\{x \in R \colon \operatorname{dist}(x, \Gamma) < \delta\big\}. \tag{1.10}$$

Note that condition (1.5) holds.

Precisely, our results will apply to the following stronger reconstructibility condition over $\Gamma(\delta)$.

Definition 1.6. Let $u_0 \in \mathrm{Im}(R)$ and let D be an open subset of R. Then u_0 is said to be *stably reconstructible* over D if there exists $\varepsilon > 0$ such that all $u \in \mathrm{Im}(R)$ of the form

$$u = \sum_{k=1}^{N} \xi_k' 1_{\Omega_k}, \quad \text{with } \max_{1 \leqslant k \leqslant N} |\xi_k' - \xi_k| < \varepsilon,$$

are reconstructible over D.

For damaged regions of the type (1.10), and when Γ is piecewise C^1, we give a sufficient condition for stable reconstructibility in terms of an explicit algebraic inequality that involves the values of the colors and the angles of the corners of Γ, if any.

Definition 1.7. A color image $u_0 \in \mathrm{Im}(R)$ is a *regular image* if for every $k = 1, \ldots, N$, the ith connected component $\Sigma_k^{(i)}$ of the kth contour Σ_k, $i = 1, \ldots, m_k$, is given by the image of a piecewise C^1 curve $\gamma_{k,i}$ parametrized by arc-length, and oriented so that $(\gamma_{k,i}')^\perp$ agrees with the inner unit normal vector field ν_k of Ω_k.

Given a regular image u_0, using the notation just introduced, for each curve $\gamma_{k,i}$ consider the set of those $s \in [0, \ell_{k,i}]$, where $\ell_{k,i} := \mathrm{length}(\gamma_{k,i})$, such that Γ has a corner at $\gamma_{k,i}(s)$, precisely,

$$\mathcal{S}_{k,i} := \{ s \in [0, \ell_{k,i}) \colon \gamma_{k,i}'(s^+) \neq (\gamma_{k,i}')(s^-) \}. \tag{1.11}$$

Here we use the following convention: If the curve $\gamma_{k,i}$ is closed, then we define $\gamma_{k,i}(0^-) := \gamma_{k,i}(\ell_{k,i}^-)$ and $\gamma_{k,i}'(0^-) := \gamma_{k,i}'(\ell_{k,i}^-)$; if $\gamma_{k,i}$ is not closed, then we extend $\gamma_{k,i}$ to $[-\epsilon, 0)$ in a C^1 way, so that $\gamma_{k,i}'(0^-)$ is well-defined.

When (1.4) holds, we will show that the stable reconstructibility of u_0 on the damaged region $\Gamma(\delta)$ depends only on some compatibility conditions between the vector fields $z_k \in L^\infty(\Sigma_k; \langle e\rangle^\perp)$ and the tangent vectors $\gamma_{k,i}'$ at points in $\mathcal{S}_{k,i}$. Using a blow-up argument, and in view of Proposition 1.4 and Remark 1.5, in Theorems 1.8 and 1.10 below we show that the analysis may be reformulated in terms of a Lipschitz extension-type problem to $\mathbb{R}^2$ of a function $g : C \to \langle e\rangle^\perp$, where

$$C := \{ x \in \mathbb{R}^2 \colon x = -sv,\ s \geqslant 0 \} \cup \{ x \in \mathbb{R}^2 \colon x = sw,\ s \geqslant 0 \}, \tag{1.12}$$

with $v, w \in S^1$ linearly independent. Recall that we write $g = g^{(1)}\varepsilon_1 + g^{(2)}\varepsilon_2$, where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis of $\langle e\rangle^\perp$ so that each $g^{(i)}$, $i = 1, 2$, is of the form

$$g^{(i)}(x) = \begin{cases} -sr^{(i)} & \text{if } x = -sv \text{ for } s \geqslant 0, \\ st^{(i)} & \text{if } x = sw \text{ for } s \geqslant 0, \end{cases}$$

for some $r^{(i)}, t^{(i)} \in [-1, 1]$. Note that

$$\begin{aligned} \mathrm{Lip}(g^{(i)}, C) &:= \sup_{x, y \in C,\, x \neq y} \frac{|g^{(i)}(x) - g^{(i)}(y)|}{|x - y|} \\ &= \sup_{s_1, s_2 > 0} \frac{|s_1 r^{(i)} + s_2 t^{(i)}|}{|s_1 v + s_2 w|} =: G(r^{(i)}, t^{(i)}, v, w). \end{aligned} \tag{1.13}$$

The exact expression of G may be found in Proposition A.1 in Appendix A. The next theorem provides a sufficient condition for stable reconstructibility in terms of the Lipschitz constant (1.13).

Theorem 1.8. *Let $u_0 \in \mathrm{Im}(R)$ be a regular image. Assume that* (1.4) *holds and*

$$[G(z_0^{(1)}, z_1^{(1)}, v, w)]^2 + [G(z_0^{(2)}, z_1^{(2)}, v, w)]^2 < 1, \tag{1.14}$$

whenever

$$z_0 = z_k(\gamma_{k,i}(s^-)), \qquad z_1 = z_k(\gamma_{k,i}(s^+)), \qquad v = \gamma_{k,i}'(s^-), \qquad w = \gamma_{k,i}'(s^+), \tag{1.15}$$

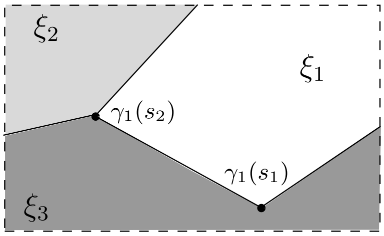

Fig. 2. If we chose ξ_1, ξ_2, ξ_3 so that ζ_1 is continuous at the corner point $\gamma_1(s_1)$ and it jumps at $\gamma_1(s_2)$, then condition (1.16) is imposed at $\gamma_1(s_1)$, while condition (1.14) is imposed at $\gamma_1(s_2)$.

with $k = 1, \ldots, N$, $i = 1, \ldots, m_k$, and $s \in \mathcal{S}_{k,i}$. Then u_0 is stably reconstructible on the damaged region $\Gamma(\delta)$ for some $\delta > 0$.

Remark 1.9. (Cf. Fig. 2.)

(i) In Example 6.2 we exhibit an image u_0 consisting of just two colors separated by a corner, with parameters chosen in such a way that (1.14) fails. We will show that in this case minimizers present an additional (artificial) contour.

(ii) Note that if z_k is constant near $\gamma_k(s)$, i.e., $z_0 = z_1 =: z$, then (1.14) takes the particularly simple form

$$|z|^2 < \frac{1 + v \cdot w}{2}. \tag{1.16}$$

In the limit case $v = w$, condition (1.16) reduces to the trivial fact $|z| < 1$.

(iii) If $z_0 = -z_1$, then condition (1.16) becomes $|z| < 1$ (see (A.2) in Appendix A), and therefore it is always satisfied.

Next we give a necessary condition for $u_0 \in \operatorname{Im}(R)$ to be stably reconstructible on the damaged region $\Gamma(\delta)$.

Theorem 1.10. *Let $u_0 \in \operatorname{Im}(R)$ be a regular image.*

(i) *If u_0 is reconstructible on the damaged region $\Gamma(\delta)$ for some $\delta > 0$, then whenever z_0, z_1, v, and w are as in* (1.15),

$$\min\bigl\{\|\nabla g\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\colon\ g \in W^{1,\infty}\bigl(\mathbb{R}^2; \langle e\rangle^\perp\bigr),\ g(y) = B[y] \text{ for } y \in C\bigr\} \leqslant 1, \tag{1.17}$$

where C is the set defined in (1.12) *and $B \in \langle e\rangle^\perp \otimes \mathbb{R}^2$ is the tensor uniquely determined by*

$$B[v] = -z_0 \quad \text{and} \quad B[w] = -z_1. \tag{1.18}$$

(ii) *If u_0 is stably reconstructible on the damaged region $\Gamma(\delta)$ for some $\delta > 0$, then the inequality in* (1.17) *is strict.*

Remark 1.11. Note that in some cases condition (1.17) coincides with (1.14) with $<$ replaced by $\leqslant$. For instance, if $z_0 = z_1$, let $g \in W^{1,\infty}(\mathbb{R}^2; \langle e\rangle^\perp)$ be a solution of the minimization problem (1.17) satisfying $\|\nabla g\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)} \leqslant 1$. Consider the points $x = -v$ and $y = w$. Then by (1.18),

$$\begin{aligned}2|z_0| = |z_0 + z_1| &= \bigl|g(-v) - g(w)\bigr| \leqslant \|\nabla g\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)}|v + w| \\ &\leqslant |v + w| = \sqrt{2}\sqrt{1 + v \cdot w},\end{aligned}$$

which is (1.16) with $<$ replaced by $\leqslant$. In particular, by part (ii) of Theorem 1.10, if $z_0 = z_1$, then (1.16) is both necessary and sufficient for the stable reconstructibility.

It is important to observe that in the previous theorem we are using the Euclidean norm of $\nabla g(x)$, that is,

$$\|\nabla g\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} \sqrt{\bigl|\nabla g^{(1)}(x)\bigr|^2 + \bigl|\nabla g^{(2)}(x)\bigr|^2},$$

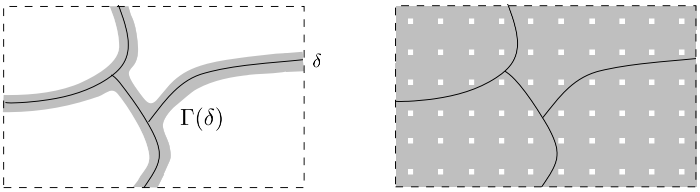

Fig. 3. On the left figure, $\Gamma(\delta)$ is a neighborhood of width δ of the contour Γ of $u_0 \in \mathrm{Ad}(R)$. On the right figure, the color is given only inside small and uniformly distributed squares. Note that here the damaged region contains a small neighborhood of the contour Γ.

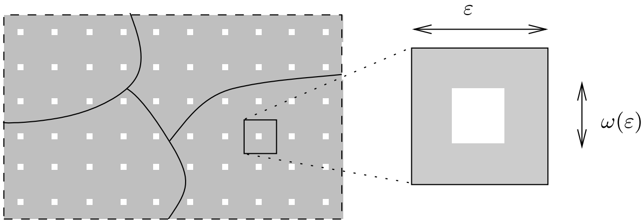

Fig. 4. An example of an ε-uniformly distributed (undamaged) region.

where $g = g^{(1)}\varepsilon_1 + g^{(2)}\varepsilon_2$. Hence, the minimization problem (1.17) is different from the classical problem of finding an extension with minimal Lipschitz constant (see [3,4,19,21,25]). Indeed, in the vectorial case the Lipschitz constant of an admissible function $g \in W^{1,\infty}(\mathbb{R}^2; \langle e\rangle^\perp)$ in (1.17) is

$$\mathrm{Lip}\big(g, \mathbb{R}^2\big) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^2} \sup_{\nu\in S^1} \big|\nabla g(x)[\nu]\big| \leqslant \|\nabla g\|_{L^\infty(\mathbb{R}^2; \langle e\rangle^\perp \otimes \mathbb{R}^2)},$$

where the inequality is in general strict.

Next we focus on images for which neighboring colors do not share the same gray level and when the undamaged region $R \setminus D$ has small area but is uniformly distributed in R.

Let $\omega : (0,\infty) \to (0,\infty)$ be such that

$$\lim_{\varepsilon\to 0^+} \frac{\omega(\varepsilon)}{\varepsilon^2} = \infty, \qquad \lim_{\varepsilon\to 0^+} \frac{\omega(\varepsilon)}{\varepsilon} = 0. \tag{1.19}$$

Given $\varepsilon > 0$ and an open set $U \subset R$ with Lipschitz boundary, we say that U is an *ε-uniformly distributed* (*undamaged*) *region* if

$$U \supset R \cap \bigg(\bigcup_{x\in\varepsilon\mathbb{Z}^2} \overline{Q\big(x, \omega(\varepsilon)\big)} \bigg), \tag{1.20}$$

where for $x \in \mathbb{R}^2$ and $r > 0$, $Q(x,r) := x + (-\frac{r}{2}, \frac{r}{2})^2$. Hence, in this case, the damaged region $D := R \setminus U$ is contained in

$$R \cap \bigg(\bigcup_{x\in\varepsilon\mathbb{Z}^2} \overline{Q(x,\varepsilon)} \setminus \overline{Q\big(x, \omega(\varepsilon)\big)} \bigg)$$

(see Fig. 4).

For damaged regions of this type it is always possible to obtain an asymptotic reconstruction result for arbitrary color images $u_0 \in BV(R; \mathbb{R}^3)$. Precisely,

Theorem 1.12. *Let $u_0 \in BV(R; \mathbb{R}^3) \cap L^\infty(R; \mathbb{R}^3)$ and let*

$$D_\varepsilon = R \cap \bigcup_{x\in\varepsilon\mathbb{Z}^2} \overline{Q(x,\varepsilon)} \setminus \overline{Q\big(x, \omega(\varepsilon)\big)}, \tag{1.21}$$

where ω satisfies (1.19). *For every $\varepsilon > 0$ the variational problem*

$$\inf\big\{|Du|(R)\colon\ u \in BV\big(R; \mathbb{R}^3\big),\ u = u_0\ \mathcal{L}^2\text{-a.e. on } R \setminus D_\varepsilon\big\} \tag{1.22}$$

admits a minimizer. In addition, if $u_\varepsilon \in BV(R; \mathbb{R}^3)$ is a minimizer for (1.22)*, then $u_\varepsilon \to u_0$ in $L^1(R; \mathbb{R}^3)$ as $\varepsilon \to 0^+$.*

Under additional assumptions, Theorems 1.13 and 1.14 provide exact reconstruction for sufficiently small values of ε.

Theorem 1.13. *Let $u_0 \in \mathrm{Im}(R)$ be a stably reconstructible image over the damaged region $\Gamma(\delta)$ for some $\delta > 0$. If* (1.4) *holds, then there exists $\varepsilon_0 > 0$ such that u_0 is reconstructible on the complement of every ε-uniformly distributed* (*undamaged*) *region, with $\varepsilon < \varepsilon_0$.*

We note that the condition $(1.19)_1$ is sharp, in the sense that if $\omega(\varepsilon) \leqslant c\varepsilon^2$ for some $c > 0$, then we cannot expect, in general, to attain exact reconstruction (see Example 6.4).

It is possible to treat more general non-periodic geometries, in particular, the case in which each cube $Q(x, \omega(\varepsilon))$ in (1.20) is replaced by a closed connected set whose diameter is of order $\omega(\varepsilon)$. More precisely, given $\varepsilon > 0$ and $\theta \in (0, 1)$, we consider the class $\mathcal{D}_{\varepsilon,\theta}$ of open sets $D \subset R$ with Lipschitz boundary such that

$$D \subset R \cap \bigcup_{x \in \varepsilon\mathbb{Z}^2} \left(\overline{Q(x, \varepsilon)} \setminus C_{x,\varepsilon}\right),$$

where, for every $x \in \varepsilon\mathbb{Z}^2$, $C_{x,\varepsilon}$ is a connected closed set contained in $Q(x, \theta\varepsilon)$, with $\mathrm{diam}(C_{x,\varepsilon}) \geqslant \omega(\varepsilon)$. Then the following generalization of Theorem 1.13 holds.

Theorem 1.14. *Let $u_0 \in \mathrm{Im}(R)$ be a stably reconstructible image over the damaged region $\Gamma(\delta)$ for some $\delta > 0$. If* (1.4) *holds, then for every $\theta \in (0, 1)$ there exists $\varepsilon_0 > 0$ depending only on u_0, θ, and ω, such that for every $0 < \varepsilon < \varepsilon_0$, u_0 is reconstructible over all damaged regions $D \in \mathcal{D}_{\varepsilon,\theta}$.*

This paper is organized as follows. In Section 2 we prove some preliminary results. In Section 3 we prove Theorem 1.2. The proofs of Theorems 1.8, 1.10 and of Theorems 1.12, 1.13, and 1.14 are presented in Section 4 and Section 5, respectively. In Section 6 we give some explicit examples in which exact reconstruction fails, and minimizers display different, spurious contours.

2. Existence of minimizers and an alternative formulation of the model

We begin by proving that the minimization problem (1.2) admits always a solution.

Proposition 2.1. *Let $u_0 \in BV(R; \mathbb{R}^3)$ and let $D \subset R$ be an open set with Lipschitz boundary. Then there exists a minimizer of* (1.2).

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset BV(R; \mathbb{R}^3)$ be a minimizing sequence for (1.2). In particular,

$$\sup_{n\in\mathbb{N}} |Du_n|(R) \leqslant |Du_0|(R) < \infty,$$

where we have used the fact that $u_0 \in \mathrm{Ad}(u_0, D)$. Moreover,

$$|Du_n|(R) = |Du_n|(D) + |Du_n|(R \setminus \overline{D}) + \int_{\partial D} |u_n - u_0| \, d\mathcal{H}^1, \tag{2.1}$$

where in the boundary integral u_n stands for the trace of u_n on ∂D as an element of $BV(D; \mathbb{R}^3)$, and u_0 stands for the trace of u_0 on ∂D as an element of $BV(R \setminus \overline{D}; \mathbb{R}^3)$ (cf. [1, Theorem 3.84]). Let

$$v_n := \begin{cases} u_n - u_0 & \text{on } D, \\ 0 & \text{on } \mathbb{R}^2 \setminus D. \end{cases}$$

Then $v_n \in BV(\mathbb{R}^2; \mathbb{R}^3)$ and, by the Sobolev–Gagliardo–Nirenberg inequality,

$$\begin{aligned} 2\sqrt{\pi}\left(\int_D |u_n - u_0|^2 \, dx\right)^{1/2} &= 2\sqrt{\pi}\, \|v_n\|_{L^2(\mathbb{R}^2;\mathbb{R}^3)} \leqslant |Dv_n|\big(\mathbb{R}^2\big) \\ &= \big|D(u_n - u_0)\big|(D) + \int_{\partial D} |u_n - u_0| \, d\mathcal{H}^1 \leqslant |Du_n|(R) + |Du_0|(D), \end{aligned}$$

where in the last inequality we used (2.1). Therefore $\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(R;\mathbb{R}^3)} < \infty$ and, in turn, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $BV(R;\mathbb{R}^3)$. By [1, Corollary 3.49], up to a subsequence, $\{u_n\}$ converges strongly in $L^1(R;\mathbb{R}^3)$ to some $\bar{u} \in BV(R;\mathbb{R}^3)$. Using the lower semicontinuity of the total variation and the fact that the class $\mathrm{Ad}(u_0, D)$ is closed with respect to weak star convergence in BV, we have that $\bar{u}$ belongs to $\mathrm{Ad}(u_0, D)$ and is a minimizer. □

If the damaged region D is an open set with Lipschitz boundary and satisfies the geometric condition (1.5) and if $u_0 \in \mathrm{Im}(R)$, then it is possible to reformulate the minimization problem (1.2) as

$$\inf\big\{F(u, D)\colon\ u \in BV\big(D;\mathbb{R}^3\big),\ u\cdot e = u_0\cdot e\ \mathcal{L}^2\text{-a.e. in } D\big\}, \tag{2.2}$$

where

$$F(u, D) := |Du|(D) + \sum_{k=1}^{N} \int_{\partial D\cap\Omega_k} |u-\xi_k|\,d\mathcal{H}^1.$$

In the boundary terms of F we integrate the trace of $u \in BV(D;\mathbb{R}^3)$ on ∂D.

Remark 2.2. Note that the Euler–Lagrange equation of the functional F are given, formally, by the 1-Laplacian Neumann problem

$$\begin{cases} \operatorname{div}\frac{Du}{|Du|} \parallel e & \text{on } D,\\ P\big(\frac{Du}{|Du|}[\nu_D]\big) = -z & \text{on } \partial D, \end{cases} \tag{2.3}$$

where $z := P(\frac{u-\xi_k}{|u-\xi_k|})$ in $\partial D\cap\Omega_k$. Since this equation is in general not well-defined, $\frac{Du}{|Du|}$ is replaced by the tensor field M given in Theorem 1.2. Hence, the conditions on M can be considered as a weak formulation of the Euler–Lagrange equations of F. For similar results see, e.g., [6,11,18].

Proposition 2.3. *Let $u_0 \in \mathrm{Im}(R)$ and let $D \subset R$ be an open set with Lipschitz boundary such that* (1.5) *holds. If $u \in BV(D;\mathbb{R}^3)$ is a minimizer of* (2.2)*, then the function*

$$\bar{u} = \begin{cases} u_0 & \text{on } R\setminus D,\\ u & \text{in } D, \end{cases}$$

is a minimizer of (1.2)*. Conversely, if $u \in BV(R;\mathbb{R}^3)$ is a minimizer of* (1.2)*, then its restriction $\bar{u}$ to D is a minimizer of* (2.2)*.*

Proof. For every $v \in \mathrm{Ad}(u_0, R)$, we have

$$|Dv|(R) = |Dv|(R\setminus\overline{D}) + |Dv|(D) + |Dv|(R\cap\partial D).$$

Since $v = u_0$ in the open set $R\setminus\overline{D}$, it follows that $|Dv|(R\setminus\overline{D}) = |Du_0|(R\setminus\overline{D})$. Moreover, $|Dv|(D) = |D\tilde{v}|(D)$, where $\tilde{v}$ is the restriction of v to D. Thus

$$|Dv|(R) = |Du_0|(R\setminus\overline{D}) + |D\tilde{v}|(D) + |Dv|(R\cap\partial D).$$

We now consider the term $|Dv|(R\cap\partial D)$. Using (1.5), we obtain

$$|Dv|(R\cap\partial D) = |Dv|\big((R\setminus\Gamma)\cap\partial D\big) = \sum_{k=1}^{N} |Dv|(\Omega_k\cap\partial D).$$

By [1, Theorem 3.84],

$$|Dv|(\Omega_k\cap\partial D) = \int_{\partial D\cap\Omega_k} |\xi_k - \tilde{v}|\,d\mathcal{H}^1,$$

where in the last integral $\tilde{v}$ denotes the trace on ∂D of $\tilde{v}$ as an element of $BV(D;\mathbb{R}^3)$. Therefore,

$$|Dv|(R) = |Du_0|(R\setminus\overline{D}) + F(\tilde{v}; D),$$

and the statement follows. □

Remark 2.4. Note that $u \in BV(D;\mathbb{R}^3)$ is a minimizer of (2.2) if and only if for every $\varphi \in BV(D, \langle e\rangle^\perp)$ the convex function $f:[0,\infty)\to[0,\infty)$ defined by

$$f(t) := F(u_0 + t\varphi, D), \quad t \geqslant 0, \tag{2.4}$$

has a minimum in $t=0$, or, equivalently,

$$0 \leqslant \lim_{t\to 0^+} \frac{f(t)-f(0)}{t}. \tag{2.5}$$

This is an immediate consequence of the fact that $v \in BV(D;\mathbb{R}^3)$ is such that $v\cdot e = u_0 \cdot e$ on D if and only if $v = u_0+\varphi$ with $\varphi \in BV(D;\langle e\rangle^\perp)$.

3. Reconstructibility and tensor-valued calibrations: Proof of Theorem 1.2

In (2.2) we minimize $u \mapsto F(u,D)$ under the constraint that $u\cdot e = u_0\cdot e$ on D. In view of Remark 2.4, it is natural to consider variations of the form $u+\varepsilon\varphi$, where $\varphi : D \to \langle e\rangle^\perp$. Writing $\varphi = \varphi_1\varepsilon_1 + \varphi_2\varepsilon_2$, with $\varphi_i : D\to\mathbb{R}$, $i=1,2$, we have that

$$\nabla\varphi = \varepsilon_1 \otimes \nabla\varphi_1 + \varepsilon_2\otimes\nabla\varphi_2,$$

i.e., $\nabla\varphi : D \to \langle e\rangle^\perp \otimes \mathbb{R}^2$.

Lemma 3.1. *Let D be an open set with Lipschitz boundary, let Γ_1, Γ_2 be Borel subsets of ∂D such that*

$$\partial D = \Gamma_1 \cup \Gamma_2, \qquad \mathcal{H}^1(\Gamma_1\cap\Gamma_2) = 0,$$

and let $z \in L^\infty(\Gamma_2;\langle e\rangle^\perp)$. Then the following two conditions are equivalent:

(i) *for every $\varphi \in BV(D;\langle e\rangle^\perp)$,*

$$0 \leqslant |D\varphi|(D) + \int_{\Gamma_1} |\varphi|\,d\mathcal{H}^1 + \int_{\Gamma_2} \varphi\cdot z\,d\mathcal{H}^1;$$

(ii) *there exists $M \in \mathcal{F}(D)$ such that*

$$M[\nu_D] = -z \quad \text{on } \Gamma_2.$$

Proof.

Step 1. We prove that (ii) implies (i). Since D has Lipschitz boundary, it suffices to verify (i) for $\varphi \in C^\infty(\overline{D};\langle e\rangle^\perp)$. By (1.6), for every such φ we have

$$|D\varphi|(D) = \int_D |\nabla\varphi|\,dx \geqslant \int_D \nabla\varphi : M\,dx = \int_{\partial D} M[\nu_D]\cdot\varphi\,d\mathcal{H}^1.$$

Since $\mathcal{H}^1(\Gamma_1\cap\Gamma_2)=0$, we have that

$$\int_{\partial D} M[\nu_D]\cdot\varphi\,d\mathcal{H}^1 = \int_{\Gamma_1} M[\nu_D]\cdot\varphi\,d\mathcal{H}^1 - \int_{\Gamma_2} z\cdot\varphi\,d\mathcal{H}^1 \geqslant -\int_{\Gamma_1}|\varphi|\,d\mathcal{H}^1 - \int_{\Gamma_2} z\cdot\varphi\,d\mathcal{H}^1,$$

where in the last inequality we have used (1.6), and so (i) follows.

Step 2. We prove that (i) implies (ii). Although this implication could be derived from general theorems in convex analysis (see [12]), for the convenience of the reader, we give a direct proof. For every $\varphi \in BV(D;\langle e\rangle^\perp)$ and $\mu \in X$ we define

$$\Psi(\varphi) := \int_{\Gamma_1} |\varphi|\, d\mathcal{H}^1 + \int_{\Gamma_2} \varphi \cdot z\, d\mathcal{H}^1,$$
$$\Phi(\varphi, \mu) := |D\varphi - \mu|(D) + \Psi(\varphi),$$

where $X := \mathcal{M}_b(D; \langle e\rangle^\perp \otimes \mathbb{R}^2)$ is the Banach space of all bounded Radon measures with values in $\langle e\rangle^\perp \otimes \mathbb{R}^2$.

Consider the function $f : X \to [-\infty, \infty]$ defined as

$$f(\mu) := \inf\bigl\{\Phi(\varphi, \mu):\ \varphi \in BV\bigl(D; \langle e\rangle^\perp\bigr)\bigr\}, \quad \mu \in X.$$

We claim that

$$\bigl|f(\mu)\bigr| \leqslant |\mu|(D) \quad \text{for all } \mu \in X. \tag{3.1}$$

Indeed, $f(\mu) \leqslant \Phi(0, \mu) = |\mu|(D)$, while, in view of (i), for every $\varphi \in BV(D; \langle e\rangle^\perp)$,

$$\Phi(\varphi, \mu) \geqslant |D\varphi|(D) - |\mu|(D) + \Psi(\varphi) \geqslant -|\mu|(D).$$

In particular, f is real-valued and locally bounded. Moreover, f is convex. Hence, the sub-differential of f at every $\mu \in X$ is nonempty (see, e.g., Theorems 4.43 and 4.51 in [13]). Therefore, since $f(0) = 0$, there exists μ^* in the topological dual X^* of X such that

$$\bigl\langle \mu^*, \mu\bigr\rangle \leqslant f(\mu) \quad \text{for all } \mu \in X. \tag{3.2}$$

Since the restriction of μ^* to $L^1(D; \langle e\rangle^\perp \otimes \mathbb{R}^2)$ is still a continuous linear functional, there exists $M \in L^\infty(D; \langle e\rangle^\perp \otimes \mathbb{R}^2)$ such that

$$\bigl\langle \mu^*, \mu\bigr\rangle = -\int_D M' : M\, dx,$$

whenever $\mu = M'\mathcal{L}^2 \lfloor D$ for some $M' \in L^1(D; \langle e\rangle^\perp \otimes \mathbb{R}^2)$. By (3.1) and (3.2), it follows that

$$\|M\|_{L^\infty(D;\langle e\rangle^\perp \otimes \mathbb{R}^2)} \leqslant 1.$$

Restricting (3.2) to measures of the form $\mu = \nabla\varphi\mathcal{L}^2\lfloor D$, where $\varphi \in C^\infty(\overline{D}; \langle e\rangle^\perp)$, we obtain

$$\begin{aligned} -\int_D \nabla\varphi : M\, dx &\leqslant f\bigl(\nabla\varphi\mathcal{L}^2\lfloor D\bigr) \leqslant \Phi\bigl(\varphi, \nabla\varphi\mathcal{L}^2\lfloor D\bigr) \\ &= \Psi(\varphi) \quad \text{for all } \varphi \in C^\infty\bigl(\overline{D}; \langle e\rangle^\perp\bigr). \end{aligned} \tag{3.3}$$

In particular, taking $\pm\varphi \in C_c^\infty(D; \langle e\rangle^\perp)$, we find that

$$\int_D \nabla\varphi : M\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty\bigl(D; \langle e\rangle^\perp\bigr),$$

and thus $M \in \mathcal{F}(D)$. Then by (1.7) and (3.3), we get

$$\int_{\Gamma_1} |\varphi|\, d\mathcal{H}^1 + \int_{\Gamma_2} \varphi \cdot z\, d\mathcal{H}^1 \geqslant -\int_{\partial D} M[\nu_D] \cdot \varphi\, d\mathcal{H}^1 \quad \text{for all } \varphi \in C^\infty\bigl(\overline{D}; \langle e\rangle^\perp\bigr).$$

In particular,

$$\int_{\Gamma_2} \bigl(M[\nu_D] + z\bigr) \cdot \varphi\, d\mathcal{H}^1 \geqslant 0$$

for every $\varphi \in C^\infty(\overline{D}; \langle e\rangle^\perp)$ such that $\varphi = 0$ on Γ_1. Thus, $M[\nu_D] = -z$ on Γ_2 and we conclude the proof of the lemma. □

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2.

Step 1. We start by proving that (i) is equivalent to (ii). Let f be the function defined in (2.4). Since $u_0 = \sum_{k=1}^N \xi_k 1_{\Omega_k}$, we have that

$$f(t) = \left|D(u_0 + t\varphi)\right|(D) + \sum_{k=1}^N \int_{\partial D\cap\Omega_k} \left|\xi_k - (u_0 + t\varphi)\right| d\mathcal{H}^1$$
$$= \left|D(u_0 + t\varphi)\right|(\Gamma \cap D) + \sum_{k=1}^N \left|D(u_0 + t\varphi)\right|(D \cap \Omega_k) + t \sum_{k=1}^N \int_{\partial D\cap\Omega_k} |\varphi|\, d\mathcal{H}^1.$$

If $\varphi_k \in BV(D \cap \Omega_k, \langle e\rangle^\perp)$ denotes the restriction of φ to the open set $\Omega_k \cap D$, then

$$f(t) = \left|D(u_0 + t\varphi)\right|(\Gamma \cap D) + t \sum_{k=1}^N \left(|D\varphi_k|(D \cap \Omega_k) + \int_{\partial D\cap\Omega_k} |\varphi_k|\, d\mathcal{H}^1 \right). \tag{3.4}$$

Note that

$$\left|D(u_0 + t\varphi)\right|(\Gamma \cap D) = \sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} \left|(\xi_k + t\varphi_k) - (\xi_h + t\varphi_h)\right| d\mathcal{H}^1, \tag{3.5}$$

where, we recall, $\Sigma_{k,h} = \partial\Omega_k \cap \partial\Omega_h$. Therefore, by (3.4) and (3.5),

$$f(t) = \sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} \left|(\xi_k + t\varphi_k) - (\xi_h + t\varphi_h)\right| d\mathcal{H}^1$$
$$+ t \sum_{k=1}^N \left(|D\varphi_k|(D \cap \Omega_k) + \int_{\partial D\cap\Omega_k} |\varphi_k|\, d\mathcal{H}^1 \right). \tag{3.6}$$

Hence, by Remark 2.4, condition (i) is equivalent to (2.5), which, by (3.6), becomes

$$0 \leqslant \sum_{k=1}^N |D\varphi_k|(D \cap \Omega_k) + \int_{\partial D\cap\Omega_k} |\varphi_k|\, d\mathcal{H}^1 + \sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} \frac{\xi_k - \xi_h}{|\xi_k - \xi_h|} \cdot (\varphi_k - \varphi_h)\, d\mathcal{H}^1. \tag{3.7}$$

By definition of z_k and since φ takes values in $\langle e\rangle^\perp$, we have that

$$1_{D\cap\Sigma_{k,h}} \frac{\xi_k - \xi_h}{|\xi_k - \xi_h|} \cdot (\varphi_k - \varphi_h) = 1_{D\cap\Sigma_{k,h}} (z_k \cdot \varphi_k + z_h \cdot \varphi_h).$$

Thus, after relabeling terms,

$$\sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} \frac{\xi_k - \xi_h}{|\xi_k - \xi_h|} \cdot (\varphi_k - \varphi_h)\, d\mathcal{H}^1$$
$$= \sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} (z_k \cdot \varphi_k + z_h \cdot \varphi_h)\, d\mathcal{H}^1$$
$$= \sum_{k=1}^{N-1} \sum_{h=k+1}^N \int_{D\cap\Sigma_{k,h}} z_k \cdot \varphi_k\, d\mathcal{H}^1 + \sum_{h=2}^N \sum_{k=1}^{h-1} \int_{D\cap\Sigma_{k,h}} z_h \cdot \varphi_h\, d\mathcal{H}^1$$
$$= \sum_{k=1}^N \int_{D\cap\Sigma_k} z_k \cdot \varphi_k\, d\mathcal{H}^1. \tag{3.8}$$

Finally, from (3.7) and (3.8) we conclude that (i) is equivalent to

$$0 \leqslant \sum_{k=1}^{N}\left(|D\varphi_k|(D\cap\Omega_k) + \int_{\partial D\cap\Omega_k} |\varphi_k|\,d\mathcal{H}^1 + \int_{D\cap\Sigma_k} z_k\cdot\varphi_k\,d\mathcal{H}^1\right)$$

for every $\varphi \in BV(D,\langle e\rangle^{\perp})$, and, in turn, this last inequality is equivalent to (ii).

Step 2. To prove that (ii) is equivalent to (iii), it suffices to apply Lemma 3.1 on $\Omega_k \cap D$ for every $k = 1,\dots,N$.

Step 3. Let us now assume that (1.4) holds and prove that if u_0 is a minimizer for F on D, then it is the unique minimizer. Let $u \in BV(D;\mathbb{R}^3)$ be such that $u\cdot e = u_0\cdot e$ on D. Then, we may write $u = u_0 + \varphi$, where $\varphi \in BV(D;\langle e\rangle^{\perp})$. By (3.6) with $t = 1$,

$$F(u,D) = \sum_{k=1}^{N-1}\sum_{h=k+1}^{N}\int_{D\cap\Sigma_{k,h}} \bigl|(\xi_k+\varphi_k)-(\xi_h+\varphi_h)\bigr|\,d\mathcal{H}^1 + \sum_{k=1}^{N}\left(|D\varphi_k|(D\cap\Omega_k) + \int_{\partial D\cap\Omega_k}|\varphi_k|\,d\mathcal{H}^1\right), \tag{3.9}$$

where, as before, φ_k is the restriction of φ to $\Omega_k \cap D$. Setting $\psi := u - (u_0\cdot e)e = u_0 - (u_0\cdot e)e + \varphi$, we have that $\psi \in BV(D;\langle e\rangle^{\perp})$, and in $\Omega_k \cap D$,

$$\psi = \psi_k := \varphi_k + \bigl(\xi_k - (\xi_k\cdot e)e\bigr), \quad k = 1,\dots,N.$$

Therefore,

$$F(u,D) = \sum_{k=1}^{N-1}\sum_{h=k+1}^{N}\int_{D\cap\Sigma_{k,h}} \bigl|\bigl((\xi_k\cdot e)e-(\xi_h\cdot e)e\bigr)+(\psi_k-\psi_h)\bigr|\,d\mathcal{H}^1$$
$$+ \sum_{k=1}^{N}\left(|D\psi_k|(D\cap\Omega_k) + \int_{\partial D\cap\Omega_k}\bigl|\psi_k - \bigl(\xi_k-(\xi_k\cdot e)e\bigr)\bigr|\,d\mathcal{H}^1\right).$$

By the Pythagorean theorem

$$\bigl|(\xi_k\cdot e-\xi_h\cdot e)e+(\psi_k-\psi_h)\bigr| = \sqrt{(\xi_k\cdot e-\xi_h\cdot e)^2+(\psi_k-\psi_h)^2},$$

where $(\xi_k\cdot e-\xi_h\cdot e)^2 > 0$ thanks to (1.4). Thus, the integrals

$$\int_{D\cap\Sigma_{k,h}} \bigl|(\xi_k\cdot e-\xi_h\cdot e)e+(\psi_k-\psi_h)\bigr|\,d\mathcal{H}^1$$

are strictly convex in the $\psi_k-\psi_h$ variables. In particular, if u_0 and u are both minimizers, we must have

$$\psi_k-\psi_h = \psi_{k,0}-\psi_{h,0} \quad \mathcal{H}^1\text{-a.e. on } \Sigma_{k,h},$$

i.e., $\varphi_k = \varphi_h$ $\mathcal{H}^1$-a.e. on $\Sigma_{k,h}$. Here $\psi_{k,0} = \xi_k - (\xi_k\cdot e)e$. Using the equality $F(u,D) = F(u_0,D)$ and the expression of F given in (3.9) we then obtain that

$$0 = \sum_{k=1}^{N}\left(|D\varphi_k|(D\cap\Omega_k) + \int_{\partial D\cap\Omega_k}|\varphi_k|\,d\mathcal{H}^1\right).$$

Hence, φ_k is constant on each connected component of $D\cap\Omega_k$ but this value must be zero, since $\varphi_k = 0$ $\mathcal{H}^1$-a.e. on $\partial D\cap\Omega_k$. We conclude that $u = u_0$. □

4. Reconstructibility on a δ-neighborhood of the contour: Proof of Theorems 1.8 and 1.10

We begin with the following preliminary result.

Lemma 4.1. *For $i = 1, 2$, let $D_i \subset \mathbb{R}^2$ be an open set with Lipschitz boundary and let $M_i \in \mathcal{F}(D_i)$. Let $D_1 \cap D_2 = \emptyset$, $\mathcal{H}^1(\partial D_1 \cap \partial D_2) > 0$, and define $M : D_1 \cup D_2 \to \langle e\rangle^\perp \otimes \mathbb{R}^2$ by*

$$M := M_i \quad \text{on } D_i.$$

If $M_1[\nu_{D_1}] = -M_2[\nu_{D_2}]$ on $\partial D_1 \cap \partial D_2$, then $M \in \mathcal{F}(D)$, where D is the interior of $\overline{D_1 \cup D_2}$.

Proof. If $\varphi \in C_c^\infty(D; \langle e\rangle^\perp)$, then $\varphi = 0$ on $\partial D_1 \Delta \partial D_2$ and $\varphi \in C^\infty(\overline{D_i}; \langle e\rangle^\perp)$ for $i = 1, 2$. Thus

$$\int_D M : \nabla\varphi\, dx = \sum_{i=1}^{2} \int_{D_i} M_i : \nabla\varphi\, dx = \sum_{i=1}^{2} \int_{\partial D_i} \varphi \cdot M_i[\nu_{D_i}]\, d\mathcal{H}^1$$
$$= \int_{\partial D_1 \cap \partial D_2} \sum_{i=1}^{2} \varphi \cdot M_i[\nu_{D_i}]\, d\mathcal{H}^1 = 0. \qquad \square$$

Before proving Theorem 1.8, we apply Theorem 1.2 to obtain an equivalent formulation of stable reconstructibility of $u_0 \in \mathrm{Im}(R)$ on $\Gamma(\delta)$ for some $\delta > 0$. We remark that for sufficiently small values of $\delta > 0$, the damaged region $\Gamma(\delta)$ satisfies the condition

$$\mathcal{H}^1\big(\Gamma \cap \partial\Gamma(\delta)\big) = 0,$$

which is one of the hypotheses of Theorem 1.2.

Proposition 4.2. *Let $u_0 \in \mathrm{Im}(R)$ satisfy* (1.4). *Then for all $\delta > 0$ sufficiently small, the following two statements are equivalent*:

(i) *u_0 is stably reconstructible over $\Gamma(\delta)$;*
(ii) *for every $k = 1, \ldots, N$ there exists $M_k \in \mathcal{F}(\Gamma(\delta) \cap \Omega_k)$ such that*

$$M_k[\nu_{\Gamma(\delta)\cap\Omega_k}] = -z_k \quad \text{on } \Sigma_k,$$

with the further property that

$$\|M_k\|_{L^\infty(\Gamma(\delta)\cap\Omega_k; \langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1.$$

Proof.

Step 1. We prove that (i) implies (ii). Let u_0 be stably reconstructible over $\Gamma(\delta)$ and let $\varepsilon_0 > 0$ be such that if $u = \sum_{k=1}^{N} \xi'_k 1_{\Omega_k}$ and $\max_{1\leqslant k\leqslant N} |\xi'_k - \xi_k| < \varepsilon_0$, then u is reconstructible over $\Gamma(\delta)$. We claim that there exists such an image u with the additional property that

$$z'_k = \lambda z_k, \quad k = 1, \ldots, N, \tag{4.1}$$

for some constant $\lambda > 1$, where z'_k denotes the compatibility vector field associated to u. To see this, note that by (1.4), $|P(\frac{\xi_k - \xi_h}{|\xi_k - \xi_h|})| < 1$ for all h, k with $h \neq k$ such that $\mathcal{H}^1(\Sigma_{k,h}) > 0$.

Assume first $\mathcal{H}^1(\Sigma_{k,h}) > 0$ for all h, k with $h \neq k$. The map

$$T_1 : \mathbb{R}^{3N} \to \big(\mathbb{R}^3\big)^\tau$$
$$\big(\xi'_1, \ldots, \xi'_N\big) \mapsto \big(\xi'_1 - \xi'_2, \ldots, \xi'_1 - \xi'_N, \xi'_2 - \xi'_3, \ldots, \xi'_2 - \xi'_N, \ldots, \xi'_{N-1} - \xi'_N\big)$$

where $(\mathbb{R}^3)^\tau := \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{\tau \text{ times}}$ and $\tau := \frac{N(N-1)}{2}$, is open by the surjective mapping theorem (see [5]), the map

$$T_2 : (\mathbb{R}^3)^\tau \setminus \{0\} \to (S^2)^\tau$$
$$(\eta'_1, \dots, \eta'_\tau) \mapsto \left(\frac{\eta'_1}{|\eta'_1|}, \dots, \frac{\eta'_\tau}{|\eta'_\tau|} \right)$$

is open, while the map

$$T_3 : (S^2)^\tau \to (\langle e \rangle^\perp)^\tau$$
$$(\varsigma'_1, \dots, \varsigma'_\tau) \mapsto (P(\varsigma'_1), \dots, P(\varsigma'_\tau))$$

is locally open in a neighborhood of each point $(\varsigma_1, \dots, \varsigma_\tau) \in (S^2)^\tau$ such that $|P(\varsigma_i)| < 1$ for all $i = 1, \dots, \tau$. Therefore

$$T_3 \circ T_2 \circ T_1 \big(B((\xi_1, \dots, \xi_N), \varepsilon_0)\big) \supset B\big(T_3 \circ T_2 \circ T_1((\xi_1, \dots, \xi_N)), \varepsilon_1\big)$$

for some $\varepsilon_1 > 0$. Let

$$1 < \lambda < 1 + \frac{\varepsilon_1}{1 + \|T_3 \circ T_2 \circ T_1((\xi_1, \dots, \xi_N))\|}.$$

Then there exists $(\xi'_1, \dots, \xi'_N) \in B((\xi_1, \dots, \xi_N), \varepsilon_0)$ such that

$$T_3 \circ T_2 \circ T_1 (\xi'_1, \dots, \xi'_N) = \lambda T_3 \circ T_2 \circ T_1 \big((\xi_1, \dots, \xi_N)\big).$$

This proves the claim in the case $\mathcal{H}^1(\Sigma_{k,h}) > 0$ for all h, k with $h \neq k$. The general case can be treated in an analogous way.

Define $u := \sum_{k=1}^N \xi'_k 1_{\Omega_k}$. Since u is reconstructible over $\Gamma(\delta)$, by Theorem 1.2, for every k we can find $M'_k \in \mathcal{F}(\Gamma(\delta) \cap \Omega_k)$ with $M'_k[\nu_{\Gamma(\delta)\cap\Omega_k}] = -\lambda z_k$ on Σ_k, where we used (4.1). Setting $M_k := (1/\lambda) M'_k$, it now follows that M_k satisfies the required properties.

Step 2. We prove that (ii) implies (i). By Theorem 1.2, and since each Ω_k has finitely many connected components, it suffices to show that for a fixed $k \in \{1, \dots, N\}$ and for a fixed connected component Σ of Σ_k, there is $0 < \varepsilon_0 < 1 - \|z_k\|_{L^\infty(\Sigma_k; \langle e \rangle^\perp \otimes \mathbb{R}^2)}$ such that for all $0 < \varepsilon < \varepsilon_0$ and for all $u = \sum_{j=1}^N \xi'_j 1_{\Omega_j}$ with $\max |\xi'_j - \xi_j| < \varepsilon$, there exists $M \in \mathcal{F}(\Sigma(\delta) \cap \Omega_k)$ satisfying

$$M[\nu_{\Sigma(\delta)\cap\Omega_k}] = -z' \quad \text{on } \Sigma, \tag{4.2}$$

where $z' \in L^\infty(\Sigma; \langle e \rangle^\perp)$ is the restriction of the compatibility vector fields z'_k of u to Σ. Here, and throughout the paper,

$$\Sigma(\delta) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Sigma) < \delta\}.$$

We divide the proof in two cases. Assume first that the curve Σ is open. Since Σ is Lipschitz, the geodesic distance

$$d_\Sigma(x_1, x_2) := \inf \left\{ \int_0^1 |\sigma'(t)|\, dt : \sigma \in W^{1,\infty}([0,1]; \Sigma),\ \sigma(0) = x_1,\ \sigma(1) = x_2 \right\}, \quad x_1, x_2 \in \Sigma, \tag{4.3}$$

on Σ is Lipschitz, and so there exists $L > 0$ such that

$$d_\Sigma(x_1, x_2) \leqslant L|x_1 - x_1| \quad \text{for all } x_1, x_2 \in \Sigma.$$

Using the fact that the projection P is Lipschitz, there exists $\varepsilon_0 > 0$ such that if $u = \sum_{j=1}^N \xi'_j 1_{\Omega_j}$ and $\max_{1 \leqslant j \leqslant N} |\xi'_j - \xi_j| < \varepsilon_0$, then

$$\|z_k - z'_k\|_{L^\infty(\Sigma; \langle e \rangle^\perp)} < \frac{1 - \|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k; \langle e \rangle^\perp \otimes \mathbb{R}^2)}}{\sqrt{2} L}.$$

Fix any such u, and let $f : \Sigma \to \langle e \rangle^\perp$ be determined (up to additive constants) by

$$\partial_{\tau_{\Sigma(\delta)\cap\Omega_k}} f = z_k - z' \quad \text{on } \Sigma. \tag{4.4}$$

Then

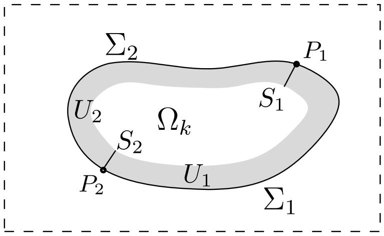

Fig. 5. The case in which Σ is a closed curve.

$$\begin{aligned}\left|f(x_1)-f(x_2)\right| &\leqslant \left\|z_k-z'\right\|_{L^\infty(\Sigma;\langle e\rangle^\perp)} d_\Sigma(x_1,x_2)\\ &\leqslant \frac{1-\|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}}{\sqrt{2}L} d_\Sigma(x_1,x_2)\\ &\leqslant \frac{1-\|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}}{\sqrt{2}}|x_1-x_2|\end{aligned}$$

for all $x_1, x_2 \in \Sigma$. Applying McShane's lemma with respect to the Euclidean distance yields a function $f \in W^{1,\infty}(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp)$ such that

$$\left|f(x_1)-f(x_2)\right| \leqslant \frac{1-\|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}}{\sqrt{2}}|x_1-x_2|$$

for all $x_1, x_2 \in \Sigma(\delta)\cap\Omega_k$. It follows that

$$\|\nabla f\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1-\|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}. \tag{4.5}$$

To conclude, setting $M^{(j)} := M_k^{(j)} - (\nabla f^{(j)})^\perp$, $j = 1, 2$, in view of (4.4) and (4.5), we have that $M \in \mathcal{F}(\Sigma(\delta)\cap\Omega_k)$ and (4.2) holds.

In the case in which the curve Σ is closed, choose two points P_1 and P_2 on Σ. These points determine the two arcs Σ_1 and Σ_2 with endpoints P_1 and P_2 and whose union is Σ. For every $i = 1, 2$, consider a segment S_i having one of the endpoints at P_i and the other on $\Omega_k \cap \partial[\Sigma(\delta)]$, so that $\Sigma(\delta)\cap\Omega_k \setminus (S_1\cup S_2)$ has two connected components U_1 and U_2. Set $\Sigma_i' := \Sigma_i \cup S_1 \cup S_2$, $i = 1, 2$ (see Fig. 5). For $i = 1, 2$, we now reason as in the previous case, considering the open curve Σ_i' and taking

$$\partial_{\tau_{U_i}} f_i := \begin{cases} z_k - z' & \text{on } \Sigma_i,\\ 0 & \text{on } S_1\cup S_2,\end{cases}$$

to find a function $f_i \in W^{1,\infty}(U_i;\langle e\rangle^\perp)$ such that

$$\|\nabla f_i\|_{L^\infty(U_i;\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1-\|M_k\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}.$$

We conclude by setting $M^{(j)} := M_k^{(j)} - (\nabla f_i^{(j)})^\perp$ in U_i, $i, j = 1, 2$. □

The following extension lemma will play an important role in the proof of Theorem 1.8. The context represented in Fig. 6 is the following: Consider a piecewise C^1 injective curve $\gamma : [r_0, r_1] \to \mathbb{R}^2$, with $\gamma \in C^1([r_0, s_1];\mathbb{R}^2) \cap C^1([s_1, r_1];\mathbb{R}^2)$ for some $s_1 \in (r_0, r_1)$ and such that $|\gamma'(t)| = 1$ for every $t \in [r_0, s_1)\cup(s_1, r_1]$. Define $\Sigma := \gamma([r_0, r_1])$.

Given $z_0, z_1 \in \langle e\rangle^\perp$, with $|z_0| < 1$, $|z_1| < 1$, in Lemma 4.3 we construct a divergence-free tensor field on a δ-neighborhood $\Sigma(\delta)$ of Σ, with normal trace taking equal to z_0 and z_1 on $\gamma([r_0, s_1))$ and $\gamma((s_1, r_1])$, respectively. We introduce some further notation. Setting

$$t_i(\delta) := \sup\left\{t > 0\colon\ \gamma(r_i) + s\gamma'(r_i)^\perp \in \Sigma(\delta) \text{ for all } s \in (0, t)\right\}, \quad i = 0, 1,$$

then $t_i(\delta) > 0$ and $p_{i,\delta} := \gamma(r_i) + t_i(\delta)\gamma'(r_i)^\perp \in \partial\Sigma(\delta)$. Consider the open segments $\Sigma_{i,\delta}$ with endpoints $\gamma(r_i)$ and $p_{i,\delta}$.

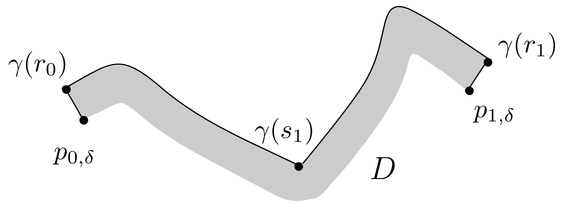

Fig. 6. The situation in Lemma 4.3.

Lemma 4.3. *Let Σ, $\Sigma_{0,\delta}$, $\Sigma_{1,\delta}$, and γ be as above. Let $z_0, z_1 \in \langle e\rangle^\perp$, with $|z_0| < 1$, $|z_1| < 1$, and let $v_0 := \gamma'(s_1^-)$, $v_1 := \gamma'(s_1^+)$ be such that*

$$\big[G\big(z_0^{(1)}, z_1^{(1)}, v_0, v_1\big)\big]^2 + \big[G\big(z_0^{(2)}, z_1^{(2)}, v_0, v_1\big)\big]^2 < 1. \tag{4.6}$$

In the case in which $v_0 = v_1$, assume further that $z_0 = z_1 =: z$. Then for all $\delta > 0$ sufficiently small there exists $M \in \mathcal{F}(\Sigma(\delta))$ such that

$$\|M\|_{L^\infty(\Sigma(\delta);\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1,$$
$$M[\nu] = 0 \quad \text{on } \Sigma_{0,\delta} \cup \Sigma_{1,\delta},$$
$$M[\nu] = -z_0 \quad \text{on } \gamma\big([r_0, s_1]\big),$$
$$M[\nu] = -z_1 \quad \text{on } \gamma\big([s_1, r_1]\big),$$

where ν is the unit normal to $\Sigma \cup \Sigma_{0,\delta} \cup \Sigma_{1,\delta}$ such that $\nu = -(\gamma')^\perp$ on Σ.

Proof.

Step 1. Endow $\Sigma(\delta)$ and Σ with their geodesic distances $d_{\Sigma(\delta)}$ and d_Σ, respectively. We claim that for every $c, \eta > 0$ sufficiently small there exists $\delta_0 > 0$ such that

$$d_\Sigma\big(\gamma(r), \gamma(s)\big) < (1+\eta) d_{\Sigma(\delta)}\big(\gamma(r) + t_1\gamma'(r)^\perp, \gamma(s) + t_2\gamma'(s)^\perp\big) \tag{4.7}$$

for all $r, s \in [r_0, s_1) \cup (s_1, r_1]$ and $0 \leqslant t_1, t_2 \leqslant \delta$ such that $|\gamma(r) + t_1\gamma'(r)^\perp - (\gamma(s) + t_2\gamma'(s)^\perp)| \geqslant c$ and for all $0 < \delta \leqslant \delta_0$. We begin by observing that

$$d_{\Sigma(\delta)}\big(\gamma(r) + t_1\gamma'(r)^\perp, \gamma(s) + t_2\gamma'(s)^\perp\big) \leqslant d_\Sigma\big(\gamma(r), \gamma(s)\big) + 2\delta \tag{4.8}$$

for all $r, s \in [r_0, s_1) \cup (s_1, r_1]$, $0 \leqslant t_1, t_2 \leqslant \delta$. To see this, observe that by adding to any Lipschitz curve admissible for $d_\Sigma(\gamma(r), \gamma(s))$ (see (4.3)) the segments joining $\gamma(r)$ to $\gamma(r) + t_1\gamma'(r)^\perp$ and $\gamma(s)$ to $\gamma(s) + t_2\gamma'(s)^\perp$, we obtain a curve admissible for $d_{\Sigma(\delta)}(\gamma(r) + t_1\gamma'(r)^\perp, \gamma(s) + t_2\gamma'(s)^\perp)$.

To prove (4.7), we argue by contradiction and assume that there exist $c, \eta > 0$, $\delta_k \to 0^+$, $0 \leqslant t_{1,k}, t_{2,k} \leqslant \delta_k$, and $r_k, s_k \in [r_0, s_1) \cup (s_1, r_1]$ such that, defining $x_k := \gamma(r_k) + t_{1,k}\gamma'(r_k)^\perp$ and $y_k := \gamma(s_k) + t_{2,k}\gamma'(s_k)^\perp$, we have $|x_k - y_k| \geqslant c$ and

$$d_\Sigma\big(\gamma(r_k), \gamma(s_k)\big) \geqslant (1+\eta)\, d_{\Sigma(\delta_k)}(x_k, y_k) \tag{4.9}$$

for all $k \in \mathbb{N}$. By extracting a subsequence, not relabelled, we may assume that $r_k \to r$ and $s_k \to s$, with $r, s \in [r_0, r_1]$, which implies that $x_k \to x := \gamma(r)$ and $y_k \to y := \gamma(s)$. By (4.3), for every $k \in \mathbb{N}$, there exists a Lipschitz curve $\sigma_k : [0, 1] \to \Sigma(\delta_k)$ joining x_k and y_k such that $|\sigma_k'(t)| = \operatorname{length}(\sigma_k)$ for $\mathcal{L}^1$-a.e. $t \in [0, 1]$ and

$$\operatorname{length}(\sigma_k) \leqslant d_{\Sigma(\delta_k)}(x_k, y_k) + \frac{1}{k} \leqslant d_\Sigma\big(\gamma(r_k), \gamma(s_k)\big) + \frac{1}{k} + 2\delta_k,$$

where in the last inequality we used (4.8). Therefore, $\{\sigma_k\}$ is a sequence of equi-Lipschitz curves, and so using the Ascoli–Arzelï theorem, there exist a subsequence, not relabeled, and a Lipschitz curve $\sigma : [0, 1] \to \Sigma$ joining x and y such that $\sigma_k \to \sigma$ uniformly, and

$$d_\Sigma(x,y) \leqslant \text{length}(\sigma) \leqslant \liminf_{k\to\infty} \text{length}(\sigma_k) \leqslant \liminf_{k\to\infty} d_{\Sigma(\delta_k)}(x_k,y_k)$$
$$\leqslant \limsup_{k\to\infty} d_{\Sigma(\delta_k)}(x_k,y_k) \leqslant \lim_{k\to\infty} d_\Sigma\big(\gamma(r_k),\gamma(s_k)\big) = d_\Sigma(x,y).$$

Hence, $d_{\Sigma(\delta_k)}(x_k,y_k) \to d_\Sigma(x,y)$ and, in turn, by (4.9),

$$d_\Sigma(x,y) \geqslant (1+\eta)\, d_\Sigma(x,y).$$

Therefore, $d_\Sigma(x,y)=0$, which is impossible since $d_\Sigma(x,y) \geqslant |x-y| \geqslant c$.

Step 2. We only prove the case in which $v_0 \neq v_1$. The proof of the case $v_0 = v_1$ follows a similar argument. Set

$$G^{(j)}(v,w) := G\big(z_0^{(j)}, z_1^{(j)}, v, w\big)$$

for all linearly independent $v, w \in S^1$, $j=1,2$, where G is the function defined by (1.13). Note that by condition (4.6), there exists $\sigma \in (0,1)$ such that

$$\big(G^{(1)}(v_0,v_1)+\sigma\big)^2 + \big(G^{(2)}(v_0,v_1)+\sigma\big)^2 < 1.$$

Since G is a continuous function, there exists $\varepsilon_0 > 0$ such that

$$\big|G^{(j)}(w_1,w_2) - G^{(j)}(v_0,v_1)\big| \leqslant \sigma \tag{4.10}$$

for all $w_1, w_2 \in S^1$ with $w_1 \in B(v_0,\varepsilon_0)$ and $w_2 \in B(v_1,\varepsilon_0)$, $j=1,2$.

Using the fact that γ is piecewise C^1, there exists $c>0$ such that if $t \in [r_0,s_1)$ and $\gamma(t) \in B(\gamma(s_1),\frac{c}{2})$, then

$$\gamma'(t) \in B(v_0,\varepsilon_0), \tag{4.11}$$

while if $t \in (s_1,r_1]$ and $\gamma(t) \in B(\gamma(s_1),\frac{c}{2})$, then

$$\gamma'(t) \in B(v_1,\varepsilon_0). \tag{4.12}$$

Let $\eta > 0$ be so small and $m > 1$ so large that

$$\frac{(1+\eta)}{\sqrt{1-\frac{1}{m^2}}}\sqrt{\big(G^{(1)}(v_0,v_1)+\sigma\big)^2 + \big(G^{(2)}(v_0,v_1)+\sigma\big)^2} < 1. \tag{4.13}$$

By taking c smaller, if necessary, and using again the fact that γ is piecewise C^1, we may also assume that if $x, y \in \gamma([r_0,s_1))$ or $x,y \in \gamma((s_1,r_1])$ with $|x-y|<c$, then

$$d_\Sigma(x,y) \leqslant (1+\eta)|x-y| \leqslant (1+\eta)\, d_{\Sigma(\delta)}(x,y), \tag{4.14}$$

and also that if $x \in \Sigma \cap B(\gamma(r_0),2c)$, then

$$\big|\gamma'(r_0)^\perp \cdot \big(x-\gamma(r_0)\big)\big| \leqslant \frac{1}{m}\big|x-\gamma(r_0)\big|, \tag{4.15}$$

while if $x \in \Sigma \cap B(\gamma(r_1),2c)$, then

$$\big|\gamma'(r_1)^\perp \cdot \big(x-\gamma(r_1)\big)\big| \leqslant \frac{1}{m}\big|x-\gamma(r_1)\big|.$$

Let $\delta_0 > 0$ be given by Step 1 corresponding to the choice of η and c, and define $f : Y_\delta \to \langle e\rangle^\perp$ as

$$f(x) := \begin{cases} -(s-r_0)z_0 & \text{if } x=\gamma(s) \text{ for } s\in[r_0,s_1],\\ -(s_1-r_0)z_0-(s-s_1)z_1 & \text{if } x=\gamma(s) \text{ for } s\in[s_1,r_1],\\ 0 & \text{if } x\in\Sigma_{0,\delta},\\ -(s_1-r_0)z_0-(r_1-s_1)z_1 & \text{if } x\in\Sigma_{1,\delta}, \end{cases}$$

where $Y_\delta := \Sigma \cup \Sigma_{0,\delta} \cup \Sigma_{1,\delta}$ so that $Y_\delta \subset \Sigma(\delta)$. In view of Proposition 1.4, it suffices to prove that for all $\delta > 0$ sufficiently small there exists a function $\bar f \in W^{1,\infty}(\Sigma(\delta);\langle e\rangle^\perp)$ such that $\bar f = f$ on Y_δ and

$$\big|\nabla \bar f(x)\big| < 1 \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Sigma(\delta). \tag{4.16}$$

Fix $x, y \in Y_\delta$. We distinguish several cases.

Case 1. If both x and y belong to Σ, then

$$\begin{aligned}\left|f^{(j)}(x)-f^{(j)}(y)\right| &\leqslant \max\left\{\left|z_0^{(j)}\right|,\left|z_1^{(j)}\right|\right\} d_\Sigma(x,y)\\ &\leqslant G^{(j)}(v_0,v_1)\, d_\Sigma(x,y),\end{aligned} \tag{4.17}$$

where $f^{(j)}: Y_\delta \to \mathbb{R}$ denotes the jth component of f for $j=1,2$ and we used the fact that $G(r,t,v,w)\geqslant \max\{|r|,|t|\}$ by (A.3).

Subcase 1a. If $|x-y|\geqslant c$, then by Step 1 (with $t_1=t_2=0$), we have for all $0<\delta\leqslant\delta_0$,

$$\left|f^{(j)}(x)-f^{(j)}(y)\right| \leqslant G^{(j)}(v_0,v_1)\, d_\Sigma(x,y) \leqslant (1+\eta) G^{(j)}(v_0,v_1)\, d_{\Sigma(\delta)}(x,y). \tag{4.18}$$

Subcase 1b. If $x,y\in\Sigma$ and $|x-y|<c$, then there are two cases. If $x,y\in\gamma([r_0,s_1))$ or $x,y\in\gamma((s_1,r_1])$, then we obtain (4.18) for all $0<\delta\leqslant\delta_0$, with the only change that the last inequality now follows from (4.14). Assume next that $x\in\gamma([r_0,s_1))\cap B(\gamma(s_1),\frac{c}{2})$ and $y\in\gamma((s_1,r_1])\cap B(\gamma(s_1),\frac{c}{2})$, with $x=\gamma(r)$ and $y=\gamma(s)$. By the mean value theorem, there exist $t_1\in[r_0,s_1)$ and $t_2\in(s_1,r_1]$ such that

$$\begin{aligned} y-x &= \int_r^{s_1}\gamma'(t)\,dt + \int_{s_1}^{s}\gamma'(t)\,dt\\ &= (s_1-r)\gamma'(t_1)+(s-s_1)\gamma'(t_2)\\ &= d_\Sigma\big(x,\gamma(s_1)\big)\gamma'(t_1)+d_\Sigma\big(y,\gamma(s_1)\big)\gamma'(t_2).\end{aligned}$$

In this case, by (A.1),

$$\begin{aligned}\left|f^{(j)}(x)-f^{(j)}(y)\right| &= \left|d_\Sigma\big(x,\gamma(s_1)\big)z_0^{(j)}+d_\Sigma\big(y,\gamma(s_1)\big)z_1^{(j)}\right|\\ &\leqslant G^{(j)}\big(\gamma'(t_1),\gamma'(t_2)\big)\left|d_\Sigma\big(x,\gamma(s_1)\big)\gamma'(t_1)+d_\Sigma\big(y,\gamma(s_1)\big)\gamma'(t_2)\right|\\ &= G^{(j)}\big(\gamma'(t_1),\gamma'(t_2)\big)|y-x| \leqslant G^{(j)}\big(\gamma'(t_1),\gamma'(t_2)\big)\, d_{\Sigma(\delta)}(x,y),\end{aligned}$$

and so by (4.10), (4.11), and (4.12),

$$\left|f^{(j)}(x)-f^{(j)}(y)\right| \leqslant \big(G^{(j)}(v_0,v_1)+\sigma\big)\, d_{\Sigma(\delta)}(x,y).$$

Case 2. Next we consider the case in which at least one point, say y, belongs to $\Sigma_{0,\delta}\cup\Sigma_{1,\delta}$. We only treat the case in which $y\in\Sigma_{0,\delta}$ and $x\in\Sigma$, the other case being analogous. By the definition of f and (4.17), we have

$$\left|f^{(j)}(x)-f^{(j)}(y)\right| = \left|f^{(j)}(x)-f^{(j)}\big(\gamma(r_0)\big)\right| \leqslant G^{(j)}(v_0,v_1)\, d_\Sigma\big(x,\gamma(r_0)\big).$$

Subcase 2a. If $|x-y|\geqslant c$, then by Step 1 (with $t_2=0$), we deduce that

$$\left|f^{(j)}(x)-f^{(j)}(y)\right| \leqslant (1+\eta)G^{(j)}(v_0,v_1)\, d_{\Sigma(\delta)}(x,y).$$

Subcase 2b. If $|x-y|<c$, then taking $\delta_0<c$, we have that

$$\left|x-\gamma(r_0)\right| \leqslant \left|y-\gamma(r_0)\right| + |x-y| < 2c.$$

By (4.14) and (4.17), we have that

$$\begin{aligned}\left|f^{(j)}(x)-f^{(j)}(y)\right| &\leqslant G^{(j)}(v_0,v_1)\, d_\Sigma\big(x,\gamma(r_0)\big)\\ &\leqslant (1+\eta)G^{(j)}(v_0,v_1)\left|x-\gamma(r_0)\right|.\end{aligned}$$

Since $y=\gamma(r_0)+t\gamma'(r_0)^\perp$ for some $t\in(0,\delta)$, by (4.15),

$$\begin{aligned}|y-x|^2 &= \left|x-\gamma(r_0)\right|^2+t^2+2t\gamma'(r_0)^\perp\cdot\big(x-\gamma(r_0)\big)\\ &\geqslant \left|x-\gamma(r_0)\right|^2+t^2-\frac{2}{m}t\left|x-\gamma(r_0)\right|\\ &\geqslant \left(1-\frac{1}{m^2}\right)\left|x-\gamma(r_0)\right|^2,\end{aligned}$$

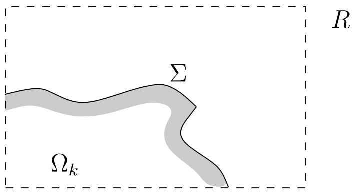

Fig. 7. The shaded region is $\Sigma(\delta)\cap\Omega_k$.

and thus,

$$\left|f^{(j)}(x)-f^{(j)}(y)\right|\leqslant\frac{(1+\eta)}{\sqrt{1-\frac{1}{m^2}}}G^{(j)}(v_0,v_1)|x-y|\leqslant\frac{(1+\eta)}{\sqrt{1-\frac{1}{m^2}}}G^{(j)}(v_0,v_1)\,d_{\Sigma(\delta)}(x,y).$$

In conclusion, collecting all the previous estimates, we deduce that

$$\left|f^{(j)}(x)-f^{(j)}(y)\right|\leqslant\frac{(1+\eta)}{\sqrt{1-\frac{1}{m^2}}}\big(G^{(j)}(v_0,v_1)+\sigma\big)\,d_{\Sigma(\delta)}(x,y) \tag{4.19}$$

for all $x, y\in Y_\delta$. We now apply McShane's lemma to extend each component $f^{(j)}$ to a function $\bar f^{(j)}:\Sigma(\delta)\to\mathbb{R}$ still satisfying (4.19) for all $x,y\in\Sigma(\delta)$. In turn,

$$\left|\bar f(x)-\bar f(y)\right|\leqslant\frac{(1+\eta)}{\sqrt{1-\frac{1}{m^2}}}\sqrt{\big(G^{(1)}(v_0,v_1)+\sigma\big)^2+\big(G^{(2)}(v_0,v_1)+\sigma\big)^2}\,d_{\Sigma(\delta)}(x,y)$$

for all $x,y\in\Sigma(\delta)$. Property (4.16) now follows from the previous inequality, (4.13), and the fact that $d_{\Sigma(\delta)}$ coincides with the Euclidean distance for sufficiently close pairs of points. □

We now turn to the proof of Theorem 1.8.

Proof of Theorem 1.8. In order to prove that u_0 is stably reconstructible over $\Gamma(\delta)$, by Proposition 4.2 it suffices to show that for every fixed k and for every connected component Σ of Σ_k, which is the support of a piecewise C^1 curve γ parameterized by its arc-length, there exists $M\in\mathcal{F}(\Sigma(\delta)\cap\Omega_k)$ such that

$$\|M\|_{L^\infty(\Sigma(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)}<1,$$
$$M[\nu_{\Sigma(\delta)\cap\Omega_k}]=-z_k\quad\text{on }\Sigma.$$

As usual, we orient γ in such a way that $(\gamma')^\perp$ is the inner unit normal to Ω_k. We denote by $\mathcal{S}$ the singular set of γ, as defined in (1.11). There are two possibilities: Either γ is closed with image contained in R or its endpoints belong to ∂R. We only consider the latter situation (see Fig. 7), since the former can be treated similarly. Fix a finite number of points in $[0,\mathcal{H}^1(\Sigma)]\setminus\mathcal{S}$,

$$r_0:=0<r_1<\cdots<r_{n+1}:=\mathcal{H}^1(\Sigma),$$

such that for each $i=0,\dots,n$, the following alternative holds: Either

$$(r_i,r_{i+1})\cap\mathcal{S}\text{ is a singleton}$$

or

$$(r_i,r_{i+1})\cap\mathcal{S}=\emptyset\quad\text{and}\quad z_k\text{ is constant on }\gamma\big((r_i,r_{i+1})\big),$$

with the latter occurring in (r_0,r_1) and (r_n,r_{n+1}). For every $i=1,\dots,n-1$, we define

$$\Sigma_i'=\big\{\gamma(r_i)+t\gamma'(r_i)^\perp:\ 0<t<t_i(\delta)\big\},$$

where $t_i(\delta):=\sup\{t>0:\ \gamma(r_i)+s\gamma'(r_i)^\perp\in\Sigma(\delta)\cap\Omega_k\text{ for all }s\in(0,t)\}$.

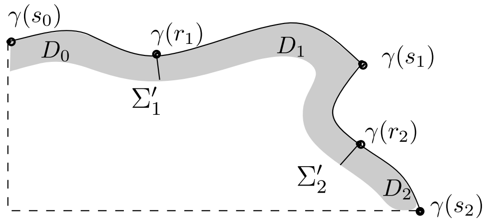

Fig. 8. The field M is obtained by gluing together the constructions of Lemma 4.3 performed on each D_i.

For every $i = 1, \dots, n-1$ we apply Lemma 4.3 with Σ, $\Sigma_{0,\delta}$, $\Sigma_{1,\delta}$, and γ replaced by $\hat{\Sigma}_i := \gamma([r_i, r_{i+1}])$, Σ'_i, Σ'_{i+1}, and $\gamma|_{[r_i,r_{i+1}]}$, respectively, to find $M_i \in \mathcal{F}(\hat{\Sigma}_i(\delta))$ such that

$$\|M_i\|_{L^\infty(\hat{\Sigma}_i(\delta);\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1,$$
$$M_i[\nu] = 0 \quad \text{on } \Sigma'_i \cup \Sigma'_{i+1},$$
$$M_i[\nu] = -z_k \quad \text{on } \gamma([r_i, r_{i+1}]).$$

Since z_k is constant on $\hat{\Sigma}_0 := \gamma((r_0, r_1))$ and on $\hat{\Sigma}_n := \gamma((r_n, r_{n+1}))$, reasoning as in Lemma 4.3, yields the existence of tensor fields $M_0 \in \mathcal{F}(\hat{\Sigma}_0(\delta))$ and $M_n \in \mathcal{F}(\hat{\Sigma}_n(\delta))$ such that

$$\|M_0\|_{L^\infty(\hat{\Sigma}_0(\delta);\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1, \qquad \|M_n\|_{L^\infty(\hat{\Sigma}_n(\delta);\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1,$$
$$M_0[\nu] = 0 \quad \text{on } \Sigma'_1, \qquad M_n[\nu] = 0 \quad \text{on } \Sigma'_n,$$
$$M_0[\nu] = -z_k \quad \text{on } \gamma([r_0, r_1]), \qquad M_n[\nu] = -z_k \quad \text{on } \gamma([r_{n-1}, r_n]).$$

For every $i = 1, \dots, n-1$, we define D_i to be the connected component of $\hat{\Sigma}_i(\delta) \setminus (\Sigma'_i \cup \Sigma'_{i+1} \cup \hat{\Sigma}_i)$ contained in Ω_k. In this way, $\Sigma(\delta) \cap \Omega_k \setminus \bigcup_{i=1}^{n-1} D_i$ has two connected components D_0 and D_n, with $\partial D_0 \cap \Sigma = \gamma([r_0, r_1])$ and $\partial D_n \cap \Sigma = \gamma([r_n, r_{n+1}])$ (cf. Fig. 8). To conclude, it suffices to define $M = M_i$ in D_i and to apply Lemma 4.1. □

Finally, we prove Theorem 1.10.

Proof of Theorem 1.10.

Step 1. We prove (i). Let $u_0 \in \mathrm{Im}(R)$ be reconstructible over $\Gamma(\delta)$ for some $\delta > 0$ (we may assume δ so small that the δ-neighborhoods of each connected component of Σ_k are all disjoint). By Theorem 1.2 and Remark 1.3, there exists a tensor field $M \in \mathcal{F}(\Gamma(\delta))$ such that

$$M[\nu_{\Omega_k}] = -z_k \quad \text{on } \Sigma_k,$$

where z_k is the compatibility vector field of u_0. Fix a connected component Σ of Σ_k, and let γ and $\mathcal{S}$ be as in the proof of the previous theorem. Let $s_0 \in \mathcal{S}$ and set

$$z^- := z_k\big(\gamma(s_0^-)\big), \qquad z^+ := z_k\big(\gamma(s_0^+)\big), \qquad v^- := \gamma'\big(s_0^-\big), \qquad v^+ := \gamma'\big(s_0^+\big).$$

By Proposition 1.4 there exists a Lipschitz function $f : B(\gamma(s_0), \varepsilon) \to \langle e\rangle^\perp$ such that

$$\big|\nabla f(x)\big| \leqslant 1 \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in B\big(\gamma(s_0), \varepsilon\big),$$
$$f\big(\gamma(s_0)\big) = 0 \quad \text{and} \quad -\big(\nabla f^{(j)}\big)^\perp = M^{(j)}, \quad j = 1, 2, \text{ in } B\big(\gamma(s_0), \varepsilon\big),$$

where $0 < \varepsilon < \delta$ is chosen so small that z_k takes only the values z^- and z^+ on $\Sigma \cap B(\gamma(s_0), \varepsilon)$. In particular, we have

$$f\big(\gamma(t)\big) = \begin{cases} -(t - s_0)z^- & \text{if } t < s_0 \text{ and } \gamma(t) \in B(\gamma(s_0), \varepsilon), \\ -(t - s_0)z^+ & \text{if } t > s_0 \text{ and } \gamma(t) \in B(\gamma(s_0), \varepsilon). \end{cases} \tag{4.20}$$

We now consider the sequence of Lipschitz functions

$$f_n(x) := nf\left(\frac{x}{n} + \gamma(s_0)\right).$$

Since $f_n(0)=0$ and $\|\nabla f_n\|_{L^\infty(B(0,n\varepsilon);\langle e\rangle^\perp\otimes\mathbb{R}^2)}=\|\nabla f\|_{L^\infty(B(\gamma(s_0),\varepsilon);\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant 1$, we may extract a subsequence (not relabelled) such that

$$f_n \overset{*}{\rightharpoonup} f_\infty \quad \text{in } W^{1,\infty}_{\mathrm{loc}}\big(\mathbb{R}^2,\langle e\rangle^\perp\big) \tag{4.21}$$

and

$$\|\nabla f_\infty\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant 1. \tag{4.22}$$

Let

$$C:=\big\{x\in\mathbb{R}^2\colon\ x=-tv^-,\ t\geqslant 0\big\}\cup\big\{x\in\mathbb{R}^2\colon\ x=tv^+,\ t\geqslant 0\big\}.$$

Let $y\in C$ be of the form $y=tv^+$ for some $t>0$. Then, using (4.20), the definition of f_n, and the Lipschitz continuity of f, we have

$$\begin{aligned}\big|f_n\big(tv^+\big)+tz^+\big|&=\left|nf\left(\frac{tv^+}{n}+\gamma(s_0)\right)-nf\left(\gamma\left(\frac{t}{n}+s_0\right)\right)\right|\\&\leqslant \mathrm{Lip}(f)\left|tv^+ +n\gamma(s_0)-n\gamma\left(\frac{t}{n}+s_0\right)\right|\\&\leqslant no\left(\frac{1}{n}\right)\to 0\quad \text{as } n\to\infty,\end{aligned} \tag{4.23}$$

where in the last equality we have used the fact that $\gamma(\frac{t}{n}+s_0)=\gamma(s_0)+v^+\frac{t}{n}+o(\frac{1}{n})$. Similarly, we can prove that

$$\big|f_n\big(-tv^-\big)-tz^-\big|\to 0\quad \text{as } n\to\infty \text{ for all } t>0. \tag{4.24}$$

Let $B\in\langle e\rangle^\perp\otimes\mathbb{R}^2$ be the tensor uniquely determined by

$$B\big[v^-\big]=-z^-\quad\text{and}\quad B\big[v^+\big]=-z^+.$$

It follows from (4.21), (4.23), and (4.24) that

$$f_\infty(y)=B[y]\quad \text{for all } y\in C.$$

Hence, the function f_∞ is admissible for the minimization problem (1.17) (with z^- and z^+ in place of z_0 of z_1). In view of (4.22), this concludes the proof of part (i).

Step 2. To prove (ii), assume that u_0 is stably reconstructible over $\Gamma(\delta)$, for some $\delta>0$. We claim that the inequality (1.17) is strict. Let $\varepsilon_0>0$ be such that if $u=\sum_{k=1}^N \xi_k' 1_{\Omega_k}$ and $\max_{1\leqslant k\leqslant N}|\xi_k'-\xi_k|<\varepsilon_0$, then u is also reconstructible over $\Gamma(\delta)$. Reasoning as in Step 1 of Proposition 4.2, we may find such an image u with the additional property that

$$z_k'=\lambda z_k,\quad k=1,\dots,N,$$

for some fixed constant $\lambda>1$, where z_k' denotes the compatibility vector field associated to u. As in Step 1 of the present proof, fix a connected component Σ of Σ_k and let $s_0\in\mathcal{S}$. Defining

$$\big(z'\big)^-:=z_k'\big(\gamma\big(s_0^-\big)\big)=\lambda z^-,\qquad \big(z'\big)^+:=z_k'\big(\gamma\big(s_0^+\big)\big)=\lambda z^+,$$

by part (i) of the theorem there exists $f_\infty\in W^{1,\infty}(\mathbb{R}^2,\langle e\rangle^\perp)$ such that $\|\nabla f_\infty\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant 1$ and

$$f_\infty\big(-tv^-\big)=t\big(z'\big)^-=t\lambda z^-,\qquad f_\infty\big(tv^+\big)=-t\big(z'\big)^+=-t\lambda z^+.$$

Hence the function $\frac{1}{\lambda}f_\infty$ is admissible for the minimization problem (1.17) and

$$\left\|\frac{1}{\lambda}\nabla f_\infty\right\|_{L^\infty(\mathbb{R}^2;\langle e\rangle^\perp\otimes\mathbb{R}^2)}\leqslant\frac{1}{\lambda}<1,$$

which implies that the inequality (1.17) is strict. □

5. Reconstructibility from small uniformly distributed undamaged regions: Proof of Theorems 1.12, 1.13, and 1.14

In this section we prove Theorems 1.12, 1.13, and 1.14.

Proof of Theorem 1.12. The existence of u_ε may proved as in the proof of Proposition 2.1. Moreover, by a truncation argument and by the minimality of u_ε, we have that $\sup_\varepsilon \|u_\varepsilon\|_\infty < \infty$. Again by minimality,

$$|Du_\varepsilon|(R) \leqslant |Du_0|(R) \quad \text{for all } \varepsilon > 0. \tag{5.1}$$

Hence, for every subsequence $\varepsilon_n \to 0$ we may extract a further subsequence (not relabelled) such that $u_{\varepsilon_n} \to v$ in $L^1(R;\mathbb{R}^3)$ for some $v \in BV(R;\mathbb{R}^3)$. We claim that $v = u_0$. Assume by contradiction that this is not the case. Then by Egoroff's theorem we may find $\eta > 0$ and a compact set $K \subset R$ of positive $\mathcal{L}^2$-measure such that

$$K \subset \bigl\{x \in R\colon \bigl|v(x) - u_0(x)\bigr| > \eta\bigr\} \quad \text{and} \quad u_{\varepsilon_n} \to v \quad \text{uniformly on } K. \tag{5.2}$$

Let $x_0 \in R$ be a point of density one for K and fix $0 < \delta < \operatorname{dist}(x_0, \partial R)$ so small that

$$\frac{\mathcal{L}^2(Q(x_0,\delta) \cap K)}{\delta^2} > \frac{3}{4}.$$

Setting $A_n := \{x \in \varepsilon_n\mathbb{Z}^2\colon Q(x,\varepsilon_n) \subset Q(x_0,\delta)\}$ and $Q_{\varepsilon_n,\delta} := \bigcup_{x\in A_n} Q(x,\varepsilon_n)$, we have

$$\operatorname{card}(A_n) \leqslant \frac{\delta^2}{\varepsilon_n^2} \quad \text{and} \quad \frac{\mathcal{L}^2(Q_{\varepsilon_n,\delta} \cap K)}{\delta^2} > \frac{3}{4} \tag{5.3}$$

for n large enough. Denoting $A_n^+ := \{x \in A_n\colon \mathcal{L}^2(Q(x,\varepsilon_n) \cap K) > \frac{1}{2}\mathcal{L}^2(Q(x,\varepsilon_n))\}$ and $A_n^- := \{x \in A_n\colon \mathcal{L}^2(Q(x,\varepsilon_n) \cap K) \leqslant \frac{1}{2}\mathcal{L}^2(Q(x,\varepsilon_n))\}$, we claim that

$$\operatorname{card}\bigl(A_n^+\bigr) > \frac{1}{4}\frac{\delta^2}{\varepsilon_n^2}. \tag{5.4}$$

Indeed, if not, we would have $\operatorname{card}(A_n^+) \leqslant \frac{1}{4}\frac{\delta^2}{\varepsilon_n^2}$ and, by the first inequality in (5.3), $\operatorname{card}(A_n^-) \leqslant \frac{\delta^2}{\varepsilon_n^2}$. Hence,

$$\mathcal{L}^2(Q_{\varepsilon_n,\delta} \cap K) \leqslant \sum_{x\in A_n} \mathcal{L}^2\bigl(Q(x,\varepsilon_n) \cap K\bigr) \leqslant \varepsilon_n^2 \operatorname{card}\bigl(A_n^+\bigr) + \frac{1}{2}\varepsilon_n^2 \operatorname{card}\bigl(A_n^-\bigr) \leqslant \frac{1}{4}\delta^2 + \frac{1}{2}\delta^2 = \frac{3}{4}\delta^2,$$

which contradicts the second inequality in (5.3).

Fix $x \in A_n^+$ and recall that by (5.2),

$$|u_{\varepsilon_n} - v| > \frac{\eta}{2} \quad \text{on } Q(x,\varepsilon_n) \cap K \tag{5.5}$$

for all $n \geqslant n_0$ for a suitable $n_0 \in \mathbb{N}$ independent of x. Let (ϱ, θ) denote the polar coordinates centered at x and for $\theta \in (0, 2\pi)$ and $n \in \mathbb{N}$ define

$$g_{\theta,n}(\rho) := u_{\varepsilon_n}\bigl(x + \rho(\cos\theta, \sin\theta)\bigr), \qquad g_\theta(\rho) := v\bigl(x + \rho(\cos\theta, \sin\theta)\bigr), \quad \rho \in I_\theta,$$

where $I_\theta := \{\rho > 0\colon x + \rho(\cos\theta, \sin\theta) \in R\}$. Let $T \subset (0, 2\pi)$ be the set of all θ's such that the set

$$U_\theta := \left\{\rho \in I_\theta\colon \bigl|g_{\theta,n}(\rho) - g_\theta(\rho)\bigr| > \frac{\eta}{2}\right\}$$

has positive $\mathcal{L}^1$-measure. By (5.5) and the definition of A_n^+, we have

$$\frac{1}{2}\mathcal{L}^2\bigl(Q(x,\varepsilon_n)\bigr) < \mathcal{L}^2\bigl(Q(x,\varepsilon_n) \cap K\bigr) \leqslant \mathcal{L}^2\left(Q(x,\varepsilon_n) \cap \left\{|u_{\varepsilon_n} - v| > \frac{\eta}{2}\right\}\right)$$
$$= \int_T\int_{U_\theta} \varrho\, d\varrho\, d\theta \leqslant \int_T \int_0^{\varepsilon_n\sqrt{2}} \varrho\, d\varrho\, d\theta = \mathcal{L}^1(T)\varepsilon_n^2,$$

which implies that

$$\mathcal{L}^1(T) > \frac{1}{2}.$$

By (5.5) and the fact that u_{ε_n} and v coincide in $Q(x, \omega(\varepsilon_n))$ (see (1.21)), for all $\theta \in T$ we have (see Sections 3.2 and 3.11 in [1])

$$\left|D(g_{\theta,n} - g_\theta)\right|\left(\frac{\omega(\varepsilon_n)}{2}, \ell_\theta\right) > \frac{\eta}{2},$$

where ℓ_θ denotes the length of the set $\{\rho > 0\colon\ x + \rho(\cos\theta, \sin\theta) \in Q(x, \varepsilon_n)\}$. Hence,

$$\begin{aligned}
\left|D(u_{\varepsilon_n} - v)\right|\left(Q(x, \varepsilon_n)\right) &\geqslant \int_T d\theta \int_{\frac{\omega(\varepsilon_n)}{2}}^{\ell_\theta} \varrho\, d\left|D(g_{\theta,n} - g_\theta)\right| \\
&\geqslant \frac{\omega(\varepsilon_n)}{2} \int_T \left|D(g_{\theta,n} - g_\theta)\right|\left(\frac{\omega(\varepsilon_n)}{2}, \ell_\theta\right) d\theta \\
&> \frac{\omega(\varepsilon_n)\eta}{4} \mathcal{L}^1(T) > \frac{\omega(\varepsilon_n)\eta}{8}.
\end{aligned}$$

Summing the last inequality over all $x \in A_n^+$ and using (5.4) and (1.19), we obtain

$$\left|D(u_{\varepsilon_n} - v)\right|(Q_{\varepsilon_n,\delta}) > \frac{\omega(\varepsilon_n)\eta}{8} \operatorname{card}\left(A_n^+\right) > \frac{\eta\delta^2}{32} \frac{\omega(\varepsilon_n)}{\varepsilon_n^2} \to \infty,$$

which contradicts (5.1). This concludes the proof of the theorem. □

Next we prove Theorem 1.13 and Theorem 1.14. Given an image u_0, which is stably reconstructible over $\Gamma(\delta)$ for some small $\delta > 0$, we show that it is reconstructible over the complement D of an ε-uniformly distributed (undamaged) region for ε sufficiently small. In view of Theorem 1.2, it suffices to construct a divergence-free tensor field M on D whose normal trace on $D \cap \partial\Omega_k$ agrees with $-z_k$. Again by Theorem 1.2, we know that such a construction is possible in the thin layer $\Gamma(\delta) \cap \Omega_k$ for δ small enough. Therefore, it remains to extend this tensor field to the whole $D \cap \Omega_k$, keeping it divergence-free and preserving its boundary values. The following lemma is at the core of this extension. In the remainder of this section, we denote a generic point x of $\mathbb{R}^2$ as $x = (s, t)$.

Lemma 5.1. *(Cf. Fig. 9.) Let $\lambda, \sigma > 0$ be such that*

$$\lambda + 4\sigma \leqslant 1 \tag{5.6}$$

and let $\eta_0 > 0$. Fix $\eta_0 < \eta < 2\eta_0$ and assume that there exist $T_1, T_2 \in C^0([0, \eta]; \langle e\rangle^\perp)$ such that

$$\max_{i=1,2} \sup_{0<r<\eta} \left|T_i(r) - \overline{T_i}\right| \leqslant \sigma, \qquad |\overline{T_1}|^2 + |\overline{T_2}|^2 \leqslant \lambda^2, \tag{5.7}$$

where

$$\overline{T_1} := \fint_0^\eta T_1(r)\, dr, \qquad \overline{T_2} := \fint_0^\eta T_2(r)\, dr.$$

For $n \in \mathbb{N}$, $0 \leqslant h, k \leqslant n - 1$, consider the squares

$$Q_{h,k} := \left\{(s, t) \in \mathbb{R}^2\colon\ \frac{h}{n}\eta < s < \frac{h+1}{n}\eta,\ \frac{k}{n}\eta < t < \frac{k+1}{n}\eta\right\} \subset (0, \eta)^2,$$

and let $Q'_{h,k}$ be the (*closed*) *square concentric to $Q_{h,k}$ with side length $\omega(\eta/n)$, where ω satisfies* (1.19). *Finally, consider the open set*

$$D_n := \bigcup_{0\leqslant h,k\leqslant n-1} \left(\overline{Q_{h,k}} \setminus Q'_{h,k}\right).$$

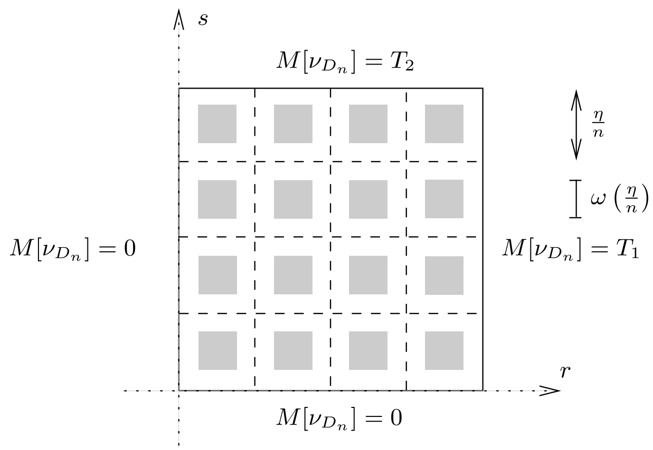

Fig. 9. The situation in Lemma 5.1.

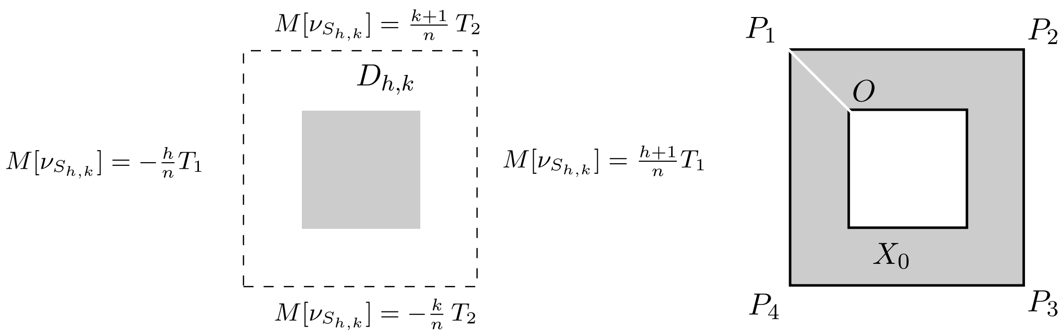

Fig. 10. Construction of M, and the set X.

Then there exists $n_0 \in \mathbb{N}$, depending only on η_0, σ, and ω, with the following property: For all $n \geqslant n_0$ there exists $M \in \mathcal{F}(D_n)$ such that

$$M[\nu_{D_n}](s,\eta) = T_2(s), \qquad M[\nu_{D_n}](\eta,t) = T_1(t), \qquad M[\nu_{D_n}](s,0) = M[\nu_{D_n}](0,t) = 0,$$

for every $s \in (0,\eta)$ and $t \in (0,\eta)$.

Proof. By Lemma 4.1, it suffices to show that for every $0 \leqslant h, k \leqslant n-1$, setting $D_{h,k} := Q_{h,k} \setminus Q'_{h,k}$, there exists $M \in \mathcal{F}(D_{h,k})$ with $M[\nu_{D_{h,k}}] = \Psi_{h,k}$ on $\partial Q_{h,k}$, where

$$\Psi_{h,k}\left(s, \frac{k+1}{n}\eta\right) := \frac{k+1}{n} T_2(s), \qquad \Psi_{h,k}\left(\frac{h+1}{n}\eta, t\right) := \frac{h+1}{n} T_1(t),$$
$$\Psi_{h,k}\left(s, \frac{k}{n}\eta\right) := -\frac{k}{n} T_2(s), \qquad \Psi_{h,k}\left(\frac{h}{n}\eta, t\right) := -\frac{h}{n} T_1(t)$$

for all $s \in (\frac{h}{n}\eta, \frac{h+1}{n}\eta)$ and $t \in (\frac{k}{n}\eta, \frac{k+1}{n}\eta)$.

Let $\sigma_{h,k}$ be the closed segment joining the upper left corner of $Q_{h,k}$ with the upper left corner of $Q'_{h,k}$, let $X_0 := (\overline{Q_{h,k}} \setminus Q'_{h,k}) \setminus \sigma_{h,k}$ (see Fig. 10), and let d_{X_0} be the geodesic distance on X_0. Given a point $x \in \sigma_{h,k}$ and a sequence $\{x_n\}$, with $x_n \to x$, we write $x_n \to x\pm$ if $(x_n - x)\cdot(1,1) \gtrless 0$ for all n sufficiently large. We now introduce a metric space (X, d_X), where X is given by the disjoint union of X_0 and two distinct copies $\sigma^+_{h,k}$ and $\sigma^-_{h,k}$ of the segment $\sigma_{h,k}$.

Roughly speaking X can be obtained from X_0 by removing an ε-neighborhood of $\sigma_{h,k}$, considering the closure of this set, and interpret X as a "limit" of this sequence of metric spaces as $\varepsilon \to 0^+$. The metric d_X is defined by

$$d_X(x_1,x_2) := \begin{cases} d_{X_0}(x_1,x_2) & \text{if } x_1,x_2 \in X_0, \\ \lim_{n\to\infty} d_{X_0}(y_n,x_2) & \text{if } x_1 \in \sigma^{\pm}_{h,k},\ x_2 \in X_0,\ y_n \to x_1\pm, \\ \lim_{n\to\infty} d_{X_0}(w_n,y_n) & \text{if } x_1 \in \sigma^{\pm}_{h,k},\ x_2 \in \sigma^{\pm}_{h,k},\ w_n \to x_1\pm,\ y_n \to x_2\pm, \\ \lim_{n\to\infty} d_{X_0}(w_n,y_n) & \text{if } x_1 \in \sigma^{+}_{h,k},\ x_2 \in \sigma^{-}_{h,k},\ w_n \to x_1+, y_n \to x_2-. \end{cases}$$

It can be shown that the definition does not depend on the choice of the sequences $\{w_n\}$ and $\{y_n\}$.

Note that the set $Y := \partial Q_{h,k} \cup \sigma^+_{h,k} \cup \sigma^-_{h,k}$ can be identified with the oriented polygonal path $OP_1P_2P_3P_4P_1O$, where P_1, P_2, P_3, P_4, and O are the points in Fig. 10.

The strategy of the proof is the following: We define a function f on Y such that

$$\partial_{\tau_{D_{h,k}}} f = \Psi_{h,k} \quad \text{on } \partial Q_{h,k}, \qquad \partial_{\tau_{\sigma_{h,k}}} f|_{\sigma^+_{h,k}} = \partial_{\tau_{\sigma_{h,k}}} f|_{\sigma^-_{h,k}} \tag{5.8}$$

and we prove that for $j = 1, 2$ the component $f^{(j)}$ is L_j-Lipschitz with respect to the distance d_X, where $L_1^2 + L_2^2 \leqslant 1$. Then we use McShane's lemma to extend $f^{(j)}$ to a L_j-Lipschitz function (still denoted by $f^{(j)}$) defined on all of X. By restricting f to X_0 and since d_X coincides with the Euclidean distance locally in X_0, this will imply, in particular, that

$$\begin{aligned} \|\nabla f\|_{L^\infty(X_0;\langle e\rangle^\perp\otimes\mathbb{R}^2)} &\leqslant \sqrt{\|\nabla f^{(1)}\|^2_{L^\infty(X_0;\mathbb{R}^2)} + \|\nabla f^{(2)}\|^2_{L^\infty(X_0;\mathbb{R}^2)}} \\ &\leqslant \sqrt{L_1^2 + L_2^2} \leqslant 1. \end{aligned}$$

Note that we can identify the restrictions of f to $\sigma^+_{h,k}$ and to $\sigma^-_{h,k}$ with the traces f^+ and f^- of f on $\sigma_{h,k}$. Next, we define the tensor field $M : X \to \langle e\rangle^\perp \otimes \mathbb{R}^2$ by $M^{(j)} := -(\nabla f^{(j)})^\perp$. By (5.8), we have that the normal trace of M is continuous across $\sigma_{h,k}$. Hence, by Proposition 1.4 and by (5.8), M belongs to $\mathcal{F}(D_{h,k})$ and $M[\nu_{D_{h,k}}] = \Psi_{h,k}$, as desired.

Step 1. Parameterize the polygonal path $OP_1P_2P_3P_4P_1O$ with constant velocity 1. For every y_1, $y_2 \in Y$, define the distance

$$d_Y(y_1,y_2) := |s_1 - s_2|,$$

where s_1 and s_2 are the parameters corresponding to y_1 and y_2, respectively. We claim that there exists $n_0 \in \mathbb{N}$, depending only on η_0, σ, and ω, such that

$$\frac{1}{n} d_Y(y_1,y_2) \leqslant \sigma\, d_X(y_1,y_2) \tag{5.9}$$

for all y_1, $y_2 \in Y$ and $n \geqslant n_0$. To prove this, we distinguish two cases.

If $d_X(y_1,y_2) \geqslant 4\omega(\eta/n)$, then, since $d_Y(y_1,y_2) \leqslant (4+\sqrt{2})\frac{\eta}{n}$, we have

$$\frac{1}{n} d_Y(y_1,y_2) \leqslant \frac{4+\sqrt{2}}{4\eta} \frac{\eta^2}{n^2\omega(\frac{\eta}{n})} d_X(y_1,y_2).$$

Hence, (5.9) follows in this case by (1.19).

If $d_X(y_1,y_2) < 4\omega(\eta/n)$ and y_1 and y_2 are contained in the same segment of the oriented polygonal path Y, then (5.9) is immediate.

Finally, consider the case in which $d_X(y_1,y_2) < 4\omega(\eta/n)$ and y_1 and y_2 belong to two different segments. We first show that these segments are consecutive and that the segment joining y_1 and y_2 is contained in X. Indeed, if not, then the Euclidean distance and, in turn, the geodesic distance in X, between y_1 and y_2 is at least $\eta/(2n)$ for n sufficiently large, unless one point is in $\sigma^+_{h,k}$ and the other is in $\sigma^-_{h,k}$, in which case the geodesic distance is at least $4\omega(\eta/n)$. In both cases, by (1.19), for n large enough we have a contradiction. Hence we have proved that the segments are consecutive segments in the oriented polygonal path. Next, we show that the segment σ_0 joining y_1 and y_2 is contained in X, provided that n is sufficiently large. Indeed, if not, then σ_0 intersects $\partial Q'_{h,k}$, thus the Euclidean distance and, in turn, the geodesic distance between y_1 and y_2 is at least $\operatorname{dist}(\partial Q_{h,k}, \partial Q'_{h,k}) > \frac{\eta}{4n} > 4\omega(\frac{\eta}{n})$ for n sufficiently large, where in the last inequality we have used (1.19). Hence, σ_0 is contained in X, and so $d_X(y_1, y_2)$ reduces to the Euclidean distance.

Let $y_0 \in \{P_1, P_2, P_3, P_4\}$ be the vertex between y_1 and y_2, so that $d_Y(y_1, y_2)$ is given by $|y_1 - y_0| + |y_2 - y_0| \leqslant c|y_1 - y_2|$, where c depends only on the angles of the polygonal path. Inequality (5.9) now follows.

Step 2. Let $f_0 : \partial Q_{h,k} \to \langle e\rangle^\perp$, $g : Y \to \langle e\rangle^\perp$ be the functions (uniquely) determined by

$$f_0\left(\frac{h}{n}\eta, \frac{k+1}{n}\eta\right) = 0, \qquad g\left(\frac{h}{n}\eta, \frac{k+1}{n}\eta\right) = 0$$

and

$$\begin{aligned}
&\partial_{\tau_{D_{h,k}}} f_0\left(s, \frac{k+1}{n}\eta\right) = \frac{k}{n}T_2(s), &&\partial_{\tau_{D_{h,k}}} g\left(s, \frac{k+1}{n}\eta\right) = \frac{1}{n}T_2(s),\\
&\partial_{\tau_{D_{h,k}}} f_0\left(\frac{h+1}{n}\eta, t\right) = \frac{h}{n}T_1(t), &&\partial_{\tau_{D_{h,k}}} g\left(\frac{h+1}{n}\eta, t\right) = \frac{1}{n}T_1(t),\\
&\partial_{\tau_{D_{h,k}}} f_0\left(s, \frac{k}{n}\eta\right) = -\frac{k}{n}T_2(s), &&\partial_{\tau_{D_{h,k}}} g\left(s, \frac{k}{n}\eta\right) = 0,\\
&\partial_{\tau_{D_{h,k}}} f_0\left(\frac{h}{n}\eta, t\right) = -\frac{h}{n}T_1(t), &&\partial_{\tau_{D_{h,k}}} g\left(\frac{h}{n}\eta, t\right) = 0
\end{aligned} \tag{5.10}$$

for all $s \in (\frac{h}{n}\eta, \frac{h+1}{n}\eta)$ and $t \in (\frac{k}{n}\eta, \frac{k+1}{n}\eta)$, and

$$\partial_{\tau_{\sigma_{h,k}}} g = 0 \quad \text{on } \sigma_{h,k}, \tag{5.11}$$

where $\tau_{\sigma_{h,k}} := (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. We set

$$\lambda_j =: \sqrt{\left|\overline{T_1}^{(j)}\right|^2 + \left|\overline{T_2}^{(j)}\right|^2}.$$

By (5.7),

$$\lambda_1^2 + \lambda_2^2 \leqslant \lambda^2.$$

We claim that $f_0^{(j)}$ is Lipschitz with $\operatorname{Lip}(f_0) \leqslant \lambda_j + \sqrt{2}\sigma$. To see this, we fix $x_1 = (s_1, t_1)$ and $x_2 = (s_2, t_2)$ on $\partial Q_{h,k}$. If x_1 and x_2 are on the same side, this follows from (5.7). Assume next that x_1 and x_2 are on two consecutive sides, say $x_1 = (s_1, \frac{k+1}{n}\eta)$ and $x_2 = (\frac{h+1}{n}\eta, t_2)$ (the other cases are analogous). Then,

$$\begin{aligned}
\left|f_0^{(j)}(x_1) - f_0^{(j)}(x_2)\right| &\leqslant \frac{k}{n}\int_{s_1}^{\frac{h+1}{n}\eta} \left|T_2^{(j)}\right| ds + \frac{h}{n}\int_{t_1}^{\frac{k+1}{n}\eta} \left|T_1^{(j)}\right| dt\\
&\leqslant \left(\frac{h+1}{n}\eta - s_1\right)\left(\left|\overline{T_2^{(j)}}\right| + \sigma\right) + \left(\frac{k+1}{n}\eta - t_1\right)\left(\left|\overline{T_1^{(j)}}\right| + \sigma\right)\\
&\leqslant (\lambda_j + \sqrt{2}\sigma)|x_1 - x_2|,
\end{aligned}$$

where in the last inequality we have used the Cauchy–Schwarz inequality together with (5.7). If x_1 and x_2 are on two opposite sides, say $x_1 = (s_1, \frac{k+1}{n}\eta)$ and $x_2 = (s_2, \frac{k}{n}\eta)$, with $s_1 \leqslant s_2$ (the other cases are analogous), then

$$\begin{aligned}
\left|f_0^{(j)}(x_1) - f_0^{(j)}(x_2)\right| &\leqslant \frac{k}{n}\int_{s_1}^{s_2} \left|T_2^{(j)}\right| ds + \frac{h}{n}\int_{\frac{k}{n}\eta}^{\frac{k+1}{n}\eta} \left|T_1^{(j)}\right| dt\\
&\leqslant (s_2 - s_1)\left(\left|\overline{T_2^{(j)}}\right| + \sigma\right) + \frac{\eta}{n}\left(\left|\overline{T_1^{(j)}}\right| + \sigma\right)\\
&\leqslant (\lambda_j + \sqrt{2}\sigma)|x_1 - x_2|,
\end{aligned}$$

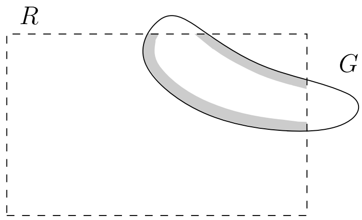

Fig. 11. The situation in Lemma 5.2. The region $R \cap G^\delta$ is colored in gray.

again by the Cauchy–Schwarz inequality and (5.7). Hence $f_0^{(j)}$ is Lipschitz with $\mathrm{Lip}(f_0) \leqslant \lambda_j + \sqrt{2}\sigma$, and so by McShane's lemma we can extend it to a function defined on all of $Q_{h,k}$ with the same Lipschitz constant and still denoted by f_0. Moreover,

$$\|\nabla f_0\|_\infty \leqslant \sqrt{\lambda_1^2 + \lambda_2^2 + 4\sigma^2} \leqslant \sqrt{\lambda^2 + 4\sigma^2} \leqslant \lambda + 2\sigma. \tag{5.12}$$

Next, since by (5.10) we have $\|\partial_{\tau_{\sigma_{h,k}}} g\|_\infty < \frac{1}{n}$, using (5.9) we obtain

$$\left|g(y_1) - g(y_2)\right| < \frac{1}{n} d_Y(y_1, y_2) \leqslant \sigma\, d_X(y_1, y_2) \quad \text{for all } y_1, y_2 \in Y. \tag{5.13}$$

Using McShane's lemma with respect to d_X, we can extend g to X in such a way that

$$\left|g^{(j)}(x_1) - g^{(j)}(x_2)\right| \leqslant \sigma\, d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X \text{ and } j = 1, 2.$$

Since in a neighborhood of every point in the interior of X the distance d_X coincides with the Euclidean distance, it follows that $\|\nabla g\|_\infty \leqslant \sqrt{2}\sigma$. We now define $f : X \to \langle e\rangle^\perp$, $f := f_0 + g$. By (5.10) we obtain $(5.8)_1$. Also, by (5.6) and (5.12), $\|\nabla f\|_\infty \leqslant \lambda + 2\sigma + \sqrt{2}\sigma < \lambda + 4\sigma \leqslant 1$. Moreover, by (5.11) and the fact that f_0 is defined on the whole $Q_{h,k}$, we obtain $(5.8)_2$. This concludes the proof of the lemma. □

In the next lemma we give an approximation result for elements of $\mathcal{F}(D)$ by more regular divergence-free fields.

Lemma 5.2. *Let $G \subset \mathbb{R}^2$ be an open set with C^∞ boundary such that $\partial R \cap G$ is the union of finitely many segments whose endpoints are not corners of R, and for $\delta > 0$ let*

$$G^\delta := \left\{x \in G\colon \operatorname{dist}(x, \partial G) < \delta\right\}.$$

If $M \in \mathcal{F}(R \cap G^\delta)$ and if $\sigma \in (0, \frac{1}{2})$, then there exist $0 < \delta' < \delta$ and $M_\sigma \in \mathcal{F}(R \cap G^{\delta'})$ such that M_σ is continuous in $\overline{R \cap G^{\delta'}} \setminus (\partial G \cap R)$ and

$$\sup_{R \cap G^{\delta'}} |M_\sigma| \leqslant (1 + \sigma)\|M\|_{L^\infty(R \cap G^\delta; \langle e\rangle^\perp \otimes \mathbb{R}^2)},$$
$$M_\sigma[\nu_G] = M[\nu_G] \quad \text{on } \partial G \cap R. \tag{5.14}$$

Proof. Without loss of generality, we may assume that $\delta > 0$ is so small that for all $0 < \delta' < \frac{1}{2}\delta$, $G^{\delta'}$ has C^∞ boundary, the function $d_\sigma(x) := \sigma \operatorname{dist}(x, \partial G)$ is smooth on the closure of $G^{\delta'}$, and that $\partial R \cap G^{\delta'}$ is the finite union of segments $\sigma_1, \ldots, \sigma_\ell$ whose endpoints are not the corners of R (see Fig. 11).

Define $z := M[\nu_{R \cap G^{\delta'}}]$ on $\partial(R \cap G^{\delta'})$, where we have used the fact that $M \in \mathcal{F}(R \cap G^{\delta'})$. To each segment σ_i we associate the (open) square $Q_i \subset \mathbb{R}^2 \setminus R$ with one side coinciding with σ_i. We now extend M to Q_i, $i = 1, \ldots, \ell$, according to the following rules. If Q_i corresponds to a vertical segment σ_i, then M is the tensor field uniquely determined by the conditions

$$M[e_2] = 0 \quad \text{in } Q_i,$$
$$M[\nu_{Q_i}] = -z \quad \text{on } \sigma_i,$$
M is constant in the e_1-direction.

If Q_i corresponds to a horizontal segment σ_i, then we perform an analogous construction. Note that in this way we have a tensor field M belonging to $\mathcal{F}(\bigcup_i Q_i \cup [\overline{R} \cap G^{\delta'}])$. For $x \in R \cap G^{\delta'}$ we define

$$M_\sigma^{(j)}(x) := \frac{1}{\pi} \int\limits_{B(0,1)} \big\{ M^{(j)}\big(x + d_\sigma(x)w\big) - \big[w \cdot \big(M^{(j)}\big(x + d_\sigma(x)w\big)\big)^\perp\big]\big(\nabla d_\sigma(x)\big)^\perp \big\}\, dw,$$

$j = 1, 2$. Since for each $x \in R \cap G^{\delta'}$ the ball $B(x, d_\sigma(x))$ is contained $\bigcup_i Q_i \cup [\overline{R} \cap G^{\delta}]$, the function $M_\sigma^{(j)}$ is well-defined. Moreover, by construction M_σ is continuous and (5.14) holds. Thus, it remains to prove that M_σ is a divergence-free vector field having the same normal trace of M on $\partial G \cap R$.

Step 1. To prove that M_σ is divergence-free, we will show that in every ball $B(x_0, r) \subset R \cap G^{\delta'}$, with $0 < r < \frac{1}{4}\operatorname{dist}(x_0, \partial G)$, the tensor field M_σ is the rotated gradient of a vector-valued function and then use Proposition 1.4. Since the domain may not be simply connected, this function may change from ball to ball. Fix any such $B(x_0, r)$, and note that for every $x \in B(x_0, r)$ the ball $B(x, d_\sigma(x))$ is contained in the set $B(x_0, \operatorname{dist}(x_0, \partial G))$, which, in turn, is contained in $\bigcup_i Q_i \cup [\overline{R} \cap G^{\delta}]$. Let $f : B(x_0, \operatorname{dist}(x_0, \partial G)) \to \mathbb{R}$ be a 1-Lipschitz function such that

$$(\nabla f)^\perp = -M^{(1)} \quad \text{in } B\big(x_0, \operatorname{dist}(x_0, \partial G)\big). \tag{5.15}$$

For $x \in B(x_0, r)$ define

$$f_\sigma(x) := \fint\limits_{B(x, d_\sigma(x))} f(y)\, dy = \frac{1}{\pi} \int\limits_{B(0,1)} f\big(x + d_\sigma(x)w\big)\, dw, \tag{5.16}$$

and note that

$$\nabla f_\sigma(x) = \frac{1}{\pi} \int\limits_{B(0,1)} \big\{ \nabla f\big(x + d_\sigma(x)w\big) + \big[w \cdot \nabla f\big(x + d_\sigma(x)w\big)\big] \nabla d_\sigma(x) \big\}\, dw.$$

Then

$$\begin{aligned}(\nabla f_\sigma)^\perp(x) &= \frac{1}{\pi} \int\limits_{B(0,1)} \big\{ (\nabla f)^\perp\big(x + d_\sigma(x)w\big) + \big[w \cdot \nabla f\big(x + d_\sigma(x)w\big)\big](\nabla d_\sigma)^\perp(x) \big\}\, dw \\ &= \frac{1}{\pi} \int\limits_{B(0,1)} \big\{ M^{(1)}\big(x + d_\sigma(x)w\big) - \big[w \cdot \big(M^{(1)}\big(x + d_\sigma(x)w\big)\big)^\perp\big]\big(\nabla d_\sigma(x)\big)^\perp \big\}\, dw,\end{aligned}$$

where we used (5.15). We deduce that $M_\sigma^{(1)} = -(\nabla f_\sigma)^\perp$ in $B(x_0, r)$, which, in turn, implies $\operatorname{div} M_\sigma^{(1)} = 0$ in the sense of distributions. Since the argument for $M_\sigma^{(2)}$ is clearly the same, we have proved that M_σ is a divergence-free tensor field.

Step 2. We finally show that M_σ and M share the same normal trace. It will be enough to prove that for every $x_0 \in \partial G \cap R$,

$$M_\sigma[\nu_G] = M[\nu_G] \quad \mathcal{H}^1\text{-a.e. on } \partial G \cap R \cap \overline{B(x_0, r)} \tag{5.17}$$

where $r > 0$ is taken so small that $B(x_0, 2r) \subset R$, for every $x \in B(x_0, r) \cap G^{\delta'}$ the ball $B(x, d_\sigma(x))$ is contained in the set $B(x_0, 2r) \cap G^{\delta'}$, and the set $B(x_0, 2r) \cap G^{\delta'}$ is simply connected. For any such $B(x_0, r)$, as in the previous step, let $f : B(x_0, 2r) \cap G^{\delta'} \to \mathbb{R}$ be a 1-Lipschitz function satisfying (5.15) and let f_σ be as in (5.16). From the previous step

$$M_\sigma^{(1)} = -(\nabla f_\sigma)^\perp \quad \text{in } B(x_0, r) \cap G^{\delta'}. \tag{5.18}$$

Now observe that for all $x \in B(x_0, r) \cap G^{\delta'}$,

$$\big|f_\sigma(x) - f(x)\big| \leqslant \fint\limits_{B(x, d_\sigma(x))} \big|f(y) - f(x)\big|\, dy \leqslant \fint\limits_{B(x, d_\sigma(x))} |y - x|\, dy \leqslant d_\sigma(x).$$

Thus $f_\sigma = f$ on $\partial G \cap B(x_0, r)$ and, in turn, $\partial_{\tau_G} f_\sigma = \partial_{\tau_G} f$ $\mathcal{H}^1$-a.e. on $\partial G \cap B(x_0, r)$. Recalling (5.18), this is equivalent to $M_\sigma^{(1)} \cdot \nu_G = M^{(1)} \cdot \nu_G$ $\mathcal{H}^1$-a.e. on $\partial G \cap B(x_0, r)$. An analogous argument shows that the same holds for $M_\sigma^{(2)}$ and $M^{(2)}$. Hence, (5.17) is established and the proof of the lemma is completed. □

Proof of Theorem 1.13. Let

$$D_\varepsilon := R \cap \left(\bigcup_{x \in \varepsilon\mathbb{Z}^2} \overline{Q(x,\varepsilon)} \setminus \overline{Q\big(x, \omega(\varepsilon)\big)} \right).$$

The strategy of the proof is the following: We will show that for ε small enough it is possible to find an admissible domain $\tilde{D}_\varepsilon$, with $\mathcal{H}^1(\partial \tilde{D}_\varepsilon \cap \Gamma) = 0$ and $D_\varepsilon \subset \tilde{D}_\varepsilon$, and a tensor field $M_{k,\varepsilon} \in \mathcal{F}(\tilde{D}_\varepsilon \cap \Omega_k)$, $k = 1, \ldots, N$, such that

$$M_{k,\varepsilon}[\nu_{\Omega_k}] = -z_k \quad \text{on } \Sigma_k.$$

Then, by Theorem 1.2 the function u_0 is reconstructible on $\tilde{D}_\varepsilon$ and, a fortiori, on D_ε.

Step 1. To construct $\tilde{D}_\varepsilon$ and $M_{k,\varepsilon}$, we apply Proposition 4.2 to find $\delta > 0$ sufficiently small and tensor fields $M_k \in \mathcal{F}(\Sigma_k(\delta) \cap \Omega_k)$, $k = 1, \ldots, N$, such that

$$\|M_k\|_{L^\infty(\Sigma_k(\delta)\cap\Omega_k;\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1,$$
$$M_k[\nu_{\Omega_k}] = -z_k \quad \text{on } \Sigma_k.$$

We claim that there exists $\varepsilon_k > 0$ with the following properties: For all $0 < \varepsilon < \varepsilon_k$ there exist an admissible set $D_{k,\varepsilon}$ such that

$$\begin{aligned} &D_\varepsilon \cap \Sigma_k(\delta) \cap \Omega_k \subset D_{k,\varepsilon} \subset \Sigma_k(\delta) \cap \Omega_k, \\ &\Sigma_k\big(\delta'\big) \cap \Omega_k \subset D_{k,\varepsilon}, \quad \text{for some } \delta' \in (0, \delta), \\ &\Omega_k \cap \partial\Sigma_k(\delta) \subset \Omega_k \cap \partial D_{k,\varepsilon}, \end{aligned} \tag{5.19}$$

and $\hat{M}_{k,\varepsilon} \in \mathcal{F}(D_{k,\varepsilon})$ such that

$$\begin{aligned} &\hat{M}_{k,\varepsilon}[\nu_{D_{k,\varepsilon}}] = -z_k \quad \text{on } \Sigma_k, \\ &\hat{M}_{k,\varepsilon}[\nu_{D_{k,\varepsilon}}] = 0 \quad \text{on } \partial\Sigma_k(\delta) \cap \Omega_k. \end{aligned} \tag{5.20}$$

Note that if the claim holds, then to conclude the proof of the theorem, for $0 < \varepsilon < \varepsilon_0 := \min_k \varepsilon_k$, it suffices to define $\tilde{D}_\varepsilon$ by

$$\tilde{D}_\varepsilon := \bigcup_k \big(D_{k,\varepsilon} \cup \big(\Omega_k \setminus \overline{\Sigma_k(\delta)}\big)\big),$$

and $M_{k,\varepsilon} \in \mathcal{F}(\tilde{D}_\varepsilon \cap \Omega_k)$ by

$$M_{k,\varepsilon} := \begin{cases} \hat{M}_{k,\varepsilon} & \text{in } D_{k,\varepsilon}, \\ 0 & \text{elsewhere in } \tilde{D}_\varepsilon \cap \Omega_k. \end{cases}$$

In view of Lemma 4.1, the tensor field $M_{k,\varepsilon}$ has all the desired properties.

In the remaining of the proof we show (5.19) and (5.20) for a fixed $k = 1, \ldots, N$. For simplicity, we drop the dependence on k and ε, and we write Ω, D, Σ, M, and z in place of Ω_k, $D_{k,\varepsilon}$, Σ_k, $M_{k,\varepsilon}$, and z_k, respectively.

With this simplification, we recall that we are given a Lipschitz open set $\Omega \subset R$ and a tensor field $M \in \mathcal{F}(\Sigma(\delta) \cap \Omega)$ such that

$$\begin{aligned} &\|M\|_{L^\infty(\Sigma(\delta)\cap\Omega;\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1, \\ &M[\nu_\Omega] = -z \quad \text{on } \Sigma, \end{aligned} \tag{5.21}$$

where $\Sigma := \partial\Omega \cap R$.

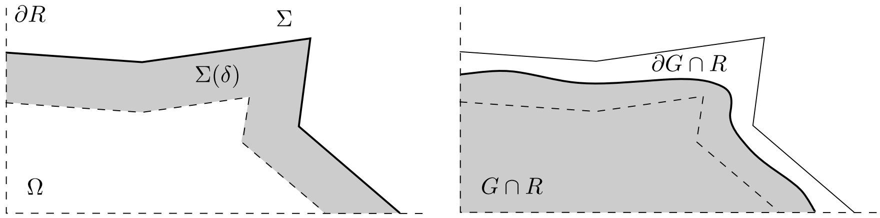

Fig. 12. On the left, the sets Ω, Σ and $\Sigma(\delta)\cap\Omega$. On the right, the sets $G\cap R$ and $\partial G\cap R$.

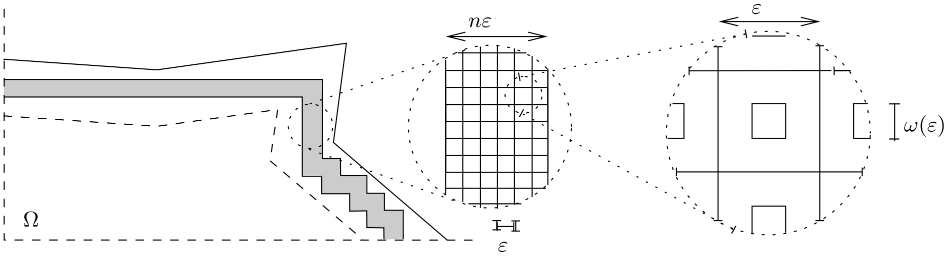

Fig. 13. The construction in Step 3. The set $D_{k,\varepsilon}$ is represented in the picture as the set $\Sigma(\delta)\cap\Omega$ after removing the colored squares contained in the shaded region.

Step 2. We claim that, without loss of generality, we may assume that $\Omega = G\cap R$, where $G\subset\mathbb{R}^2$ is a smooth open set. To see this, choose an open set $G\subset\mathbb{R}^2$ with C^∞ boundary such that $\partial R\cap G$ is the union of finitely many segments whose endpoints are not corners of R, G contains $\Omega\setminus\Sigma(\delta)$, and $\partial G\cap R$ is contained in $\Sigma(\delta)\cap\Omega$ (see Fig. 12). Since the restriction of M to $\Sigma(\delta)\cap G\cap R$ still satisfies (5.21), with a possibly smaller $\delta>0$, with Ω, Σ, and $-z$ replaced by $G\cap R$, $\partial G\cap R$, and with $M[\nu_{G\cap R}]$ on $\partial G\cap R$, respectively, we can modify M in $G\cap R$ (without changing its normal trace on $\partial G\cap R$) and then use Lemma 4.1 to glue the modified field with $M|_{\Omega\setminus(G\cap R)}$.

Step 3. (Cf. Fig. 13.) In view of the previous step and by (5.21), we are in a position to apply Lemma 5.2 (with σ sufficiently small) to find $\tilde M\in\mathcal{F}(\Sigma(\delta')\cap\Omega)$, where $0<\delta'<\delta$, such that $\tilde M$ is continuous in $\overline{\Sigma(\delta')}\cap\Omega\setminus\Sigma$ and

$$\lambda := \|\tilde M\|_{L^\infty(\Sigma(\delta')\cap\Omega;\langle e\rangle^\perp\otimes\mathbb{R}^2)} < 1,$$

$$\tilde M[\nu_\Omega] = M[\nu_\Omega] \quad \text{on } \Sigma.$$

Let σ satisfy (5.6). Since $\tilde M$ is uniformly continuous in $\overline{\Sigma(\delta')}\cap\Omega\setminus\Sigma(\delta'/2)$, we may find $\eta_0>0$ such that

$$\sup\bigl\{\bigl|\tilde M(x)-\tilde M(y)\bigr|:\ x,y\in\overline{\Sigma(\delta')}\cap\Omega\setminus\Sigma(\delta'/2),\ |x-y|\leqslant 2\eta_0\bigr\}<\sigma.$$

Let now $n_0\geqslant 3$ be the integer depending on λ, σ, and η_0 given by Lemma 5.1, and set

$$\varepsilon_0 := \frac{\eta_0}{n_0}.$$

Fix $0<\varepsilon<\varepsilon_0$, let $n>n_0$ be an odd integer such that

$$\eta_0 < n\varepsilon < 2\eta_0, \tag{5.22}$$

and consider a family $\{Q(x_k,3\varepsilon n)\}_{k\in\mathbb{N}}$ of open squares such that $x_k\in\varepsilon\mathbb{Z}^2$ for every $k\in\mathbb{N}$, $Q(x_k,3\varepsilon n)\cap Q(x_j,3\varepsilon n)=\emptyset$ for $k\neq j$, and $\mathbb{R}^2=\bigcup_k \overline{Q(x_k,3n\varepsilon)}$. Set

$$J := \bigl\{k\in\mathbb{N}:\ \overline{Q(x_k,3\varepsilon n)}\cap R\neq\emptyset \text{ and } \overline{Q(x_k,3\varepsilon n)}\cap R\subset\Omega\setminus\overline{\Sigma\bigl(\delta'/2\bigr)}\bigr\}$$

and let $\widetilde\Omega_{\varepsilon,n} := \operatorname{int}(\bigcup_{k\in J}\overline{Q(x_k,3\varepsilon n)})$. Note that $\partial\widetilde\Omega_{\varepsilon,n}\cap R$ is contained in $\Sigma(\delta')\cap\Omega\setminus\overline{\Sigma(\delta'/2)}$, provided that η_0 is sufficiently small (see (5.22)). Consider the subfamily

$$J' := \bigl\{k\in J:\ \mathcal{H}^1\bigl(\partial Q(x_k,3n\varepsilon)\cap\partial\widetilde\Omega_{\varepsilon,n}\cap R\bigr)>0\bigr\}$$

corresponding to the "boundary squares". Decompose each $Q(x_k, 3n\varepsilon)$, $k \in J'$, into the disjoint union of nine squares of side $n\varepsilon$ (still centered at points of $\varepsilon\mathbb{Z}^2$) and keep only those squares whose boundary intersects $\partial Q(x_k, 3n\varepsilon) \cap \partial\widetilde{\Omega}_{\varepsilon,n} \cap R$. In this way we obtain a new family $\{Q_h\}_h$ of disjoint squares of side $n\varepsilon$ whose boundary intersects $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$, such that each Q_h has either one or two consecutive sides in common with $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$. Moreover,

$$\left(\bigcup_h \partial Q_h\right) \cap (\partial\widetilde{\Omega}_{\varepsilon,n} \cap R) = \partial\widetilde{\Omega}_{\varepsilon,n} \cap R.$$

We are now in a position to apply Lemma 5.1 to each Q_h according to the following cases:

(i) If Q_h is contained in R and shares two consecutive sides, say σ_1^h and σ_2^h, with $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$, then we apply Lemma 5.1 with $\eta := n\varepsilon$ and

$$T_1 := -\tilde{M}[\nu_{R\setminus\overline{Q_h}}] \quad \text{on } \sigma_1^h, \qquad T_2 := -\tilde{M}[\nu_{R\setminus\overline{Q_h}}] \quad \text{on } \sigma_2^h.$$

In this way we obtain a tensor field $M_h \in \mathcal{F}(Q_h \cap D_\varepsilon)$ such that $M_h[\nu_{Q_h}] = T_i$ on σ_i^h, $i = 1, 2$, and $M_h[\nu_{Q_h}] = 0$ on the remaining two sides of Q_h.

(ii) If Q_h is contained in R and shares just one side, say σ_1^h, with $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$, then we apply Lemma 5.1 with $\eta := n\varepsilon$, $T_1 := -\tilde{M}[\nu_{R\setminus\overline{Q_h}}]$ on σ_1^h, and $T_2 := 0$ to obtain a tensor field $M_h \in \mathcal{F}(Q_h \cap D_\varepsilon)$ such that $M_h[\nu_{Q_h}] = T_1$ on σ_1^h and $M_h[\nu_{Q_h}] = 0$ on the remaining three sides of Q_h.

(iii) Next assume that Q_h intersects the complement of R and two consecutive sides, σ_1^h and σ_2^h, have nonempty intersection with $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$. Since Σ has positive distance from the vertices of R, if η_0 is sufficiently small, then $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$ has distance greater than $3\varepsilon n$ from the vertices of R. It follows that only one of σ_1^h and σ_2^h, say σ_1^h, is not contained in R. Construct a continuous function T_1 with oscillation less than σ and coinciding with $-\tilde{M}[\nu_{R\setminus\overline{Q_h}}]$ on $\sigma_1^h \cap \widetilde{\Omega}$. As in (ii), we also take $T_2 := -\tilde{M}[\nu_{R\setminus\overline{Q_h}}]$ on σ_2^h and we apply Lemma 5.1 to obtain a tensor field $M_h \in \mathcal{F}(Q_h \cap R \cap D_\varepsilon)$ such that $M_h[\nu_{Q_h}] = T_i$ on σ_i^h, $i = 1, 2$, and $M_h[\nu_{Q_h}] = 0$ on the remaining sides.

(iv) Finally, if Q_h intersects the complement of R and only one side, σ_1^h, has nonempty intersection with $\partial\widetilde{\Omega}_{\varepsilon,n} \cap R$, we proceed exactly as in (ii) when $\sigma_1^h \subset R$. If σ_1^h intersects also the complement of R, then we take T_1 as in (iii) and we proceed as in (ii).

By Lemma 4.1, setting

$$D := \left(\Omega \cap D_\varepsilon \cap \left(\bigcup_h Q_h\right)\right) \cup \left(\Omega \setminus \left(\bigcup_h Q_h\right)\right)$$

and

$$\hat{M} := \begin{cases} \tilde{M} & \text{on } \Omega \setminus \widetilde{\Omega}_{\varepsilon,n}, \\ M_h & \text{on } Q_h \cap \Omega \cap D_\varepsilon,\ h \in \mathbb{N}, \\ 0 & \text{elsewhere in } \widetilde{\Omega}_{\varepsilon,n} \cap \Omega, \end{cases}$$

we obtain a tensor field $\hat{M}$ satisfying (5.19) and (5.20). This concludes the proof of Theorem 1.13. □

Next we prove Theorem 1.14.

Proof of Theorem 1.14. The proof of this theorem is exactly the same as the one of Theorem 1.13, with the only difference that Lemma 5.1 should be replaced by Lemma 5.3 below. □

Lemma 5.3. *Let σ, λ, η_0, η, T_1, and T_2 be as in Lemma* 5.1. *For $n \in \mathbb{N}$, $\theta \in (0, 1)$, and $0 \leqslant h, k \leqslant n - 1$, consider the squares*

$$Q_{h,k} := \left\{(s, t) \in \mathbb{R}^2\colon\ \frac{h}{n}\eta < s < \frac{h+1}{n}\eta,\ \frac{k}{n}\eta < t < \frac{k+1}{n}\eta\right\} \subset (0, \eta)^2,$$

denote by $Q^\theta_{h,k}$ *the closed square concentric to* $Q_{h,k}$ *and with side length* $(\theta\eta)/n$, *and let* $C_{h,k} \subset Q^\theta_{h,k}$ *be a closed connected set such that* $\operatorname{diam}(C_{h,k}) \geqslant \omega(\eta/n)$, *where* ω *satisfies* (1.19). *Consider the open set*

$$D_n := \bigcup_{0\leqslant h,\ k\leqslant n-1} (\overline{Q_{h,k}} \setminus C_{h,k}) \cap (0,\eta)^2.$$

Then there exists $n_0 \in \mathbb{N}$ *depending only on* η_0, σ, θ, *and* ω *with the following property*: *For all* $n \geqslant n_0$ *there exists* $M \in \mathcal{F}(D_n)$ *such that*

$$M[\nu_{D_n}](s,\eta) = T_2(s), \qquad M[\nu_{D_n}](\eta,t) = T_1(t), \qquad M[\nu_{D_n}](s,0) = M[\nu_{D_n}](\eta,t) = 0,$$

for all $s \in (0,\eta)$ *and* $t \in (0,\eta)$.

Remark 5.4. Note that by replacing each $C_{h,k}$ by a smaller closed connected subset, if needed, we may assume that $\operatorname{diam}(C_{h,k}) = \omega(\eta/n)$.

Proof of Lemma 5.3. The argument is very similar to the one of Lemma 5.1 and we only indicate the main changes. The sets $D_{h,k}$ should now be replaced by $D_{h,k} := Q_{h,k} \setminus C_{h,k}$, while $\sigma_{h,k}$ is now the segment joining the upper left corner of $Q_{h,k}$ with a point O of $C_{h,k}$ in such a way that $\mathcal{H}^1(\sigma_{h,k})$ is minimal. Define $X_0 := \overline{Q_{h,k}} \setminus (\sigma_{h,k} \cup C_{h,k})$. Given a point $x \in \sigma_{h,k}$ and a sequence $\{x_n\}$, $x_n \to x$, we write $x_n \to x\pm$ if $(x_n - x)\cdot \tau_{\sigma_{h,k}} \gtrless 0$ for all n sufficiently large, where $\tau_{\sigma_{h,k}}$ is a unit tangent vector to $\sigma_{h,k}$. We now continue as in the proof of Lemma 5.1 until Step 1.

Step 1. Let Y and d_Y be as in Step 1 in the proof of Lemma 5.1. We claim that there exists $n_0 \in \mathbb{N}$ depending only on η, σ, θ, and ω such that

$$\frac{1}{n} d_Y(y_1,y_2) \leqslant \sigma d_X(y_1,y_2) \tag{5.23}$$

for all $y_1, y_2 \in Y$ and $n \geqslant n_0$. To see this, we distinguish two cases. If $d_X(y_1,y_2) \geqslant \omega(\eta/n)$, then, since $d_Y(y_1,y_2) \leqslant (4+\sqrt{2})\frac{\eta}{n}$, we have

$$\frac{1}{n} d_Y(y_1,y_2) \leqslant \frac{4+\sqrt{2}}{\eta} \frac{\eta^2}{n^2\omega(\frac{\eta}{n})} d_X(y_1,y_2).$$

Hence, (5.23) follows in this case by (1.19).

If $d_X(y_1,y_2) < \omega(\eta/n)$, then we claim that for n large enough either y_1 and y_2 are contained in the same segment of the oriented polygonal path $OP_1P_2P_3P_4P_1O$ (see Fig. 10) or y_1 and y_2 belong to two consecutive segments in the oriented polygonal path and the segment joining y_1 and y_2 is contained in X (here and in what follows we are identifying y_1 and y_2 with points on the polygonal path). If y_1 and y_2 are on the same segment, then there (5.23) follows immediately. Thus, we may assume that they belong to two different segments σ_1 and σ_2. If σ_1 and σ_2 are not consecutive, then the Euclidean and, in turn, the geodesic distance in X between y_1 and y_2, is at least $(1-\theta)\eta/(2n)$, unless one point is in $\sigma^+_{h,k}$ and the other in $\sigma^-_{h,k}$. In the latter case the curve realizing the geodesic distance between y_1 and y_2 must enclose the set $C_{h,k}$, and so its length is at least the diameter $\omega(\eta/n)$ of $C_{h,k}$. In both cases, by (1.19), for n large enough we have a contradiction. Hence, we have proved that σ_1 and σ_2 are consecutive segments in the oriented polygonal path. Next, we show that the segment σ_0 joining y_1 and y_2 is contained in X, provided that n is sufficiently large. Indeed, if not, then σ_0 intersects $C_{h,k}$, thus by (1.19) the Euclidean and, in turn, the geodesic distance between y_1 and y_2 is at least $\operatorname{dist}(\partial Q_{h,k}, C_{h,k}) \geqslant (1-\theta)\eta/(2n) > \omega(\eta/n)$ for n sufficiently large. We have proved the claim.

Note that in view of the previous claim $d_X(y_1,y_2)$ reduces to the Euclidean distance. Hence, if $y_0 \in \{P_1,P_2,P_3,P_4\}$ is the vertex between y_1 and y_2, then $d_Y(y_1,y_2)$ is given by $|y_1-y_0| + |y_2-y_0| \leqslant c|y_1-y_2|$, where c depends only on the minimum angle of the polygonal path and thus on θ. The inequality (5.23) easily follows.

Step 2. The argument is analogous to that of Step 2 in the proof of Lemma 5.1. □

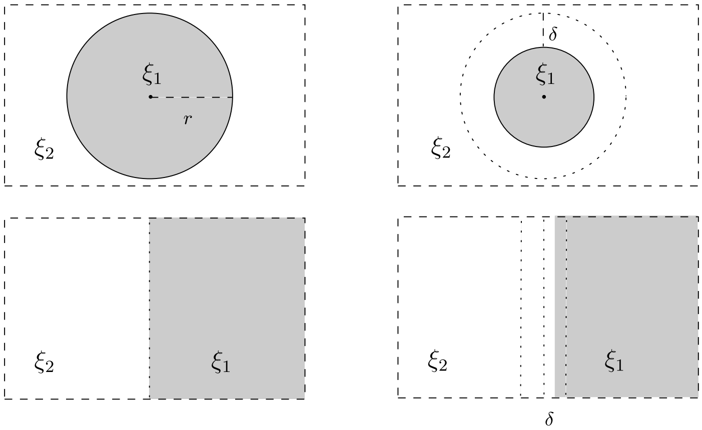

Fig. 14. Example 6.1: the two figures on top represent on the left the original u_0 and on the right an admissible field with lower energy. In this case the damaged region is the annulus of width δ. Hence, picture u_0 is not a minimizer. The two figures on the bottom represent on the left the original u_0 and on the right an admissible field with the same energy. In this case the damaged region is a rectangle of width δ. Hence, there is lack of uniqueness. In both cases the compatibility vector z between ξ_1 and ξ_2 is a unit vector.

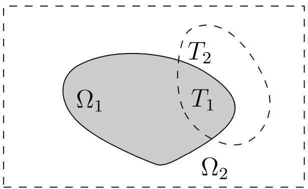

Fig. 15. The dashed line is the contour of the damaged region D.

6. Examples

In this section we present several explicit examples in which the model fails to provide exact reconstruction.

Example 6.1. This example shows that without condition (1.4) we may lose either the reconstructibility of u_0 over $\Gamma(\delta)$ for $\delta > 0$ small or the fact that u_0 is the unique minimizer (see Theorem 1.2). Consider two colors ξ_1 and ξ_2 that have the same gray level, i.e., $\xi_1 \cdot e = \xi_2 \cdot e$, and let u_0 be an admissible color image of the form $u_0 = \xi_1 1_{\Omega_1} + \xi_2 1_{R\setminus\overline{\Omega_1}}$. If $\Omega_1 = B_r$ is a ball of radius $r > 0$ contained in R (see Fig. 14), then the image $u_\delta = \xi_1 1_{B_{r-\delta}} + \xi_2 1_{R\setminus B_{r-\delta}}$ is admissible for the reconstruction problem over $\Gamma(\delta)$ for every $\delta > 0$. Since $F(u_\delta) < F(u_0)$, u_0 is not reconstructible over $\Gamma(\delta)$ for any damaged region D. Similarly, if $\Omega_1 = (0, \frac{a}{2}) \times (0, b)$, then u_0 is a minimizer for the reconstruction problem on $\Gamma(\delta)$ for every $\delta > 0$, but is not unique.

In the next two examples we consider an image

$$u_0 = \xi_1 1_{\Omega_1} + \xi_2 1_{\Omega_2} \tag{6.1}$$

$\Omega_2 = R \setminus \overline{\Omega_1}$, where the two colors have different gray levels (i.e., $\xi_1 \cdot e \neq \xi_2 \cdot e$), and the damaged region D is compactly contained in R and such that $\mathcal{H}^1(\Gamma \cap \partial D) = 0$, see Fig. 15. Set $T_1 := \Omega_1 \cap D$ and $T_2 := \Omega_2 \cap D$, so that $D = T_1 \cup T_2 \cup (D \cap \partial\Omega_1)$. We look for necessary and sufficient conditions for an image u of the form

$$u = \xi_* 1_{T_1} + \xi_1 1_{\Omega_1\setminus T_1} + \xi_2 1_{\Omega_2} \tag{6.2}$$

to be a minimizer in the reconstruction problem over D, when $\xi_* \neq \xi_1$ and $\xi_* \cdot e = \xi_1 \cdot e$. Whenever such an image is a minimizer, the model creates a new "artificial" contour, namely, $\Omega_1 \cap \partial D$.

By Remark 2.4, u is a minimizer for the reconstruction problem over D if and only if for every $\varphi \in BV(D; \langle e\rangle^\perp)$,

$$\liminf_{t\to 0^+} \frac{f(t) - f(0)}{t} \geqslant 0, \tag{6.3}$$

where $f(t) := F(u + t\varphi, D)$, $t \geqslant 0$.

We have

$$F(u,D)=\int_{D\cap\partial\Omega_1}|\xi_*-\xi_2|\,d\mathcal{H}^1+\int_{\Omega_1\cap\partial D}|\xi_*-\xi_1|\,d\mathcal{H}^1.$$

Fix $\varphi_*\in BV(T_1;\langle e\rangle^\perp)$, $\varphi_2\in BV(T_2;\langle e\rangle^\perp)$ and set $\varphi:=\varphi_*1_{T_1}+\varphi_2 1_{T_2}$, so that

$$(u+t\varphi)1_D=(\xi_*+t\varphi_*)1_{T_1}+(\xi_2+t\varphi_2)1_{T_2}.$$

Hence,

$$\begin{aligned}F(u+t\varphi,D)&=\left|D(u+t\varphi)\right|(D)+\sum_{k=1}^{2}\int_{\Omega_k\cap\partial D}\left|\xi_k-(u+t\varphi)\right|d\mathcal{H}^1\\&=t|D\varphi_*|(T_1)+t|D\varphi_2|(T_2)+\int_{D\cap\partial\Omega_1}\left|(\xi_*-\xi_2)+t(\varphi_*-\varphi_2)\right|d\mathcal{H}^1\\&\quad+\int_{\Omega_1\cap\partial D}\left|(\xi_*-\xi_1)+t\varphi_*\right|d\mathcal{H}^1+t\int_{\Omega_2\cap\partial D}|\varphi_2|\,d\mathcal{H}^1.\end{aligned}$$

Defining

$$z_{*1}:=P\left(\frac{\xi_*-\xi_1}{|\xi_*-\xi_1|}\right),\qquad z_{*2}:=P\left(\frac{\xi_*-\xi_2}{|\xi_*-\xi_2|}\right),$$

the inequality (6.3) becomes

$$0\leqslant|D\varphi_*|(T_1)+|D\varphi_2|(T_2)+\int_{D\cap\partial\Omega_1}z_{*2}\cdot(\varphi_*-\varphi_2)\,d\mathcal{H}^1+\int_{\Omega_1\cap\partial D}z_{*1}\cdot\varphi_*\,d\mathcal{H}^1+\int_{\Omega_2\cap\partial D}|\varphi_2|\,d\mathcal{H}^1,$$

and we conclude that u is a minimizer if and only if

$$0\leqslant|D\varphi_*|(T_1)+\int_{D\cap\partial\Omega_1}z_{*2}\cdot\varphi_*\,d\mathcal{H}^1+\int_{\Omega_1\cap\partial D}z_{*1}\cdot\varphi_*\,d\mathcal{H}^1,$$

$$0\leqslant|D\varphi_2|(T_2)+\int_{\Omega_2\cap\partial D}|\varphi_2|\,d\mathcal{H}^1-\int_{D\cap\partial\Omega_1}z_{*2}\cdot\varphi_2\,d\mathcal{H}^1$$

for all $\varphi_*\in BV(T_1;\langle e\rangle^\perp)$ and $\varphi_2\in BV(T_2;\langle e\rangle^\perp)$. By Lemma 3.1, these two inequalities are equivalent to the existence of tensor-fields $M_i\in\mathcal{F}(T_i)$, $i=1,2$, such that

$$M_1[\nu_{T_1}]=\begin{cases}-z_{*2} & \text{on } D\cap\partial\Omega_1,\\ -z_{*1} & \text{on } \Omega_1\cap\partial D,\end{cases}\tag{6.4}$$

$$M_2[\nu_{T_2}]=z_{*2}\quad\text{on } D\cap\partial\Omega_1.\tag{6.5}$$

In the next two examples we consider the colors

$$\xi_2=0,\qquad\xi_1=s\varepsilon_1+e,\qquad\xi_*=t\varepsilon_1+e,\tag{6.6}$$

where $0<t<s$. Note that $\xi_*\neq\xi_1$, $\xi_*\cdot e=\xi_1\cdot e$, and

$$z_{12}=\frac{s}{\sqrt{1+s^2}}\varepsilon_1,\qquad z_{*2}=\frac{t}{\sqrt{1+t^2}}\varepsilon_1,\qquad z_{*1}=-\varepsilon_1,\tag{6.7}$$

where z_{12} is the compatibility vector between ξ_1 and ξ_2, i.e.,

$$z_{12}=P\left(\frac{\xi_1-\xi_2}{|\xi_1-\xi_2|}\right).$$

The situation is represented in Fig. 16.

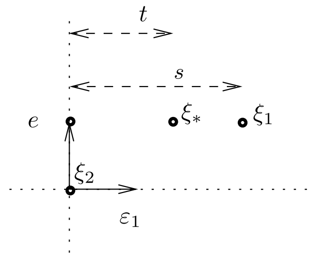

Fig. 16. The mutual positions of the colors in u.

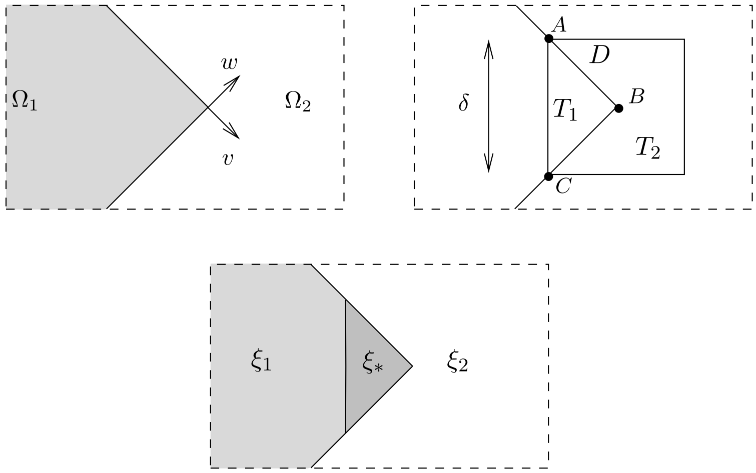

Fig. 17. The situation in Example 6.2. Since $|\zeta_{12}| > 1/\sqrt{2}$, the minimizer u over the damaged region D is different from u_0 and presents a new contour, no matter how small the side length of D.

Example 6.2. This example shows that if an image does not satisfy (1.14), then the reconstructed image may present an additional spurious contour even if the inpainting region D is an arbitrarily small neighborhood of a point of the contour Γ.

Let Ω_1 be as in Fig. 17, where the vectors v and w are two orthogonal, and consider as the damaged region D the square of side length δ with center at the corner point and sides parallel to $v+w$ and $v-w$ (see Fig. 17). Then, by Theorem 1.8 and (1.16), u_0 is stably reconstructible over $\Gamma(\delta)$ if $|z_{12}| < 1/\sqrt{2}$, i.e., if $s < 1$. Hence, if $s > 1$, (1.17) does not hold. In this case, we claim that if $t = 1$ in (6.7), then the function u in (6.2) is the unique minimizer. Note that, since $\xi_* \neq \xi_1$, u has a larger contour than u_0. To prove the claim, define $M_i \in \mathcal{F}(T_i)$, $i = 1, 2$, by

$$M_i := -\varepsilon_1 \otimes \frac{v+w}{|v+w|}.$$

Then $\operatorname{div} M_i = 0$, $\|M_i\|_{L^\infty(T_i;\varepsilon^\perp\otimes\mathbb{R}^2)} = 1$. Moreover, using the identities $\nu_{T_1} = -\frac{v+w}{|v+w|}$ on $\Omega_1 \cap \partial D = [AC]$, $\nu_{T_1} = w$ on $[AB]$, $\nu_{T_1} = v$ on $[BC]$, $\nu_{T_2} = -\nu_{T_1}$ on $[AB] \cup [BC]$, and

$$M_i[\nu_{T_i}] = \left(-\frac{v+w}{|v+w|} \cdot \nu_{T_i}\right)\varepsilon_1, \quad i = 1, 2,$$

we have that (6.4) and (6.5) hold.

This shows that u is a minimizer for the reconstruction problem over D. Moreover, arguing as in the proof of Theorem 1.2, it can be seen that it is the unique minimizer. Hence, the claim holds. Finally, we remark that ξ_* does not depend on s.

Example 6.3. This example shows that if the undamaged regions is not ε-uniformly distributed, then an image u_0 needs not be reconstructible over a region D even if it is stably reconstructible over $\Gamma(\delta)$ for some small $\delta > 0$.

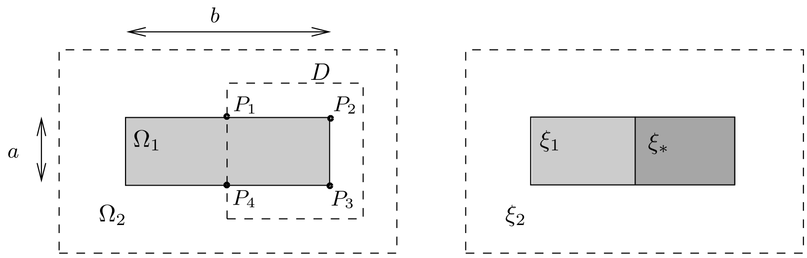

Fig. 18. The situation in Example 6.3.

Let u_0 be as in (6.1), with ξ_1 and ξ_2 satisfying (6.6). Take Ω_1 to be an open rectangle of sides $a, b > 0$, compactly contained in R, and let D be another open rectangle, as in Fig. 18. We assume that $0 < s < 1$, so that, as in the previous example, u_0 satisfies (1.14) with strict inequality. Hence, u_0 is stably reconstructible by Theorem 1.8. We claim that if b/a is large enough, then u given by (6.2) is a minimizer for the reconstruction problem over D for a suitable value of t.

To see this, consider the points P_1, P_2, P_3, P_4 as in Fig. 18. To construct the tensor fields M_1 and M_2 satisfying (6.4) and (6.5), it is sufficient to show the existence of 1-Lipschitz functions $f_i : \overline{T_i} \to \mathbb{R}$ such that

$$\begin{aligned}
&\partial_\tau f_1 = -\frac{t}{\sqrt{1+t^2}} && \text{on } [P_1P_2] \cup [P_2P_3] \cup [P_3P_4],\\
&\partial_\tau f_1 = 1 && \text{on } [P_1P_4],\\
&\partial_\tau f_2 = \frac{t}{\sqrt{1+t^2}} && \text{on } [P_1P_2] \cup [P_2P_3] \cup [P_3P_4],\\
&\partial_\tau f_2 = -1 && \text{on } [P_1P_4].
\end{aligned} \tag{6.8}$$

Here $\tau := \tau_{T_1} = (\nu_{T_1})^\perp$, so that we are orienting ∂T_1 clockwise. Note that $(6.8)_1$ and $(6.8)_2$ define f_1 over ∂T_1 up to an additive constant. The resulting function is continuous over ∂T_1 if and only if

$$-\frac{t}{\sqrt{1+t^2}}(b+a) + a = 0, \quad \text{i.e.,} \quad t = \frac{1}{\sqrt{(m+1)^2 - 1}}, \quad m := \frac{b}{a}. \tag{6.9}$$

Note that this value of t is compatible with the constraint $t < s$ (used in computing $z_{*1} = -\varepsilon_1$) if and only if

$$m = \frac{b}{a} \geqslant \sqrt{1 + \frac{1}{s^2}} - 1. \tag{6.10}$$

Since f_1 is piecewise affine with slopes smaller than or equal to 1, to verify that it is 1-Lipschitz over ∂T_1, it remains to study what happens near the corners P_1, P_2, P_3, P_4. Let v and w denote the tangent vector τ on $[P_1P_2]$ and $[P_2P_3]$, respectively. Then f_1 is 1-Lipschitz if

$$\begin{aligned}
&\left(-\alpha\frac{t}{\sqrt{1+t^2}} - \beta\frac{t}{\sqrt{1+t^2}}\right)^2 \leqslant |\alpha v + \beta w|^2 \quad \text{for all } \alpha, \beta > 0,\\
&\left(\alpha - \beta\frac{t}{\sqrt{1+t^2}}\right)^2 \leqslant |-\alpha w + \beta v|^2 \quad \text{for all } \alpha, \beta > 0.
\end{aligned} \tag{6.11}$$

The first condition is equivalent to $t \leqslant 1$, which is satisfied, since $t < s < 1$ by (6.10), while the second one is always satisfied.

Similarly, condition (6.11) guarantees that f_2 is 1-Lipschitz over $[P_1P_2] \cup [P_2P_3] \cup [P_3P_4]$. By McShane's lemma we can extend f_i to 1-Lipschitz functions $f_i : \overline{T_i} \to \mathbb{R}$, $i = 1, 2$. Define

$$M_i := -\varepsilon_1 \otimes (\nabla f_i)^\perp,$$

$i = 1, 2$. Then (6.4), (6.5) hold by construction, and so u is a minimizer. The uniqueness can be proved arguing as in the last part of the proof of Theorem 1.2.

In conclusion, whenever Ω_1 is such that (6.10) holds, then the image u (corresponding to the value of ξ_* defined by (6.9)) is a minimizer for the reconstruction problem of u_0 over the damaged region D. The image u has a larger contour than u_0, that nevertheless is reconstructible over $\Gamma(\delta)$ for suitably small values of δ. Note also that the value of s enters only in determining the ratio between a and b through (6.10): Once Ω_1 satisfies this constraint, the new color ξ_* appearing in u is the same for all the compatible values of s.

Example 6.4. The following example shows that the first assumption in (1.19) is optimal. For a constant $c > 0$ to be chosen later, set

$$\omega(\varepsilon) := c\varepsilon^2, \tag{6.12}$$

and let $R := (0,3) \times (0,3)$, $\Omega := (1,2) \times (1,2)$. We consider the image $u_0 := \xi_0 1_\Omega$, with $\xi_0 \neq 0$ and $\xi_0 \neq \xi_1$, where

$$|\xi_1| := \min\{|\xi|: \ \xi \in \mathbb{R}^3, \ \xi \cdot e = \xi_0 \cdot e\}.$$

For $i, j = 1, \dots, n$, set $Q_{ij,n} := (\frac{i-1}{n}, \frac{i}{n}) \times (\frac{k-1}{n}, \frac{k}{n})$, let x_{ij}^n be the center of $Q_{ij,n}$, and define $Q'_{ij,n} := Q(x_{ij}, \omega(\frac{1}{n}))$. We then set

$$D_n := \bigcup_{i,j=1}^{n} \big(Q_{ij,n} \setminus \overline{Q'_{ij,n}}\big).$$

We now show that with the present choice of ω, the result of Theorem 1.13 may fail. Indeed, define

$$u_n(x) := \begin{cases} 0 & \text{if } x \in R \setminus \Omega, \\ \xi_0 & \text{if } x \in \Omega \cap Q'_{ij,n} \text{ for some } i, j \in \{1, \dots, n\}, \\ \xi_1 & \text{otherwise.} \end{cases}$$

Then, recalling (6.12), a straightforward computation gives

$$F(u_n, D_n) = 4|\xi_1| + 4n^2\omega\left(\frac{1}{n}\right)|\xi_1 - \xi_0| = 4|\xi_1| + 4c|\xi_1 - \xi_0| < 4|\xi_0| = F(u_0, D_n),$$

where the last inequality holds provided that the constant c is small enough.

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Appendix A

Proposition A.1. *Let G be the function defined in* (1.13)*, that is,*

$$G(r,t,v,w) := \sup_{s_1,s_2>0} \frac{|s_1 r + s_2 t|}{|s_1 v + s_2 w|} = \sup_{s>0} \frac{|r + st|}{|v + sw|}, \tag{A.1}$$

where $r, t \in [-1,1]$ and $v, w \in S^1$ are linearly independent. Then

$$G(r,t,v,w) = \begin{cases} |r| & \text{if } |r| \geqslant |t| \text{ and } r^2(v \cdot w) - rt \geqslant 0, \\ |t| & \text{if } |t| \geqslant |r| \text{ and } t^2(v \cdot w) - rt \geqslant 0, \\ \sqrt{\frac{r^2+t^2-2rt(v\cdot w)}{1-(v\cdot w)^2}} & \text{otherwise.} \end{cases} \tag{A.2}$$

Moreover, in the third case,

$$G(r,t,v,w)=\sup_{s\in\mathbb{R}}\frac{|r+st|}{|v+sw|}=\sqrt{\frac{r^2+t^2-2rt(v\cdot w)}{1-(v\cdot w)^2}}\geqslant\max\{|r|,|t|\}. \tag{A.3}$$

Proof. Consider the function

$$\varphi(s):=\frac{|r+st|^2}{|v+sw|^2}=\frac{r^2+s^2t^2+2rts}{1+s^2+2sc},\quad r>0,$$

where $c:=v\cdot w$ and we used the fact that $v,w\in S^1$. Assume that $|r|\geqslant|t|$. Then it can be checked that

$$\varphi(s)=\frac{r^2+s^2t^2+2rts}{1+s^2+2sc}\geqslant r^2=\varphi(0)$$

for all $s\geqslant 0$ if and only if $r^2(v\cdot w)-rt\geqslant 0$. Hence, if $|r|\geqslant|t|$ and $r^2(v\cdot w)-rt\geqslant 0$, then $G(r,t,v,w)=|r|$. On the other hand, if $|r|\leqslant|t|$, then

$$\varphi(s)=\frac{r^2+s^2t^2+2rts}{1+s^2+2sc}\geqslant t^2=\lim_{s\to\infty}\varphi(s)$$

for all $s\geqslant 0$ if and only if $t^2(v\cdot w)-rt\geqslant 0$. It follows that if $|r|\leqslant|t|$ and $t^2(v\cdot w)-rt\geqslant 0$, then $G(r,t,v,w)=|t|$. Finally, in the remaining cases, we have that

$$M:=\sup_{s>0}\varphi(s)>\max\Big\{\varphi(0),\lim_{s\to\infty}\varphi(s)\Big\}.$$

Hence, there exists $s_0>0$ such that $M=\varphi(s_0)$. Define

$$p(s):=\big(M^2-t^2\big)s^2+2(Mc-rt)s+M^2-r^2,\quad s\in\mathbb{R}.$$

Then $p(s)\geqslant 0$ for all $s>0$ and $p(s_0)=0$. Since p is a convex parabola, it follows that s_0 is the unique global minimizer of p, and so $p(s)>0$ for all $s\neq s_0$. Hence, its discriminant vanishes, that is,

$$M=\frac{r^2+t^2-2rt(v\cdot w)}{1-(v\cdot w)^2},$$

which shows that

$$G(r,t,v,w)=\sup_{s\in\mathbb{R}}\frac{|r+st|}{|v+sw|}=\sqrt{\frac{r^2+t^2-2rt(v\cdot w)}{1-(v\cdot w)^2}}.\qquad\square$$

We conclude the appendix by recalling McShane's Lemma (see [21,25]):

Lemma A.2 *(McShane). If (X,d) is a metric space, $Y\subset X$, and $f:Y\to\mathbb{R}$ is r-Lipschitz with respect to the distance d, i.e.,*

$$\big|f(y_1)-f(y_2)\big|\leqslant rd(y_1,y_2)\quad\textit{for all } y_1,y_2\in Y,$$

then there exists $\bar f:X\to\mathbb{R}$ such that $\bar f=f$ on Y and

$$\big|\bar f(x_1)-\bar f(x_2)\big|\leqslant rd(x_1,x_2)\quad\textit{for all } x_1,x_2\in X.$$

One such extension $\bar f$ is given by

$$\bar f(x):=\inf\big\{f(y)+rd(x,y):\ y\in Y\big\},\quad x\in X.$$

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. 135 (1983) 293–318.

[3] G. Aronsson, Extension of functions satisfying Lipschitz conditions, Ark. Mat. 6 (1967) 551–556.

[4] G. Aronsson, M.G. Crandall, P. Juutinen, A tour of the theory of absolutely minimizing functions, Bull. Amer. Math. Soc. 41 (2004) 439–505.

[5] R.G. Bartle, The Elements of Real Analysis, second ed., John Wiley & Sons, New York, London, Sydney, 1976.

[6] G. Bellettini, V. Caselles, M. Novaga, The total variation flow in $\mathbb{R}^N$, J. Differ. Equations 184 (2002) 475–525.

[7] J. Buriánek, D. Sýkora, J. Žára, Unsupervised colorization of black-and-white cartoons, in: Proc. 3rd Int. Symp. Non-Photorealistic Animation and Rendering, 2004, pp. 121–127.

[8] T.F. Chan, J. Shen, Mathematical models for local nontexture inpaintings, SIAM J. Appl. Math. 62 (2001/2002) 1019–1043.

[9] T.F. Chan, J. Shen, Variational image inpainting, Comm. Pure Appl. Math. 58 (2005) 579–619.

[10] D. Cohen-Or, R. Irony, D. Lischinski, Colorization by example, in: Proc. Eurograph. Symp. Rendering, 2005, pp. 201–210.

[11] F. Demengel, Functions locally almost 1-harmonic, J. Appl. Anal. 83 (2004) 865–896.

[12] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, Classics in Applied Mathematics, vol. 28, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[13] I. Fonseca, G. Leoni, Modern Methods in the Calculus of Variations: L^p Spaces, Springer Monographs in Mathematics, Springer, New York, 2007.

[14] M. Fornasier, Nonlinear projection recovery in digital inpainting for color image restoration, J. Math. Imaging Vision 24 (2006) 359–373.

[15] M. Fornasier, Faithful recovery of vector valued functions from incomplete data, in: Scale Space and Variational Methods in Computer Vision, in: Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2009, pp. 116–127.

[16] M. Fornasier, R. March, Restoration of color images by vector valued BV functions and variational calculus, SIAM J. Appl. Math. 68 (2007) 437–460.

[17] S.H. Kang, R. March, Variational models for image colorization via chromaticity and brightness decomposition, IEEE Trans. Image Process. 16 (2007) 2251–2261.

[18] B. Kawohl, F. Schuricht, Dirichlet problems for the 1-Laplace operator, including the eigenvalue problem, Commun. Contemp. Math. 9 (2007) 515–543.

[19] M.D. Kirszbraun, Ber die zusammenziehende und Lipschitzsche Transformationen, Fund. Math. 22 (1934) 77–108.

[20] A. Levin, D. Lischinsk, Y. Weiss, Colorization using optimization, in: Proc. SIGGRAPH Conf., vol. 23, 2004, pp. 689–694.

[21] E.J. McShane, Extension of range of functions, Bull. Amer. Math. Soc. 40 (1934) 837–842.

[22] L.I. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, Phys. D 60 (1992) 259–268.

[23] G. Sapiro, Inpainting the colors, in: Proc. IEEE Int. Conf. Image Processing, vol. 2, 2005, pp. 698–701.

[24] G. Sapiro, L. Yatziv, Fast image and video colorization using chrominance blending, IEEE Trans. Image Process. 15 (2006) 1120–1129.

[25] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934) 63–89.