

# A local symmetry result for linear elliptic problems with solutions changing sign

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## Abstract

We prove that the only domain  $\Omega$  such that there exists a solution to the following problem  $\Delta u + \omega^2 u = -1$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , and  $\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \partial_{\mathbf{n}} u = c$ , for a given constant  $c$ , is the unit ball  $B_1$ , if we assume that  $\Omega$  lies in an appropriate class of Lipschitz domains.

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## 1. Introduction

Let us consider the following problem: for  $\omega \in \mathbb{R}$ , is it true that the only domain  $\Omega$  such that there exists a solution  $u$  to the problem

$$\begin{cases} \Delta u + \omega^2 u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

with

$$\partial_{\mathbf{n}} u = c \quad \text{on } \partial\Omega, \quad (1.2)$$

is a ball? Here  $\Omega$  is a sufficiently smooth bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $\partial_{\mathbf{n}} u$  is the external normal derivative to the boundary  $\partial\Omega$ , and  $c$  is a given constant. By using the Alexandrov method of moving planes J. Serrin [20] has proved that if there exists a solution  $u$  to (1.1), (1.2), and if  $u$  has a *sign* in  $\Omega$ , then  $\Omega = B_1$  (for example for  $\omega = 0$ , by the maximum principle it follows that  $u$  is positive in  $\Omega$ ). For the particular case  $\omega = 0$  see also the proofs of H. Weinberger [23], based on a Rellich-type identity and on the maximum principle, and M. Choulli, A. Henrot [7], which use the technique of domain derivative. We point out that Serrin in [20] has studied the same type of problem for more general nonlinear elliptic equations. For further references concerning symmetry (and non-symmetry) results for overdetermined elliptic problems, see also [1–4, 8–19, 21, 22]. All these results need hypothesis on the sign of  $u$ . In [5] the authors have given a positive answer to the above question by supposing that

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- (i)  $\omega^2 \notin \{\lambda_n\}_{n \geq 1}$  ( $\{\lambda_n\}_{n \geq 1}$  being the sequence, in increasing order, of eigenvalues of  $-\Delta$  in  $B_1$  with Dirichlet boundary conditions),
- (ii)  $\omega \notin \Lambda$ , where  $\Lambda$  is an enumerable set of  $\mathbb{R}^+$ , whose limit points are the values  $\lambda_{1m}$ , for some integer  $m \geq 1$ ,  $\lambda_{1m}$  being the  $m$ th-zero of the first-order Bessel function  $I_1$ ,
- (iii)  $\Omega$  is such that the  $\ker(\Delta + \omega^2) = \{0\}$  in  $\Omega$ ,
- (iv) the boundary  $\partial\Omega$  is a Lipschitz perturbation of the unit sphere  $\partial B_1$  of  $\mathbb{R}^N$ .

We point out that in [5] no hypothesis are required on the sign of the solution  $u$ . We can say that paper [6] can be considered as preparatory of [5] (in the sense that some ideas developed in [6] are used in [5]). In the present paper we give a new proof of the result proved in [5], which let us permit to avoid hypothesis (i)–(iii) above.

We recall that if let us denote by  $(\lambda_n)_{n \geq 1}$  the sequence, in increasing order, of eigenvalues of  $-\Delta$  in  $B_1$  with Dirichlet boundary conditions, we have that the eigenvalue  $\lambda_n$ , for some  $n \in \mathbb{N}$ , coincides, for some integers  $\ell \geq 0$  and  $m \geq 1$ , with  $\lambda_{\ell m}^2$ . Here and in what follows  $\lambda_{\ell m}$  will denote the  $m$ th-zero of the so-called  $N$ -dimensional  $\ell$ -order Bessel function of the first kind  $I_\ell$ , i.e.  $I_\ell(\lambda_{\ell m}) = 0$  (see Section 2). We recall in particular that (see [5, Lemma 3.5])

$$I'_0 = -I_1 \quad \text{in } \mathbb{R}.$$

From these remarks it follows that the function  $u^{(0)}$  given by

$$u^{(0)}(x) = \frac{1}{\omega^2} \left( \frac{I_0(\omega r)}{I_0(\omega)} - 1 \right) \quad \text{in } B_1, \tag{1.3}$$

solves (1.1), (1.2) when  $\Omega = B_1$ . Here  $r = |x|$ ,  $|\cdot|$  denoting the Euclidean norm in  $\mathbb{R}^N$ . We observe that if the constant  $\omega$  is smaller or equal than  $\lambda_{11}$ , the solution  $u^{(0)}$  is positive in  $B_1$ , while if  $\omega$  is bigger than  $\lambda_{11}$ , then  $u^{(0)}$  changes sign. In the rest of the paper we will assume  $\omega \geq 0$ . The same conclusions hold true for  $\omega < 0$ , since the coefficient  $\omega^2$  is even in (1.1). We stress out that in order that (1.3) makes sense, in the rest of the paper we will suppose that

$$\omega \notin \{\lambda_{0m}\}_{m \geq 1}.$$

Here and in what follows  $c = \partial_n u^{(0)}$  on  $\partial B_1$ . By (1.3), we obtain that

$$c = \frac{I'_0(\omega)}{\omega I_0(\omega)}. \tag{1.4}$$

In the present paper we prove the following

**Theorem 1.1.** *For  $\omega \notin \{\lambda_{0m}\}_{m \geq 1}$ , there exists a class  $\mathcal{D}$  of  $C^{2,\alpha}$ -domains such that if  $u$  is a solution to (1.1) verifying*

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \partial_n u = c,$$

*with  $\Omega \in \mathcal{D}$ , and  $c$  given by (1.4), then  $\Omega = B_1$ , and  $u = u^{(0)}$ .*

The idea underlying the proof of Theorem 1.1 is the following. Let  $E$  be the vector space of  $C^{2,\alpha}$  functions defined on the unit sphere  $\partial B_1$ , i.e.

$$E = \{k \in C^{2,\alpha}(\partial B_1)\},$$

$0 < \alpha < 1$ . For  $k \in E$ , let  $\Omega_k$  be the domain whose boundary  $\partial\Omega_k$  can be written as perturbation of  $\partial B_1$ , i.e.

$$\partial\Omega_k = \{x = (1+k)y, y \in \partial B_1\}$$

(in particular for  $k \equiv 0$  on  $\partial B_1$ ,  $\Omega_0 = B_1$ ). We denote by  $\Phi$  the following operator

$$\Phi : E \mapsto \mathbb{R},$$

defined by

$$\Phi(k) = \int_{\partial\Omega_k} \partial_n u_p - c \int_{\partial\Omega_k},$$

where  $u_p$  is a particular solution to (1.1), when  $\Omega = \Omega_k$  ( $u_p$  will be defined in Section 3 below). We observe that  $\Phi$  has not a sign in a neighborhood of 0 in  $E$  (i.e.  $\Phi$  is neither positive nor negative). In fact  $\Phi(0) = 0$  (since  $u_p = u^{(0)}$  when  $\Omega = B_1$ ). Moreover since the unit sphere centered at the point  $x_0 \in \mathbb{R}^N$  is parametrized by

$$\partial B_1(x_0) = \{x = (1 + k')y, y \in \partial B_1\},$$

where  $k'$  is given by

$$k'(y) = x_0 \cdot y - 1 + \sqrt{1 + |x_0 \cdot y|^2 - |x_0|^2}, \tag{1.5}$$

we have that  $\Phi(k') = 0$ , with

$$k' \rightarrow 0 \quad \text{in } E, \quad \text{as } x_0 \rightarrow 0.$$

So the best one can expect is that  $\Phi$  is different to 0 in  $\mathcal{O} \setminus \{k \in E; k = k'\}$ , for some neighborhood  $\mathcal{O}$  of 0 in  $E$ . By studying the behavior of the operator  $\Phi$  at 0, we prove that if  $\omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1}$ , with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ , then  $\Phi$  is differentiable at zero in  $E$ . On the other hand if  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$  (with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ ), then  $\Phi$  is differentiable at zero in the vector space

$$E_\ell = \{k \in E; k_{\ell q} = 0, k_{pq'} = 0, p \in I\} \tag{1.6}$$

of functions  $k \in E$  which don't have either the frequency  $\ell$  or the frequency  $p$ ,  $I$  being a (eventually empty) finite set of positive integer such that  $I_p(\lambda_{\ell m}) = 0$  (the cardinality of  $I$  depending on the multiplicity of the eigenvalue  $\lambda_{\ell m}^2$ , see Section 2 for more details). Here and in what follows  $k_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} k Y_{st}$  is the  $s$ -order (Fourier) coefficient of  $k$ , and  $Y_{st}$  is the spherical harmonic of degree  $s$ , with  $t = 1, \dots, d_s$ . More precisely we have that the differential at zero in the direction  $k$  has a sign if  $k_0 \neq 0$  (see Lemma 3.3),  $k_0$  being the zeroth-order coefficient of  $k$  (i.e.  $k_0 = \frac{1}{|\partial B_1|} \int_{\partial B_1} k$ ). We can show then that there exists a neighborhood  $\mathcal{O}$  of 0 in  $E$  such that  $\Phi$  is positive in  $\mathcal{O} \cap E^+$ , and  $\Phi$  is negative in  $\mathcal{O} \cap E^-$ , where  $E^+$  and  $E^-$  are two circular sectors respectively in the subset  $\{k \in E; k_0 < 0\}$ , and  $\{k \in E; k_0 > 0\}$ . Now, since if there exists a solution  $u$  to (1.1), when  $\Omega = \Omega_k$ , verifying  $\frac{1}{|\partial \Omega_k|} \int_{\partial \Omega_k} \partial_{\mathbf{n}} u = c$ , one can prove that  $\Phi(k) = 0$ , we obtain that  $k = 0$ , if we assume that  $k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})$ . Finally, since the operator  $\Phi$  is invariant up to isometries, we obtain that the class  $\mathcal{D}$  in Theorem 1.1 is defined as

$$\mathcal{D} = \{\Omega; \Omega = \sigma(\Omega_k)\},$$

for some  $\sigma \in \Sigma$ , and some  $\Omega_k \in \mathcal{G}$ , where  $\Sigma$  is the set of isometries of  $\mathbb{R}^N$ , and

$$\mathcal{G} = \{\Omega_k; k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})\}.$$

We stress out that  $E$  through the paper is the space of functions of class  $C^{2,\alpha}$  on  $\partial B_1$  (this means that we consider only regular perturbations of the unit sphere), but, up to obvious changes, the same conclusions hold true in the case where  $E$  is the space of functions of class  $C^{0,1}$  on  $\partial B_1$ , i.e. the boundary  $\partial \Omega_k$  is of Lipschitz class. The paper is organized as follows: in the next section we give some notations used through the paper, in Section 3 we give the first-order approximation of the operator  $\Phi$  in a neighborhood of 0, and in Section 4 we prove Theorem 1.1, and we consider the Lipschitz case. Finally in Section 5 counter-examples to Theorem 1.1 are given.

## 2. Preliminaries and notations

Let us denote by  $B_1$  the ball of radius 1 in  $\mathbb{R}^N$  centered at zero. By  $\bar{B}_1$  we define the Euclidean closure of  $B_1$ . Let us denote by  $I_\ell$  the so-called  $N$ -dimensional  $\ell$ -order Bessel function of the first kind, i.e.

$$I_\ell(r) = r^{-\nu} J_{\nu+\ell}(r),$$

where  $\nu = \frac{N}{2} - 1$ , and  $J_{\nu+\ell}$  is the well-known  $(\nu + \ell)$ -order Bessel function of the first kind (we observe that for  $N = 2$ ,  $I_\ell$  coincides with the  $\ell$ -order Bessel function of the first kind  $J_\ell$ ).  $I_\ell$  solves the following Bessel equation

$$I_\ell'' + \frac{N-1}{r} I_\ell' + \left(1 - \frac{\ell(\ell+N-2)}{r^2}\right) I_\ell = 0 \quad \text{in } \mathbb{R}.$$

Let  $\lambda_{\ell m}$  be the  $m$ th-zero of the  $\ell$ -order Bessel function  $I_\ell$ . Let  $(\lambda_n)_{n \geq 1}$  be the sequence, in increasing order, of eigenvalues of  $-\Delta$  in  $B_1$  with Dirichlet boundary conditions. An eigenvalue  $\lambda_n$ , for some  $n \in \mathbb{N}$ , coincides, for some integer  $\ell \geq 0$ , and  $m \geq 1$ , with  $\lambda_{\ell m}^2$ . The corresponding eigenfunctions can be written as (in polar coordinates)

$$\begin{aligned} \varphi_1 &= I_\ell(\lambda_{\ell m} r) Y_{\ell 1}(\theta), \\ &\vdots \\ \varphi_{d_\ell} &= I_\ell(\lambda_{\ell m} r) Y_{\ell d_\ell}(\theta), \\ \varphi_{p_q} &= I_p(\lambda_{\ell m} r) Y_{p_q}(\theta), \end{aligned}$$

where  $p \in I$ , and  $I$  is a (eventually empty) finite set (by Fredholm theorem) of integer such that  $I_p(\lambda_{\ell m}) = 0$ , i.e.

$$I = \{p \in \mathbb{N}, p \neq \ell; I_p(\lambda_{\ell m}) = 0\}. \tag{2.1}$$

Here  $Y_{st}$  is the spherical harmonic of degree  $s$ , with  $t = 1, \dots, d_s$ , and

$$d_s = \begin{cases} 1 & \text{if } s = 0, \\ \frac{(2s+N-2)(s+N-3)!}{s!(N-2)!} & \text{if } s \geq 1. \end{cases}$$

We will use the following convention: we say that a function  $f$  has the frequency  $s$ , if the  $s$ -order coefficient of  $f$ , i.e.  $f_{st} = \frac{1}{|\partial B_1|} \int_{\partial B_1} f Y_{st}$ , is different to zero. And similarly we say that a function  $f$  doesn't have the frequency  $s$ , if the  $s$ -order coefficient of  $f$  vanishes.

Let  $\tilde{k}$  be a  $C^{2,\alpha}$ -extension of  $k$  into  $\bar{B}_1$ . Let us call  $A$  the Jacobian matrix of change of variable

$$x = (1 + k(y))y, \quad y \in \bar{B}_1 \tag{2.2}$$

(where we denote  $\tilde{k}$  by  $k$ ). The matrix  $A$  is given by

$$A_{ij} = \begin{bmatrix} 1 + k + y_1 \partial_1 k & y_1 \partial_2 k & \dots & y_1 \partial_N k \\ y_2 \partial_1 k & 1 + k + y_2 \partial_2 k & \dots & y_2 \partial_N k \\ \vdots & \vdots & \ddots & \vdots \\ y_N \partial_1 k & \dots & \dots & 1 + k + y_N \partial_N k \end{bmatrix}.$$

Let  $G = A^T A$ . The matrix  $G$  can be written as

$$G = I_N + G^{(1)} + o(\|k\|),$$

where  $I_N$  is the  $N$ -order identity matrix, and the matrix  $G^{(1)}$  depends linearly on  $k$  and  $\nabla k$ . Following [5], the matrix  $G^{(1)}$  is given by

$$G_{ij}^{(1)} = 2k I_N + \begin{bmatrix} 2x_1 \partial_1 k & x_1 \partial_2 k + x_2 \partial_1 k & \dots & x_1 \partial_N k + x_N \partial_1 k \\ x_1 \partial_2 k + x_2 \partial_1 k & 2x_2 \partial_2 k & \dots & x_2 \partial_N k + x_N \partial_2 k \\ \vdots & \vdots & \ddots & \vdots \\ x_1 \partial_N k + x_N \partial_1 k & \dots & \dots & 2x_N \partial_N k \end{bmatrix}. \tag{2.3}$$

### 3. The first-order expansion of the operator $\Phi$

A function  $k \in E$  can be written, in Fourier series expansion, as

$$k = k_0 + \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{p_q} Y_{p_q} \quad \text{on } \partial B_1.$$

We recall that problem (1.1) cannot have solutions or, if a solution exists, it cannot be unique. This happens all times the kernel  $\ker(\Delta + \omega^2) \neq \{0\}$  in  $\Omega$ . More precisely by Fredholm theorem there exists a solution to (1.1) if and only if

$$-1 \in \ker(\Delta + \omega^2)^\perp \quad \text{in } \Omega.$$

We can write a solution  $u$  as

$$u = u_p + u_h,$$

where  $u_p$  is a particular solution to (1.1) such that

$$u_p \in \ker(\Delta + \omega^2)^\perp \quad \text{in } \Omega, \tag{3.1}$$

and  $u_h$  solves the corresponding homogeneous problem. We observe that  $u_p$  is unique and can be written as

$$u_p = \sum_{p \in I^C} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq},$$

where  $\alpha_{pq} = \frac{\int_\Omega \psi_{pq}}{\mu - \lambda_p}$  is the  $p$ -order Fourier coefficient of  $u$ . Here  $\lambda_p$  and  $\psi_{pq}$  are respectively the  $p$ th-eigenvalue and a corresponding eigenfunction of  $-\Delta$  in  $\Omega$  (with Dirichlet boundary conditions), and  $n_p$  is the dimension of the corresponding eigenspace.  $I$  is a finite set of integer (by Fredholm theorem), and  $I^C$  is the complementary of  $I$ . On the other hand if the kernel  $\ker(\Delta + \omega^2) = \{0\}$ , then a solution  $u$  exists and is unique. For example for  $\omega = \lambda_{\ell m}$ , for some  $\ell, m \geq 1$ , then  $u_p = \frac{1}{\lambda_{\ell m}^2} (\frac{I_0(\lambda_{\ell m} r)}{I_0(\lambda_{\ell m})} - 1)$  is a particular solution to (1.1) when  $\Omega = B_1$  (lying in the  $\ker(\Delta + \lambda_{\ell m}^2)^\perp$  in  $B_1$ ), and  $u_h$  has the form (in polar coordinates)

$$u_h = \sum_{q=1}^{d_\ell} \alpha_{\ell q} I_\ell(\lambda_{\ell m} r) Y_{\ell q}(\theta) + \sum_{p \in I} \sum_{q=1}^{d_p} \alpha_{pq} I_p(\lambda_{\ell m} r) Y_{pq}(\theta),$$

where  $I$  is defined in (2.1), and  $\alpha_{\ell 1}, \dots, \alpha_{\ell d_\ell}, \alpha_{pq} \in \mathbb{R}$ . We denote by  $\Phi$  the following operator

$$\Phi : E \mapsto \mathbb{R},$$

defined by

$$\Phi(k) := \int_{\partial \Omega_k} \partial_{\mathbf{n}} u_p - c \int_{\partial \Omega_k},$$

where  $u_p$  is a particular solution to (1.1), verifying (3.1), when  $\Omega = \Omega_k$ . The operator  $\Phi$  is well-defined, since we suppose that a solution  $u$  exists for  $k$  lying in some neighborhood of 0 in  $E$ . Using (2.2), we have that the function  $\tilde{u}$  defined by

$$\tilde{u}(y) = u((1+k)y) \quad \text{in } \bar{B}_1,$$

solves

$$\begin{cases} \operatorname{div}(\sqrt{g} G^{-1} \nabla \tilde{u}) + \omega^2 \sqrt{g} \tilde{u} = -\sqrt{g} & \text{in } B_1, \\ \tilde{u} = 0 & \text{on } \partial B_1, \end{cases} \tag{3.2}$$

where  $g = |\det G|$ . Following [5], the external normal derivative of  $u$  at the point  $x = (1+k)y \in \partial \Omega_k$  is given by

$$\partial_{\mathbf{n}} u((1+k)y) = (G^{-1} y \cdot y)^{-1/2} G^{-1} \nabla \tilde{u} \cdot y.$$

The operator  $\Phi$  then becomes

$$\Phi(k) = \int_{\partial B_1} (G^{-1} y \cdot y)^{-1/2} G^{-1} \nabla \tilde{u}_p \cdot y \sqrt{g} - c \int_{\partial B_1} \sqrt{g},$$

where  $\tilde{u}_p(y) = u_p((1+k)y)$ , and  $\sqrt{g}$  is the surface element of the new variable  $y$ . Let us denote  $\tilde{u}_p$  by  $u_p$ , and  $y$  by  $x$ . We begin by proving the following

**Lemma 3.1.** *We have*

$$u_p \rightarrow u^{(0)} \quad \text{as } k \rightarrow 0.$$

**Proof of Lemma 3.1.** Let  $z = u_p - u^{(0)}$ . By writing the matrix  $\sqrt{g} G^{-1}$  in (3.2) as

$$\sqrt{g} G^{-1} = I_N + K, \tag{3.3}$$

it follows that  $z$  solves

$$\begin{cases} \Delta w + \omega^2 w = (1 - \sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K \nabla u_p) & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1. \end{cases} \tag{3.4}$$

Let assume that the  $\ker(\Delta + \omega^2) = \{0\}$  in  $B_1$ . The solution  $w$  to (3.4) can be written as

$$w = \sum_{p=1}^{+\infty} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq},$$

where the  $p$ -order Fourier coefficient

$$\alpha_{pq} = \frac{\int_{B_1} ((1 - \sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K \nabla u_p)) \psi_{pq}}{\omega^2 - \lambda_p}.$$

Since

$$\sqrt{g} = 1 + Nk + x \cdot \nabla k + o(\|k\|), \tag{3.5}$$

we obtain

$$w \rightarrow 0 \quad \text{as } k \rightarrow 0.$$

On the other hand, if the  $\ker(\Delta + \omega^2) \neq \{0\}$  in  $B_1$ , i.e.  $\omega^2 = \lambda_n$ , for some  $n \geq 2$  (we recall that  $\lambda_n \notin \{\lambda_{0m}^2\}_{m \geq 1}$ ), then a solution  $w$  to (3.4) can be written as

$$w = w_p + w_h,$$

where

$$w_p = \sum_{p \in I^c} \sum_{q=1}^{n_p} \alpha_{pq} \psi_{pq}.$$

We claim that  $w_p = z$ . We have that the function  $w_p - z$  solves

$$\begin{cases} \Delta(w_p - z) + \lambda_n(w_p - z) = 0 & \text{in } B_1, \\ w_p - z = 0 & \text{on } \partial B_1. \end{cases}$$

So we obtain

$$w_p - z = \sum_{p \in I} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq},$$

i.e.

$$u_p = u^{(0)} + w_p + \sum_{p \in I} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq},$$

for all  $\beta_{pq} \in \mathbb{R}$ . Since  $u_p$  is a solution to (3.2), it follows that

$$\begin{aligned} -\sqrt{g} &= \operatorname{div}(\sqrt{g} G^{-1} \nabla u_p) + \lambda_n \sqrt{g} u_p \\ &= \operatorname{div}(\sqrt{g} G^{-1} \nabla (u^{(0)} + w_p)) + \lambda_n \sqrt{g} (u^{(0)} + w_p) \\ &\quad + \sum_{p \in I} \sum_{q=1}^{n_p} \beta_{pq} \operatorname{div}(\sqrt{g} G^{-1} \nabla \psi_{pq}) + \lambda_n \sqrt{g} \sum_{p \in I} \sum_{q=1}^{n_p} \beta_{pq} \psi_{pq} \\ &= -\sqrt{g} + \sum_{p \in I} \sum_{q=1}^{n_p} \beta_{pq} (\operatorname{div}(\sqrt{g} G^{-1} \nabla \psi_{pq}) + \lambda_n \sqrt{g} \psi_{pq}). \end{aligned}$$

In particular we obtain

$$\beta_{pq} (\operatorname{div}(\sqrt{g}G^{-1}\nabla\psi_{pq}) + \lambda_n\sqrt{g}\psi_{pq}) = 0.$$

We claim that

$$\operatorname{div}(\sqrt{g}G^{-1}\nabla\psi_{pq}) + \lambda_n\sqrt{g}\psi_{pq} \neq 0 \quad \text{in } B_1.$$

By contradiction let assume that there exists a  $p \in I$  and a  $q \in \{1, \dots, n_p\}$  such that

$$\operatorname{div}(\sqrt{g}G^{-1}\nabla\psi_{pq}) + \lambda_n\sqrt{g}\psi_{pq} = 0 \quad \text{in } B_1.$$

By defining by  $y = y(x)$  the inverse of the change of variable (2.2), we obtain that

$$\tilde{\psi}_{pq}(x) = \psi_{pq}(y(x)), \quad x \in \Omega_k,$$

solves

$$\Delta\tilde{\psi}_{pq} + \lambda_n\tilde{\psi}_{pq} = 0 \quad \text{in } \Omega_k, \quad \tilde{\psi}_{pq} = 0 \quad \text{on } \partial\Omega_k.$$

This implies that  $\lambda_n$  is an eigenvalue of  $-\Delta$  in  $\Omega_k$ . Then  $u_p$  doesn't lie in  $\ker(\Delta + \lambda_n)^\perp$  in  $\Omega_k$ , which yields a contradiction. This yields that  $\beta_{pq} = 0$ , for all  $p \in I$ , and  $q = 1, \dots, n_p$ , and then  $u_p = u^{(0)} + w_p$ .  $\square$

By (3.3) it follows that

$$\sqrt{g}I_N - G = KG = (K^{(1)} + o(\|k\|))(I_N + G^{(1)} + o(\|k\|)),$$

where  $K^{(1)}$  denotes the one-order term of the matrix  $K$  (the matrix  $G^{(1)}$  is given by (2.3)). In particular the matrix

$$K^{(1)} = g^{(1)}I_N - G^{(1)}, \tag{3.6}$$

where  $g^{(1)}$ , the one-order term of  $\sqrt{g}$ , is given by

$$g^{(1)} = Nk + x \cdot \nabla k. \tag{3.7}$$

By (3.5) we have

$$\frac{1}{\sqrt{g}} = 1 - Nk - x \cdot \nabla k + o(\|k\|),$$

and by (3.3), (3.6), and (3.7), we obtain

$$\begin{aligned} G^{-1} &= \frac{I_N}{\sqrt{g}} + \frac{1}{\sqrt{g}}K^{(1)} + \dots \\ &= I_N - G^{(1)} + o(\|k\|). \end{aligned} \tag{3.8}$$

**Lemma 3.2.** *If  $\omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1}$ , with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ , then  $u_p$  has the form*

$$u_p = u^{(0)} + u^{(1)} + o(\|k\|) \quad \text{in } E, \tag{3.9}$$

where  $u^{(1)}$  solves

$$\begin{cases} \Delta u^{(1)} + \omega^2 u^{(1)} = f^{(1)} & \text{in } B_1, \\ u^{(1)} = 0 & \text{on } \partial B_1, \end{cases} \tag{3.10}$$

and  $f^{(1)}$  is given by

$$f^{(1)} = -(Nk + x \cdot \nabla k)(1 + \omega^2 u^{(0)}) - \operatorname{div}(K^{(1)}\nabla u^{(0)}).$$

If  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$  (with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ ), the same holds true by changing  $E$  with  $E_\ell$ , where  $E_\ell$  is defined in (1.6).

To prove Lemma 3.2, we observe that if the  $\ker(\Delta + \omega^2) = \{0\}$  in  $B_1$ , then  $u_p$  admits a one-order expansion in  $E$ . The same holds true if the  $\ker(\Delta + \omega^2) \neq \{0\}$  in  $B_1$ , with  $\omega = \lambda_{1m}$ , for some  $m \geq 1$ . On the other hand, if the  $\ker(\Delta + \omega^2) = \{0\}$  in  $B_1$ , i.e.  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$ , then  $u_p$  admits a one-order expansion in the vector space  $E_\ell$  of functions  $k \in E$  which don't have either the frequency  $\ell$  or the frequency  $p$ , with  $p \in I$ , the set  $I$  being defined in (2.1).

**Proof of Lemma 3.2.** Let  $\omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1}$ , with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ . Let assume that  $u_p$  can be written as in (3.9). Then  $u_p$  solves

$$\begin{cases} \Delta u_p + \operatorname{div}(K \nabla u_p) + \omega^2 \sqrt{g} u_p = -\sqrt{g} & \text{in } B_1, \\ u_p = 0 & \text{on } \partial B_1. \end{cases} \tag{3.11}$$

We have

$$\begin{aligned} \operatorname{div}(K \nabla u_p) + \sqrt{g}(\omega^2 u_p + 1) &= \operatorname{div}(K^{(1)}(\nabla u^{(0)} + \nabla u^{(1)})) \\ &\quad + (1 + Nk + x \cdot \nabla k)(\omega^2(u^{(0)} + u^{(1)} + 1) + \dots. \end{aligned} \tag{3.12}$$

The one-order terms in (3.12) are given by

$$(Nk + x \cdot \nabla k)(1 + \omega^2 u^{(0)}) + \omega^2 u^{(1)} + \operatorname{div}(K^{(1)} \nabla u^{(0)}).$$

By taking the one-order terms in (3.11), we obtain that  $u^{(1)}$  solves (3.10). By a direct calculation  $u^{(1)}$  has the form

$$u^{(1)} = \frac{I'_0(\lambda_{1m} r)}{\lambda_{1m} I_0(\lambda_{1m})} r k,$$

if  $\omega = \lambda_{1m}$ , since  $I'_0 = -I_1$ . Otherwise, for  $\omega \neq \lambda_{1m}$ , then  $u^{(1)}$  has the form

$$u^{(1)} = \frac{I'_0(\omega r)}{\omega I_0(\omega)} r k + \bar{u},$$

where  $\bar{u}$  solves

$$\begin{cases} \Delta \bar{u} + \omega^2 \bar{u} = 0 & \text{in } B_1, \\ \bar{u} = \frac{I_1(\omega)}{\omega I_0(\omega)} k & \text{on } \partial B_1. \end{cases}$$

The solution  $\bar{u}$  (in polar coordinates) can be written as

$$\bar{u}(r, \theta) = -c \left( k_0 I_0(\omega r) / I_0(\omega) + \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq} I_p(\omega r) / I_p(\omega) Y_{pq}(\theta) \right). \tag{3.13}$$

Now obviously (3.13) is well-defined for all  $\omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1}$ . Let us define by

$$w = u_p - u^{(0)} - u^{(1)}.$$

The function  $w$  solves

$$\begin{cases} \Delta w + \omega^2 w = (1 - \sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K \nabla u_p) - f^{(1)} & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

By writing  $u_p$  as

$$u_p = u^{(0)} + f,$$

with  $f(k) = o(1)$  as  $k \rightarrow 0$  in  $E$ , we obtain

$$(1 - \sqrt{g})(\omega^2 u_p + 1) - \operatorname{div}(K \nabla u_p) - f^{(1)} = o(\|k\|).$$

By standard  $C^{2,\alpha}$ -estimates we obtain

$$\|w\|_{C^{2,\alpha}(B_1)} = o(\|k\|).$$

Now if  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$ , then (3.13) makes sense if and only if  $k \in E_\ell$ , and the same above conclusions hold true, by substituting  $E$  with  $E_\ell$ .  $\square$



**Lemma 3.3.** *If  $\omega \notin \{\lambda_{\ell m}\}_{\ell \geq 2, m \geq 1}$ , with  $\lambda_{\ell m} \neq \lambda_{1m'}$ , for all  $m' \geq 1$ , then the operator  $\Phi$  is differentiable at 0 in  $E$ , and*

$$\langle d\Phi(0) | k \rangle = -k_0 \left( \frac{I'_1(\omega)}{I_0(\omega)} + \frac{I'_0(\omega)^2}{I_0(\omega)^2} \right) |\partial B_1|.$$

Otherwise if  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$ , the same holds true by changing  $E$  with  $E_\ell$ .

The previous lemma means that if  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$ , then  $\Phi$  is not differentiable at 0 in  $k$ , with  $k$  having the form

$$k = \sum_{m=1}^{d_\ell} k_{\ell m} Y_{\ell m}(\theta) + \sum_{p \in I} \sum_{q=1}^{d_p} k_{pq} Y_{pq}(\theta). \tag{3.14}$$

**Proof of Lemma 3.3.** By (2.3), (3.8), and (3.9), we obtain

$$\begin{aligned} \Phi(k) &= \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} G^{-1} \nabla u_p \cdot x \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} \\ &= \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} G^{-1} \nabla u^{(0)} \cdot x \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} + \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} G^{-1} \nabla u^{(1)} \cdot x \sqrt{\tilde{g}} + \dots \\ &= c \int_{\partial B_1} (1 - 2k - 2\partial_{\mathbf{n}}k)^{1/2} \sqrt{\tilde{g}} - c \int_{\partial B_1} \sqrt{\tilde{g}} \\ &\quad + \int_{\partial B_1} (1 - 2k - 2\partial_{\mathbf{n}}k)^{-1/2} (\partial_{\mathbf{n}}u^{(1)} - G^{(1)} \nabla u^{(1)} \cdot x) \sqrt{\tilde{g}} + \dots \end{aligned} \tag{3.15}$$

Since the surface element  $\sqrt{\tilde{g}}$  can be written as

$$\sqrt{\tilde{g}} = 1 + o(\|k\|),$$

by taking the one-order terms in (3.15), we obtain

$$\langle d\Phi(0) | k \rangle = -c \int_{\partial B_1} (k + \partial_{\mathbf{n}}k) + \int_{\partial B_1} \partial_{\mathbf{n}}u^{(1)}.$$

Since

$$\partial_{\mathbf{n}}u^{(1)} = \left( \frac{I''_0(\omega)}{I_0(\omega)} + c \right) k + c \partial_{\mathbf{n}}k + \partial_{\mathbf{n}}\bar{u},$$

and

$$\partial_{\mathbf{n}}\bar{u} = -c\omega \left( k_0 I'_0(\omega) / I_0(\omega) + \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq} I'_p(\omega) / I_p(\omega) Y_{pq}(\theta) \right),$$

we obtain

$$\begin{aligned} \langle d\Phi(0) | k \rangle &= -c \int_{\partial B_1} (k + \partial_{\mathbf{n}}k) + \left( c - \frac{I'_1(\omega)}{I_0(\omega)} \right) \int_{\partial B_1} k + c \int_{\partial B_1} \partial_{\mathbf{n}}k + \int_{\partial B_1} \partial_{\mathbf{n}}\bar{u} \\ &= -\frac{I'_1(\omega)}{I_0(\omega)} \int_{\partial B_1} k - c\omega \frac{I'_0(\omega)}{I_0(\omega)} k_0 |\partial B_1| \\ &= -k_0 \left( \frac{I'_1(\omega)}{I_0(\omega)} + \frac{I'_0(\omega)^2}{I_0(\omega)^2} \right) |\partial B_1|, \end{aligned}$$

being  $c = \frac{I'_0(\omega)}{\omega I_0(\omega)}$ .  $\square$

**Lemma 3.4.** *The number*

$$\frac{I_1'(\omega)}{I_0(\omega)} + \frac{I_0'(\omega)^2}{I_0(\omega)^2} > 0. \quad (3.16)$$

**Proof of Lemma 3.4.** We have

$$\Phi(k_0) = \int_{\partial B_{1+k_0}} \partial_{\mathbf{n}} u_p - c \int_{\partial B_{1+k_0}} = \left( \frac{I_0'((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I_0'(\omega)}{I_0(\omega)} \right) \frac{|\partial B_{1+k_0}|}{\omega}.$$

Now since the function

$$\frac{I_0'(\omega)}{I_0(\omega)}$$

is decreasing in  $\omega$ , it follows that for  $k_0 > 0$  sufficiently small, the function

$$\frac{I_0'((1+k_0)\omega)}{I_0((1+k_0)\omega)} - \frac{I_0'(\omega)}{I_0(\omega)} < 0.$$

So  $\Phi$  is decreasing in the direction  $tk_0$ , for some  $t \in I$ , and then

$$\langle d\Phi(0) | k_0 \rangle < 0,$$

which yields (3.16).  $\square$

#### 4. Proof of Theorem 1.1

Before proceeding with the proof of Theorem 1.1, we need the following

**Lemma 4.1.** *There exists a neighborhood  $\mathcal{O}$  of the origin in  $E$ , such that if  $k \in \mathcal{O} \cap E_1^C$ , then the mass center  $\bar{x}$  of  $\Omega_k$  is different to zero.*

Here  $E_1$  is the vector space

$$E_1 = \{k \in E; k_{1q} = 0\},$$

of functions  $k \in E$  which don't have the frequency 1, and

$$E_1^C = \{k \in E; k_{1q} \neq 0 \text{ for s}$$

$$\begin{aligned}
 F_i(k) &= \frac{1}{|\Omega_k|} \int_{\Omega_k} x_i = \frac{1}{\int_{B_1} \sqrt{g}} \int_{B_1} (1+k)x_i \sqrt{g} \\
 &= \int_{B_1} (1 - Nk - x \cdot \nabla k + \dots) \int_{B_1} (x_i + (N+1)kx_i + x \cdot \nabla kx_i + \dots) \\
 &= \int_{B_1} (1 - Nk - x \cdot \nabla k + \dots) \int_{B_1} ((N+1)kx_i + x \cdot \nabla kx_i + \dots).
 \end{aligned}$$

By taking the one-order terms, we have that the differential of  $F_i$  at zero in  $k$  is given by

$$\begin{aligned}
 \langle dF_i(0) | k \rangle &= (N+1) \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq} \int_0^1 r^{p+N} \int_{\partial B_1} Y_{pq} Y_{1i} + \sum_{p \geq 1} \sum_{q=1}^{d_p} p k_{pq} \int_0^1 r^{p+N-1} \int_{\partial B_1} Y_{pq} Y_{1i} \\
 &= (N+1)k_{1i} \int_0^1 r^{N+1} + k_{1i} \int_0^1 r^N \\
 &= 1
 \end{aligned}$$

where  $C = \frac{I'_1(\omega)}{I_0(\omega)} + \frac{I'_0(\omega)^2}{I_0(\omega)^2}$ . So there exists a sufficiently small interval  $I$  of  $0$  in  $\mathbb{R}^+$  such that  $\Phi$  is positive in

$$E^+ = \{tk; t \in I, k \in E_\epsilon^+\}, \tag{4.1}$$

and  $\Phi$  is negative in

$$E^- = \{tk; t \in I, k \in E_\epsilon^-\}. \tag{4.2}$$

Let  $\mathcal{O}$  be a neighborhood of  $0$  in  $E$  such that  $\mathcal{O} \cap E^+ \cup \{0\}$  is contained in  $E^+ \cup \{0\}$ , and  $\mathcal{O} \cap E^- \cup \{0\}$  is contained in  $E^- \cup \{0\}$ . Now if  $\omega = \lambda_{\ell m}$ , for some  $\ell \geq 2$ , and  $m \geq 1$ , the same above conclusions hold true by changing  $E_1$  with the subspace

$$E_\ell = \{k \in E_1; k_{\ell q} = 0, k_{pq'} = 0, p \in I\}$$

of  $E_1$ . Now since for example  $\Phi$  is positive in  $E^+ \cap E_\ell$  and is continuous in  $E^+$ , and  $E_\ell$  is finite dimensional, it follows that  $\Phi$  is positive in  $E^+$ .

*Step 2.* Let  $\mathcal{D}$  be the class of  $C^{2,\alpha}$ -domains defined as

$$\mathcal{D} = \{\Omega; \Omega = \sigma(\Omega_k)\},$$

for some  $\sigma \in \Sigma$ , and some  $\Omega_k \in \mathcal{G}$ , where  $\Sigma$  is the set of isometries of  $\mathbb{R}^N$ , and

$$\mathcal{G} = \{\Omega_k; k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})\}.$$

Let assume that there exists a  $\Omega \in \mathcal{D}$  such that  $\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \partial_{\mathbf{n}} u = c$ . Since the problem is invariant up to isometries we have that  $\frac{1}{|\partial\Omega_k|} \int_{\partial\Omega_k} \partial_{\mathbf{n}} u = c$ , for some  $k \in \mathcal{O} \cap (E^+ \cup E^- \cup \{0\})$ .

*Step 3.* Let assume that the kernel  $\ker(\Delta + \omega^2) = \{0\}$  in  $\Omega_k$ . Then  $u$  coincides with  $u_p$ , and

$$\Phi(k) = 0.$$

Let assume that  $k \in \mathcal{O} \cap E^+ \cup \{0\}$ . This yields that  $k = 0$ , since  $\Phi$  is positive in  $\mathcal{O} \cap E^+$ . Now if the kernel  $\ker(\Delta + \omega^2) \neq \{0\}$  in  $\Omega_k$ , then  $u$  can be written as

$$u = u_p + u_h \quad \text{in } \Omega_k.$$

Since by Fredholm theorem  $-1 \in \ker(\Delta + \omega^2)^\perp$ , by divergence theorem we obtain

$$0 = \int_{\Omega_k} u_h = -\frac{1}{\omega^2} \int_{\Omega_k} \Delta u_h = -\frac{1}{\omega^2} \int_{\partial\Omega_k} \partial_{\mathbf{n}} u_h.$$

Then we have

$$\Phi(k) = \int_{\partial\Omega_k} \partial_{\mathbf{n}} u_p - c \int_{\partial\Omega_k} = \int_{\partial\Omega_k} \partial_{\mathbf{n}} u - c \int_{\partial\Omega_k} = 0. \quad \square$$

We conclude this section by examining briefly the Lipschitz case. Let us define by

$$E = \{k \in C^{0,1}(\partial B_1)\}.$$

Let  $u \in H^1(\Omega_k)$  be a weak solution to (1.1), when  $\Omega = \Omega_k$ , and  $k \in E$ . Then  $u$  solves

$$\int_{\Omega_k} \nabla u \cdot \nabla \phi - \omega^2 \int_{\Omega_k} u \phi = \int_{\Omega_k} \phi,$$

for all  $\phi \in C_c^\infty(\Omega_k)$ . Since, by regularity results,  $u \in C^{0,1}(\overline{\Omega_k})$ , the operator  $\Phi$  is well-defined in  $E$ . By repeating the same arguments as in the regular case, one can prove the following

**Theorem 4.2.** For  $\omega \notin \{\lambda_{0m}\}_{m \geq 1}$ , there exists a class  $\mathcal{D}$  of Lipschitz domains, such that if  $u \in H^1(\Omega)$  is a weak solution to (1.1) verifying

$$\frac{1}{|\partial\Omega|} \int_{\partial\Omega} \partial_{\mathbf{n}} u = c,$$

with  $\Omega \in \mathcal{D}$ , and  $c$  given by (1.4), then  $\Omega = B_1$ , and  $u = u^{(0)}$ .

## 5. Concluding remark

We recall that by the proof of Theorem 1.1 it follows that  $\Phi$  is positive in the circular sector  $E^+$  in  $\{k \in E; k_0 < 0\}$ , and is negative in the circular sector  $E^-$  in  $\{k \in E; k_0 > 0\}$ . So the operator  $\Phi$  must vanish somewhere. In fact let  $\epsilon > 0$  be fixed. Let  $k \in E^-$ . Then  $\Phi(k)$  is negative. Now the domain  $\tilde{\Omega}_k$ , whose boundary is given by

$$\partial\tilde{\Omega}_k = \{x = (1 + (a + k))y, y \in \partial B_1\},$$

with  $-1 < a < 0$ , is a contraction of the domain  $\Omega_k$ . We can find then a value  $a$  such that  $a + k \in E^+$ . But  $\Phi(a + k)$  is positive. Then there exists a  $\bar{k}$  such that  $\Phi(\bar{k}) = 0$ . By repeating the same argument for all  $\epsilon > 0$ , and for all  $k \in E^-$ , we can find a variety  $\mathcal{M}$  in  $E_1$  (whose tangent space at 0 is contained or coincides with  $E_0 = \{k; k_0 = 0\}$ ), such that  $\Phi$  vanishes identically on  $\mathcal{M}$ . In particular we obtain that all domains  $\Omega$  lying in the class

$$\mathcal{D} = \{\Omega; \Omega = \sigma(\Omega_k)\},$$

for some  $\sigma \in \Sigma$ , and some  $k \in \mathcal{M}$ , are counter-examples to Theorem 1.1.

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