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On the uniqueness of sign changing bound state solutions of a semilinear equation ☆

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Abstract

We establish the uniqueness of the higher radial bound state solutions of

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^n. \tag{P}$$

We assume that the nonlinearity $f \in C(-\infty, \infty)$ is an odd function satisfying some convexity and growth conditions, and has one zero at $b > 0$, is nonpositive and not-identically 0 in $(0, b)$, positive in $[b, \infty)$, and is differentiable in $(0, \infty)$.

1. Introduction and main results

In this paper we establish the uniqueness of higher bound state solutions to

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^n, \tag{P}$$

in the radial situation. That is, we give conditions on f under which

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0, \quad r > 0,\ n \geqslant 2, \\ &u'(0) = 0, \qquad \lim_{r\to\infty} u(r) = 0, \end{aligned} \tag{1}$$

has at most two solutions, one with $u(0) > 0$ and one with $u(0) < 0$, having a certain number of zeros.

Any nonconstant solution to (1) is called a bound state solution. Bound state solutions such that $u(r) > 0$ for all $r > 0$, are referred to as a first bound state solution, or a ground state solution. The uniqueness of the first bound state solution of (1) or for the quasilinear situation involving the m-Laplacian operator $\nabla \cdot (|\nabla u|^{m-2}\nabla u)$, $m > 1$, has been exhaustively studied during the last thirty years, see for example the works [2–7,10,11,13–15,17–20].

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We will assume that the function $f:\mathbb{R}\to\mathbb{R}$ is continuous, and that f satisfies (f_1)–(f_2), where:

(f_1) f is odd, $f(0)=0$, and there exist $\beta>b>0$ such that $f(s)>0$ for $s>b$, $f(s)\leqslant 0$, $f(s)\not\equiv 0$ for $s\in[0,b]$,[1] $F(\beta)=0$, and $\lim_{s\to\infty}F(s)=\infty$, where $F(s):=\int_0^s f(t)\,dt$.
(f_2) f is continuous in $[0,\infty)$, continuously differentiable in $(0,\infty)$ and $f'\in L^1(0,1)$.

Our first result deals with the uniqueness of the k-th bound state in space dimension $2\leqslant n\leqslant 4$:

Theorem 1.1. *Let $2\leqslant n\leqslant 4$, $k\in\mathbb{N}$, and assume that f satisfies (f_1)–(f_2). If in addition f satisfies*

(f_*) $(\frac{F}{f})'(s)\geqslant\frac{n-2}{2}$ *for all $s>\beta$,*

then problem (1) *has at most one solution satisfying $u(0)>0$ which has exactly k sign changes in* $(0,\infty)$. *Moreover, if there exists a solution with $k>1$ sign changes, then there exists exactly one solution with j sign changes for $j=1,\dots,k-1$ such that $u(0)>0$.*

Our second result is a strong improvement of the one in [8], where we established uniqueness of the second bound state solution in the superlinear case. The uniqueness of the first bound state solution under more general assumptions than those of Theorem 1.2 below is already known, see [10,20].

Theorem 1.2. *Assume that f satisfies (f_1)–(f_2). If f satisfies*

(f_3) $f(s)\geqslant f'(s)(s-\beta)$, *for all $s\geqslant\beta$, and*
(f_4) $(\frac{F}{f})'(s)\geqslant\frac{n-2}{2n}$ *for all $s>\beta$,*

then problem (1) *has at most one solution satisfying $u(0)>0$ which has exactly one sign change in* $(0,\infty)$. *The same conclusion holds if instead of (f_3)–(f_4), f satisfies*

(f_5) $\frac{sf'(s)}{f(s)}$ *decreases for all $s\geqslant\beta$, and*
(f_6) $\frac{\beta f'(\beta)}{f(\beta)}\leqslant\frac{n}{n-2}$, *when $n>2$.*

As will be seen in Section 4.2, the following result is an immediate consequence of Theorem 1.2:

Corollary 1.1. *If f satisfies (f_1)–(f_2) and f' decreases in (β,∞), then problem* (1) *has at most one solution satisfying $u(0)>0$ which has exactly one sign change in* $(0,\infty)$.

To the best of our knowledge, there is only one work (besides [8]) concerning the uniqueness of higher bound states: Troy, see [21, Theorems 1.1, Theorem 1.3] studied the existence and uniqueness of the solution to (1) having exactly one sign change in dimension $n=3$ for

$$f(s)=\begin{cases} s+1, & s\leqslant -1/2,\\ -s, & s\in(-1/2,1/2),\\ s-1, & s\geqslant 1/2.\end{cases}$$

Note that in this case $b=1$, $\beta=1+\sqrt{2}/2$, and for $s>\beta$,

$$\left(\frac{F}{f}\right)'(s)=\frac{1}{2}+\frac{1}{4(s-1)^2}\geqslant\frac{1}{2}=\left.\frac{n-2}{2}\right|_{n=3}\qquad\text{for all } s\geqslant\beta.$$

[1] The oddness of f is not essential, this assumption can be relaxed to a sign condition: $f(0)=0$, and there exist $b^+>0>b^-$ such that $f(u)>0$ for $u>b^+$, $f(u)<0$ for $u<b^-$, and $f(u)\leqslant 0$, $f(u)\not\equiv 0$, for $u\in(0,b^+)$ and $f(u)\geqslant 0$, $f(u)\not\equiv 0$, for $u\in(b^-,0)$.

Hence, according to our Theorem 1.1, in this case problem (1) has at most one solution with exactly k zeros in $(0,\infty)$ for any $k\in\mathbb{N}$. Other typical example of a function f satisfying the assumptions of Theorem 1.1 is

$$f(s)=s^p-s^q,\quad p>q>0,$$

with no other restriction if $n=2$, and with $p^2+q^2\leqslant 1$ when $n=3$. We note also that this function satisfies (f_5) for any $p>q>0$, and satisfies (f_6) if $p+q\leqslant 2/(n-2)$ (which reads $p+q\leqslant 2$ when $n=3$). Hence from Theorem 1.2 we obtain that problem (1) has at most one solution with exactly one sign change in $(0,\infty)$ when $p+q\leqslant 2/(n-2)$.

The existence problem has been treated by several authors. We can mention the work of Berestycki and Lions [1], where the existence of infinitely many radially symmetric solutions of our problem when f is an odd function is established by using variational methods, and the work of Jones and Küpper [12] where the authors use a dynamical systems approach and the Conley index. Also, we mention the work of McLeod, Troy and Weissler in [16], where they established the existence of solutions with a prescribed number of zeros when $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous and satisfies appropriate sign conditions and is of subcritical growth:

$$f(u)=C|u|^{p-1}u+g(u),\quad u>0,\ 1<p<\frac{n+2}{n-2},$$
$$\text{where } C \text{ is a positive constant, and } g(u)=o\big(u^p\big) \text{ as } u\to\infty, \tag{2}$$

i.e., it is superlinear and subcritical. We will treat the existence problem in a forthcoming paper, where we will prove existence of solutions having any prescribed number of sign changes under a more general asymptotic assumption than the one contained in (2) and also for the sublinear case.

Finally we describe our approach. In order to prove our results we will study the behavior of the solutions to the initial value problem

$$u''(r)+\frac{n-1}{r}u'(r)+f(u)=0,\quad r>0,\ n\geqslant 2,$$
$$u(0)=\alpha,\qquad u'(0)=0 \tag{3}$$

for $\alpha\in(0,\infty)$. As usual, we will denote by $u(r,\alpha)$ a C^2 solution of (3).

Our theorems will follow after a series of comparison results between solutions to (3) with initial value in some small neighborhood of α^*, where $u(\cdot,\alpha^*)$ is a k-th bound state, that is, $u(\cdot,\alpha^*)$ is a solution to (3) which has exactly $k-1$ sign changes in $(0,\infty)$ and $\lim_{r\to\infty}u(r,\alpha^*)=0$. The crucial property used to prove our uniqueness results is the following:

Key Property. *There exists a left neighborhood of α^* such that for any α in this neighborhood, the solution $u(\cdot,\alpha)$ has exactly $k-1$ simple zeros in $(0,\infty)$, and there exists a right neighborhood of α^* such that for any α in this set, the solution $u(\cdot,\alpha)$ has exactly k simple zeros in $(0,\infty)$ (see Fig. 1).*

In Section 3 we follow the ideas of Coffman, see [3], and use the function $\varphi(r,\alpha)=\frac{\partial}{\partial\alpha}u(r,\alpha)$ to study the behavior of the solutions between two consecutive extremal points.

In Section 4.1 we prove that under the assumptions of Theorem 1.1, the *Key Property* holds for any $k\in\mathbb{N}$. The main tool we use is the functional

$$Q(s,\alpha)=-4\frac{F}{f}(s)\frac{r(s,\alpha)}{r'(s,\alpha)}-\frac{r^2(s,\alpha)}{(r'(s,\alpha))^2}-2r^2(s,\alpha)F(s)+H(s),\quad s\neq b,$$

where $H(s)$ is chosen appropriately so that

$$Q'(s,\alpha)=\frac{\partial Q}{\partial s}(s,\alpha)=\left(2(n-2)-4\left(\frac{F}{f}\right)'(s)\right)\frac{r(s,\alpha)}{r'(s,\alpha)},$$

and the functional W defined by

$$W(s,\alpha)=r(s,\alpha)\sqrt{\big(u'\big(r(s,\alpha),\alpha\big)\big)^2+2F(s)},$$

introduced in [10]. Here $r(s,\alpha)$ denotes the inverse of u between two consecutive extremal points.

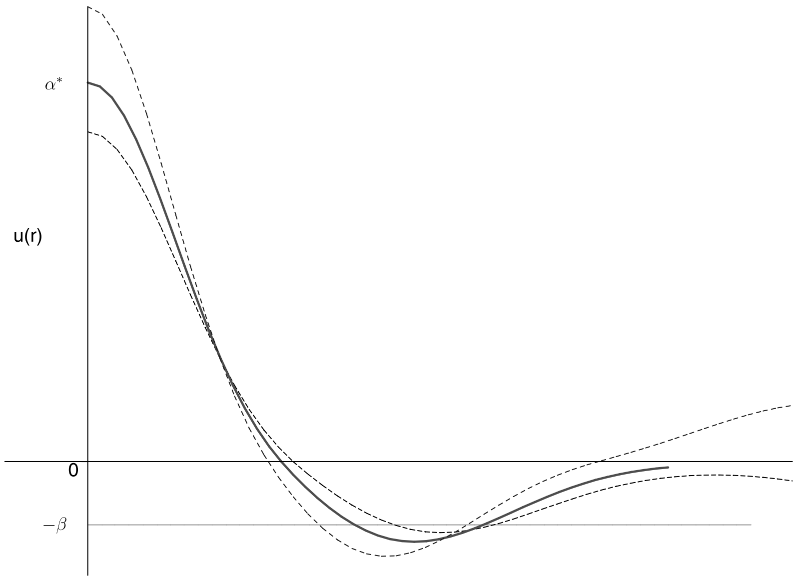

Fig. 1. Illustration of the Key Property for $k = 2$.

Section 4.2 is devoted to the proof of Theorem 1.2. We do so by considering the celebrated functional introduced first by Erbe and Tang in [9]:

$$P(s,\alpha) = -2n\frac{F}{f}(s)\frac{r^{n-1}(s,\alpha)}{r'(s,\alpha)} - \frac{r^n(s,\alpha)}{(r'(s,\alpha))^2} - 2r^n(s,\alpha)F(s), \quad s \neq b$$

(also used by [19,20] to establish uniqueness of the first bound state) and the functional $\widetilde{W}$ (used also in [18]) defined by

$$\widetilde{W}(s,\alpha) = r^{n-1}(s,\alpha)\sqrt{\big(u'\big(r(s,\alpha),\alpha\big)\big)^2 + 2F(s)},$$

where $r(s,\alpha)$ denotes the inverse of u before the first minimum point. Under the assumptions of Theorem 1.2 we can prove the *Key Property* only for $k = 2$. (For $k = 1$ it is proved in [6] for the superlinear case.)

2. Preliminaries

The aim of this section is to establish several properties of the solutions to the initial value problem (3). Clearly, this solution is unique at least until it reaches a double zero.

Proposition 2.1. *Let f satisfy (f_1)–(f_2) and let $u(\cdot,\alpha)$ be a solution of (3) which is defined in $(0,\infty)$. Then*:

(i) *There exists $C(\alpha) > 0$ such that $|u(r,\alpha)| \leqslant C(\alpha)$ for all $r > 0$.*
(ii) *If $u(\cdot,\alpha)$ is monotone in some interval (r_0,∞), then*

$$\lim_{r\to\infty}\big|u(r,\alpha)\big| = \ell \quad \text{where } \ell \text{ is either } b \text{ or } 0, \quad \text{and} \quad \lim_{r\to\infty} u'(r,\alpha) = 0.$$

Proof. Let $u(r,\alpha)$ be any solution of (3) which is defined and of class $C^2(0,\infty)$ and set

$$I(r,\alpha) = \big(u'(r,\alpha)\big)^2 + 2F\big(u(r,\alpha)\big). \tag{4}$$

A simple calculation yields

$$I'(r,\alpha) = -\frac{2(n-1)}{r}\big(u'(r,\alpha)\big)^2, \tag{5}$$

and therefore, as $n \geqslant 2$, we have that I is decreasing in r. Moreover, since $F(s)$ is bounded below by $F(b)$, we have that

$$F(\alpha) \geqslant F\big(u(r,\alpha)\big) \geqslant F(b)$$

and thus (i) follows from the assumption that $F(s) \to \infty$ as $|s| \to \infty$.

Assume next that u is monotone in (r_0, ∞) for some $r_0 > 0$. Then $\lim_{r\to\infty} u(r,\alpha) = L$ and from (i) L is finite. As I decreases and $F(u(r,\alpha)) \to F(L)$, we have that $\lim_{r\to\infty} u'(r,\alpha) = 0$. Moreover, from the equation and applying L'Hôpital's rule twice, we conclude that

$$0 = \lim_{r\to\infty} \frac{u(r,\alpha) - L}{r^2} = \lim_{r\to\infty} \frac{u'(r,\alpha)}{2r} = \lim_{r\to\infty} \frac{r^{n-1}u'(r,\alpha)}{2r^n} = \lim_{r\to\infty} \frac{-r^{n-1}f(u)}{2nr^{n-1}} = -\frac{f(L)}{2n}.$$

Hence $L = 0, \pm b$ and (ii) follows. $\square$

It can be seen that for $\alpha \in (b,\infty)$, one has $u(r,\alpha) > 0$ and $u'(r,\alpha) < 0$ for r small enough, and thus we can define the extended real number

$$Z_1(\alpha) := \sup\big\{r > 0 \,\big|\, u(s,\alpha) > 0 \text{ and } u'(s,\alpha) < 0 \text{ for all } s \in (0,r)\big\}.$$

Following [17,18] we set

$$\mathcal{N}_1 = \big\{\alpha > b\colon\ u\big(Z_1(\alpha),\alpha\big) = 0 \text{ and } u'\big(Z_1(\alpha),\alpha\big) < 0\big\},$$
$$\mathcal{G}_1 = \big\{\alpha > b\colon\ u\big(Z_1(\alpha),\alpha\big) = 0 \text{ and } u'\big(Z_1(\alpha),\alpha\big) = 0\big\},$$
$$\mathcal{P}_1 = \big\{\alpha > b\colon\ u\big(Z_1(\alpha),\alpha\big) > 0\big\}.$$

If our problems have a solution, then $\mathcal{N}_1 \neq \emptyset$. Let

$$\widetilde{\mathcal{F}}_2 = \big\{\alpha \in \mathcal{N}_1\colon\ u'(r,\alpha) < 0 \text{ for all } r > Z_1(\alpha)\big\}.$$

For $\alpha \notin \widetilde{\mathcal{F}}_2$ we define

$$T_1(\alpha) := \inf\big\{r > Z_1(\alpha)\colon\ u'(r,\alpha) = 0\big\},$$

and if $\alpha \in \widetilde{\mathcal{F}}_2$, we set $T_1(\alpha) = \infty$. Also, for $\alpha \in \mathcal{N}_1 \setminus \widetilde{\mathcal{F}}_2$ we can define the extended real number

$$Z_2(\alpha) := \sup\big\{r > T_1(\alpha) \,\big|\, u(s,\alpha) < 0 \text{ and } u'(s,\alpha) > 0 \text{ for all } s \in \big(T_1(\alpha), r\big)\big\}.$$

Let now

$$\mathcal{F}_2 = \big\{\alpha \in \mathcal{N}_1 \setminus \widetilde{\mathcal{F}}_2\colon\ u\big(Z_2(\alpha),\alpha\big) < 0\big\},$$
$$\mathcal{N}_2 = \big\{\alpha \in \mathcal{N}_1 \setminus \widetilde{\mathcal{F}}_2\colon\ u\big(Z_2(\alpha),\alpha\big) = 0 \text{ and } u'\big(Z_2(\alpha),\alpha\big) > 0\big\},$$
$$\mathcal{G}_2 = \big\{\alpha \in \mathcal{N}_1 \setminus \widetilde{\mathcal{F}}_2\colon\ u\big(Z_2(\alpha),\alpha\big) = 0 \text{ and } u'\big(Z_2(\alpha),\alpha\big) = 0\big\},$$
$$\mathcal{P}_2 = \widetilde{\mathcal{F}}_2 \cup \mathcal{F}_2.$$

For $k \geqslant 3$, and if $\mathcal{N}_{k-1} \neq \emptyset$, we set

$$\widetilde{\mathcal{F}}_k = \big\{\alpha \in \mathcal{N}_{k-1}\colon\ (-1)^k u'(r,\alpha) < 0 \text{ for all } r > Z_{k-1}(\alpha)\big\}.$$

For $\alpha \in \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k$, we set

$$T_{k-1}(\alpha) := \inf\big\{r > Z_{k-1}(\alpha)\colon\ u'(r,\alpha) = 0\big\},$$

and if $\alpha \in \widetilde{\mathcal{F}}_k$, we set $T_{k-1}(\alpha) = \infty$. Next, for $\alpha \in \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k$, we define the extended real number

$$Z_k(\alpha) := \sup\big\{r > T_{k-1}(\alpha) \,\big|\, (-1)^k u(s,\alpha) < 0 \text{ and } (-1)^k u'(s,\alpha) > 0 \text{ for all } s \in \big(T_{k-1}(\alpha), r\big)\big\}.$$

Finally we set

$$\mathcal{F}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k\colon\ (-1)^k u(Z_k(\alpha),\alpha) < 0\},$$
$$\mathcal{N}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k\colon\ u(Z_k(\alpha),\alpha) = 0 \text{ and } (-1)^k u'(Z_k(\alpha),\alpha) > 0\},$$
$$\mathcal{G}_k = \{\alpha \in \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k\colon\ u(Z_k(\alpha),\alpha) = 0 \text{ and } u'(Z_k(\alpha),\alpha) = 0\},$$
$$\mathcal{P}_k = \widetilde{\mathcal{F}}_k \cup \mathcal{F}_k.$$

Concerning the sets $\mathcal{N}_k$ and $\mathcal{P}_k$ we have:

Proposition 2.2. *The sets $\mathcal{N}_k$ and $\mathcal{P}_k$ are open.*

Proof. The proof that $\mathcal{N}_k$ is open is by continuity and follows as in [6] with obvious modifications, so we omit it.

The proof that $\mathcal{P}_k$ is open is based in the fact that the functional I defined in (4) is decreasing in r, and $\alpha \in \mathcal{P}_k$ if and only if $\alpha \in \mathcal{N}_{k-1}$ and $I(r_1,\alpha) < 0$ for some $r_1 \in (0, Z_k(\alpha))$.

Let $\alpha \in \mathcal{P}_k$ and assume first that $Z_k(\alpha) = \infty$. From Proposition 2.1, $\lim_{r\to\infty} u(r,\alpha) = \pm b$, implying that

$$\lim_{r\to\infty} I(r,\alpha) = 2F(\pm b) < 0.$$

Assume next $Z_k(\alpha) < \infty$ and hence $\alpha \in \mathcal{F}_k$. Then $Z_k(\alpha)$ is either a maximum point for u with $u(Z_k(\alpha),\alpha) < 0$, or a minimum point of u with $u(Z_k(\alpha),\alpha) > 0$ implying that either

$$0 \leqslant -u''(Z_k(\alpha),\alpha) = f(u(Z_k(\alpha),\alpha))$$

and hence $-b < u(Z_k(\alpha),\alpha) < 0$, or

$$0 \geqslant -u''(Z_k(\alpha),\alpha) = f(u(Z_k(\alpha),\alpha))$$

and thus $0 < u(Z_k(\alpha),\alpha) < b$ ($u(Z_k(\alpha),\alpha) \neq \pm b$ from the uniqueness of the solutions and since $u(0,\alpha) = \alpha$). Therefore

$$I(Z_k(\alpha),\alpha) = 2F(u(Z_k(\alpha),\alpha)) < 0.$$

Conversely, if $\alpha \notin \mathcal{P}_k$ and $\alpha \in \mathcal{N}_{k-1}$, then $\alpha \in \mathcal{G}_k \cup \mathcal{N}_k$, and thus the claim follows from the fact that $I(r,\alpha) \geqslant I(Z_k(\alpha),\alpha) \geqslant 0$ for all $r \in (0, Z_k(\alpha))$. Hence the openness of $\mathcal{P}_k$ follows from the continuous dependence of solutions to (3) in the initial value α and from the openness of $\mathcal{N}_{k-1}$. □

Finally in this section we establish the existence of a neighborhood of $\alpha^* \in \mathcal{G}_k$ so that solutions with initial value in this interval cannot be decreasing for all $r > 0$.

Proposition 2.3. *Let $\alpha^* \in \mathcal{G}_k$, $k \geqslant 2$. Then there exists $\delta_0 > 0$ such that $(\alpha^* - \delta_0, \alpha^* + \delta_0) \subseteq \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k$.*

Proof. Since $\alpha^* \in \mathcal{G}_k$, there exists $\tau > T_{k-1}(\alpha^*)$ such that $(-1)^k u'(\tau,\alpha^*) > 0$. By continuity, there exists $\delta_0 > 0$ such that

$$(-1)^k u'(\tau,\alpha) > 0 \quad \text{for all } \alpha \in (\alpha^* - \delta_0, \alpha^* + \delta_0),$$

implying that

$$T_{k-1}(\alpha) < \tau \quad \text{for all } \alpha \in (\alpha^* - \delta_0, \alpha^* + \delta_0),$$

and thus

$$(\alpha^* - \delta_0, \alpha^* + \delta_0) \subset \mathcal{N}_{k-1} \setminus \widetilde{\mathcal{F}}_k. \qquad \square$$

Lemma 2.1. *Assume that f satisfies (f_1)–(f_2) and let $k \in \mathbb{N}$. If $\alpha^* \in \mathcal{G}_k$ and there exists $\eta_1 > 0$ such that*

$$(\alpha^*, \alpha^* + \eta_1) \subset \mathcal{N}_k, \tag{6}$$

then there exists $\eta_2 > 0$ such that $(\alpha^*, \alpha^* + \eta_2) \subset \mathcal{P}_{k+1}$.

Proof. Let $\alpha \in (\alpha^*, \alpha^* + \eta_1)$ so that $T_k(\alpha_i)$ is defined. If $T_k(\alpha) = \infty$, then $\alpha \in \mathcal{P}_{k+1}$ and we are done, so we may assume that $T_k(\alpha) < \infty$. Without loss of generality we may assume that $u(\cdot, \alpha^*)$ is decreasing in $(T_{k-1}(\alpha^*), Z_k(\alpha^*))$ so that $T_k(\alpha)$ is a minimum point for $u(\cdot, \alpha)$ and therefore $u(T_k(\alpha), \alpha) < 0$ and

$$0 > -u''\big(T_k(\alpha), \alpha\big) = f\big(u\big(T_k(\alpha), \alpha\big)\big),$$

implying that $u(T_k(\alpha), \alpha) < -b$. Let us denote by $r(\cdot, \alpha)$ the inverse of $u(\cdot, \alpha)$ in $(T_{k-1}(\alpha), T_k(\alpha))$.

Let now $\varepsilon > 0$. Since

$$\lim_{r \to Z_k(\alpha^*)} I\big(r, \alpha^*\big) = 0,$$

there exists $r^* > 0$ such that

$$I\big(r^*, \alpha^*\big) < \varepsilon, \qquad u\big(r^*, \alpha^*\big) > 0,$$

and therefore by continuity, there exists $\eta_1' \in (0, \eta_1)$ such that for all $\alpha \in (\alpha^*, \alpha^* + \eta_1')$, $Z_k(\alpha) > r^*$ $(u(r^*, \alpha) > 0)$ and

$$I\big(r^*, \alpha\big) < 2\varepsilon.$$

Since I is decreasing in r, we have that

$$I(r, \alpha) < 2\varepsilon \quad \text{for all } r \geqslant r^* \text{ and all } \alpha \in \big(\alpha^*, \alpha^* + \eta_1'\big),$$

hence

$$\big|u'(r, \alpha)\big| \leqslant \sqrt{2\big|F(b)\big| + 2\varepsilon} \quad \text{for all } r \geqslant r^*,$$

and thus, from the mean value theorem we obtain that

$$r(-b, \alpha) - r\left(-\frac{b}{2}, \alpha\right) \geqslant \frac{b}{2K}$$

for some positive constant K. Let now

$$E(r, \alpha) = r^{2(n-1)} I(r, \alpha).$$

Then

$$E'(r, \alpha) = 4(n-1) r^{2n-3} F\big(u(r, \alpha)\big),$$

implying that

$$E\big(r, \alpha^*\big) \downarrow L \geqslant 0$$

as $r \to Z_k(\alpha^*)$ and thus we may assume that

$$E\big(Z_k(\alpha), \alpha\big) \leqslant L + \varepsilon \quad \text{for } i \text{ large enough.}$$

Integrating E' over $(Z_k(\alpha), r(-b, \alpha))$, we find that

$$E\big(r(-b, \alpha), \alpha\big) - E\big(Z_k(\alpha), \alpha\big) = -4(n-1) \int_{Z_k(\alpha)}^{r(-b,\alpha)} t^{2n-3} \big|F\big(u(t, \alpha)\big)\big| \, dt$$

and thus

$$\begin{aligned}
E\big(r(-b,\alpha),\alpha\big) &\leqslant L+\varepsilon-4(n-1)\big(Z_k(\alpha)\big)^{2n-3}\int_{Z_k(\alpha)}^{r(-b,\alpha)}\big|F\big(u(t,\alpha)\big)\big|\,dt\\
&\leqslant L+\varepsilon-4(n-1)\big(Z_k(\alpha)\big)^{2n-3}\int_{r(-\frac{b}{2},\alpha)}^{r(-b,\alpha)}\big|F\big(u(t,\alpha)\big)\big|\,dt\\
&\leqslant L+\varepsilon-4(n-1)\big(Z_k(\alpha)\big)^{2n-3}\left(r\left(-\frac{b}{2},\alpha\right)-r(-b,\alpha)\right)\big|F(b)\big|\\
&\leqslant L+\varepsilon-2(n-1)\big(Z_k(\alpha)\big)^{2n-3}\frac{b}{K}\big|F(b)\big|.
\end{aligned}$$

If $Z_k(\alpha^*)=\infty$, by taking $\eta_2\in(0,\eta_1')$ small enough, we conclude that $E(r(-b,\alpha),\alpha)<0$ and thus $\alpha\in\mathcal{P}_{k+1}$ for all $\alpha\in(\alpha^*,\alpha^*+\eta_2)$. If $Z_k(\alpha^*)<\infty$, the same conclusion follows by observing that in this case $L=0$ and $Z_k(\alpha)$ is bounded below by $\bar r/2$, where $\bar r$ the first value of $r>0$ where $u(\cdot,\alpha^*)$ takes the value β. □

3. Behavior of the function $\varphi(r,\alpha)=\frac{\partial}{\partial\alpha}u(r,\alpha)$

We will study the behavior of the solutions to the initial value problem (3). To this end, $\alpha^*\in\mathcal{G}_k$ is fixed and $\alpha\in(\alpha^*-\delta_0,\alpha^*+\delta_0)$, where $\delta_0>0$ is given in Proposition 2.3.

Under assumptions (f_1)–(f_2), the functions $u(r,\alpha)$ and $u'(r,\alpha)=\frac{\partial u}{\partial r}(r,\alpha)$ are of class C^1 in $(0,\infty)\times(b,\infty)$. We set

$$\varphi(r,\alpha)=\frac{\partial u}{\partial\alpha}(r,\alpha).$$

Then, for any $r>0$ such that $u(r)\neq0$, φ satisfies the linear differential equation

$$\begin{aligned}
&\varphi''(r,\alpha)+\frac{n-1}{r}\varphi'(r,\alpha)+f'\big(u(r,\alpha)\big)\varphi(r,\alpha)=0,\quad n\geqslant2,\\
&\varphi(0,\alpha)=1,\qquad \varphi'(0,\alpha)=0,
\end{aligned}\tag{7}$$

where $'=\frac{\partial}{\partial r}$.

Set

$$u(r)=u(r,\alpha),\qquad \varphi(r)=\varphi(r,\alpha).$$

Proposition 3.1. *Let f satisfy (f_1)–(f_2). Then* (i) *between two consecutive zeros $r_1<r_2$ of u' there is at least one zero $r^*\in(r_1,r_2)$ of φ.* (ii) *Furthermore, if $\alpha\in\mathcal{G}_k$, then φ has at least one zero in $(T_{k-1}(\alpha),Z_k(\alpha))$.*

Proof. Let $r_1<r_2$ be two consecutive finite zeros of u' (hence u has at most one zero in (r_1,r_2)) and assume by contradiction that $\varphi(r)$ does not change sign in (r_1,r_2). Since $u\in C^2(0,\infty)$ and $\varphi\in C^1(0,\infty)$, by differentiating the equation in (1) we obtain that $v=u'$ and φ satisfy

$$v''+\frac{n-1}{r}v'+\left(f'(u)-\frac{n-1}{r^2}\right)v=0,\tag{8}$$

and

$$\varphi''+\frac{n-1}{r}\varphi'+f'(u)\varphi=0,\tag{9}$$

for all r such that $u(r)\neq0$. Hence multiplying (8) by $r^{n-1}\varphi$ and (9) by $r^{n-1}v$ and subtracting, we obtain

$$\big(r^{n-1}\big(v'\varphi-v\varphi'\big)\big)'=(n-1)r^{n-3}v\varphi.\tag{10}$$

Assume first that $v,\varphi>0$ in (r_1,r_2). Integrating (10) over (r_1,r_2) we find that

$$r_2^{n-1}v'(r_2)\varphi(r_2) > r_1^{n-1}v'(r_1)\varphi(r_1),$$

a contradiction with the fact that from our choice of the sign for v, it must be that $v'(r_2) < 0$ and $v'(r_1) > 0$. (If $u(\bar{r}) = 0$ for some $\bar{r} \in (r_1, r_2)$, we integrate (10) over $(r_1, \bar{r} - \varepsilon)$ and over $(\bar{r} + \varepsilon, r_2)$, use the continuity of v, v', φ and φ', and then let $\varepsilon \to 0$ to obtain a contradiction.) Hence φ must have a first zero in (r_1, r_2). If either v or φ are negative in (r_1, r_2) the proof follows with obvious modifications.

Let now $\alpha \in \mathcal{G}_k$. If $Z_k(\alpha) < \infty$, the claim follows from (i). If $Z_k(\alpha) = \infty$, assume by contradiction that φ does not change sign in $(T_{k-1}(\alpha), \infty)$. We may assume without loss of generality that $u'(r) > 0$ and $\varphi(r) > 0$ for all $r \in (T_{k-1}(\alpha), \infty)$. From $u'(r) > 0$ for all $r \in (T_{k-1}(\alpha), \infty)$, and $u(r) \to 0$ as $r \to \infty$, we find that there exists $r_0 > T_{k-1}(\alpha)$ such that $-b < u(r) < 0$ for all $r \in (r_0, \infty)$ implying

$$\left(r^{n-1}u'\right)' = -r^{n-1}f(u) \leqslant 0.$$

Thus $r^{n-1}u'$ decreases in (r_0, ∞) implying that

$$\lim_{r\to\infty} r^{n-1}u'(r) = L \in [0, \infty). \tag{11}$$

From the equation we find that

$$u''(r) = -\frac{n-1}{r}u'(r) - f\big(u(r)\big) < 0 \quad \text{for all } r \in (r_0, \infty),$$

and thus $v' = u'' < 0$ for all $r \in (r_0, \infty)$. On the other hand, integrating (10) over $(T_{k-1}(\alpha), r)$, for $r \in (r_0, \infty)$, we find that

$$\begin{aligned} r^{n-1}\big(v'\varphi - v\varphi'\big)(r) &= \big(T_{k-1}(\alpha)\big)^{n-1}v'\big(T_{k-1}(\alpha)\big)\varphi\big(T_{k-1}(\alpha)\big) + (n-1)\int_{T_{k-1}(\alpha)}^{r} t^{n-3}v(t)\varphi(t)\,dt \\ &\geqslant (n-1)\int_{T_{k-1}(\alpha)}^{r_0} t^{n-3}v(t)\varphi(t)\,dt = c_0 > 0 \end{aligned}$$

for some positive constant c_0. Hence,

$$0 > r^{n-1}v'(r)\varphi(r) > r^{n-1}v(r)\varphi'(r) + c_0,$$

which from (11) implies that $\varphi'(r) \leqslant -c_0/(r^{n-1}v(r)) \leqslant -c$ for some positive constant c and therefore

$$\varphi(r) \leqslant \varphi(r_0) - c(r - r_0) \to -\infty \quad \text{as } r \to \infty,$$

a contradiction. □

In what follows, $r(s, \alpha)$ denotes the inverse of $u(r, \alpha)$ in the interval $[0, T_1(\alpha)]$. We have

Proposition 3.2. *If f satisfies (f_1)–(f_3), then φ is strictly positive in $(0, r(b, \alpha))$, where $r(s, \alpha)$ denotes the inverse of $u(r, \alpha)$ in the interval $[0, T_1(\alpha)]$.*

Proof. Multiplying the equation in (7) by $r^{n-1}(u - \beta)$ and integrating by parts over $(0, r)$, $r \leqslant r(\beta, \alpha)$, we have that

$$r^{n-1}\varphi'(r)(u - \beta) - \int_0^r r^{n-1}u'(r)\varphi'(r)\,dr + \int_0^r f'\big(u(r)\big)\varphi(r)\big(u(r) - \beta\big)r^{n-1}\,dr = 0,$$

and a second integration by parts yields

$$\int_0^r \Big(f'\big(u(t)\big)\big(u(t) - \beta\big) - f\big(u(t)\big)\Big)\varphi(t)t^{n-1}\,dt = r^{n-1}\Big(u'(r)\varphi(r) - \varphi'(r)\big(u(r) - \beta\big)\Big). \tag{12}$$

Using now that from (f_3), $f'(u(r))(u(r)-\beta)-f(u(r))\leqslant 0$ for $r\in(0,r(\beta,\alpha))$, we have that if $\varphi(r)=0$ for the first time at some $r\in(0,r(\beta,\alpha))$, then $-\varphi'(r)(u(r)-\beta)\leqslant 0$, which is a contradiction since $\varphi'(r)<0$ at such point. Therefore $\varphi(r)>0$ in $[0,r(\beta,\alpha))$.

Let now $\tilde f$ be continuous in $[0,\infty)$, continuously differentiable in $(0,\infty)$ with $\tilde f'\in L^1(0,1)$, satisfying (f_1) with $\tilde b=b$, $\tilde f=f$ in (b,∞), and $b<\tilde\beta<\beta$. Let us denote by $\tilde u(\cdot,\alpha)$ the solution of the initial value problem (3) with f replaced by $\tilde f$. By the previous argument, $\tilde\varphi(r)>0$ in $[0,r(\tilde\beta,\alpha))$. Since $\tilde u(\cdot,\alpha)=u(\cdot,\alpha)$ as long as they are greater than b, and $b<\tilde\beta<\beta$, we obtain that $r(\tilde\beta,\alpha)>r(\beta,\alpha)$ and thus $\varphi(r,\alpha)=\tilde\varphi(r,\alpha)>0$ in $[0,r(\tilde\beta,\alpha))$. Since $\tilde\beta$ is any number in (b,β), the result follows. □

Our next result is an improvement of [8, Lemma 3.1], where we proved it under an additional superlinear growth assumption on f.

Proposition 3.3. *Let f satisfy (f_1)–(f_2) and (f_5)–(f_6). If the first zero $z>0$ of φ occurs in $(0,r(\beta,\alpha)]$, then $\varphi(r)<0$ for $r\in(z,r(b,\alpha))$ and $\varphi'(r(b,\alpha))\leqslant 0$.*

Proof. The proof follows step by step the ideas in [8]. Let the first zero $z>0$ of φ occur in $(0,r(\beta,\alpha)]$, set $U_z:=u(z)$ and assume $U_z\geqslant\beta$. We will show that

$$\frac{U_zf'(U_z)}{f(U_z)}>1.$$

If not, then by (f_5) we have that

$$\frac{(s-U_z)f'(s)}{f(s)}<\frac{sf'(s)}{f(s)}\leqslant 1\quad\text{for all } s\geqslant U_z,$$

and we can argue as in the proof of Proposition 3.2 (with β replaced by U_z) to obtain the contradiction

$$\int_0^z\big(f'\big(u(t)\big)\big(u(t)-U_z\big)-f\big(u(t)\big)\big)\varphi(t)t^{n-1}\,dt=z^{n-1}\big(u'(z)\varphi(z)-\varphi'(z)\big(u(z)-U_z\big)\big)=0.$$

We conclude that there exists $c>0$ such that

$$\frac{U_zf'(U_z)}{f(U_z)}=1+\frac{2}{c}.$$

Moreover, from (f_5)–(f_6), it must be that $c\geqslant n-2$. Then, since by (f_5), the function

$$r\to c\frac{u(r)f'(u(r))}{f(u(r))}-c-2$$

is increasing in $(0,r(b,\alpha))$, we have that

$$\phi(r):=f\big(u(r)\big)\left(c\frac{u(r)f'(u(r))}{f(u(r))}-c-2\right)$$

is nonpositive in $(0,z)$ and nonnegative in $(z,r(b,\alpha))$.

Let us set $v(r)=ru'(r)+cu(r)$. Then v satisfies

$$v''+\frac{n-1}{r}v'+f'\big(u(r)\big)v=\phi(r),$$

and, as long as $\varphi(r)$ does not change sign in (z,r), with $r\in(z,r(b,\alpha))$, we have

$$\begin{aligned}0\geqslant\int_0^r t^{n-1}\varphi(t)\phi(t)\,dt&=\int_0^r t^{n-1}(\varphi\Delta v-v\Delta\varphi)\,dt\\&=r^{n-1}\big(\varphi(r)v'(r)-\varphi'(r)v(r)\big),\end{aligned}\tag{13}$$

and therefore

$$\varphi(r)v'(r) - \varphi'(r)v(r) \leqslant 0, \tag{14}$$

implying in particular that $v(z) \leqslant 0$. On the other hand, using that $c \geqslant n-2$ we have that

$$v'(r) = ru''(r) + (c+1)u'(r) \leqslant ru''(r) + (n-1)u'(r) = -rf\big(u(r)\big) < 0$$

for all $r \in (0, r(b,\alpha))$. Now we can prove that z is the only zero of φ in $(0, r(b,\alpha))$. Indeed, if φ has a second zero at $z_1 \in r(b,\alpha)$, then from (14), it must be that $v(z_1) \geqslant 0$, contradicting $v'(r) < 0$ in $(0, r(b,\alpha))$. Hence φ has exactly one zero in $(0, r(b,\alpha)]$.

Finally, evaluating (14) at $r = r(b,\alpha)$, we find that

$$\varphi\big(r(b,\alpha)\big)v'\big(r(b,\alpha)\big) - \varphi'\big(r(b,\alpha)\big)v\big(r(b,\alpha)\big) \leqslant 0,$$

implying $\varphi'(r(b,\alpha)) \leqslant 0$. □

4. Uniqueness of bound states

Assume that $\alpha^* \in \mathcal{G}_k$. The following result deals with the existence of a neighborhood V of α^* such that any solution to (3) with $\alpha \in V$ has its minimum values satisfying $U < -\beta$ and its maximum values satisfying $U > \beta$.

We observe that $u(\cdot,\alpha)$ is invertible in each interval $(T_{i-1}(\alpha), T_i(\alpha))$, $T_0(\alpha) = 0$, $i = 1, 2, \ldots, k-1$, and we denote by $r(\cdot,\alpha)$ its inverse at the intervals where u decreases and by $\bar{r}(\cdot,\alpha)$ its inverse at intervals where u increases.

Lemma 4.1. *Let f satisfy (f_1)–(f_2), and let $\alpha^* \in \mathcal{G}_k$. Then, there exist $a > 0$ and $\delta_1 > 0$, such that for any $\alpha \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$, $u(\cdot,\alpha)$ has exactly k extremal points in $[0, T_{k-1}(\alpha^*) + a]$. The extremal values E of $u(\cdot,\alpha)$ satisfy $E < -\beta$ if E is a minimum value, while $E > \beta$ if E is a maximum value. Moreover, if $\alpha_1 < \alpha_2$ are two values in $(\alpha^* - \delta_1, \alpha^* + \delta_1)$, then*

(i) *the corresponding solutions u_1 and u_2 intersect between any two of their consecutive extremal points, and*
(ii) *there exists an intersection point in $(T_{k-1}(\alpha^*), Z_k(\alpha^*))$.*

Proof. Let δ_0 be given as in Proposition 2.3. The assumption $\alpha^* \in \mathcal{G}_k$ implies that the functional defined in (4) satisfies

$$I\big(Z_k(\alpha^*), \alpha^*\big) = 0,$$

and thus $I(r,\alpha^*) > 0$ for all $r \in (0, Z_k(\alpha^*))$. In particular, for any $i = 1, 2, \ldots, k-1$, we have

$$2F\big(u\big(T_i(\alpha^*), \alpha^*\big)\big) = I\big(T_i(\alpha^*), \alpha^*\big) > 0,$$

implying that $|u(T_i(\alpha^*), \alpha^*)| > \beta$. Hence, from the continuity of u and $T_i(\alpha)$ for $\alpha \in (\alpha^* - \delta_0, \alpha^* + \delta_0)$, we conclude that there exists $\bar{\delta}_1 < \delta_0$ such that the first assertion of the lemma holds.

From Proposition 3.1, for each $i = 1, 2, \ldots, k-1$, there exists $r^* \in (T_{i-1}(\alpha^*), T_i(\alpha^*))$ such that $\varphi(r^*, \alpha^*) = 0$. Hence without loss of generality we may assume that there exist $r^- < r^* < r^+$ such that $\varphi(r^+, \alpha^*) < 0 < \varphi(r^-, \alpha^*)$. By continuity, there exists $\delta_1 \in (0, \bar{\delta}_1)$ such that $\varphi(r^-, \alpha) > 0$ and $\varphi(r^+, \alpha) < 0$ for all $\alpha \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$. Since

$$u(r,\alpha_2) - u(r,\alpha_1) = \int_{\alpha_1}^{\alpha_2} \varphi(r,\alpha)\, d\alpha,$$

which is positive at $r = r^-$ and negative at $r = r^+$, and thus (i) is proved. (ii) follows in the same way. □

4.1. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need to establish several properties of the solutions to (3) in the neighborhood of a bound state solution. We recall that in Theorem 1.1, $2 \leqslant n \leqslant 4$.

Let $m < M$ be such that $r(s,\alpha)$, the inverse of $u(r,\alpha)$, is defined and decreasing in $[m,M]$. For $s \in [m,M]$ we set

$$Q(s,\alpha) = -4\frac{F}{f}(s)\frac{r(s,\alpha)}{r'(s,\alpha)} - \frac{r^2(s,\alpha)}{(r'(s,\alpha))^2} - 2r^2(s,\alpha)F(s) + H(s),$$

where $r'(s,\alpha) = \frac{d}{ds}r(s,\alpha)$ and

$$H'(s) = -4(n-2)\frac{F}{f}(s).$$

Then,

$$Q'(s,\alpha) = \frac{\partial Q}{\partial s}(s,\alpha) = \left(2(n-2) - 4\left(\frac{F}{f}\right)'(s)\right)\frac{r(s,\alpha)}{r'(s,\alpha)}. \tag{15}$$

Similarly, for $\overline{m} < \overline{M}$ such that $\bar{r}(s,\alpha)$ the inverse of $u(r,\alpha)$, is defined and increasing in $[\overline{m},\overline{M}]$, we define

$$\bar{Q}(s,\alpha) = -4\frac{F}{f}(s)\frac{\bar{r}(s,\alpha)}{\bar{r}'(s,\alpha)} - \frac{\bar{r}^2(s,\alpha)}{(\bar{r}'(s,\alpha))^2} - 2\bar{r}^2(s,\alpha)F(s) + \bar{H}(s),$$

$\bar{r}'(s,\alpha) = \frac{d}{ds}\bar{r}(s,\alpha)$ and

$$\bar{H}'(s) = -4(n-2)\frac{F}{f}(s).$$

Note that if (f_*) holds, then $Q'(s,\alpha) \geqslant 0$ for all $s \in [m,M]$ and $\bar{Q}'(s,\alpha) \leqslant 0$ for all $s \in [\overline{m},\overline{M}]$.

Let now a and δ_1 be as in Lemma 4.1, let $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$, with $\alpha_1 < \alpha_2$, and for $j = 1, 2$ set

$$u_j(r) = u(r,\alpha_j), \qquad r_j(s) = r(s,\alpha_j), \quad \text{and} \quad Q_j(s) = Q(s,\alpha_j).$$

Let

$M_1,\ m_1$ be the i-th consecutive local maximum and minumum values of u_1,

and

$M_2,\ m_2$ be the i-th consecutive local maximum and minumum values of u_2

for $r \in [0, T_{k-1}(\alpha^*) + a]$. The behavior of the solutions for $r > T_{k-1}(\alpha^*)$ will be studied separately. We have

Proposition 4.1. *Assume that f satisfies (f_1)–(f_2) and (f_*), and let $\alpha^* \in \mathcal{G}_k$. Then, there exists $\delta_{2,i} \in (0,\delta_1)$, with δ_1 as in Lemma 4.1, such that for any $\alpha_1, \alpha_2 \in (\alpha^* - \delta_{2,i}, \alpha^* + \delta_{2,i})$ with $\alpha_1 < \alpha_2$ we have that if*

$$M_1 < M_2 \quad \text{and} \quad Q_1(M_1) > Q_2(M_2),$$

then

$$m_1 > m_2 \quad \text{and} \quad Q_1(m_1) > Q_2(m_2).$$

In order to prove this result we need a separation lemma, so for $j = 1, 2$ we consider the functional W_j defined below, introduced in [10]:

$$W_j(s) = r_j(s)\sqrt{\left(u_j'\left(r_j(s)\right)\right)^2 + 2F(s)}, \quad s \in [m_j, M_j].$$

The functional W_j is well defined in this interval, since $(u_j'(r))^2 + 2F(u_j(r)) > 0$ for $r \in [0, T_{k-1}(\alpha^*) + a]$.

Lemma 4.2. *Assume that f satisfies (f_1)–(f_2), and let $\alpha^* \in \mathcal{G}_k$. Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$ with $\alpha_1 < \alpha_2$ and δ_1 as in Lemma 4.1. Assume that there exists $U \in [-\beta, \beta]$ such that*

$$r_1(U) \geqslant r_2(U) \quad \text{and} \quad W_1(U) < W_2(U). \tag{16}$$

Then

$$r_1(s) > r_2(s), \qquad W_1(s) < W_2(s), \quad \text{for all } s \in [-\beta, U].$$

Proof. Clearly, $|r_1'(U)| > |r_2'(U)|$, and thus $r_1 > r_2$ in some small left neighborhood of U. Hence, there exists $c \in [-\beta, U)$ such that

$$W_1 \leqslant W_2, \qquad r_1 > r_2, \quad \text{and} \quad r_1' < r_2' \quad \text{in } [c, U).$$

Next, we will show that $W_1 - W_2$ is increasing in $[c, U)$. This will imply that the infimum of such c is $-\beta$, proving the lemma.

From the definition of $W_j(s)$ we have

$$\frac{\partial W_j}{\partial s}(s) = \frac{-2F(s) + (n-2)(u_j'(r_j(s)))^2}{|u_j'(r_j(s))|\sqrt{(u_j'(r_j(s)))^2 + 2F(s)}}.$$

As $F(s) \leqslant 0$ for $s \in [-\beta, \beta]$, we have that the function

$$h(p) = \frac{-2F(s)}{p\sqrt{p^2 + 2F(s)}} + \frac{(n-2)p}{\sqrt{p^2 + 2F(s)}}, \quad p > 0,$$

is decreasing, and thus, for $s \in [c, U)$, and using that $|u_1'(r_1(s))| < |u_2'(r_2(s))|$, we obtain

$$\left(\frac{\partial W_1}{\partial s} - \frac{\partial W_2}{\partial s}\right)(s) = h\big(\big|u_1'\big(r_1(s)\big)\big|\big) - h\big(\big|u_2'\big(r_2(s)\big)\big|\big) > 0$$

as we claimed. □

Proof of Proposition 4.1. We will first show that $r_1(M_1) < r_2(M_1)$. As Q_2 is strictly increasing, $M_1 < M_2$, and $Q_1(M_1) > Q_2(M_2)$, it holds that $Q_1(M_1) > Q_2(M_1)$.

Let M^* denote the i-th maximum value of $u(\cdot, \alpha^*)$. Since $u'(r(M^*, \alpha^*), \alpha^*) = 0$ and $4\frac{F}{f}(M^*) > 0$, by continuity there exists $\delta_{2,i} < \delta_1$ such that for any $\alpha_1, \alpha_2 \in (\alpha^* - \delta_{2,i}, \alpha^* + \delta_{2,i})$, we have

$$4\frac{F}{f}(M_1) > -r_2(M_1)u_2'\big(r_2(M_1)\big),$$

and hence

$$4\frac{F}{f}(M_1)r_2(M_1)u_2'\big(r_2(M_1)\big) + \big(r_2(M_1)\big)^2\big(u_2'\big(r_2(M_1)\big)\big)^2 < 0.$$

Therefore,

$$\begin{aligned} 0 &< (Q_1 - Q_2)(M_1) \\ &= 4\frac{F}{f}(M_1)r_2(M_1)u_2'\big(r_2(M_1)\big) + \big(r_2(M_1)\big)^2\big(u_2'\big(r_2(M_1)\big)\big)^2 + 2F(M_1)\big(r_2^2 - r_1^2\big)(M_1) \\ &< 2F(M_1)\big(r_2^2 - r_1^2\big)(M_1), \end{aligned}$$

implying

$$r_1(M_1) < r_2(M_1)$$

as claimed.

From Lemma 4.1 there exists a greatest intersection point U_I of r_1 and r_2 in $[\max\{m_1, m_2\}, M_1]$.

Let us set

$$U = \min\{-\beta, U_I\}.$$

We will show that

$$(Q_1 - Q_2)(U) > 0, \quad \text{and} \quad \frac{r_1}{|r_1'|}(U) < \frac{r_2}{|r_2'|}(U). \tag{17}$$

We distinguish the following cases according to the position of U_I:

Case 1. $U_I \in [\beta, M_1]$. We will prove first that

$$\frac{r_1}{|r_1'|}(s) < \frac{r_2}{|r_2'|}(s), \quad \text{for all } s \in [U_I, M_1].$$

Indeed, since $u_1'(r_1(M_1)) = 0$, we have that this inequality holds for $s = M_1$. Assume now that there exists $t \in (U_I, M_1)$ such that

$$\frac{r_1}{|r_1'|}(s) < \frac{r_2}{|r_2'|}(s), \quad \text{for all } s \in (t, M_1) \quad \text{and} \quad \frac{r_1}{|r_1'|}(t) = \frac{r_2}{|r_2'|}(t).$$

As

$$\frac{d}{ds}\left(\frac{r_1}{|r_1'|} - \frac{r_2}{|r_2'|}\right)(t) = f(t)\big(r_2|r_2'| - r_1|r_1'|\big)(t) = f(t)\frac{|r_1'|}{r_1}(t)\big(r_2^2 - r_1^2\big)(t) > 0,$$

we obtain a contradiction.

Assume next that there exists $t \in [\beta, U_I)$ such that

$$\frac{r_1}{|r_1'|}(s) < \frac{r_2}{|r_2'|}(s), \quad \text{for all } s \in (t, M_1) \quad \text{and} \quad \frac{r_1}{|r_1'|}(t) = \frac{r_2}{|r_2'|}(t).$$

Then, from (f_*),

$$(Q_1 - Q_2)'(s) = 4\left(\frac{r_1}{|r_1'|}(s) - \frac{r_2}{|r_2'|}(s)\right)\left(\left(\frac{F}{f}\right)'(s) - \frac{n-2}{2}\right) < 0, \quad s \in (t, M_1)$$

implying that

$$0 > -2F(t)\big(r_1^2(t) - r_2^2(t)\big) = (Q_1 - Q_2)(t) > (Q_1 - Q_2)(M_1) > 0,$$

a contradiction. We conclude that

$$(Q_1 - Q_2)(\beta) > (Q_1 - Q_2)(M_1) > 0$$

implying

$$\frac{r_1}{|r_1'|}(\beta) < \frac{r_2}{|r_2'|}(\beta) \quad \text{and} \quad r_1(\beta) \geqslant r_2(\beta).$$

Thus (16) in Lemma 4.2 is satisfied with $U = \beta$, and we conclude that $r_1(-\beta) > r_2(-\beta)$ and $W_1(-\beta) < W_2(-\beta)$, implying (17) at $U = -\beta$.

Case 2. $U_I \in [-\beta, \beta]$. In this case $W_1(U_I) < W_2(U_I)$ and $r_1(U_I) = r_2(U_I)$, hence by Lemma 4.2, we conclude $W_1(-\beta) < W_2(-\beta)$ implying that (17) holds.

Case 3. $U_I \in [\max\{m_1, m_2\}, -\beta]$. In this case it is straightforward to verify that

$$(Q_1 - Q_2)(U_I) > 0,$$

and hence in this case (17) also holds.

To end the proof, assume that there exists $\tau \in (\max\{m_1, m_2\}, U]$ such that

$$\frac{r_1}{|r_1'|}(s) < \frac{r_2}{|r_2'|}(s), \quad \text{for all } s \in (\tau, U],$$

and

$$\frac{r_1}{|r_1'|}(\tau) = \frac{r_2}{|r_2'|}(\tau).$$

Then,

$$(Q_1 - Q_2)'(s) = 4\left(\frac{r_1}{|r_1'|}(s) - \frac{r_2}{|r_2'|}(s)\right)\left(\left(\frac{F}{f}\right)'(s) - \frac{n-2}{2}\right) < 0, \quad s \in (\tau, U]$$

implying that

$$0 > -2F(\tau)\big(r_1^2(\tau) - r_2^2(\tau)\big) = (Q_1 - Q_2)(\tau) > (Q_1 - Q_2)(U) > 0,$$

a contradiction, and thus

$$\frac{r_1}{|r_1'|}(s) < \frac{r_2}{|r_2'|}(s), \quad \text{for all } s \in \big[\max\{m_1, m_2\}, U\big).$$

Therefore,

$$\max\{m_1, m_2\} = m_1, \qquad (Q_1 - Q_2)'(s) > 0, \quad \text{for all } s \in [m_1, U),$$

which yields $Q_1(m_1) > Q_2(m_1)$. Since Q_2 increases and $m_1 > m_2$, it follows that $Q_1(m_1) > Q_2(m_2)$, ending the proof of the proposition. □

Similarly we set

$$\bar{m}_1,\ \bar{M}_1 \text{ the } i\text{-th consecutive local minumum and maximum of } u_1,$$

and

$$\bar{m}_2,\ \bar{M}_2 \text{ the } i\text{-th consecutive local minumum and maximum of } u_2,$$

for $r \in [0, T_{k-1}(\alpha^*) + a]$.

We have the following result.

Proposition 4.2. *Assume that f satisfies (f_1)–(f_2) and (f_*), and let $\alpha^* \in \mathcal{G}_k$. Then, there exists $\bar{\delta}_{2,i} \in (0, \delta_1)$, with δ_1 as in Lemma* 4.1*, such that for any $\alpha_1, \alpha_2 \in (\alpha^* - \bar{\delta}_{2,i}, \alpha^* + \bar{\delta}_{2,i})$ with $\alpha_1 < \alpha_2$ we have that if*

$$\bar{m}_1 > \bar{m}_2 \quad \text{and} \quad \bar{Q}_1(\bar{m}_1) > \bar{Q}_2(\bar{m}_2),$$

then

$$\bar{M}_1 < \bar{M}_2 \quad \text{and} \quad \bar{Q}_1(\bar{M}_1) > \bar{Q}_2(\bar{M}_2).$$

Proof. It follows from Proposition 4.1 considering $v(r, \alpha_j) = -u(r, \alpha_j)$. □

Combining Propositions 4.1 and 4.2 we obtain the following result.

Proposition 4.3. *Assume that f satisfies (f_1)–(f_2) and (f_*), and let $\alpha^* \in \mathcal{G}_k$. Let $\delta = \min_i\{\delta_{2,i}, \bar{\delta}_{2,i}\}$, and let $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$.*

(i) *If k is even, then the k-th extremal points $T_{k-1}(\alpha_j)$ are minima,*

$$m_1 > m_2 \quad \text{and} \quad Q_1(m_1) > Q_2(m_2),$$

where $m_j = u_j(T_{k-1}(\alpha_j))$, $j = 1, 2$.

(ii) *If k is odd, then the k-th extremal points $T_{k-1}(\alpha_j)$ are maxima,*

$$M_1 < M_2 \quad \text{and} \quad Q_1(M_1) > Q_2(M_2),$$

where $M_j = u_i(T_{k-1}(\alpha_j))$, $j = 1, 2$.

Proof. As $T_0(\alpha_j) = 0$ is the first extremal point of u_j, we have

$$u_1\big(T_0(\alpha_1)\big) = \alpha_1 < \alpha_2 = u_2\big(T_0(\alpha_2)\big).$$

Moreover, as $\alpha_j > \beta$, H is decreasing in $[\beta, \infty)$ and therefore

$$Q_1(\alpha_1) = H(\alpha_1) > H(\alpha_2) = Q_2(\alpha_2).$$

Hence, for the first extremal points, the assumption of Proposition 4.1 holds and thus,

$$u_1\big(T_1(\alpha_1)\big) > u_2\big(T_1(\alpha_2)\big), \quad \text{and} \quad Q_1\big(u_1\big(T_1(\alpha_1)\big)\big) > Q_2\big(u_2\big(T_1(\alpha_2)\big)\big).$$

Applying alternatively Proposition 4.2 and Proposition 4.1 we obtain the result. □

We proceed now to our final step. To this end, we may assume without loss of generality that k is odd, so that $T_{k-1}(\alpha_j)$ is a maximum point, and we fix δ as given in Proposition 4.3.

Proposition 4.4. *Assume that f satisfies (f_1)–(f_2) and (f_*), and let $\alpha^* \in \mathcal{G}_k$. Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$.*

If $\alpha_1 \in \mathcal{G}_k \cup \mathcal{N}_k$, then $\alpha_2 \in \mathcal{N}_k$,

$$Z_k(\alpha_1) > Z_k(\alpha_2) \quad \textit{and} \quad \big|u_1'\big(Z_k(\alpha_1)\big)\big| < \big|u_2'\big(Z_k(\alpha_2)\big)\big|. \tag{18}$$

If $\alpha_2 \in \mathcal{G}_k$, then $\alpha_1 \in \mathcal{F}_k$.

In order to prove this result we need the following separation lemma which can be found in [8, Lemma 4.4.1]. Its proof is very similar to that of Lemma 4.2 and thus we omit it. For $j = 1, 2$, let

$$S_j := \inf\big\{s \in \big(u_j\big(Z_k(\alpha_j)\big), M_j\big)\colon \big|u_j'\big(r_j(s)\big)\big|^2 + 2F(s) > 0\big\},$$

where $M_j = u_j(T_{k-1}(\alpha_j))$. We note that $S_j = 0$ if and only if $\alpha_j \in \mathcal{G}_k \cup \mathcal{N}_k$.

Lemma 4.3. *Assume that f satisfies (f_1)–(f_2), and let $\alpha^* \in \mathcal{G}_k$. Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. Assume that there exists $U \in [0, \beta]$ such that*

$$r_1(U) \geqslant r_2(U) \quad \textit{and} \quad W_1(U) < W_2(U). \tag{19}$$

Then, $S_1 \geqslant S_2$ and

$$r_1(s) > r_2(s), \qquad W_1(s) < W_2(s), \quad \textit{and} \quad \big|u_1'\big(r_1(s)\big)\big| < \big|u_2'\big(r_2(s)\big)\big|, \quad s \in [S_1, U).$$

Proof of Proposition 4.4. Let r_I denote the first intersection point of u_1 and u_2 in $(T_{k-1}(\alpha^*), Z_k(\alpha^*))$ guaranteed by Lemma 4.1(ii) and $U_I = u_j(r_I)$. Arguing as in the proof of Proposition 4.1, Cases 1 and 2, this time with $U = \min\{\beta, U_I\}$, we obtain that (19) holds. Hence, by Lemma 4.3, we have $S_1 \geqslant S_2$,

$$r_1(s) > r_2(s), \qquad W_1(s) < W_2(s), \quad \text{and} \quad \big|u_1'\big(r_1(s)\big)\big| < \big|u_2'\big(r_2(s)\big)\big| \quad \text{for all } s \in [S_1, U).$$

If $\alpha_1 \in \mathcal{G}_k \cup \mathcal{N}_k$, then $S_1 = 0$ implying $S_2 = 0$ and $\alpha_2 \in \mathcal{G}_k \cup \mathcal{N}_k$. As $Z_k(\alpha_1) = r_1(0) > r_2(0) = Z_k(\alpha_2)$ and $|u_1'(Z_k(\alpha_1))| < |u_2'(Z_k(\alpha_2))|$ we conclude that $\alpha_2 \in \mathcal{N}_k$.

If $\alpha_2 \in \mathcal{G}_k$, then $S_2 = 0$. As $|u_2'(Z_k(\alpha_2))| = 0$, we conclude that $S_1 > 0$ implying $\alpha_1 \in \mathcal{F}_k$. □

Proof of Theorem 1.1. Suppose $\alpha^* \in \mathcal{G}_1$. We will prove that $\mathcal{N}_1 = (\alpha^*, \infty)$ and $\mathcal{G}_1 = \{\alpha^*\}$. Indeed, assume that $(\beta, \alpha^*) \cap \mathcal{N}_1 \neq \emptyset$ and let

$$\gamma = \sup\big\{\alpha \in \big(\beta, \alpha^*\big) \bigm| \alpha \in \mathcal{N}_1\big\}.$$

By Proposition 4.4, $\gamma < \alpha^*$. By Proposition 2.2, $\gamma \notin \mathcal{N}_1$ and $\gamma \notin \mathcal{P}_1$. Also, $\gamma \notin \mathcal{G}_1$ by Proposition 4.4. Hence, $\mathcal{N}_1 \subset (\alpha^*, \infty)$. Consider the set

$$\big\{\alpha > \alpha^* \bigm| \big(\alpha^*, \alpha\big) \subset \mathcal{N}_1\big\}.$$

From Proposition 4.4, this set in nonempty. Let $\bar\gamma$ denote its supremum. If $\bar\gamma < \infty$, by Proposition 2.2, $\bar\gamma \notin \mathcal{N}_1$ and $\bar\gamma \notin \mathcal{P}_1$. Since $\bar\gamma \notin \mathcal{G}_1$ by Proposition 4.4, we have a contradiction and hence, $\bar\gamma = \infty$ and $\mathcal{N}_1 = (\alpha^*, \infty)$. Since this holds for any $\alpha^* \in \mathcal{G}_1$, we conclude $\mathcal{G}_1 = \{\alpha^*\}$.

Let $k > 1$ and let $\alpha^* \in \mathcal{G}_k$. We will prove that $\mathcal{G}_k = \{\alpha^*\}$ and $\mathcal{N}_k = (\alpha^*, \infty)$.

First we note that $\alpha^* \in \mathcal{G}_k$ implies $\alpha^* \in \mathcal{N}_i$ for $i = 1, 2, \dots, k-1$. Set

$$\alpha_1^* = \sup\{\alpha \in [\beta, \alpha^*] \mid \alpha \in \mathcal{P}_1\}.$$

Since $\beta \in \mathcal{P}_1$, α_1^* is well defined. Moreover, as $\mathcal{P}_1$ is open and $\alpha^* \in \mathcal{N}_1$, we have that $\beta < \alpha_1^* < \alpha^*$. As $\mathcal{P}_1$ and $\mathcal{N}_1$ are open, $\alpha_1^* \notin \mathcal{N}_1 \cup \mathcal{P}_1$. Hence, $\alpha_1^* \in \mathcal{G}_1$, and arguing as above, we can prove that

$$\mathcal{G}_1 = \{\alpha_1^*\} \quad \text{and} \quad \mathcal{N}_1 = (\alpha_1^*, \infty). \tag{20}$$

If $k > 2$, we set

$$\alpha_2^* = \sup\{\alpha \in [\alpha_1^*, \alpha^*] \mid \alpha \in \mathcal{P}_2\}.$$

From (20) we can use Lemma 2.1 to obtain that the set $\{\alpha \in [\alpha_1^*, \alpha^*] \mid \alpha \in \mathcal{P}_2\} \neq \emptyset$ and thus α_2^* is well defined and $\alpha_1^* < \alpha_2^*$. Since $k > 2$, $\alpha^* \in \mathcal{N}_2$ and as $\mathcal{N}_2$ is open, we also have $\alpha_2^* < \alpha^*$. Using again that $\mathcal{N}_2$ and $\mathcal{P}_2$ are open we obtain $\alpha_2^* \notin \mathcal{N}_2 \cup \mathcal{P}_2$. Hence, as $\alpha_2^* \in \mathcal{N}_1$, we have that $\alpha_2^* \in \mathcal{G}_2$. Now we can argue as above we to prove that

$$\beta < \alpha_1^* < \alpha_2^* < \alpha^*, \quad \mathcal{G}_2 = \{\alpha_2^*\} \quad \text{and} \quad \mathcal{N}_2 = (\alpha_2^*, \infty). \tag{21}$$

Repeating this process, for $i < k$, we can define

$$\alpha_i^* = \sup\{\alpha \in [\alpha_{i-1}^*, \alpha^*] \mid \alpha \in \mathcal{P}_i\},$$

and we have

$$\beta < \alpha_1^* < \alpha_2^* < \cdots < \alpha_i^* < \alpha^*, \quad \mathcal{G}_i = \{\alpha_i^*\}, \qquad \mathcal{N}_i = (\alpha_i^*, \infty).$$

For $i = k$, we set $\alpha_i^* = \alpha^*$, and arguing as above we conclude

$$\mathcal{G}_k = \{\alpha_k^*\} \quad \text{and} \quad \mathcal{N}_k = (\alpha_k^*, \infty). \tag{22}$$

Hence, there exists at most one solution of (1) with exactly $k-1$ sign changes in $(0, \infty)$. □

4.2. *Proof of Theorem 1.2*

In what follows we use the ideas of Pucci, Serrin and Tang in [19,20]. For $s \in (u_1(T_1(\alpha_1)), -\beta]$ we set

$$P(s,\alpha) = -2n\frac{F}{f}(s)\frac{r^{n-1}(s,\alpha)}{r'(s,\alpha)} - \frac{r^n(s,\alpha)}{(r'(s,\alpha))^2} - 2r^n(s,\alpha)F(s).$$

Then,

$$P'(s,\alpha) = \frac{\partial P}{\partial s}(s,\alpha) = \left(n - 2 - 2n\left(\frac{F}{f}\right)'(s)\right)\frac{r^{n-1}(s,\alpha)}{r'(s,\alpha)}. \tag{23}$$

By (f_4) it holds that $P'(s,\alpha) \geqslant 0$ for all $s \in (u_1(T_1(\alpha_1)), -\beta]$.

In this case we can prove the analogue of Proposition 4.1 but only for the first maximal and minimal points of u_1 and u_2. Let now $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$, with $\alpha_1 < \alpha_2$, and set

$$P_1(s) = P(s,\alpha_1), \qquad P_2(s) = P(s,\alpha_2),$$
$$m_1 = u_1(T_1(\alpha_1)), \qquad m_2 = u_2(T_1(\alpha_2)).$$

We have

Proposition 4.5. *Assume that f satisfies (f_1)–(f_4), or (f_1)–(f_2) and (f_5)–(f_6), and let $\alpha^* \in \mathcal{G}_k$. Then there exists $\delta_2 \in (0, \delta_1)$, δ_1 as in Lemma* 4.1, *such that for any $\alpha_1, \alpha_2 \in (\alpha^* - \delta_2, \alpha^* + \delta_2)$ with $\alpha_1 < \alpha_2$ it holds that*

$$m_1 > m_2 \quad \text{and} \quad P_1(m_1) > P_2(m_2). \tag{24}$$

In order to prove this result we need the following variations of Lemma 4.2, so for $j = 1, 2$ we consider the functional $\widetilde{W}_j$ defined below,

$$\widetilde{W}_j(s) = r_j^{n-1}(s)\sqrt{\big(u_j'\big(r_j(s)\big)\big)^2 + 2F(s)}, \quad s \in [m_j, \alpha_j].$$

From Lemma 4.1, the solutions u_1 and u_2 intersect at a first $r_I > 0$. Set $U_I = u_1(r_I) = u_2(r_I)$.

Lemma 4.4. *Let f satisfy (f_1)–(f_2). Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$ with $\alpha_1 < \alpha_2$ and δ_1 as in Lemma* 4.1. *If $U_I \in [-\beta, \beta]$ then*

$$r_1(s) > r_2(s) \quad \text{and} \quad \widetilde{W}_1(s) < \widetilde{W}_2(s), \quad \text{for all } s \in [-\beta, U_I).$$

Proof. Clearly, $|r_1'(U_I)| > |r_2'(U_I)|$, and thus $r_1 > r_2$ in some small left neighborhood of U_I. Hence, there exists $c \in [-\beta, U_I)$ such that

$$\widetilde{W}_1 \leqslant \widetilde{W}_2, \qquad r_1 > r_2, \quad \text{and} \quad r_1' < r_2' \quad \text{in } [c, U_I).$$

Next, we will show that $\widetilde{W}_1 - \widetilde{W}_2$ is increasing in $[c, U_I)$. This will imply that the infimum of such c is $-\beta$, proving the theorem.

From the definition of $\widetilde{W}(s, \alpha)$ we have

$$\frac{\partial \widetilde{W}}{\partial s}(s, \alpha) = \frac{2(n-1)r^{n-2}(s, \alpha)F(s)}{u'(r(s,\alpha),\alpha)\sqrt{(u'(r(s,\alpha),\alpha))^2 + 2F(s)}},$$

and thus, for $s \in [c, U_I)$,

$$\begin{aligned}
&\frac{1}{2(n-1)}\left(\frac{\partial \widetilde{W}_1}{\partial s}(s) - \frac{\partial \widetilde{W}_2}{\partial s}(s)\right) \\
&\quad = F(s)\left(\frac{r_1^{n-2}(s)}{u_1'(r_1(s))\sqrt{(u_1'(r_1(s))^2 + 2F(s)}} - \frac{r_2^{n-2}(s)}{u_2'(r_2(s))\sqrt{(u_2'(r_2(s)))^2 + 2F(s)}}\right) \\
&\quad \geqslant r_2^{n-2}(s)\big|F(s)\big|\left(\frac{1}{|u_1'(r_1(s))|\sqrt{(u_1'(r_1(s))^2 + 2F(s)}} - \frac{1}{|u_2'(r_2(s))|\sqrt{(u_2'(r_2(s)))^2 + 2F(s)}}\right) \\
&\quad \geqslant 0. \qquad \square
\end{aligned}$$

For the case when f satisfies (f_5)–(f_6) we use [8, Proposition 4.1.2]. Even though in this proposition we assumed f superlinear, this assumption is not used in the proof, so we state it here without proof.

Lemma 4.5. *Let f satisfy (f_1)–(f_2) and (f_5)–(f_6). Then there exists $\delta_2 \in (0, \delta_1]$ such that for all $\alpha_1, \alpha_2 \in (\alpha^* - \delta_2, \alpha^* + \delta_2)$ with $\alpha_1 < \alpha_2$ it holds that*

$$r_1(s) > r_2(s) \quad \text{and} \quad \widetilde{W}_1(s) < \widetilde{W}_2(s), \quad \text{for all } s \in [-\beta, U_{bI}),$$

where $U_{bI} = b$ if $U_I > \beta$, and $U_{bI} = U_I$ if $U_I \leqslant \beta$.

Proof of Proposition 4.5. Let δ_2 be as in Lemma 4.5. As in [9,20], we set

$$S_{12}(s) = \frac{r_1^{n-1}r_2'}{r_2^{n-1}r_1'}(s).$$

Then

$$S_{12}'(s) = S_{12}(s)f(s)\big(\big(r_2'(s)\big)^2 - \big(r_1'(s)\big)^2\big). \tag{25}$$

Let

$$U = \min\{-\beta, U_I\}.$$

We will prove first that $m_1 > m_2$ and that for all $s \in [m_1, U)$ we have

$$S_{12}(s) < 1, \qquad |r_1'(s)| > |r_2'(s)|, \qquad r_1(s) > r_2(s). \tag{26}$$

If $U_I > -\beta$ then $U = -\beta$. If f satisfies (f_3), then from Proposition 3.2(i), $U_I \leqslant \beta$ and from Lemma 4.4, using that $F(-\beta) = 0$, we have that $S_{12}(U) \leqslant 1$ and $r_1(U) > r_2(U)$. If f satisfies (f_5)–(f_6), we use Lemma 4.5 to obtain the same conclusion. Thus, $|r_1'(U)| > |r_2'(U)|$. On the other hand, if $U = U_I$, we also have that $S_{12}(U) < 1$ and $|r_1'(U)| > |r_2'(U)|$.

From (25) we have that $S_{12}(s)$ is increasing as long as $|r_1'(s)| > |r_2'(s)|$, for $s < U$. If (26) does not hold for all $s \in (\max\{m_1, m_2\}, U)$, then at the largest point s_0 where it fails, we must have that $|r_1'(s_0)| = |r_2'(s_0)|$ and $r_1(s_0) > r_2(s_0)$ implying that $S_{12}(s_0) > 1$, a contradiction. Thus (26) holds in $(\max\{m_1, m_2\}, U)$, and hence $m_1 = \max\{m_1, m_2\}$.

Next we prove that $P_1 > P_2$ in $[m_1, U]$. From the definition of P_1 and P_2 we have

$$\begin{aligned}
(P_1 - P_2)(U) &= \left(\frac{r_2^n}{(r_2')^2} - \frac{r_1^n}{(r_1')^2}\right)(U) + 2n\frac{F}{f}(U)\left(\frac{r_1^{n-1}(U)}{|r_1'(U)|} - \frac{r_2^{n-1}(U)}{|r_2'(U)|}\right) \\
&\geqslant \left(\frac{r_2^n}{(r_2')^2} - \frac{r_1^n}{(r_1')^2}\right)(U) \\
&= \left(\frac{r_2^n}{(r_2')^2}\left[1 - S_{12}^2\frac{r_2^{n-2}}{r_1^{n-2}}\right]\right)(U) > 0.
\end{aligned}$$

In order to finish our proof, we note that from the proof of [8, Proposition 4.2.1], it follows that (f_5)–(f_6) imply (f_4). Hence, from (f_4) and (26),

$$(P_1 - P_2)'(s) = \big(S_{12}(s) - 1\big)\left(n - 2 - 2n\left(\frac{F}{f}\right)'(s)\right)\frac{r_2^{n-1}}{r_2'}(s) < 0,$$

implying that $P_1 > P_2$ in $[m_1, U]$. In particular, $P_1(m_1) > P_2(m_1)$. Now, since $P_2' > 0$, we have that $P_2(m_1) > P_2(m_2)$, and thus $P_1(m_1) > P_2(m_2)$, ending the proof of the proposition. □

The analogue of Lemma 4.3 for the case $k = 2$ can be found in [8, Lemma 4.4.1], we state it below for the sake of completeness. For $j = 1, 2$, we set

$$\bar W_j(s) = \bar r_j(s)\sqrt{\big(u_j'(\bar r_j(s))\big)^2 + 2F(s)}, \quad s \in \big[m_1(\alpha_j), \bar S_j\big),$$

where

$$\bar S_j := \sup\big\{s \in \big(m_j, u_2(Z_2(\alpha_j))\big)\colon \big(u_j'(\bar r_j(s))\big)^2 + 2F(s) > 0\big\}.$$

Lemma 4.6. *Assume that f satisfies (f_1)–(f_2), and let $\alpha^* \in \mathcal{G}_2$. Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta_2, \alpha^* + \delta_2)$ with $\alpha_1 < \alpha_2$ and δ_2 as in Lemma 4.5. Assume that there exists $U \in [-\beta, 0]$ such that*

$$r_1(U) \geqslant r_2(U) \quad \text{and} \quad \bar W_1(U) < \bar W_2(U). \tag{27}$$

Then,

$$\bar S_1 \leqslant \bar S_2$$

and

$$\bar r_1(s) > \bar r_2(s), \qquad \bar W_1(s) < \bar W_2(s), \quad \text{and} \quad u_1'\big(\bar r_1(s)\big) < u_2'\big(\bar r_2(s)\big), \quad s \in (U, \bar S_1].$$

We define

$$\bar P(s,\alpha) = -2n\frac{F}{f}(s)\frac{\bar r^{n-1}(s,\alpha)}{\bar r'(s,\alpha)} - \frac{\bar r^n(s,\alpha)}{(\bar r'(s,\alpha))^2} - 2\bar r^n(s,\alpha)F(s),$$
$$\bar P'(s,\alpha) = \left(n-2-2n\left(\frac{F}{f}\right)'(s)\right)\frac{\bar r^{n-1}(s,\alpha)}{\bar r'(s,\alpha)},$$
$$\bar S_{12}(s) = \frac{\bar r_1^{n-1}\bar r_2'}{\bar r_2^{n-1}\bar r_1'}(s),$$
$$\bar S_{12}'(s) = \bar S_{12}(s)f(s)\big(\big(\bar r_2'(s)\big)^2 - \big(\bar r_1'(s)\big)^2\big). \tag{28}$$

Proposition 4.6. *Assume that f satisfies (f_1)–(f_4), or (f_1)–(f_2) and (f_5)–(f_6), and let $\alpha^* \in \mathcal{G}_2$. Then there exists $\delta \in (0,\delta_2)$ such that for $\alpha_1, \alpha_2 \in (\alpha^*-\delta, \alpha^*+\delta)$ with $\alpha_1 < \alpha_2$ it holds that: if $\alpha_1 \in \mathcal{G}_2 \cup \mathcal{N}_2$, then $\alpha_2 \in \mathcal{N}_2$,*

$$Z_2(\alpha_1) > Z_2(\alpha_2) \quad \text{and} \quad \big|u_1'\big(Z_2(\alpha_1)\big)\big| < \big|u_2'\big(Z_2(\alpha_2)\big)\big|, \tag{29}$$

and if $\alpha_2 \in \mathcal{G}_2$, then $\alpha_1 \in \mathcal{F}_2$.

Proof. We prove the proposition only in the case that f satisfies (f_1)–(f_4), the other case corresponds to [8, Proposition 4.4.1]. Let m^* denote the minimum value of $u(\cdot,\alpha^*)$. Since $u'(r(m^*,\alpha^*),\alpha^*) = 0$ and $-2n\frac{F}{f}(m^*) > 0$, by continuity we may choose $\delta \in (0,\delta_2)$ small enough so that

$$-2n\frac{F}{f}(m_1) > \bar r_2(m_1)u_2'\big(\bar r_2(m_1)\big),$$

for all $\alpha_1, \alpha_2 \in (\alpha^*-\delta, \alpha^*+\delta)$ and hence

$$-2n\frac{F}{f}(m_1)\big(\bar r_2(m_1)\big)^{n-1}u_2'\big(\bar r_2(m_1)\big) - \big(\bar r_2(m_1)\big)^n\big(u_2'\big(\bar r_2(m_1)\big)\big)^2 > 0. \tag{30}$$

On the other hand, from (24) in Proposition 4.5, we have that $P_1(m_1) > P_2(m_2)$ and thus, using $m_2 < m_1$ and the fact that $\bar P_2$ decreases, we find that

$$\bar P_1(m_1) = P_1(m_1) > P_2(m_2) = \bar P_2(m_2) > \bar P_2(m_1).$$

Therefore,

$$0 > (\bar P_2 - \bar P_1)(m_1)$$
$$= -2n\frac{F}{f}(m_1)\big(\bar r_2(m_1)\big)^{n-1}u_2'\big(\bar r_2(m_1)\big) - \big(\bar r_2(m_1)\big)^n\big(u_2'\big(\bar r_2(m_1)\big)\big)^2 - 2F(m_1)\big(\bar r_2^n - \bar r_1^n\big)(m_1)$$

implying, by (30),

$$\bar r_1(m_1) < \bar r_2(m_1).$$

From Lemma 4.1(ii), there exists an intersection point in $(T_1(\alpha^*), Z_2(\alpha^*))$. If $\bar r_I$ denotes the first of such points and if $\bar U_I = u_1(\bar r_I) = u_2(\bar r_I)$, then $\bar U_I \in (u_1(T_1(\alpha^*)), 0]$. Let us set

$$U = \max\{-\beta, \bar U_I\}.$$

We will show that U satisfies (27) in Lemma 4.6, that is,

$$\bar r_1(U) \geqslant \bar r_2(U), \quad \text{and} \quad \bar W_1(U) < \bar W_2(U). \tag{31}$$

We distinguish two cases:

Case 1. $\bar U_I \in [m_1, -\beta]$. We will first prove

$$\frac{\bar r_1^{n-1}}{\bar r_1'}(s) < \frac{\bar r_2^{n-1}}{\bar r_2'}(s) \quad \text{and} \quad \bar P_1(s) > \bar P_2(s) \quad \text{for all } s \in [m_1, \bar U_I]. \tag{32}$$

Observe first that $\bar S_{12}(m_1)=0$ and $\bar S_{12}(\bar U_I)<1$. If there exists a point $t\in(m_1,\bar U_I)$ such that $\bar S'_{12}(t)=0$, then $\bar r'_1(t)=\bar r'_2(t)$ and hence, from the definition of $\bar U_I$,

$$\bar S_{12}(t)=\frac{\bar r_1^{n-1}}{\bar r_2^{n-1}}(t)<1,$$

implying $\bar S_{12}(s)<1$ for $s\in[m_1,\bar U_I]$.

On the other hand, from the second equation in (28), using that $\bar S_{12}(s)<1$ and (f_4), we obtain

$$(\bar P_1-\bar P_2)'(s)=\left((\bar S_{12}-1)\left(n-2-2n\left(\frac{F}{f}\right)'\right)\frac{\bar r_2^{n-1}}{\bar r'_2}\right)(s)>0.$$

Hence, for all $s\in(m_1,\bar U_I)$, $\bar P_1(s)-\bar P_2(s)>\bar P_1(m_1)-\bar P_2(m_1)>0$.

Next we will prove that

$$\bar r_1(s)>\bar r_2(s),\quad\text{and}\quad\frac{\bar r_1}{\bar r'_1}(s)<\frac{\bar r_2}{\bar r'_2}(s)\quad\text{for all } s\in(\bar U_I,-\beta].\tag{33}$$

From the definition of $\bar U_I$, $\frac{\bar r_1}{\bar r'_1}<\frac{\bar r_2}{\bar r'_2}$ at $\bar U_I$. Assume by contradiction that (33) does not hold. Then, there exists a first point $t\in(\bar U_I,-\beta)$ such that

$$\frac{\bar r_1}{\bar r'_1}(t)=\frac{\bar r_2}{\bar r'_2}(t)\quad\text{and}\quad\bar r_1(s)>\bar r_2(s),\quad\text{for all } s\in(\bar U_I,t],$$

implying

$$\bar S_{12}(t)=\left(\frac{\bar r_1(t)}{\bar r_2(t)}\right)^{n-2}=D>1.$$

From the definition of $\bar P_1$ and $\bar P_2$, we have that

$$(\bar P_1-D\bar P_2)(t)=2\big(D\bar r_2^n-\bar r_1^n\big)F(t)=2\bar r_1^{n-2}\big(\bar r_2^2-\bar r_1^2\big)F(t)<0.$$

On the other hand, from (32), we have that $(\bar P_1-\bar P_2)(\bar U_I)>0$. Since $\bar P_2(m_2)<0$ and $\bar P_2$ decreases in $(m_2,-\beta)$, we have that $\bar P_2(\bar U_I)<0$. Hence, as $D>1$, we conclude that

$$(\bar P_1-D\bar P_2)(\bar U_I)>0.$$

From the last equation in (28) we obtain that $\bar S_{12}$ is increasing in $(\bar U_I,t)$ implying that $\bar S_{12}(s)<D$. Finally, using (f_4) we deduce

$$(\bar P_1-D\bar P_2)'(s)=\left((\bar S_{12}-D)\left(n-2-2n\left(\frac{F}{f}\right)'\right)\frac{\bar r_2^{n-1}}{\bar r'_2}\right)(s)>0$$

for all $s\in(\bar U_I,t)$ and thus

$$(\bar P_1-D\bar P_2)(t)>0,$$

a contradiction. Hence, (32) follows, and, since $F(-\beta)=0$, also (31).

Case 2. $\bar U_I\in[-\beta,0)$. In this case $U=\bar U_I$, and (31) trivially holds.

Hence, by Lemma 4.6, we have $\bar S_1\leqslant\bar S_2$,

$$r_1(s)>r_2(s),\qquad\bar W_1(s)<\bar W_2(s),\quad\text{and}\quad u'_1\big(r_1(s)\big)<u'_2\big(r_2(s)\big)\quad\text{for all } s\in(U,S_1].$$

If $\alpha_1\in\mathcal G_2\cup\mathcal N_2$, then $S_1=0$ implying $S_2=0$ and $\alpha_2\in\mathcal G_2\cup\mathcal N_2$. As $Z_2(\alpha_1)=\bar r_1(0)>\bar r_2(0)=Z_2(\alpha_2)$ and $u'_1(Z_2(\alpha_1))<u'_2(Z_2(\alpha_2))$ we conclude that $\alpha_2\in\mathcal N_2$.

If $\alpha_2\in\mathcal G_2$, then $\bar S_2=0$. As $u'_2(Z_2(\alpha_2))=0$, we conclude that $\bar S_1<0$ implying $\alpha_1\in\mathcal F_2$. □

Proof of Theorem 1.2. Let $\alpha^* \in \mathcal{G}_2$. Then, $\alpha^* \in \mathcal{N}_1$ and since $\beta \in \mathcal{P}_1$, we can set

$$\alpha_1^* = \sup\{\alpha \in [\beta, \alpha^*] \mid \alpha \in \mathcal{P}_1\}.$$

Arguing as in the proof of Theorem 1.1, we deduce that $\alpha_1^* \in \mathcal{G}_1$. Under assumptions (f_1)–(f_3), $\mathcal{G}_1 = \{\alpha_1^*\}$ by [10], and under assumption (f_1)–(f_2) and (f_5)–(f_6), the same result holds by [20]. Hence $\mathcal{N}_1 = (\alpha_1^*, \infty)$. Let

$$A = \{\alpha > \alpha^*\colon (\alpha^*, \alpha) \subset \mathcal{N}_2\}.$$

By Proposition 4.6, A is not empty. Let $\bar\alpha = \sup A$ and assume $\bar\alpha < \infty$. Since $\mathcal{P}_2$ and $\mathcal{N}_2$ are open, $\bar\alpha \notin \mathcal{N}_2 \cup \mathcal{P}_2$, hence, as $\mathcal{N}_1 = (\alpha_1^*, \infty)$, we have that $\bar\alpha \in \mathcal{G}_2$. But from Proposition 4.6, there exists $\delta > 0$ such that $(\bar\alpha - \delta, \bar\alpha) \subset \mathcal{P}_2$ implying that $\bar\alpha$ is not the supremum of A. Hence we conclude that $\bar\alpha = \infty$ and thus $\mathcal{N}_2 \supset (\alpha^*, \infty)$. Since this is true for any $\alpha^* \in \mathcal{G}_2$, we conclude that $\mathcal{G}_2 = \{\alpha^*\}$. □

Finally, we prove Corollary 1.1.

Proof of Corollary 1.1. We only need to prove that if f' decreases in (β, ∞), then (f_3) and (f_4) are satisfied. Indeed, we have

$$f(s) \geqslant f(s) - f(\beta) = \int_\beta^s f'(t)\,dt \geqslant f'(s)(s-\beta),$$

and (f_3) is satisfied.

Since (f_4) can equivalently written as

$$\frac{Ff'}{f^2}(s) \leqslant \frac{n+2}{2n},$$

it follows that if $f'(s) < 0$ for some $s > \beta$, (f_4) holds at such point. Now, as

$$F(s) = \int_\beta^s f(t)\,dt = f(s)(s-\beta) - \int_\beta^s (t-\beta) f'(t)\,dt \leqslant f(s)(s-\beta) - f'(s)\frac{(s-\beta)^2}{2},$$

if $f'(s) \geqslant 0$, multiplying by $f'(s)/f^2(s)$ we obtain that

$$\frac{Ff'}{f^2}(s) \leqslant \frac{(s-\beta)f'(s)}{f(s)} - \frac{1}{2}\left(\frac{(s-\beta)f'(s)}{f(s)}\right)^2 \leqslant \frac{1}{2} \leqslant \frac{n+2}{2n},$$

and thus (f_4) is always satisfied.

Hence, from Theorem 1.2, problem (1) has at most one solution with exactly one sign change. □

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