Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Ann. I. H. Poincaré – AN 28 (2011) 837–852

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Existence of self-dual non-topological solutions in the Chern–Simons Higgs model

Kwangseok Choe [a], Namkwon Kim [b,*], Chang-Shou Lin [b,c]

[a] *Department of Mathematics, Inha University, Incheon, 402-751, Republic of Korea*
[b] *Department of Mathematics, Chosun University, Kwangju, 501-759, Republic of Korea*
[c] *Department of Mathematics, Taida Institute for Mathematical Sciences (TIMS), National Taiwan University, Taiwan*

Received 18 February 2011; received in revised form 24 June 2011; accepted 28 June 2011

Available online 12 July 2011

Abstract

In this paper we investigate the existence of non-topological solutions of the Chern–Simons Higgs model in $\mathbb{R}^2$. A long standing problem for this equation is: Given N vortex points and $\beta > 8\pi(N+1)$, does there exist a non-topological solution in $\mathbb{R}^2$ such that the total magnetic flux is equal to $\beta/2$? In this paper, we prove the existence of such a solution if $\beta \notin \{8\pi N\frac{k}{k-1} \mid k = 2, \dots, N\}$. We apply the bubbling analysis and the Leray–Schauder degree theory to solve this problem.

Résumé

L'objectif de cet article est de prouver l'existence de solutions non-topologiques du modèle de Chern–Simons Higgs dans $\mathbb{R}^2$. Un problème de longue date existe pour cette équation : Soit N points vortex et $\beta > 8\pi(N+1)$, existe-t-il une solution non-topologique dans $\mathbb{R}^2$ telle que le flux magnétique total est égal à $\beta/2$? Dans cet article, nous prouvons l'existence d'une solution pour $\beta \notin \{8\pi N\frac{k}{k-1} \mid k = 2, \dots, N\}$. Nous appliquons l'analyse par bulles et la theorie de Leray–Schauder pour résoudre ce problème.

Keywords: Semi-linear PDE; Non-topological vortices; Chern–Simons Higgs model

1. Introduction

In the paper, we want to show the existence of non-topological multi-vortex solutions to the $(2+1)$-dimensional relativistic Chern–Simons gauge field theory. The Chern–Simons gauge field theory is minimal self-dual model containing the Chern–Simons term and was proposed independently by Hong et al. [11] and Jackiw and Weinberg [12] to study the anyonic superconductivity. The Chern–Simons–Higgs Lagrangian density is given by

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + D_\mu\phi\overline{D^\mu\phi} - \frac{1}{\kappa^2}|\phi|^2\big(1-|\phi|^2\big)^2, \tag{1.1}$$

* Corresponding author. Tel.: +82 62 230 6608; fax: +82 62 234 4326.
E-mail addresses: kschoe@inha.ac.kr (K. Choe), kimnamkw@chosun.ac.kr (N. Kim), cslin@math.ntu.edu.tw (C.-S. Lin).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.06.003

where A_μ ($\mu = 0, 1, 2$) is the gauge field on $\mathbb{R}^3$, $F_{\mu\nu} = \frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu$ is the curvature tensor, ϕ is the Higgs field on $\mathbb{R}^3$, $D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$ ($i = \sqrt{-1}$) is the gauge covariant derivative associated with A_μ, $\epsilon_{\mu\nu\rho}$ is the skew symmetric tensor with $\epsilon_{012} = 1$ and the constant κ is the coupling constant. When the energy for a pair (ϕ, A) is saturated, in [11] and [12], the authors independently derived the following Bogomol'nyi type equations.

$$(D_1 + iD_2)\phi = 0, \quad \text{and} \tag{1.2}$$

$$F_{12} + \frac{2}{\kappa^2}|\phi|^2\big(|\phi|^2 - 1\big) = 0. \tag{1.3}$$

Following Jaffe and Taubes [13], we can reduce the self-dual system (1.2) and (1.3) to a single elliptic equation of second order as follows. Let $p_1, \dots, p_N$ be any set of points in $\mathbb{R}^2$. Introduce a real valued function u and θ by

$$\phi = e^{\frac{1}{2}(u+i\theta)} \quad \text{and} \quad \theta = 2\sum_{j=1}^{N} \arg(z - p_j), \quad z = x_1 + ix_2 \in \mathbb{C}^1.$$

Then u satisfies

$$\Delta u + \frac{4}{\kappa^2} e^u\big(1 - e^u\big) = 4\pi \sum_{j=1}^{N} \delta(z - p_j) \quad \text{in } \mathbb{R}^2, \tag{1.4}$$

where $\delta(z - p_j)$ is the Dirac measure with the total mass at p_j.

For the details of the derivation of Eqs. (1.2)–(1.4) and recent developments of related subjects, we refer the readers to Hong et al. [11], Jackiw and Weinberg [12], Dunne [10], Lee et al. [15,16], Caffarelli and Yang [2], Choe [7,8], Choe and Kim [9], Lin and Yan [18], Spruck and Yang [21,22], Tarantello [23,24], Yang [25], Wang [26] and references therein. Eq. (1.4) has recently attracted a lot of attention because it is closely related to the mean field equation of Liouville type, see Nolasco and Tarantello [20], Chen and Lin [5] and Lin and Wang [17].

A solution u of Eq. (1.4) is called topological if $u(x) \to 0$ as $|x| \to +\infty$, and is called non-topological if $u(x) \to -\infty$ as $|x| \to \infty$. For a given set $\{p_1, \dots, p_N\}$ of vortex points, the existence of topological solution had been proved by Wang [26] long time ago. However, the existence problem of non-topological solutions is much more subtle. When $p_1 = \cdots = p_N$, Chen et al. [6] proved that for a positive number β, there exists radially symmetric solution u of (1.4) such that

$$\frac{4}{\kappa^2}\int_{\mathbb{R}^2} e^u\big(1 - e^u\big)\,dx = \beta$$

holds if and only if $\beta > 8\pi(1 + N)$.

Naturally, we will ask the question: Given any $\beta > 8\pi(1 + N)$, does Eq. (1.4) possess a non-topological solution u such that

$$\frac{4}{\kappa^2}\int_{\mathbb{R}^2} e^u\big(1 - e^u\big)\,dx = \beta?$$

Note that by (1.3), the quantity β represents twice of the total magnetic flux:

$$\int_{\mathbb{R}^2} F_{12}\,dx = \frac{2}{\kappa^2}\int_{\mathbb{R}^2} e^u\big(1 - e^u\big)\,dx = \frac{\beta}{2}. \tag{1.5}$$

It was Chae and Imanuvilov [3] who obtained the first existence result of non-topological solutions. They solved the problem by viewing Eq. (1.4) as a perturbation of the classical Liouville equation. Consequently, they could find solutions such that the order parameter $|\phi|$ is very small, and β is very close to $8\pi(N + 1)$. On the other hand, Chan et al. [4] could obtain non-topological solutions with β greater than $16\pi N$. The method of Chan et al. is to construct solutions which bubble at each vortex point p_j, therefore, in their theory the configuration of $\{p_1, \dots, p_N\}$ must have a symmetry. Those are only two existence results for multi-vortex non-topological solutions. At this moment, the answer toward understanding the structure of non-topological solutions is far from complete. In the paper, we want to answer the long standing open problem affirmatively.

Theorem 1.1. *Let $p_1, \dots, p_N \in \mathbb{R}^2$ be given. For any number $\beta > 8\pi(N+1)$ satisfying $\beta \notin \{8\pi N \frac{k}{k-1} \mid k = 2, 3, \dots, N\}$, there exists a solution u of Eq.* (1.4) *satisfying*

$$\frac{4}{\kappa^2}\int_{\mathbb{R}^2} e^u\left(1-e^u\right)dx = \beta.$$

We sketch our idea to prove Theorem 1.1. In Section 4, we consider the deformation of Eq. (1.4):

$$\begin{cases} \Delta u + \dfrac{4}{\kappa^2}e^u\left(1-e^u\right) = 4\pi \displaystyle\sum_{j=1}^{N}\delta_{\varepsilon p_j} \quad \text{in } \mathbb{R}^2, \\ \dfrac{4}{\kappa^2}\displaystyle\int_{\mathbb{R}^2} e^u\left(1-e^u\right)dx = \beta \end{cases} \tag{1.6}$$

where $\varepsilon \in [0,1]$. For $\varepsilon = 1$, (1.6) is the same equation as (1.4), and $\varepsilon = 0$, (1.6) is reduced to

$$\begin{cases} \Delta u + \dfrac{4}{\kappa^2}e^u\left(1-e^u\right) = 4\pi N\delta_0, \\ \dfrac{4}{\kappa^2}\displaystyle\int_{\mathbb{R}^2} e^u\left(1-e^u\right)dx = \beta \end{cases} \tag{1.7}$$

where 0 is the origin in $\mathbb{R}^2$. The major work in this paper is to establish the uniform bound of any solution $u_\varepsilon(x)$ of (1.6). This apriori estimate is surprising because in most nonlinear problems, the collapsing of vortices would cause the bubbling phenomenon.

After establishing the apriori bounds, we will apply the classical Leray–Schauder degree theory to solve Eq. (1.4). Note that for any radial solution $u(z)$ for (1.7), $z = x_1 + ix_2$, $u_\theta(z) = u(e^{i\theta}z)$ is also a solution, and the orbit $\{u_\theta(z)\}$, $\theta \in [0, 2\pi]$, is S^1, whose Euler characteristic vanishes. Thus the contribution of each orbit to the topological degree is 0. Hence due to a result of Wang [27], we conclude that the computation of the degree for Eq. (1.7) is reduced to those radially symmetric solution of (1.7). By the result for radial solution of Eq. (1.7) in [4], the topological degree for Eq. (1.7), thus for (1.4) also, is equal to -1. Then Theorem 1.1 follows immediately.

2. Preliminaries

Without loss of generality, we may assume that $\kappa^2 = 4$ in (1.4) and (1.6) in the sequel. Now suppose u is a solution of (1.6). Then it is easy to see that

$$u(x) = -2\alpha \ln|x| + O(1) \quad \text{as } |x| \to +\infty, \tag{2.1}$$

where

$$\alpha = \frac{\beta}{4\pi} - N. \tag{2.2}$$

Applying the maximum principle to Eq. (1.6), we have $u(x) < 0$, $\forall x \in \mathbb{R}^2$. Write

$$u(x) = v(x) + f_\varepsilon(x), \tag{2.3}$$

where $f_\varepsilon(x) = \sum_{j=1}^{N} \ln|x - \varepsilon p_j|^2$. Then $v(x)$ satisfies

$$\Delta v + e^{v+f_\varepsilon}\left(1 - e^{v+f_\varepsilon}\right) = 0. \tag{2.4}$$

Lemma 2.1. *Let u be a solution of* (1.6). *Then u satisfies*

$$\int_{\mathbb{R}^2} e^u\left(2-e^u\right)dx = 4\pi\left(\alpha^2 - N^2\right) - 4\pi\sum_{j=1}^{N}\varepsilon p_j \cdot \nabla v(\varepsilon p_j), \tag{2.5}$$

where $v \equiv u - f_\varepsilon$.

Proof. In terms of v, (2.4) becomes

$$-\Delta v = -e^u\big(e^u - 1\big).$$

Multiplying the above by $x \cdot \nabla(v + f_\varepsilon)$ and integrating over $\Omega = B_R$, we have

$$\int_{\partial\Omega} \left(\frac{1}{2}(x \cdot \nu)|\nabla v|^2 - (x \cdot \nabla v)(\nu \cdot \nabla v)\right) d\sigma - \int_\Omega (x \cdot \nabla f_\varepsilon)\Delta v\, dx$$
$$= \int_{\partial\Omega} (x \cdot \nu)\left(e^u - \frac{1}{2}e^{2u}\right) d\sigma - \int_\Omega \big(2e^u - e^{2u}\big)\, dx.$$

Here $\nu = x/|x|$. Since $v(x) = -2(\alpha + N)\ln|x| + O(1)$ near ∞,

$$LHS = -4\pi(\alpha + N)^2 - \int_\Omega \sum_{j=1}^N 2x \cdot \nabla \ln|x - \varepsilon p_j|\, \Delta v\, dx + o(1)$$
$$= -4\pi(\alpha + N)^2 + 8\pi N(\alpha + N) - \sum_{j=1}^N 2\int_\Omega \varepsilon p_j \cdot \nabla \ln|x - \varepsilon p_j|\, \Delta v\, dx + o(1)$$

as $R \to \infty$. By the repeated use of the integration by parts and the fact that $\ln|x - \varepsilon p_j|$ is the Green function,

$$\int_\Omega \varepsilon p_j \cdot \nabla \ln|x - \varepsilon p_j|\Delta v\, dx$$
$$= \lim_{r\to 0}\left[\left(\int_{\partial\Omega} - \int_{\partial B_r(\varepsilon p_j)}\right)\varepsilon p_j \cdot \frac{(x - \varepsilon p_j)}{|x - \varepsilon p_j|^2}\nu \cdot \nabla v - \left(\int_{\partial\Omega} - \int_{\partial B_r(\varepsilon p_j)}\right)\nu \cdot \nabla\left(\varepsilon p_j \cdot \frac{(x - \varepsilon p_j)}{|x - \varepsilon p_j|^2}\right)v\right]$$
$$= \lim_{r\to 0}\int_{\partial B_r(\varepsilon p_j)} \left[-\varepsilon p_j \cdot \frac{(x - \varepsilon p_j)}{|x - \varepsilon p_j|^2}\nu \cdot \nabla v + \nu \cdot \nabla\left(\varepsilon p_j \cdot \frac{(x - \varepsilon p_j)}{|x - \varepsilon p_j|^2}\right)v\right] + o(1)$$
$$= -2\pi \varepsilon p_j \cdot \nabla v(\varepsilon p_j) + o(1).$$

Then we have the desired estimates. □

The following corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let u be a solution of* (1.6). *Then u satisfies*

$$\int_{\mathbb{R}^2} e^u\, dx = 4\pi(\alpha + N)(\alpha - N - 1) - 4\pi \sum_{j=1}^N \varepsilon p_j \cdot \nabla(u - f_\varepsilon)(\varepsilon p_j),$$
$$\int_{\mathbb{R}^2} e^{2u}\, dx = 4\pi(\alpha + N)(\alpha - N - 2) - 4\pi \sum_{j=1}^N \varepsilon p_j \cdot \nabla(u - f_\varepsilon)(\varepsilon p_j).$$

We recall some useful results on radially symmetric solutions of (1.7) with $\kappa^2 = 4$. If $N = 0$ then every solution of (1.7) is called a 0-vortex solution, and after a translation this solution must be radially symmetric. Also, u is a decreasing function of $|x|$ [21]. More generally, when $N \geqslant 0$, there exists a unique radially symmetric solution of (1.7) if and only if $\beta > 8\pi(N + 1)$ (see [4]). Actually, there exists a 1-parameter family of radially symmetric solution $u(r; s)$ of (1.7) which satisfies

$$-u''(r; s) - \frac{1}{r}u'(r; s) = e^{u(r;s)}\big(1 - e^{u(r;s)}\big), \quad r > 0,$$
$$u(r; s) = 2N\ln r + s + o(1) \quad \text{near } r = 0.$$

There exists a number $s_* \in \mathbb{R}$ such that $u(r;s)$ is a non-topological solution if and only if $s < s_*$, and

$$\beta(s) \equiv \int_{\mathbb{R}^2} e^{u(r;s)}\big(1 - e^{u(r;s)}\big)\,dx \tag{2.6}$$

is a well-defined continuously differentiable function on $(-\infty, s_*)$. It is proved in [4] that $\beta(\cdot)$ is monotonically increasing on $(-\infty, s_*)$. Moreover, $\beta(s)$ tends to ∞ as $s \nearrow s_*$, and $\beta(s) \to 8\pi(N+1)$ as $s \to -\infty$. It is obvious that $s_* = 0$ if $N = 0$.

Lemma 2.3. *Let u be a non-topological solution of* (1.6) *with β fixed. There exists a constant $\nu = \nu(\max|p_j|) > 0$ such that $u < -\nu$ in $\mathbb{R}^2$.*

Proof. We know that $u < 0$. So, $u - f_{\varepsilon_n} < -f_{\varepsilon_n} \leqslant C_r$ on $\partial B_r = \{x \mid |x| = r\}$ for any $r > \max_j |p_j| + 1$.

Suppose now that there exist a sequence of solutions $u_n := u_{\varepsilon_n}$ and a sequence of points $\{x_n\}$ such that $u_n(x_n) \to 0$. Since $\omega_n := u_n(x) - f_{\varepsilon_n}(x) \leqslant C_r$ on ∂B_r, $r > \max_j |p_j| + 1$, and ω_n satisfies $\Delta\omega_n + 1 \geqslant 0$ in B_r, by the maximum principle, we have $\omega_n(x) \leqslant C_r$ in B_r. Thus if $|x_n - \varepsilon_n p_j| \to 0$ for some $j \in \{1, 2, \ldots, N\}$, then $u_n(x_n) = \omega_n(x_n) + f_{\varepsilon_n}(x_n) \to -\infty$, which yields a contradiction. Hence we must have

$$\liminf_{n\to\infty} \operatorname{dist}\big(x_n, \{\varepsilon_n p_j \mid j = 1, \ldots, N\}\big) > C > 0.$$

We claim that $|x_n| \to \infty$. Indeed, if there exists a bounded subsequence of $\{x_n\}$, then, by the locally uniform Hölder estimate of u_n outside p_j's, u_n must converge locally uniformly to a function $u^* \leqslant 0$ up to subsequences. But then u^* must be a solution of (1.6) having zero as a maximum, which contradicts the strong maximum principle. This proves our claim.

Now, choose a number $s_0 < 0$ such that $\beta_0(s_0) > \beta = 4\pi(\alpha + N)$, where $\beta_0(\cdot)$ is $\beta(\cdot)$ in (2.6) for the 0-vortex solution. Since $|x_n| \to \infty$ and $u_n(x_n) \to 0$, there exists a sequence $\{y_n\}$ such that $|y_n| \geqslant |x_n|$ and $u_n(y_n) = s_0$. This holds true because $u_n(x) \to -\infty$ as $|x| \to \infty$.

Let $\bar{u}_n(x) = u_n(x + y_n)$ for $|x| \leqslant |y_n|/2$. $\bar{u}_n$ satisfies

$$-\Delta\bar{u}_n = e^{\bar{u}_n}\big(1 - e^{\bar{u}_n}\big) - \sum_{j=1}^{N} 4\pi\delta_{\varepsilon_n p_j - y_n}.$$

Then, along a subsequence, $\bar{u}_n$ converges in $C^2_{loc}(\mathbb{R}^2)$ to $\bar{u} \leqslant 0$ with $\bar{u}(0) = s_0 < 0$, satisfying $\Delta\bar{u} + e^{\bar{u}}(1 - e^{\bar{u}}) = 0$ in $\mathbb{R}^2$, and

$$\int_{\mathbb{R}^2} e^{\bar{u}}\big(1 - e^{\bar{u}}\big)\,dx \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^2} e^{\bar{u}_n}\big(1 - e^{\bar{u}_n}\big)\,dx = \beta.$$

Since $\bar{u}$ is a 0-vortex non-topological solution of (1.7), it is radially symmetric [21]. Since $M_* := \max_{\mathbb{R}^2} \bar{u} \geqslant \bar{u}(0) = s_0$,

$$\beta < \beta_0(s_0) \leqslant \beta_0(M_*) = \int_{\mathbb{R}^2} e^{\bar{u}}\big(1 - e^{\bar{u}}\big)\,dx \leqslant \beta,$$

which leads to a contradiction and the lemma is proved. □

Now, let $\varphi(r;s) = \frac{d}{ds}u(r;s)$ for $s < s_*$. We borrow the following lemma for $\varphi(r;s)$ from [4].

Lemma 2.4. *$\varphi(\cdot;s)$ has only one zero in $\mathbb{R}_+ = (0,\infty)$ and $\lim_{r\to\infty} \frac{\varphi(r;s)}{\ln r} = -c_0$ for some constant $c_0 = c_0(s) > 0$.*

Lemma 2.4 will be used in the calculation of degree in the final section.

3. Boundedness of solutions

In this section, we show that the solutions of the problem (1.6) are uniformly bounded by the blow-up analysis if $\beta > 8\pi(N+1)$ and $\beta \neq 8\pi N\frac{k}{k-1}$ for any integer $k = 2, \dots, N$. In what follows we always assume that $N \geqslant 1$. Recall that

$$f_\varepsilon(x) = \sum_{j=1}^{N} \ln|x - \varepsilon p_j|^2 \quad \text{with } 0 \leqslant \varepsilon \leqslant 1.$$

Theorem 3.1. *Let $p_1, \dots, p_N \in \mathbb{R}^2$, $\beta > 8\pi(N+1)$ and $\beta \notin \{8\pi N\frac{k}{k-1} \mid k = 2, \dots, N\}$, and u be a solution of (1.6). Then, for each compact subset K, there exists a constant $C = C(\beta, N, \max_j|p_j|, K)$ independent of ε such that*

$$\|u - f_\varepsilon\|_{L^\infty(K)} \leqslant C. \tag{3.1}$$

We will show Theorem 3.1 by contradiction. Since $u(x) < 0$ in $\mathbb{R}^2$, (1.6) can be rewritten as

$$\Delta u(x) + d(x)u(x) = 0$$

in $\mathbb{R}^2 \setminus \{p_1, \dots, p_N\}$, where $|d(x)| = |\frac{e^u(1-e^u)}{u(x)}| \leqslant C$. Thus for any compact set $K \subset \mathbb{R}^2 \setminus \{p_1, \dots, p_N\}$, by the Harnack inequality, there exists a constant $C(K) > 0$ such that

$$\sup_K |u(x)| \leqslant C \inf_K |u(x)|.$$

Therefore, if there exists a sequence $u_n := u_{\varepsilon_n}$ of solutions which do not satisfy (3.1), then $u_n \to -\infty$ uniformly on each compact subset. We will show that $\{u_n\}$ exhibits a specific concentration near ∞, which leads to a contradiction. In what follows, u_n is assumed to satisfy $u_n(x) = -2\alpha \ln|x| + O(1)$ near ∞, where α is given in (2.2). We start with the following lemma.

Lemma 3.2. *Let x_n be a maximum point of u_n. Then, $|x_n| \to \infty$ as $n \to \infty$ and $u_n(x_n) > -C$ uniformly for some constant $C > 0$.*

Proof. Suppose that $|x_n|$ is bounded up to subsequences. Then, $u_n(x_n) \to -\infty$ since $u_n \to -\infty$ locally. Now, we let

$$s_n = e^{\frac{1}{2}u_n(x_n)}, \qquad \tilde{u}_n(x) = u_n(x/s_n) - \ln s_n^2, \qquad v_n = \tilde{u}_n - f_{s_n\varepsilon_n}.$$

Then $\tilde{u}_n \leqslant \tilde{u}_n(s_n x_n) = 0$ in $\mathbb{R}^2$, and

$$-\Delta v_n = e^{v_n + f_{s_n\varepsilon_n}}\big(1 - s_n^2 e^{v_n + f_{s_n\varepsilon_n}}\big).$$

Clearly, $v_n(x) = (\tilde{u}_n - f_{s_n\varepsilon_n})(x) \leqslant -f_{s_n\varepsilon_n}(x) \leqslant C$ for $|x| \geqslant \max_j|p_j| + 1$. Also, since the right-hand side of the above equation is bounded, $v_n \leqslant C$ on $B_R(0)$ for $R = \max_j|p_j| + 1$. Thus, we have $v_n \leqslant C$ in $\mathbb{R}^2$ for some constant C. Then it follows that

$$-2N\ln s_n - f_{\varepsilon_n}(x_n) = -f_{s_n\varepsilon_n}(s_n x_n) = v_n(s_n x_n) \leqslant C.$$

Since $s_n \to 0$, it follows that $f_{\varepsilon_n}(x_n) \to +\infty$ as $n \to \infty$, which yields a contradiction. Therefore $|x_n| \to \infty$.

Next, since u_n converge locally uniformly to $-\infty$, Green's representation formula for $\{u_n\}$ implies that $\nabla(u_n - f_{\varepsilon_n}) \to 0$ in $C^1_{loc}(\mathbb{R}^2)$. Then by (2.2), Corollary 2.2 and the above convergence, we have

$$\begin{aligned} 4\pi(\alpha+N)(\alpha-N-2) + o(1) &= \int_{\mathbb{R}^2} e^{2u_n}\,dx \leqslant e^{u_n(x_n)} \int_{\mathbb{R}^2} e^{u_n}\,dx \\ &= e^{u_n(x_n)}\big(4\pi(\alpha+N)(\alpha-N-1) + o(1)\big). \end{aligned}$$

Since $\alpha > N+2$, $u_n(x_n)$ is bounded from below uniformly. □

From now on, we let

$$w_n(x) = u_n(x/r_n) - \ln r_n^2, \quad r_n = 1/|x_n| \to 0,$$

where x_n is a maximum point of u_n as before. Then w_n satisfies

$$-\Delta w_n = e^{w_n}\left(1 - r_n^2 e^{w_n}\right) - \sum_{j=1}^{N} 4\pi \delta_{r_n \varepsilon_n p_j} \quad \text{in } \mathbb{R}^2. \tag{3.2}$$

Note that $r_n \varepsilon_n p_j \to 0$ as $n \to +\infty$. By Lemma 2.3, the Brezis–Merle type alternatives for $\{w_n\}$ in [9] are still valid in any open set D satisfying $\overline{D} \Subset \mathbb{R}^2 \backslash \{0\}$. Since $w_n(r_n x_n) \to \infty$ by Lemma 3.2, $\{w_n\}$ has a blow-up point q on the unit circle, namely, there exists a sequence $\{y_{q,n}\}$ such that $y_{q,n} \to q$ and $w_n(y_{q,n}) \to \infty$. Thus, $\{w_n\}$ satisfies the blow up case in the alternatives: along a subsequence, there exists a non-empty finite set $\mathcal{S}$ of nonzero points such that $w_n \to -\infty$ uniformly on each $K \Subset \mathbb{R}^2 \backslash (\mathcal{S} \cup \{0\})$ and

$$e^{w_n}\left(1 - r_n^2 e^{w_n}\right) \to \sum_{q \in \mathcal{S}} 2\pi M_q \delta_q,$$

on any $D \Subset \mathbb{R}^2 \backslash \{0\}$ with $\mathcal{S} \subset D$ in the distribution sense.

For any $q \in \mathcal{S}$, let d be small, and

$$M_{q,n} = \frac{1}{2\pi} \int_{B_d(q)} e^{w_n}\left(1 - r_n^2 e^{w_n}\right) dx.$$

Clearly $M_{q,n} \to M_q$. By Pohozaev's identity, we have

$$\int_{B_d(q)} e^{w_n}\, dx = \pi M_q(M_q - 2) + o(1), \qquad \int_{B_d(q)} r_n^2 e^{2w_n}\, dx = \pi M_q(M_q - 4) + o(1). \tag{3.3}$$

Indeed, multiplying by $(x - q) \cdot \nabla w_n$ both sides of (3.2) and integrating over $B_d(q)$, we can repeat the Pohozaev argument as in Lemma 2.1. Therefore $M_q \geqslant 4$ for all $q \in \mathcal{S}$. Then we have $|\mathcal{S}| \leqslant (\alpha + N)/2$. Furthermore, we can prove that all M_q's are the same in the following lemma.

Lemma 3.3. *Let $\{w_n\}$ and M_q as before. Then $M_p = M_q$ for all $p, q \in \mathcal{S}$. Moreover $\max_{|x-q|<d} w_n(x) + \ln r_n^2 > -C$ uniformly for any small constant $d > 0$.*

Proof. Choose a small constant $d > 0$ such that $B_{2d}(q) \cap (\mathcal{S} \cup \{0\}) = \{q\}$ for any $q \in \mathcal{S}$. Due to Green's representation formula for w_n, the local estimates for periodic blow-up solutions of the Chern–Simons Higgs equation in [7] are still valid. Actually, following the proof of existence of profiles for the mean field equation [1], we obtain

$$\left| w_n(x) - w_n(q_n) + \frac{M_{q,n}}{2} \ln\left(1 + e^{w_n(q_n)}|x - q_n|^2\right) \right| \leqslant C \tag{3.4}$$

in $|x - q_n| \leqslant d$ for some constant C. Here $w_n(q_n) = \max_{|x-q| \leqslant d} w_n(x) \to \infty$ with $q_n \in B_d(q)$. Clearly $q_n \to q$.

We claim that

$$w_n(q_n) = -\ln r_n^2 + O(1) \quad \text{if } M_q > 4.$$

Indeed, we note that $w_n + \ln r_n^2 \leqslant 0$ in $\mathbb{R}^2$. Then our claim is an immediate consequence of the following inequality.

$$\max_{B_d(q)}\left[r_n^2 e^{w_n}\right] \int_{B_d(q)} e^{w_n}\, dx \geqslant \int_{B_d(q)} r_n^2 e^{2w_n}\, dx = \pi M_q(M_q - 4) + o(1).$$

We now prove that M_q are all the same. For this purpose, we suppose that $M_p > M_q$ for some $p, q \in \mathcal{S}$. Choose two points $x \in \partial B_d(p)$ and $y \in \partial B_d(q)$, respectively. It also follows from Green's representation formula for w_n that there exists a constant C such that $|w_n(x) - w_n(y)| \leqslant C$. Since $M_p > 4$, we have $w_n(p_n) = -\ln r_n^2 + O(1)$, where p_n is a maximum point of w_n in $B_d(p)$. Since $w_n(q_n) \leqslant -\ln r_n^2$, it follows from (3.4) that

$$w_n(x) - w_n(y) = \left(1 - \frac{M_{p,n}}{2}\right) w_n(p_n) - \left(1 - \frac{M_{q,n}}{2}\right) w_n(q_n) + O(1)$$
$$\leqslant \left(1 - \frac{M_{p,n}}{2}\right)\left(-\ln r_n^2\right) - \left(1 - \frac{M_{q,n}}{2}\right)\left(-\ln r_n^2\right) + C$$
$$\leqslant \frac{1}{2}(M_{q,n} - M_{p,n})\left(-\ln r_n^2\right) + C \to -\infty,$$

which yields a contradiction. Therefore $M_p = M_q$ for all $p, q \in \mathcal{S}$.

Finally, we show $M_q > 4$ for all $q \in \mathcal{S}$. Let x_n be a maximum point of u_n in $\mathbb{R}^2$. Since the limit points of $x_n/|x_n|$ belong to $\mathcal{S}$, it is enough to show

$$\liminf_{n\to\infty} \int\limits_{|x-x_n|<R} e^{u_n}\left(1 - e^{u_n}\right) dx > 8\pi \tag{3.5}$$

for some large enough $R > 0$. Actually, since $u_n(x_n)$ is bounded from below, it follows from Harnack's inequality that $\{u_n\}$ is bounded in $L^\infty(B_r(x_n))$ for any $r > 0$. Along a subsequence, $u_n(x + x_n)$ converges in $C^2_{loc}(\mathbb{R}^2)$ to a 0-vortex non-topological solution V of (1.7). Then (3.5) immediately follows from the fact that $\|e^V(1 - e^V)\|_{L^1(\mathbb{R}^2)} > 8\pi$ [4,6,21]. This finishes the proof. □

In what follows, we let M denote the mass at $q \in \mathcal{S}$ instead of M_q. In the following two lemmas, we show that concentration may occur only for special values of α. In particular, the following lemma implies that the origin is not a blow-up point for $\{w_n\}$.

Lemma 3.4. *For each constant $0 < s < \min_{q\in\mathcal{S}} |q|$, we have*

$$\lim_{n\to\infty} \int\limits_{|x|\leqslant s} e^{w_n}\left(1 - r_n^2 e^{w_n}\right) dx = 0, \tag{3.6}$$

and $w_n \to -\infty$ uniformly on $B_s(0)$. Moreover, $|\mathcal{S}| \geqslant 2$ and $M = 4N/(|\mathcal{S}| - 1)$.

Proof. Let $s < \min_{q\in\mathcal{S}} |q|$ be a small positive number. We first show (3.6). To see this, we argue by contradiction and suppose that, along a subsequence,

$$M_0 := \lim_{n\to\infty} \int\limits_{|x|\leqslant s} \frac{1}{2\pi} e^{w_n}\left(1 - r_n^2 e^{w_n}\right) dx > 0.$$

Let $\xi_n = w_n - f_{r_n\varepsilon_n}$. Note that $\xi_n \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2 \backslash (\mathcal{S} \cup \{0\})$. By Eq. (3.2), Green's representation formula for ξ_n implies that

$$\xi_n(x) - b_n \to -M_0 \ln|x| - \sum_{q\in\mathcal{S}} M \ln|x - q| \quad \text{in } C^1_{loc}\big(\mathbb{R}^2 \setminus \big(\mathcal{S} \cup \{0\}\big)\big)$$

for some real sequence $b_n \to -\infty$. The Pohozaev-type identity shows that

$$\int\limits_{\partial\Omega} \left(\frac{1}{2}(x\cdot\nu)|\nabla\xi_n|^2 - (x\cdot\nabla\xi_n)(\nu\cdot\nabla\xi_n)\right) d\sigma$$
$$= \int\limits_{\partial\Omega} (x\cdot\nu)\left(e^{w_n} - \frac{r_n^2}{2} e^{2w_n}\right) d\sigma - \int\limits_{\Omega} \left(2e^{w_n} - r_n^2 e^{2w_n}\right) dx - \int\limits_{\Omega} x\cdot\nabla f_{r_n\varepsilon_n}\left(e^{w_n} - r_n^2 e^{2w_n}\right) dx$$

where Ω is an open set with smooth boundary. To simplify the notations, we let

$$I_n = \int_{\partial\Omega} \left(\frac{1}{2}(x\cdot\nu)|\nabla\xi_n|^2 - (x\cdot\nabla\xi_n)(\nu\cdot\nabla\xi_n)\right) d\sigma,$$
$$J_n = \int_{\Omega} x\cdot\nabla f_{r_n\varepsilon_n}\big(e^{w_n} - r_n^2 e^{2w_n}\big)\,dx.$$

If we take $\Omega = \{x \mid |x| \leqslant s\}$, then we obtain that

$$I_n = -\pi M_0^2 + o(1)$$

as $n\to\infty$. Moreover

$$\begin{aligned} J_n &= 4\pi N M_0 + \int_{\Omega} \sum_{j=1}^{N} \frac{2r_n\varepsilon_n p_j\cdot(x - r_n\varepsilon_n p_j)}{|x - r_n\varepsilon_n p_j|^2}\big(e^{w_n} - r_n^2 e^{2w_n}\big)\,dx + o(1) \\ &= 4\pi N M_0 + \int_{|x|\leqslant s/r_n} \sum_{j=1}^{N} \frac{2\varepsilon_n p_j\cdot(x-\varepsilon_n p_j)}{|x-\varepsilon_n p_j|^2}\big(e^{u_n} - e^{2u_n}\big)\,dx + o(1) \\ &= 4\pi N M_0 + o(1) \end{aligned}$$

as $n\to\infty$. Therefore we obtain that

$$-\pi M_0^2 + o(1) = -4\pi M_0 - \int_{|x|\leqslant s} r_n^2 e^{2w_n}\,dx - 4\pi N M_0 + o(1).$$

In particular $M_0 \geqslant 4 + 4N$.

Next, we take $\Omega = \bigcup_{q\in\mathcal{S}} B_r(q)$ with small enough $r>0$. Note that

$$\nabla\xi_n(x) \to \nabla\xi(x) = -\frac{M(x-q)}{|x-q|^2} + \nabla H_q(x), \qquad \nabla H_q(x) = -\frac{M_0 x}{|x|^2} - \sum_{p\in\mathcal{S}\setminus\{q\}} \frac{M(x-p)}{|x-p|^2}$$

uniformly on any compact subset of $\Omega\setminus\mathcal{S}$. Denoting $x \equiv q + r\nu$ on $|x-q| = r$, we have $\nabla\xi = -\frac{M}{r}\nu + \nabla H_q$ on $|x-q| = r$. As $n\to\infty$,

$$\begin{aligned} I_n &\to \sum_{q\in\mathcal{S}} \int_{|x-q|=r} \left(\frac{1}{2}(x\cdot\nu)|\nabla\xi|^2 - (x\cdot\nabla\xi)(\nu\cdot\nabla\xi)\right) d\sigma \\ &= \sum_{q\in\mathcal{S}} \int_{|x-q|=r} \left(-\frac{M^2}{2r} + \frac{M}{r} q\cdot\nabla H_q(x)\right) d\sigma \\ &= -\pi|\mathcal{S}|M^2 - 2\pi|\mathcal{S}|MM_0 - 2\pi M^2 \sum_{q\in\mathcal{S}} \sum_{p\in\mathcal{S}\setminus\{q\}} \frac{q\cdot(q-p)}{|q-p|^2} + O(r) \\ &= -\pi|\mathcal{S}|M\big(|\mathcal{S}|M + 2M_0\big) + O(r). \end{aligned}$$

Moreover, it follows from (3.3) that

$$\begin{aligned} \int_{\Omega} \big(2e^{w_n} - r_n^2 e^{2w_n}\big)\,dx &= \int_{\Omega} r_n^2 e^{2w_n}\,dx + \int_{\Omega} 2\big(e^{w_n} - r_n^2 e^{2w_n}\big)\,dx \\ &= \sum_{q\in\mathcal{S}} \int_{|x-q|\leqslant r} r_n^2 e^{2w_n}\,dx + 4\pi|\mathcal{S}|M + o(1) \\ &= \pi|\mathcal{S}|M(M-4) + 4\pi|\mathcal{S}|M + o(1). \end{aligned}$$

Finally we obtain that

$$J_n = 4\pi N|\mathcal{S}|M + \sum_{j=1}^{N}\int_{\Omega} \frac{2r_n\varepsilon_n p_j \cdot (x - r_n\varepsilon_n p_j)}{|x - r_n\varepsilon_n p_j|^2}\big(e^{w_n} - r_n^2 e^{2w_n}\big)\,dx + o(1)$$
$$= 4\pi N|\mathcal{S}|M + o(1).$$

Then we conclude that

$$4N = 2M_0 + M\big(|\mathcal{S}| - 1\big) \geqslant 8 + 8N + M\big(|\mathcal{S}| - 1\big), \tag{3.7}$$

which leads to a contradiction. Therefore (3.6) is proved. By letting $M_0 = 0$ in (3.7), we conclude that $4N = M(|\mathcal{S}| - 1)$.

Next, we show that $w_n \to -\infty$ uniformly on $B_s(0)$. It follows from Lemma 3.2, (3.4) and Green's representation formula for w_n that, for $|x| \leqslant s/r_n$,

$$u_n(x) = f_{\varepsilon_n}(x) + C_n + \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln\frac{|y|}{|x-y|} e^{u_n}\big(1 - e^{u_n}\big)\,dy$$
$$= f_{\varepsilon_n}(x) + C_n + \frac{1}{2\pi}\int_{|y|\leqslant s/r_n} \ln\frac{|y|}{|x-y|} e^{u_n}\big(1 - e^{u_n}\big)\,dy + O(1).$$

Standard argument ([1]) shows that

$$u_n(x) - f_{\varepsilon_n}(x) - C_n = o(1)\ln r_n + O(1) \quad \text{for } |x| \leqslant s/r_n.$$

Then it follows from (3.4) that $C_n = (M + 2N + o(1))\ln r_n + O(1)$, and consequently, $w_n(x) = f_{r_n\varepsilon_n}(x) + (M - 2 + o(1))\ln r_n + O(1)$ for $|x| \leqslant s$. This completes the proof of Lemma 3.4. □

Lemma 3.5. *For any constant $R > \max_{q\in\mathcal{S}}|q|$, $w_n \to -\infty$ uniformly for $|x| \geqslant R$, and*

$$\lim_{n\to\infty}\int_{|x|\geqslant R} e^{w_n}\big(1 - r_n^2 e^{w_n}\big)\,dx = 0. \tag{3.8}$$

Moreover $(|\mathcal{S}| + 1)M = 4\alpha$.

Proof. Let

$$\varphi_n(x) = w_n\big(x/|x|^2\big) - 2\alpha\ln|x|.$$

Choose a constant $R > (\min_{q\in\mathcal{S}}|q|)^{-1}$. Then φ_n satisfies

$$-\Delta\varphi_n = |x|^{2\alpha-4}e^{\varphi_n}\big(1 - r_n^2|x|^{2\alpha}e^{\varphi_n}\big) \quad \text{for } |x| < R, \tag{3.9}$$

and $\varphi_n \to -\infty$ uniformly on any closed subset of $\mathbb{R}^2 \setminus (\tilde{\mathcal{S}} \cup \{0\})$, where we set $\tilde{\mathcal{S}} = \{q/|q|^2 \mid q \in \mathcal{S}\}$. Each φ_n is of class C^2 in a neighborhood of the origin.

We now show that $\varphi_n \to -\infty$ uniformly on any compact subset of $\mathbb{R}^2 \setminus \tilde{\mathcal{S}}$. Choose a number $0 < s < (\max_{q\in\mathcal{S}}|q|)^{-1}$. We argue by contradiction and suppose that there exists a subsequence still denoted by $\{\varphi_n\}$ such that $\sup_{|x|\leqslant s}\varphi_n(x)$ is bounded below. Elliptic estimates show that $\sup_{|x|\leqslant s}\varphi_n(x) \to \infty$. Otherwise, along a subsequence, the right-hand side of (3.9) would be uniformly bounded on $B_s(0)$. Since $\varphi_n(x) \to -\infty$ uniformly for $|x| = s$, we would have $\sup_{|x|\leqslant s}\varphi_n(x) \to -\infty$, which yields a contradiction.

We claim that

$$\liminf_{n\to\infty}\int_{|x|\leqslant s} |x|^{2\alpha-4}e^{\varphi_n}\big(1 - r_n^2|x|^{2\alpha}e^{\varphi_n}\big)\,dx \geqslant 8\pi \tag{3.10}$$

under the assumption that $\sup_{|x|\leqslant s}\varphi_n(x) \to \infty$. To see this, we choose a point $y_n \in B_s(0)$ such that $\varphi_n(y_n) = \sup_{|x|\leqslant s}\varphi_n(x)$. It is obvious that $|y_n| \to 0$. Choose a number t_n such that

$$\varphi_n(y_n) + (2\alpha - 2)\ln t_n = 0.$$

Along a subsequence, we have two cases:

Case 1. If $|y_n|/t_n$ is bounded above, then we consider the function

$$\overline{\varphi}_n(x) = \varphi_n(t_n x) - \varphi(y_n) \quad \text{for } |x| \leqslant s/t_n.$$

Note that $\overline{\varphi}_n(x) \leqslant \overline{\varphi}_n(y_n/t_n) = 0$ for $|x| \leqslant s/t_n$. Then elliptic estimates show that, along a subsequence, $\overline{\varphi}_n$ converges in $C^2_{loc}(\mathbb{R}^2)$ to a solution $\overline{\varphi}$ of

$$-\Delta\overline{\varphi} = |x|^{2\alpha-4}e^{\overline{\varphi}} \quad \text{in } \mathbb{R}^2,\ |x|^{2\alpha-4}e^{\overline{\varphi}} \in L^1(\mathbb{R}^2).$$

Consequently we find that

$$\liminf_{n\to\infty} \int_{B_s(0)} |x|^{2\alpha-4}e^{\varphi_n}\left(1 - r_n^2|x|^{2\alpha}e^{\varphi_n}\right)dx \geqslant \int_{\mathbb{R}^2} |x|^{2\alpha-4}e^{\overline{\varphi}}\,dx = 8\pi(\alpha-1).$$

Case 2. If $|y_n|/t_n \to \infty$ then we consider the function

$$\overline{w}_n(x) = w_n\left(x/|y_n|\right) - \ln|y_n|^2. \tag{3.11}$$

Then $\overline{w}_n$ satisfies

$$-\Delta\overline{w}_n = e^{\overline{w}_n}\left(1 - r_n^2|y_n|^2e^{\overline{w}_n}\right) - \sum_{j=1}^{N} 4\pi\delta_{r_n|y_n|\varepsilon_n p_j} \quad \text{in } \mathbb{R}^2.$$

Since $\overline{w}_n(y_n/|y_n|) = \varphi_n(y_n) + (2\alpha-2)\ln|y_n| \to \infty$, along a subsequence, $\{\overline{w}_n\}$ has a blow-up point q_* on the unit circle. Consequently we have

$$\liminf_{n\to\infty} \int_{B_s(0)} (-\Delta\varphi_n)\,dx \geqslant \liminf_{n\to\infty} \int_{B_r(q_*)} e^{\overline{w}_n}\left(1 - r_n^2|y_n|^2e^{\overline{w}_n}\right)dx \geqslant 8\pi$$

for any small constant $r > 0$. This proves (3.10).

Without loss of generality we may assume that

$$\lim_{n\to\infty} \int_{|x|\leqslant s} |x|^{2\alpha-4}e^{\varphi_n}\left(1 - r_n^2|x|^{2\alpha}e^{\varphi_n}\right)dx = 2\pi M_1, \quad M_1 \geqslant 4.$$

Then the local estimates for $\{w_n\}$ show that

$$|x|^{2\alpha-4}e^{\varphi_n}\left(1 - r_n^2|x|^{2\alpha}e^{\varphi_n}\right) \to 2\pi M_1\delta_{p=0} + \sum_{q\in\tilde{S}} 2\pi M\delta_q$$

in the distribution sense. Green's representation formula for φ_n implies that

$$\varphi_n(x) - c_n \to -M_1\ln|x| - \sum_{p\in\tilde{S}} M\ln|x-p|$$

in $C^1_{loc}(\mathbb{R}^2 \setminus (\tilde{S} \cup \{0\}))$ for some real sequence $c_n \to -\infty$. We repeat the calculations used in the proof of Lemma 3.4. The Pohozaev-type identity implies that

$$\int_{\partial\Omega} \left((x\cdot\nabla\varphi_n)(\nu\cdot\nabla\varphi_n) - \frac{1}{2}(x\cdot\nu)|\nabla\varphi_n|^2\right)d\sigma + a_n$$

$$= (2\alpha-2)\int_{\Omega} |x|^{2\alpha-4}e^{\varphi_n}\,dx - (2\alpha-1)\int_{\Omega} r_n^2|x|^{4\alpha-4}e^{2\varphi_n}\,dx \tag{3.12}$$

where $a_n \to 0$, and $\Omega = \bigcup_{p \in \tilde{\mathcal{S}}} B_r(p)$. Here $r > 0$ is a small number. Note that

$$\int_{\Omega} |x|^{2\alpha-4} e^{\varphi_n}\, dx = \int_{\tilde{\Omega}} e^{w_n}\, dy = \pi|\mathcal{S}|M(M-2) + o(1),$$

$$\int_{\Omega} r_n^2 |x|^{4\alpha-4} e^{2\varphi_n}\, dx = \int_{\tilde{\Omega}} r_n^2 e^{2w_n}\, dy = \pi|\mathcal{S}|M(M-4) + o(1),$$

where we set $\tilde{\Omega} = \{y \mid y/|y|^2 \in \Omega\}$. As $n \to \infty$, we obtain from (3.12) that

$$\pi|\mathcal{S}|^2 M^2 + 2\pi|\mathcal{S}|M_1 M + O(r) = \pi|\mathcal{S}|M(-M + 4\alpha).$$

Letting $r \to 0$, we find that

$$2M_1 + \big(|\mathcal{S}| + 1\big)M = 4\alpha. \tag{3.13}$$

Recall that $\varphi_n(y_n) = \sup_{|x| \leqslant s} \varphi_n(x)$ and $|y_n| \leqslant s$. It follows from (3.13) that $\varphi_n(y_n) + (2\alpha - 2)\ln|y_n| \to \infty$. Then we consider the function $\overline{w}_n$ given in (3.11). Along a subsequence, $\overline{w}_n$ has finitely many nonzero blow-up points as we mentioned above.

On the other hand, the proof of Lemma 3.4 implies that

$$o(1) = \int_{|x| \leqslant d} e^{\overline{w}_n}\big(1 - r_n^2 |y_n|^2 e^{\overline{w}_n}\big)\, dx = \int_{|y| \leqslant d/|y_n|} e^{w_n}\big(1 - r_n^2 e^{w_n}\big)\, dy \geqslant 2\pi|\mathcal{S}|M$$

for any small number $d > 0$, which yields a contradiction.

Therefore we conclude that $\varphi_n \to -\infty$ uniformly on $B_s(0)$, and (3.8) holds. Letting $M_1 = 0$ in (3.13), we find that $(|\mathcal{S}| + 1)M = 4\alpha$. □

We are now in a position to prove Theorem 3.1. In Lemmas 3.4 and 3.5, we have proved that if $u_n \to -\infty$ uniformly on any bounded set then

$$\begin{aligned}\int_{\mathbb{R}^2} e^{u_n}\big(1 - e^{u_n}\big)\, dx &= \sum_{q \in \mathcal{S}} \int_{|x-q| \leqslant d} e^{w_n}\big(1 - r_n^2 e^{w_n}\big)\, dx + o(1) \\ &= \frac{4|\mathcal{S}|N}{|\mathcal{S}| - 1} + o(1) = \frac{4|\mathcal{S}|\alpha}{|\mathcal{S}| + 1} + o(1).\end{aligned}$$

In particular $\alpha \in \{N(k+1)/(k-1) \mid k = 2, \ldots, N\}$ since $\alpha > N + 2$. Then Theorem 3.1 immediately follows from (2.2).

To apply the degree theory to the problem (1.6), the first step is to determine an appropriate space in which we should seek the solution v_ε (see below). To this aim we introduce the following space:

$$\mathcal{D} \equiv \left\{ v : \mathbb{R}^2 \to \mathbb{R} \;\middle|\; \|v\|_{\mathcal{D}}^2 = \int_{\mathbb{R}^2} |\nabla v|^2\, dx + \int_{\mathbb{R}^2} \frac{v^2}{(1 + |x|^2)^2}\, dx < \infty \right\}.$$

$\mathcal{D}$ is clearly a Hilbert space, and every $v \in \mathcal{D}$ satisfies

$$\ln \int_{\mathbb{R}^2} \frac{e^v}{(1 + |x|^2)^2}\, dx \leqslant \frac{1}{16\pi}\|\nabla v\|_{L^2}^2 + \bar{v} + C. \tag{3.14}$$

Here,

$$\bar{v} = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{v}{(1 + |x|^2)^2}\, dx.$$

The above inequality is obtained by pulling back the Moser–Trudinger inequality on S^2 through the stereographic projection (see for example [14]). Now let u_ε be a solution of (1.6) with $\kappa^2 = 4$. Recall that α is given in (2.2). We set

$$v_\varepsilon = u_\varepsilon - h_\varepsilon, \quad \text{where } h_\varepsilon(x) = f_\varepsilon(x) - (N+\alpha)\ln\bigl(1+|x|^2\bigr). \tag{3.15}$$

Then, from (3.14) and the maximum principle, we can see $v_\varepsilon \in \mathcal{D}$ and it satisfies

$$\Delta v_\varepsilon + e^{v_\varepsilon + h_\varepsilon}\bigl(1 - e^{v_\varepsilon + h_\varepsilon}\bigr) = g, \tag{3.16}$$

where $g(x) = \frac{4(N+\alpha)}{(1+|x|^2)}$.

Theorem 3.6. *Let v_ε be given in* (3.15)*. Under the assumptions of Theorem* 3.1*, we have*

$$\|v_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leqslant C \tag{3.17}$$

for some constant C independent of ε. Consequently, $\{v_\varepsilon\}$ is bounded uniformly in $\mathcal{D}$.

Proof. $\{v_\varepsilon\}$ is bounded in $L^\infty_{loc}(\mathbb{R}^2)$ by Theorem 3.1. So, we need to prove that $\{v_\varepsilon\}$ is bounded in $L^\infty(|x| \geqslant 1)$. Consider the function

$$\xi_\varepsilon(x) = u_\varepsilon\bigl(x/|x|^2\bigr) - 2\alpha \ln|x|, \qquad |x| \leqslant R_0 := \Bigl(1 + \max_j |p_j|\Bigr)^{-1}.$$

It follows from Theorem 3.1 that $\{\xi_\varepsilon\}$ is bounded in $L^\infty_{loc}(B_{R_0}(0) \setminus \{0\})$.

We now show that $\{\xi_\varepsilon\}$ is bounded from above in $B_{R_0}(0)$. We argue by contradiction and suppose that there is a sequence $\xi_n := \xi_{\varepsilon_n}$ such that $\xi_n(z_n) = \max_{|x| \leqslant R_0} \xi_n(x) \to \infty$ with $|z_n| \leqslant R_0$. Clearly $|z_n| \to 0$ and

$$-\Delta \xi_n = |x|^{2\alpha-4} e^{\xi_n}\bigl(1 - |x|^{2\alpha} e^{\xi_n}\bigr) \quad \text{for } |x| < R_0.$$

Therefore, the origin is a blow-up point for $\{\xi_n\}$ in this case.

We claim that $\xi_n(z_n) + (2\alpha - 2)\ln|z_n| \to \infty$. Indeed, if along a subsequence $\xi_n(z_n) + (2\alpha - 2)\ln|z_n| \leqslant C$, then we consider the scaled function $\xi_n(t_n x + z_n) - \xi_n(z_n)$, where t_n satisfies $\xi_n(z_n) + (2\alpha - 2)\ln t_n = 0$. Then the proof of Lemma 3.5 implies that

$$8\pi(\alpha - 1) \leqslant \liminf_{n\to\infty} \int\limits_{|x| \leqslant R_0} |x|^{2\alpha-4} e^{\xi_n}\bigl(1 - |x|^{2\alpha} e^{\xi_n}\bigr)\,dx \leqslant 4\pi(\alpha + N),$$

which contradicts the assumption $\alpha > N + 2$. This proves our claim.

We let $\bar{u}_n(x) = u_{\varepsilon_n}(x/|z_n|) - 2\ln|z_n|$ for $x \in \mathbb{R}^2$. Since $\bar{u}_n(z_n/|z_n|) = \xi_n(z_n) + (2\alpha - 2)\ln|z_n| \to \infty$, along a subsequence, $\{\bar{u}_n\}$ has a nonzero blow-up point on the unit circle. Then the proof of Lemma 3.4 implies that

$$\int\limits_{|x| \leqslant d} e^{\bar{u}_n}\bigl(1 - |z_n|^2 e^{\bar{u}_n}\bigr)\,dx = \int\limits_{|y| \leqslant d/|z_n|} e^{u_{\varepsilon_n}}\bigl(1 - e^{u_{\varepsilon_n}}\bigr)\,dy = o(1)$$

for any small constant $d > 0$. However, by Theorem 3.1, u_{ε_n} is locally uniformly bounded outside the vortex points. Therefore we conclude that $\{\xi_\varepsilon\}$ is bounded from above uniformly. Elliptic estimates and Theorem 3.1 further show that $\{\xi_\varepsilon\}$ is also bounded from below uniformly. Therefore (3.17) is proved.

Notice that $v_\varepsilon \in \mathcal{D}$ satisfies (3.16). Multiplying by v_ε both sides of (3.16) and using integration by parts after integration, we have $\|\nabla v_\varepsilon\|_{L^2} \leqslant C$ uniformly, which completes the proof. □

4. Existence

Due to Theorem 3.6, we can calculate the Leray–Schauder degree for the problem (1.6) as follows. Let $u(r)$ be the unique radially symmetric solution of (1.7). Recall that $u(r) = -2\alpha \ln r + O(1)$ near ∞, where α is given in (2.2). Choose a constant C_α such that

$$Q(r) := \frac{C_\alpha}{(1+r^2)^2} + e^{u(r)}\bigl(1 - 2e^{u(r)}\bigr) > 0 \quad \text{for } r \geqslant 0.$$

We define a map $T_\varepsilon : \mathcal{D} \to \mathcal{D}$ by

$$T_\varepsilon(v) = (-\Delta + \sigma)^{-1}\bigl(e^{v+h_\varepsilon}\bigl(1 - e^{v+h_\varepsilon}\bigr) + \sigma v - g\bigr),$$

where $\sigma(x) = C_\alpha(1+|x|^2)^{-2}$, and $h_\varepsilon(x) = f_\varepsilon(x) - (N+\alpha)\ln(1+|x|^2)$.

It is obvious that $T_\varepsilon : \mathcal{D} \to \mathcal{D}$ is compact, and there exists a constant $R > 0$ such that every zero of $I - T_\varepsilon$ is contained in a ball $\Omega_R := \{v \in \mathcal{D} \mid \|v\|_{\mathcal{D}} < R\}$ due to Theorem 3.6. Then the degree $\deg(I - T_\varepsilon, \Omega_R, 0)$ is well defined. Moreover, $I - T_\varepsilon$ is a continuous homotopy with respect to ε, and it preserves the degree by Theorem 3.6.

Now, we calculate the Leray–Schauder degree of $I - T_0$. It is well known that non-radial solutions of $(I - T_0)(v) = 0$, if they exist, do not affect the calculation of $\deg(I - T_0, \Omega_R, 0)$. See [27] and references therein. Then it is enough to calculate the index of $I - T_0$ at $v(r) = u(r) - h_0(r)$ in the subspace of radially symmetric functions in $\mathcal{D}$.

Lemma 4.1. *If $I - T_0$ is restricted to the subspace of radially symmetric functions in $\mathcal{D}$ then the index of $I - T_0$ at $v(r)$ is -1.*

Proof. Recall that $Q(r) = \sigma(r) + e^{u(r)}(1 - 2e^{u(r)}) > 0$ for $r > 0$, where $u(r) = v(r) + h_0(r)$ and $DT_0(v(r))w = (-\Delta + \sigma)^{-1}(Q(r)w)$ for $w \in \mathcal{D}$. Let Λ be the set of real eigenvalues of $DT_0(v(r))$ bigger than 1. We are interested in the radially symmetric eigenfunctions corresponding to $\lambda \in \Lambda$. If λ is an eigenvalue of $DT_0(v(r))$, and $\xi = \xi(r)$ is an eigenfunction corresponding to λ, then ξ satisfies

$$-\xi'' - \frac{1}{r}\xi' = e^{u(r)}\big(1 - 2e^{u(r)}\big)\xi + \frac{1-\lambda}{\lambda}Q(r)\xi. \tag{4.1}$$

Without loss of generality, we may assume that $\xi(0) = 1$. Otherwise, by the uniqueness for the initial value problem of (4.1), $\xi = 0$ identically. As before, let $\varphi(r)$ be the solution of the linearized equation

$$-\varphi'' - \frac{1}{r}\varphi' = e^{u(r)}\big(1 - 2e^{u(r)}\big)\varphi \quad \text{for } r > 0, \qquad \varphi(0) = 1. \tag{4.2}$$

Recall that φ has only one zero, and $\lim_{r\to\infty} \frac{\varphi(r)}{\ln r} = -c$ for some number $c > 0$.

First, we claim that $\lambda = 1$ cannot be an eigenvalue for a radially symmetric eigenfunction. Indeed, otherwise, $\xi(r)$ and $\varphi(r)$ would satisfy the same equation and $\xi(0) = \varphi(0)$, $\xi'(0) = \varphi'(0) = 0$. Therefore, again by the uniqueness, we have $\xi = \varphi \notin \mathcal{D}$, which yields a contradiction.

Second, we claim $\Lambda \neq \emptyset$. Indeed, if we put φ^+, the positive part of φ into the energy

$$E(\xi) = \int_0^\infty \big(r\big(\xi'\big)^2 - re^u\big(1 - 2e^u\big)\xi^2\big)\, dr,$$

then we have $E(\varphi^+) = 0$, which means that E has negative minimum in $\mathcal{D}$ and consequently $\Lambda \neq \emptyset$ since $\lambda = 1$ is not an eigenvalue of $DT_0(v(r))$.

Third, $\Lambda = \{\lambda\}$. Moreover, up to a multiplicative constant, there exists only one radial eigenfunction corresponding to $\lambda \in \Lambda$. To see this, we argue by contradiction. Suppose there exists a sign-changing eigenfunction corresponding to $\lambda \in \Lambda$. We denote it by ξ and let $a > 0$ be the first zero of ξ. We may assume that $\xi > 0$ on $(0, a)$. We also let $b > 0$ be the unique zero of $\varphi(r)$. Then the comparison theorem to (4.1) and (4.2) implies that $a > b$. Indeed if $a \leqslant b$, integration by parts would give

$$0 \leqslant -a\xi'(a)\varphi(a) = \int_0^a \frac{1-\lambda}{\lambda} Q(r)\xi(r)\varphi(r)r\, dr < 0,$$

which yields a contradiction. The comparison theorem also shows that ξ has only one zero. Otherwise, ξ would have the second zero $c > a$. Then integration by parts and Hopf lemma would give

$$0 < a\xi'(a)\varphi(a) - c\xi'(c)\varphi(c) = \int_a^c \frac{1-\lambda}{\lambda} Q(r)\xi(r)\varphi(r)r\, dr < 0,$$

which leads to a contradiction.

It is easy to check that $(r\xi')' < 0$ for r sufficiently large. Since $\xi \in \mathcal{D}$, we have $r\xi'(r) \to 0$, and $\xi(r)$ converges to a non-positive constant as $r \to \infty$. Therefore $r\xi'(r) = O(r^{-2})$ near ∞. Since $a > b$, integration by parts shows that

$$0 < a\xi'(a)\varphi(a) \leqslant \int_a^\infty \frac{1-\lambda}{\lambda} Q(r)\xi(r)\varphi(r)r\,dr < 0,$$

which yields a contradiction again. Therefore, by orthogonality, $\Lambda = \{\lambda\}$ for some $\lambda > 1$, and the sum of the algebraic multiplicities of λ is equal to 1.

As a conclusion, the index of $I - T_0$ at $v(r)$ is -1 [19]. □

In conclusion, we have $\deg(I - T_1, \Omega_R, 0) = \deg(I - T_0, \Omega_R, 0) = -1$, which implies that $I - T_1$ has a zero in $\Omega_R \subset \mathcal{D}$. Theorem 1.1 is proved.

Acknowledgements

The first and second authors would like to thank Taida Institute for Mathematical Sciences (TIMS), National Taiwan University for the warm hospitality where this work was partially done. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0086382) and the second author was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-521-C00021).

References

[1] D. Bartolucci, C.-C. Chen, C.-S. Lin, G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, Comm. Partial Differential Equations 29 (2004) 1241–1265.
[2] L.A. Caffarelli, Y. Yang, Vortex condensation in the Chern–Simons–Higgs model: An existence theorem, Comm. Math. Phys. 168 (1995) 321–336.
[3] D. Chae, O.Y. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, Comm. Math. Phys. 215 (2000) 119–142.
[4] H. Chan, C.-C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equation, Comm. Math. Phys. 231 (2002) 189–221.
[5] C.-C. Chen, C.-S. Lin, Mean field equations of Liouville type with singular data: Sharper estimates, Discrete Contin. Dyn. Syst. 28 (2010) 1667–1727.
[6] X. Chen, S. Hastings, J.B. McLeod, Y. Yang, A nonlinear elliptic equation arising from gauge theory and cosmology, Proc. R. Soc. Lond. A 446 (1994) 453–478.
[7] K. Choe, Asymptotic behavior of condensate solutions in the Chern–Simons–Higgs theory, J. Math. Phys. 48 (2007) 103501.
[8] K. Choe, Multiple existence results for the self-dual Chern–Simons–Higgs vortex equation, Comm. Partial Differential Equations 34 (2009) 1465–1507.
[9] K. Choe, N. Kim, Blow-up solutions of the self-dual Chern–Simons–Higgs vortex equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008) 313–338.
[10] G. Dunne, Self-Dual Chern–Simons Theories, Springer Lecture Note Physics, vol. M36, Springer, Berlin, 1995.
[11] J. Hong, Y. Kim, P.Y. Pac, Multivortex solutions of the abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 64 (1990) 2230–2233.
[12] R. Jackiw, E.J. Weinberg, Self-dual Chern–Simons vortices, Phys. Rev. Lett. 64 (1990) 2234–2237.
[13] A. Jaffe, C.H. Taubes, Vortices and Monopoles, Birkhäuser, Boston, 1980.
[14] N. Kim, Existence of vortices in a self-dual gauged linear sigma model and its singular limit, Nonlinearity 19 (2006) 721–739.
[15] B.H. Lee, C. Lee, H. Min, Supersymmetric Chern–Simons vortex systems and fermion zero modes, Phys. Rev. D 45 (1992) 4588–4599.
[16] C. Lee, K. Lee, E.J. Weinberg, Supersymmetry and self-dual Chern–Simons systems, Phys. Lett. B 243 (1990) 105–108.
[17] C.-S. Lin, C.L. Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. Math. II 172 (2010) 911–954.
[18] C.-S. Lin, S. Yan, Bubbling solutions for relativistic Abelian Chern–Simons model on a torus, Comm. Math. Phys. 297 (2010) 733–758.
[19] L. Nirenberg, Topics in Nonlinear Functional Analysis, Courant Lecture Notes in Mathematics, American Mathematics Society, Rhode Island, 2001.
[20] M. Nolasco, G. Tarantello, Double vortex condensates in the Chern–Simons–Higgs theory, Calc. Var. Partial Differential Equations 9 (1999) 31–94.
[21] J. Spruck, Y. Yang, The existence of nontopological solitons in the self-dual Chern–Simons theory, Comm. Math. Phys. 149 (1992) 361–376.
[22] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern–Simons theory: Existence and approximation, Ann. Inst. Henri Poincaré 12 (1995) 75–97.
[23] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, J. Math. Phys. 37 (1996) 3769–3796.

[24] G. Tarantello, Selfdual Gauge Field Vortices: An Analytical Approach, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Boston, 2008.
[25] Y. Yang, Solutions in Field-Theory and Nonlinear Analysis, Springer Monographs in Mathematics, Springer-Verlag, New York, 2001.
[26] R. Wang, The existence of Chern–Simons vortices, Comm. Math. Phys. 137 (1991) 587–597.
[27] Z. Wang, Symmetries and the calculations of degree, Chin. Ann. of Math. B 16 (1989) 520–536.