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Classical non-mass-preserving solutions of coagulation equations

M. Escobedo [a,b,*], J.J.L. Velázquez [c]

[a] *Departamento de Matemáticas, Universidad del País Vasco UPV/EHU, Apartado 644, E-48080 Bilbao, Spain*
[b] *Basque Center for Applied Mathematics (BCAM), Alameda de Mazarredo 14, E-48009 Bilbao, Spain*
[c] *Institute of Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany*

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Abstract

In this paper we construct classical solutions of a family of coagulation equations with homogeneous kernels that exhibit the behaviour known as gelation. This behaviour consists in the loss of mass due to the fact that some of the particles can become infinitely large in finite time.

1. Introduction

In this paper we prove existence of solutions of the classical coagulation equation for which the mass is not conserved in time. The coagulation equation reads as:

$$\frac{\partial f}{\partial t}(t,x) = Q[f](t,x), \quad x \geqslant 0,\ t > 0, \tag{1.1}$$

$$Q[f] = \frac{1}{2}\int_0^x K(x-y,y) f(t,x-y) f(t,y) - \int_0^\infty K(x,y) f(t,x) f(t,y)\,dy, \tag{1.2}$$

$$f(x,0) = f_0(x), \quad x > 0, \tag{1.3}$$

where the kernel K whose specific form will be precised later, satisfies $K(x,y) = K(y,x) \geqslant 0$.

The solutions of (1.1)–(1.3) satisfy formally, assuming that Fubini's Theorem can be applied, the mass conservation property:

$$\frac{d}{dt}\left(\int_0^\infty x f(t,x)\,dx\right) = 0. \tag{1.4}$$

However, it is well known that for a large class of homogeneous kernels $K(x,y)$ solutions of (1.1)–(1.3) satisfying (1.4) cannot exist globally in time (cf. [3,5,12,17]). More precisely, there exist solutions of (1.1)–(1.3) that preserve the

* Corresponding author at: Departamento de Matemáticas, Universidad del País Vasco UPV/EHU, Apartado 644, E-48080 Bilbao, Spain.
E-mail addresses: miguel.escobedo@ehu.es (M. Escobedo), velazquez@iam.uni-bonn.de (J.J.L. Velázquez).

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total mass of the particles $\int_0^\infty xf(t,x)\,dx$ during a finite time interval $0 \leqslant t \leqslant T < \infty$, but the mass is not preserved for arbitrarily long times. This phenomenon is usually termed as gelation.

In this paper we will restrict our attention to the study of kernels with the form:

$$K(x,y) = (xy)^{\frac{\lambda}{2}}, \quad 1 < \lambda < 2. \tag{1.5}$$

The range of exponents in (1.5) is the one in which changes of mass of order one can be expected in times of order one. Global weak solutions of (1.1) have been obtained in [14].

The main goal of this paper is to construct classical solutions of (1.1)–(1.3) exhibiting gelation. We will assume that the initial data behaves as a suitable power law for large values of x, and therefore that the loss of mass takes place since $t = 0$. In particular, in the classical solutions obtained in this paper, it will be possible to compute a detailed asymptotic behaviour of the solution $f(t,x)$ as $x \to \infty$, as well as the flux of mass escaping to infinity. The solutions obtained will be local in time, since we cannot avoid the possibility of discontinuities in the fluxes at infinity for positive times.

The results obtained in this paper rely heavily in the estimates obtained in the papers [8,9], where some related linear coagulation models were studied. In particular we have obtained very detailed estimates for the fundamental solution of the linear coagulation equation that results linearizing (1.1)–(1.3) around the power law $\bar{f}(x) = x^{-\frac{3+\lambda}{2}}$ in [8]. On the other hand, we have introduced in [9] some natural functional spaces to study the linearized version of (1.1)–(1.3) that results considering small deviations of a bounded initial data $f_0(x)$ behaving asymptotically as $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$. Both the fundamental solution in [8] and the functional framework introduced in [9] will be used extensively in this paper.

The power law $\bar{f}(x) = x^{-\frac{3+\lambda}{2}}$ plays a crucial role in study of solutions of Eqs. (1.1)–(1.2) having particle fluxes to infinity. Indeed, it has been explained in [8] that $\bar{f}(x)$ can be thought as a singular solution of (1.1)–(1.2) yielding a nonzero flux of particles from bounded regions to infinity. Therefore, it is natural to expect that the solutions of (1.1)–(1.2) in which particles escape towards infinity must behave asymptotically for large values as $K\bar{f}(x)$ where $K > 0$ provides a measure of the particle flux towards infinity. It is likely that solutions with the asymptotics $f(x,t) \sim \lambda(t)x^{-\alpha}$ as $x \to \infty$ with $\alpha > \frac{3+\lambda}{2}$, $\lambda(t) > 0$ might exist, at least locally in time, but they would not have a nontrivial flux of particles towards $x = \infty$. In other words, those solutions would be mass preserving, differently from the solutions considered in this paper whose main characteristic is that they lose mass. More precisely, we remark that the solutions obtained in this paper are defined in a time interval $0 \leqslant t \leqslant T$, they are globally bounded, and behave asymptotically as $K\bar{f}(x)$, henceforth they have a finite mass $M(t) = \int_0^\infty xf(x,t)\,dx$ for each time $t > 0$. Moreover, we have also $\frac{dM}{dt}(t) < 0$. Solutions of the coagulation equation with a decreasing amount of mass are usually thought in the physical literature as "post-gelation" solutions in which part of the mass escapes towards an infinitely large particle or "gel".

Notice that solutions behaving as $f(x,t) \sim \lambda(t)x^{-\alpha}$ as $x \to \infty$ with $\alpha < \frac{3+\lambda}{2}$, $\lambda(t) > 0$ and having finite mass, cannot be expected because the flux of particles towards $x \to \infty$ would be infinitely large and this would result in the instantaneous vanishing of the mass $M(t)$.

The coagulation equation is one among a large family of kinetic equations exhibiting particle fluxes for homogeneous solutions. Several examples can be found in [2]. A rigorous construction of solutions exhibiting loss of mass for small values of the energy for the so-called Uehling–Uhlenbeck equation (or quantum Boltzmann equation) has been obtained in [6,7]. The type of methods used in those papers is closely related to the ones used in this paper, although there are some technical differences.

In both cases (coagulation and Uehling–Uhlenbeck) we can think that the obtained solutions are mass preserving measure valued solutions having a singular part at some distinguished point and a regular part that is described by the integro-differential equations. In the case of coagulation the singular part of the measure (or gel) would be supported at $x = \infty$, and in the case of Uehling–Uhlenbeck such atomic measure (or Bose–Einstein condensate) would correspond to a macroscopic fraction of particles with zero energy. A natural question that arises in both cases, and in general in the study of equations with particle fluxes is to understand the interaction between the singular measure and the regular part of the measure. For the solutions obtained in [6,7] and in this paper we assume that the regular part of the measure is not affected by the singular part. However, it is well known that such interaction could be nontrivial. For instance, in the case of coagulation models, explicit examples for the kernel $K(x,y) = x \cdot y$ show that different solutions can be expected if there is interaction between the singular part and the regular part or if such interaction does not exist (cf. [11,22]). In [1] it is proved that different evolutions can be obtained for discrete systems of particles whose evolution is obtained as suitable limit processes which involve, either truncations of the kernel $K(x,y) = x \cdot y$, or a finite number of interacting particles. For more general kernels it is known that different dynamics can arise for different mass preserving regularizations of the kernel $K(x,y)$ after passing to the limit where gelation can occur (cf. [4,10]). In the case of Uehling–Uhlenbeck the computations and physical arguments in [13,19,20] suggest the existence of solutions of this equation exhibiting nontrivial interactions between the regular part of the particle distribution and the Bose–Einstein condensate. We also remark that in [15,16] a construction of global mass preserving weak solutions for the Uehling–Uhlenbeck system has been given. Such a construction begins regularizing the collision kernel for small energies and pass to the limit in the cutoff parameter. It is not known if the solutions constructed in [15,16] are the same as the ones in [6,7]. In all these problems a detailed understanding of the physical regularizations yielding cutoff mechanisms plays a crucial role (cf. also [21] for a discussion about these problems).

The plan of this paper is the following. In Section 2 we describe the functional framework used to prove the main theorem of this paper and state the main result. Section 3 gives a general sketch of the strategy of the proof. Section 4 summarizes some results that have been proved in [8,9] that will be used in this paper. Section 5 contains some auxiliary technical results concerning the functional spaces as well as the fundamental solution $g(\tau, x; 1)$ studied in [8]. Section 6 provides some estimates for the nonlinear term. Section 7 describes the asymptotics of the solutions of some linear equations as $x \to \infty$ in a detailed manner. Finally Section 8 explains the fixed point argument that concludes the proof of the theorem.

2. Functional framework and main result

In this paper we will choose the initial data in (1.3) satisfying $f_0 \in C^3(\mathbb{R}^+)$. We will assume also, as in [9], that the function f_0 is close to a power law for large x. To this end we define:

$$r = \frac{\lambda - 1}{2}. \tag{2.1}$$

We fix also $\delta > 0$ satisfying $\delta < \min\{r, \frac{2-\lambda}{2}\}$. We will then assume that f_0 has the form:

$$f_0(x) = f_1(x) + f_2(x) + f_3(x), \qquad f_1(x) = \frac{D_1 \xi(x)}{x^{\frac{3+\lambda}{2}}}, \qquad f_2(x) = \frac{D_2 \xi(x)}{x^{\frac{3+\lambda}{2}+r}}, \tag{2.2}$$

$$f_{1;2}(x) = f_1(x) + f_2(x) \tag{2.3}$$

where $D_1 > 0$, $D_2 \in \mathbb{R}$ and:

$$\xi \in C^\infty[0,\infty), \qquad \xi(x) = 1 \quad \text{for } x \geqslant 1 \quad \text{and} \quad \xi(x) = 0 \quad \text{if } 0 \leqslant x \leqslant 1/2, \qquad \xi'(x) \geqslant 0, \tag{2.4}$$

$$\left| f_3^k(x) \right| \leqslant \frac{B}{(x+1)^{\frac{3+\lambda}{2}+r+k+\delta}}, \qquad k = 0,1,2,3,4, \tag{2.5}$$

for some $B > 0$. The following auxiliary function will be used repeatedly:

$$h_0(x) = f_0(x) - f_1(x) = f_2(x) + f_3(x). \tag{2.6}$$

Notice that (2.2)–(2.5) imply:

$$\left(1 + y^{\frac{3+\lambda}{2}+r}\right)\left|h_0(y)\right| + \left(1 + y^{\frac{3+\lambda}{2}+r+1}\right)\left|h_0'(y)\right| + \left(1 + y^{\frac{3+\lambda}{2}+r+2}\right)\left|h_0''(y)\right| + \left(1 + y^{\frac{3+\lambda}{2}+r+3}\right)\left|h_0'''(y)\right| \leqslant CB \tag{2.7}$$

for some $C > 0$. We will assume in the rest of the paper that C is a generic constant that can change from line to line and that might depend only on D_1, D_2, B, λ and δ unless some additional dependence is written explicitly. Moreover, we will assume without loss of generality that $D_1 = 1$, since this parameter can be absorbed in a rescaling of t.

For any interval $I \subset (0, +\infty)$ we will denote as $L^2(I)$ the usual Lebesgue space of square integrable functions. For any $\sigma > 0$ we denote as $H^\sigma(I)$ the usual Sobolev space $W^{\sigma,2}(I)$. The corresponding norms will be denoted $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^\sigma}$. Dealing with functions depending on variables x and t we will write H_x^σ or L_t^2 in order to indicate the argument with respect to which the norm is taken.

In order to define suitable functional spaces we define, for any $T > 0$, $R > 0$ (see Fig. 1):

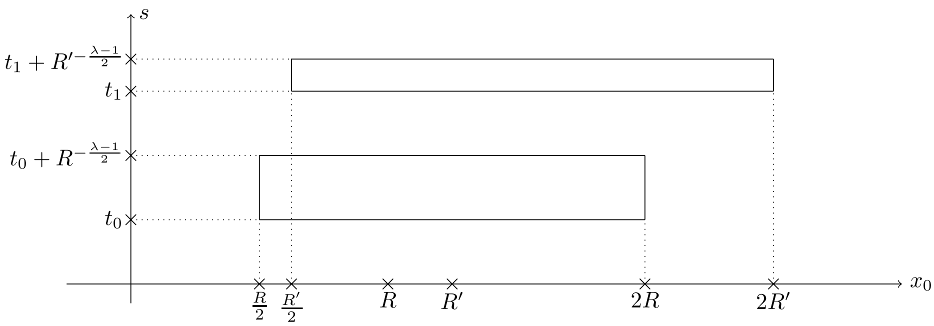

Fig. 1. Two cubes of the kind appearing in the norms $N_{2,\sigma}$ and N_∞ defined below.

$$N_{2;\sigma}(f;t_0,R)=\left(R^{\frac{\lambda-1}{2}+2\sigma-1}\int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2},T)}\left\|D_x^\sigma f(t)\right\|^2_{L^2(R/2,2R)}\,dt\right)^{1/2},\quad \sigma\geqslant 0, \tag{2.8}$$

$$M_{2;\sigma}(f;R)=\left(R^{2\sigma-1}\int_0^T\left\|D_x^\sigma f(t)\right\|^2_{L^2(R/2,2R)}\,dt\right)^{1/2},\quad \sigma\geqslant 0, \tag{2.9}$$

$$N_\infty(f;t_0,R)=\left(R^{\frac{\lambda-1}{2}}\int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2},T)}\left\|f(t)\right\|^2_{L^\infty(R/2,2R)}\,dt\right)^{1/2},$$

$$M_\infty(f;R)=\left(\int_0^T\left\|f(t)\right\|^2_{L^\infty(R/2,2R)}\,dt\right)^{1/2}.$$

Then, for any $\sigma>0$ we define the following norms:

$$\begin{aligned}
\|f\|_{Y^\sigma_{q,p}(T)}&=\sup_{0<R\leqslant 1}R^qM_{2;0}(f;R)+\sup_{0<R\leqslant 1}R^qM_{2;\sigma}(f;R)\\
&\quad+\sup_{0\leqslant t_0\leqslant T}\sup_{R\geqslant 1}R^pN_{2;0}(f;t_0,R)+\sup_{0\leqslant t_0\leqslant T}\sup_{R\geqslant 1}R^pN_{2;\sigma}(f;t_0,R),\\
\|f\|_{X_{q,p}(T)}&=\sup_{0<R\leqslant 1}R^qM_\infty(f;R)+\sup_{0\leqslant t_0\leqslant T}\sup_{R\geqslant 1}R^pN_\infty(f;t_0,R),
\end{aligned}$$

$$|||f|||_{q,p}=\sup_{0\leqslant x\leqslant 1}\left\{x^q\left|f(x)\right|\right\}+\sup_{x>1}\left\{x^p\left|f(x)\right|\right\}, \tag{2.10}$$

$$|||f|||_\sigma=\sup_{0\leqslant t\leqslant T}|||f|||_{\frac{3}{2},\frac{3+\lambda}{2}}+\|f\|_{Y^\sigma_{\frac{3}{2},\frac{3+\lambda}{2}}(T)} \tag{2.11}$$

and the following spaces:

$$Y^\sigma_{q,p}(T)=\left\{f:\ \|f\|_{Y^\sigma_{q,p}(T)}<\infty\right\},\qquad X_{q,p}(T)=\left\{f:\ \|f\|_{X_{q,p}(T)}<\infty\right\},$$
$$\mathcal{E}_{T;\sigma}=\left\{f:\ |||f|||_\sigma<\infty\right\}.$$

Throughout this paper we will assume that

$$\sigma\in(1,2). \tag{2.12}$$

Therefore, Sobolev embeddings imply $Y^\sigma_{q,p}(T)\subset X_{q,p}(T)$. Actually such embeddings would take place assuming the weaker condition $\sigma>\frac{1}{2}$. The main reason for the choice of σ as in (2.12) is purely technical, and it is due to the fact that the theorem proved in [9] to solve a suitable linearized problem (cf. for instance (3.5)) requires such a regularity. It is likely that using the "almost half-derivatives" that we introduce now would be possible to weaken the condition on σ to $\frac{1}{2}<\sigma<1$ both for the results of [9] and this paper (cf. Remark 6.4 in [9]).

We will solve (1.1)–(1.3) using a functional space that measures in a natural way the regularizing effects of the coagulation equation as $x \to \infty$ that have been studied in [9]. Let $\eta \in C^\infty(\mathbb{R}^+)$ be a cutoff function satisfying $\eta(x) = 1$ for $x \in (\frac{1}{4}, 3)$, $\eta(x) = 0$ for $x \notin (\frac{1}{8}, 4)$. Given $f \in C(\mathbb{R}^+)$, $t_0 \in [0, T]$, $R \geqslant 1$ we define:

$$F_{R,t_0}(\theta, X) = \eta(RX) f\bigl(t_0 + \theta R^{-(\lambda-1)/2}, RX\bigr) \tag{2.13}$$

and:

$$[f]_p^{\sigma;\frac{1}{2}} = \sup_{R\geqslant 1} \sup_{0\leqslant t_0\leqslant T} R^p \left(\int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2},T)} \int_{\mathbb{R}} \bigl|\hat{F}_{R,t_0}(\theta, k)\bigr|^2 Q_{R,\sigma}(k)\, dk\, d\theta \right)^{1/2} \tag{2.14}$$

where $Q_{R,\sigma}(k) = (1 + |k|^{2\sigma})(1 + \min\{|k|, R\})$,

$$\|f\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)} = \|f\|_{L^2((0,T);H_x^\sigma(0,2))} + [f]_p^{\sigma;\frac{1}{2}} + \sup_{0\leqslant t\leqslant T} |||f|||_{\frac{3}{2},p} + \|f\|_{Y^\sigma_{\frac{3}{2},p}(T)},$$

$$\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T) = \bigl\{f\colon\ \|f\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)} < \infty\bigr\}. \tag{2.15}$$

The intuition behind these spaces is the following. As it has been seen in [9] the main terms in the coagulation equation for solutions that are close to the power law $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$ can be thought as a perturbation of the half-derivative operator. However, since the integral operator $Q[f]$ in (1.2) is an integral operator Eq. (1.1) cannot be expected to have smoothing effects. Nevertheless, it has been seen in [9] that Eq. (1.1) has some kind of regularizing effect due to the fact that the right-hand side of (1.1) can be thought, for solutions close to $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$ as the half-derivative operator, if we restrict ourselves to incremental quotients with length x larger than one. This is the source of the regularizing effects that will be studied using the functionals (2.14), (2.15).

In order to gain some intuition about the spaces $X_{q,p}(T)$, $Y^\sigma_{q,p}(T)$, $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$ it is useful to think about them as functions that can be estimated like x^{-p} as $x \to \infty$ and x^{-q} as $x \to 0$ in the case of the spaces $X_{q,p}(T)$, $Y^\sigma_{q,p}(T)$ and $x^{-\frac{3}{2}}$ in the case of $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$. Concerning regularity, the functions in $X_{q,p}(T)$ are estimated pointwise, the functions in $Y^\sigma_{q,p}(T)$ have σ derivatives in space and the functions in $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$ have almost $(\sigma + \frac{1}{2})$ derivatives in the sense of the definition (2.14).

The main result of this paper is the following:

Theorem 1. *Suppose that f_0 satisfies (2.2)–(2.5), σ is as in (2.12) and K is as in (1.5). Then, there exists a classical solution $f \in \mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}$ of (1.1)–(1.3) with $f_t \in L^\infty((0,T)\times\mathbb{R}^+)$. Moreover, this solution is unique in the class of functions satisfying*:

$$f(t,x) = \lambda(t)\xi(x)x^{-\frac{3+\lambda}{2}} + h(t,x) \tag{2.16}$$

with $\lambda \in C[0,T]$, $\lambda(t) > 0$, $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$, $\lim_{\bar{t}\to 0} \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(\bar{t})} = 0$, where $\bar{p} = \frac{3+\lambda}{2} + \bar{\delta}$ with $0 < \bar{\delta} < r$, and T small enough.

Remark 2. Assumptions (2.2)–(2.5) seem a very strong condition. However, this condition is analogous to the type of compatibility conditions that must be assumed solving boundary value problems in order to obtain smooth solutions, or also to assume that the initial data has as many derivatives appear in the equation solving a parabolic problem. We notice that the assumptions (2.2)–(2.5) just state how close must be the initial data $f_0(x)$ to the power law $D_1 x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$. It is likely that (2.2)–(2.5) could be weakened to the form $f_0(x) = D_1 x^{-\frac{3+\lambda}{2}} + O(x^{-\frac{3+\lambda}{2}-\delta})$ as $x \to \infty$ for some $\delta > 0$. However, to prove this would require to obtain some delicate regularizing effects that we have preferred to avoid in this paper that is already rather technical. The specific value of r will play a role in the proof of Proposition 20 (cf. Remark 31) as well as in the proof of Proposition 29.

Remark 3. The splitting of the function $f(t,x)$ as in (2.16) just separates the part of $f(t,x)$ giving the power law asymptotics $\lambda(t)x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$ from the terms which are smaller as $x \to \infty$. The cutoff function $\xi(x)$ in the first term has been introduced in order to avoid singular terms at $x = 0$.

We also prove that the mass of the solutions in $x \in (0,\infty)$ constructed in Theorem 1 is strictly decreasing.

Theorem 4. *Suppose that f, λ and T are as in Theorem 1. Then*:

$$\frac{dM(t)}{dt} = \frac{d}{dt}\left(\int_0^\infty x f(t,x)\,dx\right) = -2\pi\big(\lambda(t)\big)^2 < 0$$

for all $t \in (0,T)$.

3. General strategy of the proof

The general plan that we will use to prove Theorem 1 is the following. We look for a solution of (1.1)–(1.3) in the form:

$$f(t,x) = \lambda(t) f_0(x) + h(t,x) \tag{3.1}$$

where f_0 is the initial data (cf. (1.3)) and h will be a small perturbation for short times. The function λ is a differentiable function to be prescribed satisfying $\lambda(0) = 1$. Then h, λ solve:

$$h_t = \lambda(t)\mathcal{L}_{f_0}[h] + Q[h] + \big(\lambda(t)\big)^2 Q[f_0] - \lambda_t f_0(x) \tag{3.2}$$

where the linear operator $\mathcal{L}_{f_0}$ is as in [9]:

$$\begin{aligned}\mathcal{L}_{f_0}[h] &= \int_0^x (x-y)^{\lambda/2} f_0(x-y) y^{\lambda/2} h(y)\,dy \\ &\quad - x^{\lambda/2} f_0(x)\int_0^\infty y^{\lambda/2} h(y)\,dy - x^{\lambda/2} h(x)\int_0^\infty y^{\lambda/2} f_0(y)\,dy.\end{aligned} \tag{3.3}$$

Our strategy is to solve (3.2) by means of a fixed point argument for a suitable operator $\mathcal{T}$ defined in $\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)$ with r as in (2.1), σ as in (2.12) and T sufficiently small (cf. (2.15)). It is convenient first, in order to apply the well-posedness results in [9] to introduce a new time scale. We will assume in all the paper that $|\lambda(t) - 1| \leqslant \frac{1}{2}$. We can then define a new time scale τ and a new function Λ by means of:

$$d\tau = \lambda(t)\,dt, \qquad \tau = 0 \quad \text{at } t = 0, \qquad \Lambda(\tau) = \lambda(t). \tag{3.4}$$

Then (3.2) becomes:

$$h_\tau = \mathcal{L}_{f_0}[h] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0] - \Lambda_\tau f_0(x)$$

where we will write $h(t,x) = h(\tau,x)$ by convenience.

Given $h \in \mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)$ and $\Lambda \in C^1([0,T])$ we will define $\tilde h = \tilde h[\Lambda]$ as the unique solution of:

$$\tilde h_\tau = \mathcal{L}_{f_0}[\tilde h] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0] - \Lambda_\tau f_0(x) \tag{3.5}$$

in $\mathcal{E}_{T;\sigma}$. The existence of such a solution will be a consequence of the results in [9]. In order to apply such results we will need to show that $Q[f_0], Q[h] \in Y^\sigma_{\frac{3}{2},(2+\bar\delta)}(T)$. In the case of $Q[f_0]$ this will be a consequence of (2.6), (2.7). In order to derive this property for $Q[h]$ we will use the decay and regularity properties of the functions $h \in \mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)$. The details will be given in Section 6.

After obtaining $\tilde{h} = \tilde{h}[\Lambda]$ we proceed to determine $\Lambda(\tau)$. To this end we will argue as follows. The asymptotic behaviour of $\tilde{h}$ as $x \to \infty$ is given by:

$$\tilde{h}(\tau, x) \sim \left[\mathcal{G}[\tau; h, \Lambda] - \int_0^\tau a(\tau - s)\Lambda_\tau(s)\, ds\right] x^{-\frac{3+\lambda}{2}} \quad \text{as } x \to \infty,\ 0 \leqslant \tau \leqslant T, \tag{3.6}$$

where $a(\cdot)$ is a function depending on f_0 and $\mathcal{G}[\cdot; h, \Lambda]$ a functional that will be precised later (cf. Propositions 26, 29 and Lemma 34 for a precise formulation of this result).

The asymptotics (3.6) will be obtained using the properties of the fundamental solution constructed in [9]. In order to close the fixed point argument, we need to choose $\Lambda(\tau)$ in such a way that $\tilde{h}(\tau, x) = o(x^{-\frac{3+\lambda}{2}})$ as $x \to \infty$. This can be achieved assuming that Λ solves the equation:

$$\int_0^\tau a(\tau - s)\Lambda_\tau(s)\, ds - \mathcal{G}[\tau; h, \Lambda] = 0, \quad 0 \leqslant \tau \leqslant T. \tag{3.7}$$

A detailed analysis of the function $a(\tau)$ (see Section 8.2) will allow to transform (3.7) in something more like a first order Volterra integral equation:

$$a(0)\Lambda(\tau) - \int_0^\tau \frac{da}{d\tau}(\tau - s)\Lambda(s)\, ds - a(\tau) - \mathcal{G}[\tau; h, \Lambda] = 0, \quad 0 \leqslant \tau \leqslant T, \tag{3.8}$$

with $a(0) = 1$. This equation can be solved by means of a standard fixed point argument, and this gives the desired Λ that will be denoted as $\tilde{\Lambda}$. We then define $\mathcal{T}[h] = \tilde{h}[\tilde{\Lambda}]$. Notice that $\mathcal{T}[h](\tau, x) = o(x^{-\frac{3+\lambda}{2}})$ as $x \to \infty$. Actually, a more careful analysis of (3.5), (3.8) shows that $\mathcal{T}[h] \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$. Moreover, the operator $\mathcal{T}$ is contractive in $\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ if T is sufficiently small and with a suitable choice of $\bar{\delta}$.

4. Summary of some of the results in [8,9]

We recall in this section several results that have been obtained in [8,9] and that will be used repeatedly in this paper.

In order to study the asymptotic behaviour of $\tilde{h}$ defined in the previous section, we will need some properties of the semigroup defined by the operator:

$$\begin{aligned} L(h) = {} & \int_0^{\frac{x}{2}} \left[(x-y)^{\lambda/2} G(x-y) - x^{\lambda/2} G(x)\right] y^{\lambda/2} h(y)\, dy \\ & + \int_0^{\frac{x}{2}} \left[(x-y)^{\lambda/2} h(x-y) - x^{\lambda/2} h(x)\right] y^{-\frac{3}{2}}\, dy - x^{-\frac{3}{2}} \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} h(y)\, dy - 2\sqrt{2} x^{\frac{\lambda-1}{2}} h(x) \end{aligned} \tag{4.1}$$

where $G(x) = \frac{1}{x^{\frac{3+\lambda}{2}}}$. We have studied in [8] the solution of the following problem:

$$\partial_\tau g(\tau, x) = L[g](\tau, x), \quad x > 0,\ \tau > 0, \quad g(0, x, x_0) = \delta(x - x_0). \tag{4.2}$$

In particular we have proved there the following results:

Theorem 5. *(Cf. Theorem 3.8 in [8].) There exists a unique solution $g(\tau, \cdot, x_0) \in C^\infty(\mathbb{R}^+)$ of (4.2) that has the following properties. There exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ depending only on λ such that, for any $0 < \varepsilon < \varepsilon_1$ the following statements hold.*

The function $g(\tau, \cdot, x_0)$ has the following self-similar structure:

$$g(\tau, x, x_0) = \frac{1}{x_0} g\left(\tau x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right). \tag{4.3}$$

For all $\tau \geqslant 1$:

$$g(\tau, x, 1) = \tau^{\frac{2}{\lambda-1}} \varphi_1(\rho) + \varphi_2(\tau, \rho), \quad \rho = \tau^{\frac{2}{\lambda-1}} x \tag{4.4}$$

with:

$$\varphi_1(\rho) = \begin{cases} a_1 \rho^{-\frac{3}{2}} + O_\varepsilon(\rho^{-\frac{4-\lambda}{2}+\varepsilon}), & 0 \leqslant \rho \leqslant 1, \\ a_2 \rho^{-\frac{3+\lambda}{2}} + O_\varepsilon(\rho^{-(1+\lambda-\varepsilon)}), & \rho > 1, \end{cases} \tag{4.5}$$

where a_1, a_2 *are two explicit constants,*

$$\varphi_2(\tau, \rho) = \begin{cases} b_1(\tau) \rho^{-\frac{3}{2}} + O(\tau^{\frac{2}{\lambda-1}-\varepsilon_2} \rho^{-\frac{3}{2}+\varepsilon_2}), & 0 \leqslant \rho \leqslant 1, \\ b_2(\tau) \rho^{-\frac{3+\lambda}{2}} + O(\tau^{\frac{2}{\lambda-1}-\varepsilon_2} \rho^{-\frac{3+\lambda}{2}-\varepsilon_2}), & \rho > 1, \end{cases} \tag{4.6}$$

where $b_1, b_2 \in$ *are two continuous functions such that* $|b_1(\tau)| + |b_2(\tau)| \leqslant C\tau^{\frac{2}{\lambda-1}-\varepsilon_2}$.

For $0 < \tau \leqslant 1$ *we have*:

$$g(\tau, x, 1) = \begin{cases} \tau x^{-\frac{3}{2}} + b_3(\tau) x^{-\frac{3}{2}} + O(\tau x^{-\frac{3}{2}+\varepsilon_2}), & 0 \leqslant x \leqslant \frac{1}{2}, \\ a_3 \tau x^{-\frac{3+\lambda}{2}} + b_4(\tau) x^{-\frac{3+\lambda}{2}} + O(\tau x^{-\frac{3+\lambda}{2}-\varepsilon_2}), & x \geqslant \frac{3}{2}, \\ O_\varepsilon\big(\frac{t^{1-2\varepsilon}}{|x-1|^{\frac{3}{2}-\varepsilon}}\big) & \text{for } t^2 < |x-1| < \frac{1}{2} \end{cases} \tag{4.7}$$

where a_3 *is an explicit numerical constant and* b_3, b_4 *are continuous functions such that* $|b_3(\tau)| + |b_4(\tau)| \leqslant C\tau^{1+\varepsilon_2}$. *Moreover*:

$$\lim_{t \to 0} t^2 g\big(t, 1 + t^2\chi, 1\big) = \Psi(\chi) \quad \text{uniformly on compact subsets of } \mathbb{R}$$

where the function Ψ *is given by*:

$$\Psi(\chi) = \frac{2}{\pi} \frac{\exp(-\frac{\pi}{\chi^{3/2}})}{\chi^{3/2}} \quad \text{for } \chi > 0, \qquad \Psi(\chi) = 0 \quad \text{for } \chi < 0. \tag{4.8}$$

Remark 6. The functions $O_\varepsilon(\cdot)$ depend on ε.

Remark 7. Notice that (4.5)–(4.7) imply the existence of a function $\Theta = \Theta(\tau)$ and $\varepsilon > 0$ such that:

$$\big|g(\tau, x, 1) - \Theta(\tau) x^{-\frac{3+\lambda}{2}}\big| \leqslant C\tau x^{-\frac{3+\lambda}{2}-\varepsilon}, \quad \tau \leqslant 1,\ x \geqslant 1, \tag{4.9}$$

$$\big|g(\tau, x, 1) - \Theta(\tau) x^{-\frac{3+\lambda}{2}}\big| \leqslant \frac{C}{\tau^{\frac{\lambda+1}{\lambda-1}+\frac{2\varepsilon}{\lambda-1}} x^{\frac{3+\lambda}{2}+\varepsilon}}, \quad \tau \geqslant 1,\ x \geqslant 1, \tag{4.10}$$

where:

$$\Theta(\tau) = \begin{cases} a_4 \tau + b_4(\tau), & |b_4(\tau)| \leqslant C\tau^{1+\varepsilon},\ \tau \leqslant 1, \\ a_2 \tau^{-\frac{\lambda+1}{\lambda-1}} + b_2(\tau), & |b_2(\tau)| \leqslant C\tau^{-\frac{\lambda+1}{\lambda-1}-\varepsilon},\ \tau \geqslant 1. \end{cases} \tag{4.11}$$

We will need improved estimates for $g(\tau, x, 1)$. More precisely we need to compute the next order in the expansion of g as $x \to \infty$. To this end we obtain the representation formulas for the function $g(\tau, x, 1)$ that we have obtained in the proof of Lemma 7.10 of [8].

Theorem 8. *(Cf. Lemma 5.1 in [8].) The function* $g(\tau, x, 1)$ *described in Theorem 5 can be written as* $g(\tau, x, 1) = G(\tau, X)$, $x = e^X$ *with*:

$$G(\tau, X) = -\frac{\mathcal{V}(2i)i}{2\pi(\lambda-1)} e^{-\frac{3+\lambda}{2}X} \int_{\operatorname{Im}(Y)=-\gamma_1} dY \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\frac{(3+\lambda)i}{2} + Y)} \Gamma\left(\frac{2iY}{\lambda-1}\right)$$
$$+ \frac{i}{\pi(\lambda-1)} \int_{\operatorname{Im}(\xi)=\beta} d\xi\, e^{i\xi X} \int_{\operatorname{Im}(Y)=-\gamma_1} dY \frac{\mathcal{V}(\xi)\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\xi+Y)} \Gamma\left(\frac{2iY}{\lambda-1}\right) \tag{4.12}$$

where $(\beta - \frac{3+\lambda}{2}) > 0$ *and* $\gamma_1 > 0$ *are sufficiently small. The function* $\mathcal{V}(\xi)$ *is given by*:

$$\mathcal{V}(\xi) = \exp\left(-\frac{2i}{\lambda-1}\int_{\operatorname{Im}(\xi)=\beta_1} \log\bigl(-\Phi(\eta)\bigr)\left[\frac{1}{1-e^{\frac{4\pi(\xi-\eta)}{\lambda-1}}} - \frac{1}{1+e^{-\frac{4\pi\eta}{\lambda-1}}}\right]d\eta\right), \quad \beta_1 \in \left(\frac{2+\lambda}{2}, \frac{3+\lambda}{2}\right),$$

$$\Phi(\eta) = -\frac{2\sqrt{\pi}\,\Gamma(i\eta + 1 + \frac{\lambda}{2})}{\Gamma(i\eta + \frac{\lambda+1}{2})}, \qquad \lim_{\operatorname{Re}(\eta)\to\infty} \arg\bigl(-\Phi(\eta)\bigr) = \frac{\pi}{4}.$$

On the other hand we have proved the following results in [9]:

Theorem 9. *(Cf. Theorem 2.1 in [9].) For any* $\sigma \in (1,2)$, $\bar{\delta} > 0$ *and any* f_0 *satisfying* (2.6), (2.7) *there exists* $T > 0$ *such that for all* $\mu \in Y^{\sigma}_{3/2,2+\bar{\delta}}$ *the Cauchy problem*

$$h_\tau = \mathcal{L}_{f_0}(h) + \mu, \quad h(0) = 0 \tag{4.13}$$

has a unique solution h *in* $\mathcal{E}_{T;\sigma}$. *Moreover* $|\!|\!| h |\!|\!|_\sigma \leqslant C\|\mu\|_{Y^{\sigma}_{3/2,(2+\bar{\delta})}}$ *for some positive constant* C *depending on* T, σ, $\bar{\delta}$ *as well as* A, B *and* γ *in* (2.6), (2.7) *but not on* μ.

Theorem 10. *(Cf. Theorem 2.2 in [9].) For any* $\sigma \in (1,2)$, $\bar{\delta} > 0$ *and for any* f_0 *satisfying* (2.6), (2.7), *the solution of the Cauchy problem* (4.13) *obtained in Theorem* 9 *satisfies*

$$[h]^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}} \leqslant C\|\mu\|_{Y^{\sigma}_{3/2,2+\bar{\delta}}}$$

for some positive constant C *depending on* T, σ, $\bar{\delta}$ *as well as* A, B *and* γ *in* (2.6), (2.7) *but not on* μ.

This is a regularity result proved in [9] that will be used repeatedly in the following:

Theorem 11. *(Cf. Theorem 3.1 in [9].)*

(i) *Suppose that* $Q \in L^2_t(0,1; H^{\sigma}_x(1/2,2))$, $P \in L^2_t(0,1; H^{\sigma-1/2}_x(1/2,2))$ *with* $\sigma \in (1/2,2)$, $\kappa \in (0,1]$ *and* $f \in L^{\infty}((1/4,2)\times(0,1)) \cap L^2(0,1; H^{1/2}(1/4,2)) \cap H^1(0,1; L^2(1/4,2))$ *is such that* $f = 0$ *if* $x < 1/8$ *or* $x > 4$ *and satisfies*

$$\frac{\partial f}{\partial t} = \kappa T_{\varepsilon,R}(M_{\lambda/2} f) + Q + P$$

for all $x \in (1/4,2)$, $t \in (0,1)$ *and* $f(x,0) = 0$. *Then*:

$$\|f\|_{L^2_t(0,1;H^{\sigma}_x(3/4,5/4))} \leqslant C\left(\|Q\|_{L^2_t(0,1;H^{\sigma}_x(1/2,2))} + \frac{1}{\varepsilon\kappa}\|P\|_{L^2_t(0,1;H^{\sigma-1/2}_x(1/2,2))} + \|f\|_{L^{\infty}((1/4,2)\times(0,1))}\right)$$

for some positive constant C *independent of* ε *and* R.

(ii) *Suppose that* $Q \in L^2_t(0,T_{\max}; H^{\sigma}_x(1/2,2))$, $P \in L^2_t(0,T_{\max}; H^{\sigma-1/2}_x(1/2,2))$, $f \in L^{\infty}((1/4,2)\times(0,T_{\max})) \cap C^1_t(0,T_{\max}; H^{1/2}_x(1/4,2))$ *for some* $T_{\max} > 0$ *is such that* $f = 0$ *if* $x < 1/8$ *or* $x > 4$ *and satisfies*

$$\frac{\partial f}{\partial t} = T_{\varepsilon,R}(M_{\lambda/2} f) + Q + P - a(x,t) f, \quad x \in (1/4,2),\ t > 0, \tag{4.14}$$

$$f(x,0) = 0 \tag{4.15}$$

for some function $a \in L^{\infty}(0,T_{\max}; H^{\sigma}(1/2,2))$, $a \geqslant A > 0$. *Then, for all* $T \in [0, T_{\max}-1]$:

$$\sup_{0\leqslant T\leqslant T_{\max}}\left(\int_T^{\min(T+1,T_{\max})} \|f(t)\|^2_{H^{\sigma}(3/4,5/4)}\,dt\right)^{1/2}$$

$$\leqslant C \sup_{0\leqslant T\leqslant T_{\max}} \left(\int_T^{\min(T+1,T_{\max})} \|Q(t)\|^2_{H^\sigma(1/2,2)}\,dt \right)^{1/2} + \frac{C}{\varepsilon} \sup_{0\leqslant T\leqslant T_{\max}} \left(\int_T^{\min(T+1,T_{\max})} \|P(t)\|^2_{H^{\sigma-1/2}(1/2,2)}\,dt \right)^{1/2} + C\|f\|_{L^\infty((1/4,2)\times(0,T_{\max}))}. \tag{4.16}$$

(iii) *Suppose that for some* $T_{\max} > 0$, $Q \in L^2_t(0, T_{\max}; H^\sigma_x(1/2, 2))$, $f \in L^\infty((1/4, 2) \times (0, T_{\max})) \cap C^1_t(0, T_{\max}; H_x^{1/2}(1/4, 2))$ *is such that* $f = 0$ *if* $x < 1/8$ *or* $x > 4$ *and satisfies* (4.14), (4.15) *with* $P = 0$ *and* $\varepsilon = 0$. *Then*

$$\left(\int_T^{\min(T+1,T_{\max})} \int_{\mathbb{R}} \left|\hat{F}(k,t)\right|^2 |k|^{2\sigma} \min\{|k|, R\}\,dk \right)^{1/2} \leqslant C \sup_{0\leqslant T\leqslant T_{\max}} \left(\int_T^{\min(T+1,T_{\max})} \|Q(t)\|^2_{H^\sigma(1/2,2)}\,dt \right)^{1/2} + C\|f\|_{L^\infty((1/4,2)\times(0,T_{\max}))} \tag{4.17}$$

where $F(x,t) = \eta(x) f(x,t)$, $\eta \in C^\infty$ *is a cutoff satisfying* $\eta(x) = 1$ *if* $x \in (\frac{3}{4}, \frac{5}{4})$ *and* $\eta(x) = 0$ *if* $x \notin (\frac{1}{8}, \frac{1}{4})$. *The constant* C *is independent of* R.

5. Some auxiliary results

In this section we collect two estimates that will be used in the proof of Theorem 1.

5.1. Remarks about notation

We will use in the arguments several different symbols. Specific letters have been reserved for quantities with precise meanings. We write them shortly here as a guide for the reader.

The letter $r = \frac{\lambda-1}{2}$ will denote the first order correction to the asymptotics of f_0 as $x \to \infty$ (cf. (2.1)–(2.5)). We will use δ to denote the exponent of the second order correction of f_0 as $x \to \infty$. It will be assumed in the whole paper that $\delta < \min\{r, \frac{2-\lambda}{2}\}$.

The parameter $\bar{\delta}$ characterizes the functional space where the solution of the equation will be obtained (cf. Theorem 1). It will be always assumed that $\bar{\delta} < \min\{r, \delta\}$. We will use also the notation $\bar{p} = \frac{3+\lambda}{2} + \bar{\delta}$.

The symbols ε's will be used for the fundamental solution associated to $g_t = L[g]$ (cf. Theorem 5).

We use σ to denote the spatial regularity of the solutions. We assume $\sigma \in (1,2)$.

5.2. A general estimate for the functions in $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$

Lemma 12. *Suppose that* $\phi \in \mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$ *for* $\sigma \in (1,2)$, $p > 0$. *Let us define*:

$$\omega(t,x) = \int_0^t \phi(s,x)\,ds, \quad x \in \mathbb{R}^+,\ 0 \leqslant t \leqslant T. \tag{5.1}$$

Then, there exists $C > 0$ *independent of* T, ϕ *such that*:

$$\|\omega\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)} \leqslant 4\sqrt{T}\|\phi\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)}. \tag{5.2}$$

Proof. Due to (2.15) to estimate $\|\omega\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)}$ we need to obtain bounds for $\|\omega\|_{L^2((0,T);H_x^\sigma(0,2))}$, $[\omega]_p^{\sigma;\frac{1}{2}}$, $\sup_{0\leqslant t\leqslant T}|||\omega|||_{\frac{3}{2},p}$, $\|\omega\|_{Y^\sigma_{\frac{3}{2},p}(T)}$. Using (5.1) and Cauchy–Schwartz we obtain:

$$\|\omega\|_{L^2((0,T);H_x^\sigma(0,2))}\leqslant T\|\phi\|_{L^2((0,T);H_x^\sigma(0,2))}. \tag{5.3}$$

Using (2.10):

$$\sup_{0\leqslant t\leqslant T}|||\omega|||_{\frac{3}{2},p}\leqslant T|||\phi|||_{\frac{3}{2},p}. \tag{5.4}$$

To estimate $\|\omega\|_{Y^\sigma_{\frac{3}{2},p}(T)}$ we need to control $N_{2;\sigma}(\omega;t_0,R)$, $M_{2;\sigma}(\omega;R)$ (cf. (2.8), (2.9)). Using again Cauchy–Schwartz inequality we arrive at:

$$N_{2;\sigma}(\omega;t_0,R)\leqslant\sqrt{T}N_{2;\sigma}(\phi;t_0,R),\quad R>1,\qquad M_{2;\sigma}(\omega;R)\leqslant\sqrt{T}M_{2;\sigma}(\phi;R),\quad R\leqslant 1. \tag{5.5}$$

Finally we can estimate $[\omega]_p^{\sigma;\frac{1}{2}}$ using also Cauchy–Schwartz for each value of R (cf. (2.14)):

$$[\omega]_p^{\sigma;\frac{1}{2}}\leqslant\sqrt{T}[\phi]_p^{\sigma;\frac{1}{2}} \tag{5.6}$$

where we use that $t_0+\theta R^{-\frac{(\lambda-1)}{2}}$ (cf. (2.13)) is bounded by T. Combining (5.2)–(5.5) we obtain (5.2). $\square$

5.3. *Improved estimates for $g(\tau,x,1)$*

We will need to compute detailed asymptotics for the function $g(\tau,x,1)$ in Theorem 5 as $x\to\infty$, since the main corrective terms coming from the asymptotics of $g(\tau,x,1)$ have the same order of magnitude as the ones due to the natural sources in the problem for the approach indicated in Section 3.

Proposition 13. *Let $g(\tau,x,1)$ be as in Theorem 5. Suppose that $\tau\geqslant 1$. Then*:

$$g(\tau,x,1)=\tau^{\frac{2}{\lambda-1}}\varphi_1(\rho)+\varphi_2(\tau,\rho),\quad \rho=\tau^{\frac{2}{\lambda-1}}x$$

with:

$$\varphi_1(\rho)=a_2\rho^{-\frac{3+\lambda}{2}}+a_5\rho^{-(\frac{3+\lambda}{2}+r)}+O\big(\rho^{-(1+\lambda+\varepsilon_1)}\big),\quad \rho>1, \tag{5.7}$$

for some $\varepsilon_1>0$. Moreover:

$$\varphi_2(\tau,\rho)=b_2(\tau)\rho^{-\frac{3+\lambda}{2}}+O_{\varepsilon_2}\big(\tau^{\frac{2}{\lambda-1}-\varepsilon_2}\rho^{-(\frac{3+\lambda}{2}+r)}\big),\quad \rho>1, \tag{5.8}$$

where $|b_2(\tau)|\leqslant C_{\varepsilon_2}\tau^{\frac{2}{\lambda-1}-\varepsilon_2}$ for any $\varepsilon_2>0$.

Suppose now that $\tau\leqslant 1$. Then:

$$g(\tau,x,1)=a_3\tau x^{-\frac{3+\lambda}{2}}+b_4(\tau)x^{-\frac{3+\lambda}{2}}+O\big(\tau x^{-(\frac{3+\lambda}{2}+r)}\big),\quad x\geqslant\frac{3}{2}, \tag{5.9}$$

where $|b_4(\tau)|\leqslant C\tau^{1+\varepsilon_3}$ for some $\varepsilon_3>0$ sufficiently small.

Proof. The argument is similar to the one in [8]. More precisely we deform the contour of integration in (4.12). Crossing the singularities of the integrand we obtain contributions using residues that yield the main terms in the asymptotics. The only difference with the argument in [8] is that we have to cross also the singularity at $\xi=(\frac{3+\lambda}{2}+r)i$. This yields the second term on the right-hand side of (5.7).

More precisely. Suppose first that $\tau\geqslant 1$. We then use the representation formula (cf. [8, Section 9.2]):

$$G(\tau,X)=(\tau)^{\frac{2}{\lambda-1}}\Psi_1(\theta)+G_1(\tau,X),\qquad \theta=X+\frac{2}{\lambda-1}\log(\tau)$$

where:

$$\Psi_1(\theta) = \frac{1}{\pi(\lambda-1)i\mathcal{V}(\frac{\lambda+1}{2}i)} \int_{\mathrm{Im}(\xi)=\beta_2} d\xi\, e^{i\xi\theta}\mathcal{V}(\xi)\Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right),$$

$$G_1(\tau,X) = \frac{i}{\pi(\lambda-1)} \int_{\mathrm{Im}(\xi)=\beta_2} d\xi\, e^{i\xi\theta} \int_{\mathrm{Im}(y)=\beta_3} dy\, \frac{\mathcal{V}(\xi)\tau^{-\frac{2iy}{\lambda-1}}}{\mathcal{V}(y)}\Gamma\left(-\frac{2i}{\lambda-1}(\xi-y)\right)$$

with $\beta_2 \in (\beta_0, \frac{3-\lambda}{2})$, $\beta_0 \in (\frac{3}{2}, 2)$, $\beta_3 \in (\frac{3-\lambda}{2}, 1)$. The asymptotics of the function $\Psi_1(\theta)$ as $\theta \to \infty$ has been obtained in [8, Proposition 9.8], moving upwards the contour of integration $\{\mathrm{Im}(\xi) = \beta_2\}$ in order to make it cross the first singularity found of $\mathcal{V}$ at $\xi = \frac{3+\lambda}{2}i$. To obtain better estimates we just move the contour of integration above the line $\{\mathrm{Im}(\xi) = \lambda + 1\}$. We then obtain the following generalization of formula (9.27) in [8]:

$$\begin{aligned}\Psi_1(\theta) = {}& -\frac{\Gamma(\frac{\lambda+1}{\lambda-1})}{2\pi i}\frac{\mathcal{V}(2i)e^{-\frac{3+\lambda}{2}\theta}}{\mathcal{V}(\frac{\lambda+1}{2}i)} - \frac{2\pi i\Gamma(\frac{\lambda}{\lambda-1})\mathcal{V}(2i)e^{-(\frac{3+\lambda}{2}+r)\theta}}{\pi(\lambda-1)i\mathcal{V}(\frac{\lambda+1}{2}i)\Phi((\lambda+1)i)\Phi'((\frac{3+\lambda}{2}i))} \\ & + \frac{1}{\pi(\lambda-1)i\mathcal{V}(\frac{\lambda+1}{2}i)} \int_{\mathrm{Im}(\xi)=1+\lambda+\varepsilon_1} d\xi\, e^{i\xi\theta}\mathcal{V}(\xi)\Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right) \end{aligned} \tag{5.10}$$

with $\varepsilon > 0$ small. We have computed $\mathrm{Res}(\mathcal{V}; \xi = (\lambda+1)i)$ using Proposition 4.1 and (5.11) in [8].

The first term on the right-hand side of (5.10) is the first one on the right-hand side of (5.7). The last one can be estimated by $Ce^{-(\frac{3+\lambda}{2}+r+\varepsilon_1)\theta}$ for $\theta > 0$. This gives (5.7). We now estimate $G_1(\tau, X)$. This can be made as the estimate of G_1 in [8, Lemma 9.9]. Deforming the contour $\{\mathrm{Im}(\xi) = \beta_2\}$ as in the derivation of (9.36) of [8], but moving it above the line $\{\mathrm{Im}(\xi) = \lambda + 1\}$ we obtain:

$$G_1(\tau,X) = b_2(\tau)e^{-\frac{3+\lambda}{2}\theta} + \tilde{b}_2(\tau)e^{-(\frac{3+\lambda}{2}+r)\theta} + \tilde{Q}_1(\tau,X)$$

where the function $b_2(\tau)$ is exactly as in [8], the function $\tilde{b}_2(\tau)$ has a similar formula, with slightly different terms arising from the integration by residues, and $\tilde{Q}_1(\tau, X)$ is similar to (9.37) in [8] with the only difference that $\beta_6 = (1+\lambda) + \varepsilon_1$, with $\varepsilon_1 > 0$. Arguing exactly as in [8] we obtain:

$$\begin{aligned}&\left|b_2(\tau)\right| + \left|\tilde{b}_2(\tau)\right| \leqslant C(\tau)^{\frac{2}{\lambda-1}-\varepsilon_2} \quad \text{for } \tau \geqslant 1,\\ &\left|\tilde{Q}_1(\tau,X)\right| \leqslant C(\tau)^{\frac{2}{\lambda-1}-\varepsilon_2}e^{-[(1+\lambda)+\varepsilon_1]\theta}.\end{aligned}$$

This gives (5.8). On the other hand, in order to derive (5.9) we argue as in [8, proof of Lemma 9.10, (9.45)]. Indeed, moving the contour of integration $\{\mathrm{Im}(\xi) = \beta_7\}$ to $\{\mathrm{Im}(\xi) = (\lambda + 1 + \varepsilon_1)\}$, with $\varepsilon_1 > 0$ we obtain:

$$\begin{aligned}G(\tau,X) = {}& -\frac{\mathcal{V}(2i)i}{2\pi(\lambda-1)}e^{-\frac{3+\lambda}{2}X} \int_{\mathrm{Im}(Y)=-\gamma_1} dY\, \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\frac{(3+\lambda)i}{2}+Y)}\Gamma\left(\frac{2iY}{\lambda-1}\right) \\ & - \frac{2\mathcal{V}(2i)e^{-(\frac{3+\lambda}{2}+r)X}}{(\lambda-1)\Phi((\lambda+1)i)\Phi'((\frac{3+\lambda}{2}i))} \int_{\mathrm{Im}(Y)=-\gamma_1} dY\, \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}((1+\lambda)i+Y)}\Gamma\left(\frac{2iY}{\lambda-1}\right) \end{aligned} \tag{5.11}$$

$$+ \frac{i}{\pi(\lambda-1)} \int_{\mathrm{Im}(\xi)=\lambda+1+\varepsilon_1} d\xi\, e^{i\xi X} \int_{\mathrm{Im}(Y)=-\gamma_1} dY\, \frac{\mathcal{V}(\xi)\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\xi+Y)}\Gamma\left(\frac{2iY}{\lambda-1}\right). \tag{5.12}$$

The time dependence of the integral terms can be obtained as in [8], since this one comes from the integration in the Y variable. □

6. Estimates for the quadratic term $Q[h]$

The following lemma will be used to show smallness of the quadratic terms $Q[h]$.

Proposition 14. *For any $\sigma \in (1,2)$ and any $\bar{\delta} > 0$, there exists $C = C(\sigma, \bar{\delta})$ such that for any $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$:*

$$\left\| Q[h] \right\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}(T)} \leqslant C \|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$$

with $Q[\cdot]$ as in (1.2), (1.5).

In order to prove Proposition 14 we rewrite $Q[h]$ as:

$$Q[h](\tau, x) = I_1 + I_2 \tag{6.1}$$

where:

$$I_1 = -\int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h(\tau, x) h(\tau, y)\, dy, \tag{6.2}$$

$$I_2 = -\int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h(\tau, x) h(\tau, y)\, dy + \int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} h(\tau, y)\left[(x-y)^{\frac{\lambda}{2}} h(\tau, x-y) - x^{\frac{\lambda}{2}} h(\tau, x)\right] dy. \tag{6.3}$$

We begin estimating I_1:

Lemma 15. *Let I_1 be as in (6.3) and $\bar{\delta} > 0$. Then:*

$$\|I_1\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta}_1)}(T)} \leqslant C \|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \tag{6.4}$$

where C is uniformly bounded for $0 \leqslant T \leqslant 1$.

Proof. We just need to estimate the functionals $N_{2;\sigma}(I_1; \tau_0, R)$ for $R \geqslant 1$ and $M_{2;\sigma}(I_1; R)$ for $R \leqslant 1$ (cf. (2.8), (2.9)). Suppose first that $R > 1$. We introduce the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $H_R(\theta, X) = h(\tau, x)$. Then:

$$\begin{aligned} I_1 &= -R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\theta, X) \int_{\frac{X}{2}}^{2} Y^{\frac{\lambda}{2}} H_R(\theta, Y)\, dY \\ &\quad - R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\theta, X) \int_{2}^{\infty} Y^{\frac{\lambda}{2}} H_R(\theta, Y)\, dY \qquad (6.5)\\ &\equiv I_{1,1} + I_{1,2}. \qquad (6.6) \end{aligned}$$

We begin estimating $I_{1,2}$. Notice that:

$$\left| R^{\frac{\lambda}{2}+1} \int_2^{\infty} Y^{\frac{\lambda}{2}} H_R(\theta, Y)\, dY \right| \leqslant \frac{C}{R^{\frac{1}{2}+\bar{\delta}_1}} \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}.$$

Then:

$$N_{2;\sigma}(I_{1,2}; \tau_0, R) \leqslant \frac{C}{R^{(2+2\bar{\delta})}} \|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}, \qquad R \geqslant 1,\ \tau_0 \in [0, T]. \tag{6.7}$$

On the other hand using the inequality:

$$\|fg\|_{H_x^{\sigma}(\frac{1}{2},2)} \leqslant C\left(\|f\|_{H_x^{\sigma}(\frac{1}{2},2)} \|g\|_{H_x^{\sigma}(\frac{1}{2},2)} + \|f\|_{H_x^{\sigma}(\frac{1}{2},2)} \|g\|_{H_x^{\sigma}(\frac{1}{2},2)}\right) \tag{6.8}$$

for $\sigma > \frac{1}{2}$ (cf. [18, Theorem 1, Section 4.6.4, p. 221]) we obtain:

$$N_{2;\sigma}(I_{1,1}; \tau_0, R) \leqslant \frac{C}{R^{(2+2\bar{\delta})}} \|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}, \qquad R \geqslant 1,\ \tau_0 \in [0, T]. \tag{6.9}$$

Therefore, combining (6.7), (6.9):

$$N_{2;\sigma}(I_1;\tau_0,R) \leqslant \frac{C}{R^{(2+2\bar{\delta})}} \|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}, \quad R \geqslant 1,\ \tau_0 \in [0,T]. \tag{6.10}$$

Suppose now that $R \leqslant 1$. We introduce now the rescaling $x = RX$, $y = RY$, $H_R(\tau,X) = h(\tau,x)$. Then:

$$I_1 = -R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\tau,X) \int_{\frac{X}{2}}^{2} Y^{\frac{\lambda}{2}} H_R(\tau,Y)\,dY - R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\tau,X) \int_{2}^{\infty} Y^{\frac{\lambda}{2}} H_R(\tau,Y)\,dY$$
$$\equiv I_{1,1} + I_{1,2}.$$

Notice that:

$$\left| R^{\frac{\lambda}{2}+1} \int_{2}^{\infty} Y^{\frac{\lambda}{2}} H_R(\tau,Y)\,dY \right| \leqslant C\|h\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}. \tag{6.11}$$

On the other hand, using $\|D_X^\sigma H_R(\tau,\cdot)\|^2_{L^2(\frac{1}{2},2)} = R^{2\sigma-1}\|D_x^\sigma h(\tau,\cdot)\|^2_{L^2(\frac{R}{2},2R)}$ we obtain:

$$\left(\int_0^T \left\| D_X^\sigma H_R(\tau,\cdot) \right\|^2_{L^2(\frac{1}{2},2)} d\tau \right)^{\frac{1}{2}} \leqslant \left(R^{2\sigma-1} \int_0^T \left\| D_x^\sigma h(\tau,\cdot) \right\|^2_{L^2(\frac{R}{2},2R)} d\tau \right)^{\frac{1}{2}}$$
$$\leqslant R^{-\frac{3}{2}} \|h\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}. \tag{6.12}$$

Moreover:

$$\left(\int_0^T \left\| H_R(\tau,\cdot) \right\|^2_{L^2(\frac{1}{2},2)} d\tau \right)^{\frac{1}{2}} \leqslant CT \sup_{0\leqslant\tau\leqslant T} \left\| H_R(\tau,\cdot) \right\|_{L^\infty(\frac{1}{2},2)} \leqslant CTR^{-\frac{3}{2}} \|h\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)} \tag{6.13}$$

henceforth, using (6.11)–(6.13):

$$R^{\frac{3}{2}} M_{2;\sigma}(I_{1,2};R) + \frac{R^{\frac{3}{2}} M_{2;0}(I_{1,2};R)}{T} \leqslant CR^{\frac{\lambda}{2}} \|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}.$$

On the other hand:

$$\left(\int_0^T \left\| D_X^\sigma \left[\int_{\frac{X}{2}}^{2} Y^{\frac{\lambda}{2}} H_R(\tau,Y)\,dY \right] \right\|^2_{L^2(\frac{1}{2},2)} d\tau \right)^{\frac{1}{2}} + \sup_{0\leqslant\tau\leqslant T} \left\| \int_{\frac{X}{2}}^{2} Y^{\frac{\lambda}{2}} H_R(\tau,Y)\,dY \right\|_{L^\infty(\frac{1}{2},2)}$$
$$\leqslant CR^{-\frac{3}{2}} \|h\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}$$

where we just estimate the L^2 norm of D_X^σ combining the estimates of the derivative and the function itself by interpolation. Then, using also (6.12), (6.13) we obtain $R^{\frac{3}{2}} M_{2;\sigma}(I_{1,1};R) + R^{\frac{3}{2}} M_{2;0}(I_{1,1};R) \leqslant CR^{\lambda-\frac{1}{2}} \|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}$ whence:

$$R^{\frac{3}{2}} M_{2;\sigma}(I_1;R) + R^{\frac{3}{2}} M_{2;0}(I_1;R) \leqslant C\|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}, \quad R \leqslant 1. \tag{6.14}$$

Combining (6.10), (6.14) we obtain (6.4) and the lemma follows. □

In order to estimate I_2 we will need the following auxiliary lemma:

Lemma 16. *Suppose that $\phi \in C_0^\infty(\mathbb{R}^+)$. There exists $C > 0$ depending only on ϕ and its derivatives such that the following inequality holds for any $R > 1$:*

$$\int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left|(\hat{\phi} * \hat{G})(\xi)\right|^2 \left(\min\left\{\sqrt{|\xi|}, \sqrt{R}\right\}\right)^2 d\xi \leqslant C \int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left|\hat{G}(\xi)\right|^2 \left(1 + \min\left\{\sqrt{|\xi|}, \sqrt{R}\right\}\right)^2 d\xi. \tag{6.15}$$

The proof of Lemma 16 will be based in the following inequality:

Lemma 17. *Let $W_R(\xi) = \min\{\sqrt{|\xi|}, \sqrt{R}\} = \sqrt{R}\min\{\sqrt{\frac{|\xi|}{R}}, 1\}$. There exists a constant $C > 0$ such that, for any $R > 0$ and any $\xi, \eta \in \mathbb{R}$:*

$$\left|W_R(\xi) - W_R(\eta)\right| \leqslant C \frac{|\xi - \eta|}{|\eta|} W_R(\eta). \tag{6.16}$$

Proof. This lemma can be thought as a particular case of Lemma 3.6 in [9]. However, we give here an elementary proof. Due to the scale invariance of the inequality (6.16) we can restrict ourselves to the case $R = 1$. The inequality is then elementary if $\max\{|\xi|, |\eta|\} \geqslant 1$. Suppose then that $\max\{|\xi|, |\eta|\} \leqslant 1$. Then (6.16) reduces to $|\sqrt{|\xi|} - \sqrt{|\eta|}| \leqslant C \frac{|\xi-\eta|}{|\eta|} \sqrt{|\eta|}$ which follows immediately multiplying both sides of the inequality by $(\sqrt{|\xi|} + \sqrt{|\eta|})$. $\square$

Proof of Lemma 16. Using the regularity properties of ϕ we have:

$$\left|(\hat{\phi} * \hat{G})(\xi)\right| \leqslant C_m \int_{\{|\eta| \leqslant 1\}} \frac{|\hat{G}(\eta)|}{1 + |\xi - \eta|^m} + C_m \int_{\{|\eta| > 1\}} \frac{|\hat{G}(\eta)|}{1 + |\xi - \eta|^m} \equiv J_1(\xi) + J_2(\xi)$$

where m can be assumed to be arbitrarily large. Using then $\frac{(1+|\xi|^\sigma) W_R(\xi)}{1+|\xi-\eta|^m} \leqslant \frac{C}{1+|\xi|}$ for $|\eta| \leqslant 1$ we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left|(\hat{\phi} * \hat{G})(\xi)\right|^2 \left(\min\left\{\sqrt{|\xi|}, \sqrt{R}\right\}\right)^2 d\xi \\
&\quad \leqslant C \int_{\{|\eta| \leqslant 1\}} \left|\hat{G}(\eta)\right|^2 d\eta + \int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left(1 + \left(W_R(\xi)\right)^2\right) \left(J_2(\xi)\right)^2 d\xi \\
&\quad \leqslant C \int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left(1 + \left(W_R(\xi)\right)^2\right) \left|\hat{G}(\xi)\right|^2 d\xi + C \int_{\mathbb{R}} \left(1 + |\xi|^\sigma\right)^2 \left(W_R(\xi)\right)^2 \left(J_2(\xi)\right)^2 d\xi \\
&\quad \equiv K_1 + K_2.
\end{aligned}$$

In order to estimate K_2 we use Lemma 17 to obtain for $|\eta| \geqslant 1$:

$$\begin{aligned}
&\left|\left(1 + |\xi|^\sigma\right) W_R(\xi) - \left(1 + |\eta|^\sigma\right) W_R(\eta)\right| \\
&\quad \leqslant C \frac{|\xi - \eta|}{|\eta| + 1} W_R(\eta) + C \frac{|\xi - \eta|}{|\eta|^{1-\sigma} + 1} W_R(\eta) + C|\xi - \eta|^\sigma W_R(\xi) \\
&\quad \leqslant C|\xi - \eta| W_R(\eta) + C|\xi - \eta|^\sigma W_R(\eta) + C \frac{|\xi - \eta|^{\sigma+1}}{|\eta| + 1} W_R(\eta) \\
&\quad \leqslant C\left(|\xi - \eta|^\sigma + |\xi - \eta|^{\sigma+1}\right) W_R(\eta).
\end{aligned}$$

Using this inequality to estimate the terms in $W_R(\xi) \cdot J_2(\xi)$ and using Young's inequality, as well as the fact that the integration $J_2(\xi)$ takes place in $|\eta| \geqslant 1$ we obtain:

$$K_2 \leqslant C \int_{\mathbb{R}} \left(1+|\xi|^{2\sigma}\right)\left(W_R(\xi)\right)^2 \left|\hat{G}(\xi)\right|^2 d\xi$$

whence Lemma 16 follows. □

We now estimate I_2. The bounds for this operator are the crucial step in the argument from the point of view of the regularity of the functions, because this operator can be estimated as some regularized version of the half-derivative operator. It will be essential to use the seminorm $[\cdot]^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}$ (cf. (2.14)).

Lemma 18. *Suppose that I_2 is as in (6.3) and $\bar{\delta} > 0$. Then:*

$$\|I_2\|_{Y^{\sigma}_{\frac{3}{2},2+\bar{\delta}}} \leqslant C R^{-(2+\bar{\delta})} \|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}.$$

Proof. Suppose first that $R \geqslant 1$. Using the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $h(\tau, x) = R^{-(\frac{3+\lambda}{2}+\bar{\delta})} G_R(\theta, X)$

$$I_2 = R^{-(2+2\bar{\delta})} \int_0^{\frac{X}{2}} Y^{\frac{\lambda}{2}} G_R(\theta, Y)\left[(X-Y)^{\frac{\lambda}{2}} G_R(\theta, X-Y) - X^{\frac{\lambda}{2}} G_R(\theta, X)\right] dY. \tag{6.17}$$

Notice that:

$$\left|G_R(\theta, Y)\right| \leqslant \|h\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)} \min\left(Y^{-(\frac{3+\lambda}{2}+\bar{\delta})}, R^{\frac{\lambda}{2}+\bar{\delta}} Y^{-\frac{3}{2}}\right). \tag{6.18}$$

We rewrite (6.17) as:

$$I_2 = I_{2,-} + \sum_{\{k=0,1,\ldots;\ \frac{2^k}{R} \leqslant \frac{1}{4}\}} I_{2,k} + I_{2,+},$$

$$I_{2,-} = R^{-(2+2\bar{\delta})} \int_0^{\frac{1}{R}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y)\, dY,$$

$$I_{2,k} = R^{-(2+2\bar{\delta})} \int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y)\, dY, \quad k = 0, 1, \ldots,$$

$$I_{2,+} = R^{-(2+2\bar{\delta})} \int_{\frac{2^{k_{\max}}}{R}}^{\frac{X}{2}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y)\, dY,$$

$$J(G; \theta, X, Y) = \left[(X-Y)^{\frac{\lambda}{2}} G(\theta, X-Y)\eta(X-Y) - X^{\frac{\lambda}{2}} G(\theta, X)\eta(X)\right] \tag{6.19}$$

where $\frac{2^{k_{\max}}}{R} \leqslant \frac{1}{4} < \frac{2^{k_{\max}+1}}{R}$ and $\eta(X)$ is the cutoff function used in (2.13). (Notice that $\eta(X) = 1$ in all the regions of integration, since $X \in (\frac{1}{2}, 2)$.) Let us write $\psi_R(\theta, X) = X^{\frac{\lambda}{2}} G_R(\theta, X)\eta(X)$. In order to estimate these terms in $H^{\sigma}_X(\frac{1}{2}, 2)$ we use Fourier:

$$\psi_R(\theta, X) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}_R(\theta, \xi) e^{i\xi X}\, d\xi.$$

Since the functions $I_{2,-}(\theta, X)$, $I_{2,k}(\theta, X)$, $I_{2,+}(\theta, X)$ are defined for $X \in \mathbb{R}$ we can compute their Fourier transforms. Using the convolution property for Fourier transforms we have:

$$\hat{I}_{2,k}(\theta,\xi) = \hat{\psi}_R(\theta,\xi)\left[R^{-(2+2\bar{\delta})}\int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} Y^{\frac{\lambda}{2}} G_R(\theta,Y)\big(e^{-i\xi Y}-1\big)\,dY\right].$$

Using (6.18):

$$\begin{aligned}\left\|D_X^\sigma I_{2,k}\right\|_{L^2(\frac{1}{2},2)} &\leqslant \left\|D_X^\sigma I_{2,k}\right\|_{L^2(\mathbb{R})}\\ &\leqslant \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \frac{R^{-(2+\bar{\delta})}}{(2^{k-1})^{\bar{\delta}}}\left(\int_{\mathbb{R}} |\xi|^{2\sigma}\left|\hat{\psi}_R(\theta,\xi)\right|^2\left(\int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}}\frac{|e^{-i\xi Y}-1|}{Y^{\frac{3}{2}}}\,dY\right)^2 d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

We now use that:

$$\int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}}\frac{|e^{-i\xi Y}-1|}{Y^{\frac{3}{2}}}\,dY \leqslant C\min\big\{\sqrt{|\xi|},\sqrt{R}\big\}$$

whence:

$$\left\|D_X^\sigma I_{2,k}\right\|_{L^2(\frac{1}{2},2)} \leqslant C\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \frac{R^{-(2+\bar{\delta})}}{(2^{k-1})^{\bar{\delta}}}\left(\int_{\mathbb{R}} |\xi|^{2\sigma}\left|\hat{\psi}_R(\theta,\xi)\right|^2\big(\min\big\{\sqrt{|\xi|},\sqrt{R}\big\}\big)^2 d\xi\right)^{\frac{1}{2}}.$$

Using Lemma 16 it follows that:

$$\left\|D_X^\sigma I_{2,k}\right\|_{L^2(\frac{1}{2},2)} \leqslant \frac{CR^{-(2+\bar{\delta})}}{(2^{k-1})^{\bar{\delta}}}\|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.20}$$

The term $I_{2,+}$ can be estimated similarly:

$$\left\|D_X^\sigma I_{2,+}\right\|_{L^2(\frac{1}{2},2)} \leqslant \frac{CR^{-(2+\bar{\delta})}}{(2^{k_{\max}-1})^{\bar{\delta}}}\|h\|^2_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.21}$$

We now estimate $I_{2,-}$. A similar argument yields:

$$\left\|D_X^\sigma I_{2,-}\right\|_{L^2(\frac{1}{2},2)} \leqslant \left\|D_X^\sigma I_{2,-}\right\|_{L^2(\mathbb{R})} \leqslant \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} R^{-(2+\bar{\delta})}\left(\int_{\mathbb{R}} |\xi|^{2\sigma}\left|\hat{\psi}_R(\theta,\xi)\right|^2\big(\Omega_R(Y)\big)^2 d\xi\right)^{\frac{1}{2}}$$

where:

$$\Omega_R(Y) = \int_0^{\frac{1}{R}}\frac{|e^{-i\xi Y}-1|}{Y^{\frac{3}{2}}}(RY)^{\frac{\lambda}{2}}\,dY = \sqrt{R}\int_0^1\frac{|e^{-i\frac{\xi}{R}y}-1|}{y^{\frac{3}{2}}}(y)^{\frac{\lambda}{2}}\,dy \leqslant CW_R(\xi) \tag{6.22}$$

with $W_R(\xi)$ as in Lemma 17. The last inequality follows computing the asymptotics of the second integral in (6.22) for $\frac{\xi}{R} \to 0$ and $\frac{\xi}{R} \to \infty$.

Therefore

$$\left\|D_X^\sigma I_{2,-}\right\|_{L^2(\frac{1}{2},2)} \leqslant C\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} R^{-(2+\bar{\delta})}\left(\int_{\mathbb{R}} |\xi|^{2\sigma}\left|\hat{\psi}_R(\theta,\xi)\right|^2\big(W_R(\xi)\big)^2 d\xi\right)^{\frac{1}{2}}$$

whence:

$$\left\| D_X^\sigma I_{2,-} \right\|_{L^2(\frac{1}{2},2)} \leqslant C\|h\|^2_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)} R^{-(2+\bar\delta)}. \tag{6.23}$$

To conclude the proof of Lemma 18 it only remains to estimate the contributions of the region where $R \leqslant 1$. The estimate of $R^{\frac{3}{2}} M_{2;\sigma}(I_2; R)$, $R^{\frac{3}{2}} M_{2;0}(I_2; R)$ can be made in exactly the same way as the estimate (6.14) for I_1. Notice that the two terms in I_2 yield integrals that converge separately since $h(\tau, y)$ can be estimated as $\frac{1}{y^{\frac{3}{2}}}$ for $y \leqslant 1$ and then, the term $y^{\frac{\lambda}{2}-\frac{3}{2}}$ gives integrability. Therefore:

$$R^{\frac{3}{2}} M_{2;\sigma}(I_2; R) + R^{\frac{3}{2}} M_{2;0}(I_2; R) \leqslant C\|h\|^2_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}, \quad R \leqslant 1. \tag{6.24}$$

Combining (6.20), (6.21), (6.23), (6.24) Lemma 18 follows. □

Proof of Proposition 14. It is just a consequence of (6.2), (6.3), Lemmas 15 and 18. □

We can also prove the following Lipschitz property for the functional $Q[\cdot]$:

Proposition 19. *For any $\sigma \in (1,2)$ and any $\bar\delta > 0$ there exists $C = C(\sigma, \bar\delta)$ such that for any $h_1, h_2 \in \mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)$:*

$$\left\| Q[h_1] - Q[h_2] \right\|_{Y^\sigma_{\frac{3}{2},(2+\bar\delta)}(T)} \leqslant C\left(\sum_{k=1}^{2} \|h_k\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}$$

with $Q[\cdot]$ as in (1.2), (1.5).

Proof. We have $Q[h_1](\tau,x) - Q[h_2](\tau,x) = I_1 + I_2$ with:

$$I_1 \equiv -\left(\int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h_1(\tau,x) h_1(\tau,y)\,dy - \int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h_2(\tau,x) h_2(\tau,y)\,dy \right),$$

$$\begin{aligned} I_2 \equiv \Bigg(&\int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} h_1(\tau,y)\left[(x-y)^{\frac{\lambda}{2}} h_1(\tau,x-y) - x^{\frac{\lambda}{2}} h_1(\tau,x)\right] dy \\ &- \int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} h_2(\tau,y)\left[(x-y)^{\frac{\lambda}{2}} h_2(\tau,x-y) - x^{\frac{\lambda}{2}} h_2(\tau,x)\right] dy \Bigg). \end{aligned}$$

To estimate I_1 we need to estimate the functionals $N_{2;\sigma}(I_1;\tau_0,R)$ for $R \geqslant 1$ and $M_{2;\sigma}(I_1;R)$ for $R \leqslant 1$ (cf. (2.8), (2.9)). Suppose first that $R > 1$. We introduce the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $H_{R,1}(\theta,X) = h_1(\tau,x)$, $H_{R,2}(\theta,X) = h_2(\tau,x)$. Then:

$$\begin{aligned} I_1 = &-R^{\lambda+1} X^{\frac{\lambda}{2}} \left[H_{R,1}(\theta,X) - H_{R,2}(\theta,X)\right] \int_{\frac{X}{2}}^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\theta,Y)\,dY \\ &- R^{\lambda+1} X^{\frac{\lambda}{2}} H_{R,2}(\theta,X) \int_{\frac{X}{2}}^{\infty} Y^{\frac{\lambda}{2}} \left[H_{R,1}(\theta,Y) - H_{R,2}(\theta,Y)\right] dY. \end{aligned} \tag{6.25}$$

Notice that:

$$\sup_{X\in(\frac{1}{2},2)}\left|\int_X^\infty Y^{\frac{\lambda}{2}}H_{R,1}(\theta,Y)\,dY\right|+\left\|\int_{(\cdot)}^\infty Y^{\frac{\lambda}{2}}H_{R,1}(\theta,Y)\,dY\right\|_{H_x^\sigma(\frac{1}{2},2)}\leqslant\frac{C}{R^{\frac{3+\lambda}{2}+\bar\delta}}\|h_1\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)},$$

$$\sup_{X\in(\frac{1}{2},2)}\left|\int_{\frac{X}{2}}^\infty Y^{\frac{\lambda}{2}}\left[H_{R,1}(\theta,Y)-H_{R,2}(\theta,Y)\right]dY\right|+\left\|\int_{(\cdot)}^\infty Y^{\frac{\lambda}{2}}\left[H_{R,1}(\theta,Y)-H_{R,2}(\theta,Y)\right]dY\right\|_{H_x^\sigma(\frac{1}{2},2)}$$
$$\leqslant\frac{C}{R^{\frac{3+\lambda}{2}+\bar\delta}}\|h_1-h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}.$$

Using (6.8) in (6.25) we obtain:

$$N_{2;\sigma}(I_1;\tau_0,R)\leqslant\frac{C}{R^{(2+2\bar\delta)}}\left(\sum_{k=1}^2\|h_k\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}\right)\|h_1-h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}\tag{6.26}$$

for $R\geqslant 1$, $\tau_0\in[0,T]$. Suppose now that $R\leqslant 1$. We introduce the rescaling $x=RX$, $y=RY$, $H_{R,1}(\tau,X)=h_1(\tau,x)$, $H_{R,2}(\tau,X)=h_2(\tau,x)$. Then:

$$I_1=I_{1,1}+I_{1,2},$$

$$I_{1,1}\equiv-\Bigg[R^{\lambda+1}X^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,X)-H_{R,2}(\tau,X)\right]\int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}}H_{R,1}(\tau,Y)\,dY$$
$$+R^{\lambda+1}X^{\frac{\lambda}{2}}H_{R,2}(\tau,X)\int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,Y)-H_{R,2}(\tau,Y)\right]dY\Bigg],\tag{6.27}$$

$$I_{1,2}\equiv-\Bigg[R^{\lambda+1}X^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,X)-H_{R,2}(\tau,X)\right]\int_2^\infty Y^{\frac{\lambda}{2}}H_{R,1}(\tau,Y)\,dY$$
$$+R^{\lambda+1}X^{\frac{\lambda}{2}}H_{R,2}(\tau,X)\int_2^\infty Y^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,Y)-H_{R,2}(\tau,Y)\right]dY\Bigg].\tag{6.28}$$

Notice that:

$$\left|R^{\frac{\lambda}{2}+1}\int_2^\infty Y^{\frac{\lambda}{2}}H_{R,1}(\tau,Y)\,dY\right|\leqslant C\|h_1\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)},\tag{6.29}$$

$$\left|R^{\frac{\lambda}{2}+1}\int_2^\infty Y^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,Y)-H_{R,2}(\tau,Y)\right]dY\right|\leqslant C\|h_1-h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}.\tag{6.30}$$

On the other hand, using the definition of $H_{R,1}$, $H_{R,2}$ we arrive at:

$$\int_0^T\left\|D_X^\sigma H_{R,2}(\tau,\cdot)\right\|_{L^2(\frac{1}{2},2)}^2\,d\tau\leqslant\|h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}^2,\tag{6.31}$$

$$\int_0^T\left\|D_X^\sigma\left[H_{R,1}(\tau,X)-H_{R,2}(\tau,X)\right]\right\|_{L^2(\frac{1}{2},2)}^2\,d\tau\leqslant\|h_1-h_2\|_{\mathcal{Z}_{\bar p}^{\sigma;\frac{1}{2}}(T)}^2.\tag{6.32}$$

Moreover:

$$\left(\int_0^T \left\|H_{R,2}(\tau,\cdot)\right\|^2_{L^2(\frac{1}{2},2)}\,d\tau\right)^{\frac{1}{2}} \leqslant CTR^{-\frac{3}{2}}\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}, \tag{6.33}$$

$$\left(\int_0^T \left\|\left[H_{R,1}(\tau,\cdot)-H_{R,2}(\tau,\cdot)\right]\right\|^2_{L^2(\frac{1}{2},2)}\,d\tau\right)^{\frac{1}{2}} \leqslant CTR^{-\frac{3}{2}}\|h_1-h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}, \tag{6.34}$$

henceforth, using (6.8), (6.28)–(6.34):

$$R^{\frac{3}{2}}M_{2;0}(I_{1,2};R)+R^{\frac{3}{2}}M_{2;\sigma}(I_{1,2};R)\leqslant R^{\frac{\lambda}{2}}\left(\sum_{k=1}^{2}\|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}\right)\|h_1-h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.35}$$

On the other hand, for $R\leqslant 1$ we have:

$$\begin{aligned}&\left(\int_0^T\left\|D_X^{\sigma}\left[\int_{\frac{X}{2}}^{2}Y^{\frac{\lambda}{2}}\left[H_{R,1}(\tau,\cdot)-H_{R,2}(\tau,\cdot)\right]dY\right]\right\|^2_{L^2(\frac{1}{2},2)}d\tau\right)^{\frac{1}{2}}\\&\quad+\sup_{0\leqslant\tau\leqslant T}\left\|\int_{\frac{X}{2}}^{2}Y^{\frac{\lambda}{2}}H_{R,1}(\tau,\cdot)-H_{R,2}(\tau,\cdot)\,dY\right\|_{L^\infty(\frac{1}{2},2)}\\&\leqslant CR^{-\frac{3}{2}}\|h_1-h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}\end{aligned}$$

where we just estimate the L^2 norm of D_X^{σ} by D_X and the function itself by interpolation. Then, using also (6.12), (6.13) as well as the fact that $R\leqslant 1$:

$$R^{\frac{3}{2}}M_{2;\sigma}(I_{1,1};R)+R^{\frac{3}{2}}M_{2;0}(I_{1,1};R)\leqslant C\left(\sum_{k=1}^{2}\|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}\right)\|h_1-h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.36}$$

Combining (6.35), (6.36):

$$\|I_1\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}(T)}\leqslant\left(\sum_{k=1}^{2}\|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}\right)C\|h_1-h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \tag{6.37}$$

where C is uniformly bounded for $0\leqslant T\leqslant 1$.

We now estimate I_2. Suppose first that $R\geqslant 1$. Using the rescaling $x=RX$, $y=RY$, $\tau=\tau_0+R^{-\frac{\lambda-1}{2}}\theta$, $h_1(\tau,x)=R^{-(\frac{3+\lambda}{2}+\bar{\delta})}G_{R,1}(\theta,X)$, $h_2(\tau,x)=R^{-(\frac{3+\lambda}{2}+\bar{\delta})}G_{R,2}(\theta,X)$ and using (6.19) we obtain:

$$\begin{aligned}&I_2=I_{2,1}+I_{2,1},\\&I_2\equiv\left(\int_0^{\frac{x}{2}}y^{\frac{\lambda}{2}}h_1(\tau,y)\left[(x-y)^{\frac{\lambda}{2}}h_1(\tau,x-y)-x^{\frac{\lambda}{2}}h_1(\tau,x)\right]dy\right.\\&\qquad\left.-\int_0^{\frac{x}{2}}y^{\frac{\lambda}{2}}h_2(\tau,y)\left[(x-y)^{\frac{\lambda}{2}}h_2(\tau,x-y)-x^{\frac{\lambda}{2}}h_2(\tau,x)\right]dy\right),\end{aligned}$$

$$I_{2,1}=R^{-(2+2\bar{\delta})}\int_0^{\frac{X}{2}}Y^{\frac{\lambda}{2}}\big(G_{R,1}(\theta,Y)-G_{R,2}(\theta,Y)\big)J(G_{R,1};\theta,X,Y)\,dY, \tag{6.38}$$

$$I_{2,2} = R^{-(2+2\bar{\delta})} \int_0^{\frac{X}{2}} Y^{\frac{\lambda}{2}} G_{R,2}(\theta, Y) J(G_{R,1} - G_{R,2}; \theta, X, Y)\, dY. \tag{6.39}$$

Notice that:

$$\left|G_{R,1}(\theta, Y) - G_{R,2}(\theta, Y)\right| \leqslant \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \min\left(Y^{-(\frac{3+\lambda}{2}+\bar{\delta})}, R^{\frac{\lambda}{2}+\bar{\delta}} Y^{-\frac{3}{2}}\right). \tag{6.40}$$

We now argue exactly as in the proof of Lemma 18 in order to estimate $I_{2,1}$, $I_{2,2}$. Notice that estimating these terms it is crucial to use the boundedness of the seminorm $[\cdot]_{\frac{3+\lambda}{2}}^{\sigma;\frac{1}{2}}$ in (2.14) for the sources. On the other hand, the argument in the proof of Lemma 18 shows that the pointwise estimate (6.40) is needed. A similar argument and estimate allows to estimate the terms $I_{2,1}$, $I_{2,2}$ in (6.38), (6.39). Therefore, after some computations:

$$N_{2,0}(I_2, \tau_0, R) + N_{2,\sigma}(I_2, \tau_0, R) \leqslant \frac{C}{R^{2+\bar{\delta}}} \left(\sum_{k=1}^{2} \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.41}$$

It only remains to estimate the region where $R \leqslant 1$. The estimate of $R^{\frac{3}{2}} M_{2;\sigma}(I_2; R)$, $R^{\frac{3}{2}} M_{2;0}(I_2; R)$ can be made exactly in the same way as the estimate of similar terms for I_1. Notice that the two terms in I_2 yield integrals that converge separately since $h(\tau, y)$ can be estimated as $\frac{1}{y^{\frac{3}{2}}}$ for $y \leqslant 1$ and then, the term $y^{\frac{\lambda}{2}-\frac{3}{2}}$ is integrable near the origin. Then:

$$R^{\frac{3}{2}} M_{2;0}(I_2; R) + R^{\frac{3}{2}} M_{2;\sigma}(I_2; R) \leqslant C \left(\sum_{k=1}^{2} \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.42}$$

Combining (6.41), (6.42) we obtain:

$$\|I_2\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}(T)} \leqslant C \left(\sum_{k=1}^{2} \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}. \tag{6.43}$$

The proof of the lemma is then concluded using (6.37) and (6.43). □

7. Derivation of the asymptotics $x^{-\frac{3+\lambda}{2}}$ as $x \to \infty$

The main result in this section is the following.

Proposition 20. *Suppose that $\varphi \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma;\frac{1}{2}}(T)$ solves:*

$$\varphi_\tau = \mathcal{L}_{f_0}[\varphi] + F(\tau, x), \quad x > 0,\ 0 \leqslant t \leqslant T, \quad \varphi(0, x) = 0 \tag{7.1}$$

where $F \in Y^{\sigma}_{\frac{3}{2},2+\bar{\delta}}(T)$ and $\bar{\delta} < r$. Then, the following asymptotics holds:

$$\varphi(\tau, x) = \mathcal{W}(\tau) x^{-\frac{3+\lambda}{2}} + \varphi_R(\tau, x) \quad \text{as } x \to \infty \tag{7.2}$$

where:

$$\mathcal{W}(\tau) = \int_0^{\tau} ds \int_0^{\infty} \frac{dx_0}{x_0} \Theta\left((\tau - s) x_0^{\frac{\lambda-1}{2}}\right) x_0^{\frac{3+\lambda}{2}} \left[F(s, x_0) + (\mathcal{L}_{f_0} - L)[\varphi](s, x_0)\right] \tag{7.3}$$

with $\Theta(\cdot)$ as in (4.11) and:

$$\varphi_R \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T). \tag{7.4}$$

The proof of Proposition 20 is based in dealing with the operator $\mathcal{L}_{f_0}$ as a perturbation of the operator L. To this end, we rewrite (7.1) as:

$$\varphi_\tau = L[\varphi] + (\mathcal{L}_{f_0} - L)[\tilde{h}] + F(\tau, x).$$

Using variations of constants and Theorem 5 we have the representation formula:

$$\varphi(\tau, x) = \int_0^\tau ds \int_0^\infty g\left((\tau - s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right)\left[(\mathcal{L}_{f_0} - L)[\varphi](s, x) + F(s, x)\right]\frac{dx_0}{x_0}. \tag{7.5}$$

In order to prove Proposition 20 we will derive some auxiliary lemmas. We begin estimating the term $(\mathcal{L}_{f_0} - L)[\varphi]$ (cf. (7.5)). Most of the estimates in the next lemma have been already obtained in [9], but we recall them here for convenience.

Lemma 21. *Suppose that f_0 satisfies (2.2)–(2.5) and $\varphi \in \mathcal{E}_{T;\sigma}$. Then:*

$$N_\infty\big((\mathcal{L}_{f_0} - L)[\varphi]; \tau_0, R\big) \leqslant \frac{C|||\varphi|||_\sigma}{R^{2+r}}, \quad \tau_0 \in (0, T),\ R \geqslant 1, \tag{7.6}$$

$$M_\infty\big((\mathcal{L}_{f_0} - L)[\varphi]; R\big) \leqslant \frac{C|||\varphi|||_\sigma}{R^{\frac{3}{2}}}, \quad R \leqslant 1, \tag{7.7}$$

$$N_{2,\sigma}\big((\mathcal{L}_{f_0} - L)[\varphi], \tau_0, R\big) \leqslant \frac{C}{R^{2+r}}\|\varphi\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}(T)}, \quad \tau_0 \in (0, T),\ R \geqslant 1, \tag{7.8}$$

$$M_{2,\sigma}\big((\mathcal{L}_{f_0} - L)[\varphi], R\big) \leqslant \frac{C}{R^{\frac{3}{2}}}\|\varphi\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}(T)}, \quad R \leqslant 1. \tag{7.9}$$

Proof. We write

$$(\mathcal{L}_{f_0} - L)[\varphi](s, x_0) = A_1 + A_2$$

where:

$$A_1 = \int_0^{\frac{x}{2}} \big(H(x - y) - H(x)\big)y^{\frac{\lambda}{2}}\varphi(\tau, y)\,dy - H(x)\int_{\frac{x}{2}}^\infty y^{\frac{\lambda}{2}}\varphi(\tau, y)\,dy - x^{\frac{\lambda}{2}}\varphi(\tau, x)\int_{\frac{x}{2}}^\infty H(y)\,dy,$$

$$A_2 = \int_0^{\frac{x}{2}} \big((x - y)^{\frac{\lambda}{2}}\varphi(\tau, x - y) - x^{\frac{\lambda}{2}}\varphi(\tau, x)\big)H(y)\,dy$$

and

$$H(y) = y^{\frac{\lambda}{2}} f_0(y) - y^{-\frac{3}{2}}.$$

It has been proved in [9] (cf. Section 5, Lemmas 5.1, 5.2, 5.4):

$$|A_1| \leqslant \frac{C|||\varphi|||_{\frac{3}{2},\frac{3+\lambda}{2}}}{x^{\frac{3}{2}}}, \quad x \leqslant 1, \qquad |A_1| \leqslant \frac{C|||\varphi|||_{\frac{3}{2},\frac{3+\lambda}{2}}}{x^{2+r}}, \quad x \geqslant 1, \tag{7.10}$$

$$N_\infty(A_2; \tau_0, R) \leqslant \frac{C|||\varphi|||_\sigma}{R^{2+r}}, \quad R \geqslant 1; \qquad M_\infty(A_2; R) \leqslant \frac{C|||\varphi|||_\sigma}{R^{\frac{3}{2}}}, \quad R \leqslant 1,\ \tau_0 \in (0, T), \tag{7.11}$$

$$N_{2,\sigma}(A_1, \tau_0, R) \leqslant \frac{C}{R^{2+r}}\|\varphi\|_{Y^\sigma_{\frac{3}{2},\frac{3+\lambda}{2}}}, \quad \tau_0 \in (0, T),\ R \geqslant 1, \tag{7.12}$$

$$M_{2,\sigma}(A_1, R) \leqslant \frac{C}{R^{\frac{3}{2}}}\|\varphi\|_{Y^\sigma_{\frac{3}{2},\frac{3+\lambda}{2}}}, \quad R \leqslant 1, \tag{7.13}$$

where r might be chosen as in (2.1) and the norms $|||\cdot|||_{\frac{3}{2},\frac{3+\lambda}{2}}$, $|||\cdot|||_\sigma$ are as in (2.10), (2.11).

On the other hand, arguing as in the proof of Lemma 18 it is possible to prove the following estimates:

$$N_{2,\sigma}(A_2,\tau_0,R) \leqslant \frac{C}{R^{2+r}}\|\varphi\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}(T)}, \quad \tau_0 \in (0,T),\ R \geqslant 1, \tag{7.14}$$

$$M_{2,\sigma}(A_2,R) \leqslant \frac{C}{R^{\frac{3}{2}}}\|\varphi\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}(T)}, \quad R \leqslant 1. \tag{7.15}$$

The only difference in the argument is that instead of (6.18) the estimate that must be used is:

$$\left|H(y)\right| \leqslant C\min\bigl(y^{-\frac{3}{2}}, y^{-(\frac{3}{2}+r)}\bigr)$$

that implies that the function $H_R(Y) = R^{(\frac{3+\lambda}{2}+r)}H(RY)$ satisfies:

$$\left|H_R(Y)\right| \leqslant C\min\bigl(Y^{-(\frac{3+\lambda}{2}+r)}, R^{\frac{\lambda}{2}+r}Y^{-\frac{3}{2}}\bigr).$$

Notice that for $R < 1$ we obtain estimates with the dependence $\frac{1}{R^{\frac{3}{2}}}$ on the right-hand side (cf. (7.11), (7.13), (7.15)) due to the fact that the term $H(x)\int_{\frac{x}{2}}^{\infty} y^{\frac{\lambda}{2}}\varphi(\tau,y)\,dy$ in the definition of A_1 yields such a power law dependence for small x. Combining (7.12)–(7.15) the lemma follows. □

Remark 22. The estimate for the term A_2 cannot be improved to the decay in the norm H_x^{σ} except if we obtain instead the decay R^{-2}. Such a decay has been obtained in [9]. The main novelty in the estimate for A_2 obtained in Lemma 21 is the decay like $R^{-(2+r)}$ in (7.14) for large R, that can be obtained using the estimate for the seminorm $[\varphi]^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}$ contained in the spaces $Z^{\sigma;\frac{1}{2}}_{(\frac{3+\lambda}{2}+\bar{\delta})}$.

In the proof of the following results the notation will become simpler using the following definitions:

$$\fint_{\tau_0-\rho}^{\tau_0} f(s)\,ds = \frac{1}{\rho}\int_{(\tau_0-\rho)_+}^{\tau_0} f(s)\,ds.$$

The next lemma shows how to compute the asymptotics as $x \to \infty$ of the solutions of:

$$J_\tau = L[J] + F(\tau,x), \quad J(0,x) = 0.$$

To this end we will use the following auxiliary functional:

$$\sup_{R\geqslant 1,\ \tau_0\in(0,T)}\bigl[N_\infty(F;\tau_0,R)R^{2+\bar{\delta}}\bigr] + \sup_{R<1}\bigl[M_\infty(F;R)R^{\frac{3}{2}}\bigr] \equiv \mathcal{H}(F). \tag{7.16}$$

Lemma 23. *Let* $0 < \bar{\delta} < \min\{\varepsilon, r\}$, *with* ε *as in* (4.10) *and* r *as in* (2.1). *Suppose that* $F \in X_{\frac{3}{2},2+\bar{\delta}}(T)$. *Let* $J = J(\tau,x)$ *be*:

$$J(\tau,x) = \int_0^{\tau} ds \int_0^{\infty} g\biggl((\tau-s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\biggr)F(s,x_0)\frac{dx_0}{x_0}. \tag{7.17}$$

Then:

$$J(\tau,x) - I(\tau;F)x^{-\frac{3+\lambda}{2}}\xi(x) = J_R(\tau,x) \tag{7.18}$$

where $\xi(\cdot)$ *is as in* (2.4) *and*:

$$I(\tau;F) = \int_0^{\infty}\frac{dx_0}{x_0}\int_0^{\tau} ds\, F(s,x_0)\Theta\bigl((\tau-s)x_0^{\frac{\lambda-1}{2}}\bigr)x_0^{\frac{3+\lambda}{2}} \tag{7.19}$$

with $\Theta(\cdot)$ as in (4.11) and

$$|||J_R|||_{\frac{3}{2},\frac{3+\lambda}{2}+\bar{\delta}} \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}. \tag{7.20}$$

Moreover:

$$\left|I(\tau;F)\right| \leqslant C\tau^{\frac{2\bar{\delta}}{\lambda-1}}\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}. \tag{7.21}$$

Proof. We split the integral in (7.17) as:

$$J = J_1 + J_2 + J_3 + J_4,$$

$$J_1 \equiv \int_{\frac{2x}{3}}^{\infty} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\,[\ldots], \qquad J_2 \equiv \int_0^{\frac{2x}{3}} dx_0 \int_{(\tau - x_0^{\frac{\lambda-1}{2}})_+}^{\tau} ds\,[\ldots], \tag{7.22}$$

$$J_3 \equiv \int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\,[\ldots], \qquad J_4 \equiv \int_0^{\frac{2x}{3}} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\,[\ldots]. \tag{7.23}$$

In the term J_1 we use the fact that (4.7) implies:

$$\left|g(\tau,x,1)\right| \leqslant C\tau x^{-\frac{3}{2}}, \quad 0 < x \leqslant \frac{1}{2},$$
$$\left|g(\tau,x,1)\right| \leqslant C\tau^{-2}\varphi\left(\frac{x-1}{\tau^2}\right), \quad |x-1| \leqslant \frac{1}{2} \text{ with } \varphi(\xi) = \frac{1}{1+\xi^{\frac{3}{2}-\varepsilon_1}}$$

for some $\varepsilon_1 > 0$. Then:

$$|J_1| \leqslant J_{1,1} + J_{1,2}$$

where:

$$J_{1,1} \equiv C\int_{\frac{x}{2}}^{\frac{3x}{2}} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} (\tau-s)^{-2}\,ds\,x_0^{-\lambda}\varphi\left(\frac{x-x_0}{(\tau-s)^2 x_0^{\lambda}}\right)\left|F(s,x_0)\right|, \tag{7.24}$$

$$J_{1,2} \equiv \frac{C}{x^{3/2}}\int_{2x}^{\infty} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} (\tau-s)\,ds\,x_0^{\frac{\lambda}{2}}\left|F(s,x_0)\right|. \tag{7.25}$$

In order to estimate J_3 we use that (4.4)–(4.6) imply:

$$\left|g(\tau,x,1)\right| \leqslant C\tau^{\frac{2}{\lambda-1}}\min\left\{\left(\tau^{\frac{2}{\lambda-1}}x\right)^{-\frac{3}{2}}, \left(\tau^{\frac{2}{\lambda-1}}x\right)^{-\frac{3+\lambda}{2}}\right\}, \quad \tau \geqslant 1.$$

Then:

$$|J_3| \leqslant C\int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\left|F(s,x_0)\right|\min\left\{(\tau-s)^{-\frac{1}{\lambda-1}}x^{-\frac{3}{2}}, (\tau-s)^{-\frac{1+\lambda}{\lambda-1}}x^{-\frac{3+\lambda}{2}}\right\}. \tag{7.26}$$

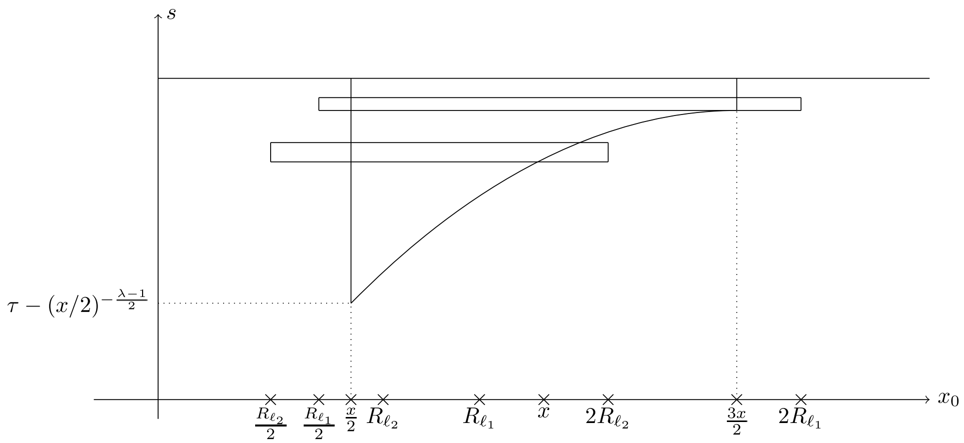

Fig. 2. Two cubes of the covering $\mathcal{C}^{(1)}_{x,\tau}$.

To obtain the leading asymptotics of J_2 as $x \to \infty$ we will use the detailed asymptotics of $g(\tau, x, 1)$. Using (4.9), (4.11) we have:

$$\left| J_2 - x^{-\frac{3+\lambda}{2}} \int_0^{\frac{2x}{3}} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, F(s, x_0) \Theta\big((\tau - s) x_0^{\frac{\lambda-1}{2}}\big) x_0^{\frac{1+\lambda}{2}} \right| = J_{2,R} \tag{7.27}$$

where, due to (5.9) in Proposition 13:

$$|J_{2,R}| \leqslant C x^{-\frac{3+\lambda}{2} - r} \int_0^{\frac{2x}{3}} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} (\tau - s)\, ds \left| F(s, x_0) \right| x_0^{\lambda + r}. \tag{7.28}$$

To estimate J_4 we use (4.4)–(4.6):

$$|J_4| \leqslant \frac{C}{x^{\frac{3+\lambda}{2}}} \int_0^{\frac{2x}{3}} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} \frac{ds}{(\tau - s)^{\frac{\lambda+1}{\lambda-1}}} \left| F(s, x_0) \right| \tag{7.29}$$

where we use that, since $x_0 \leqslant x$, we have $(\tau - s) x^{\frac{\lambda-1}{2}} \geqslant \frac{x^{\frac{\lambda-1}{2}}}{x_0^{\frac{\lambda-1}{2}}} \geqslant 1$. We now proceed to estimate all the terms J_k, $k = 1, 2, 3, 4$, in terms of $\mathcal{H}(F)$ in (7.16). We are interested in the behaviour of all these quantities for large values of x. We begin with the term $J_{1,1}$ (cf. (7.24)). Notice that we can cover the domain $\{(x_0, s)\colon x_0 \in [\frac{x}{2}, \frac{3x}{2}],\ s \in [(\tau - x_0^{-\frac{\lambda-1}{2}})_+, \tau]\}$ by a set of rectangles with the form $[\frac{R_\ell}{2}, 2R_\ell] \times [\tau_0, \tau_0 + (R_\ell)^{-\frac{\lambda-1}{2}}]$, with $R_\ell \in [\frac{x}{2}, \frac{3x}{2}]$, $\tau_0 \in [\tau - 2^{\frac{\lambda-1}{2}} x^{-\frac{\lambda-1}{2}}, \tau]$ with $\ell = 1, \ldots, N_{x,\tau}$ and $N_{x,\tau} \leqslant n_0 < \infty$ with n_0 independent of x, τ. The fact that the number of rectangles can be estimated uniformly on x follows from the self-similarity of the problem. Let us denote such finite covering as $\mathcal{C}^{(1)}_{x,\tau}$. (See Fig. 2.)

The specific rescaling chosen for the rectangles is due to the fact that they are the ones appearing in the definition of the functionals $N_\infty(F; \tau_0, R)$. We then have:

$$J_{1,1} \leqslant C \sum_{\mathcal{C}^{(1)}_{x,,\tau}} \int_{(\tau - C R_x^{-\frac{\lambda-1}{2}})_+}^{\tau} (\tau - s)^{-2}\, ds \int_{\frac{x}{2}}^{\frac{3x}{2}} dx_0\, x_0^{-\lambda} \varphi\left(\frac{x - x_0}{(\tau - s)^2 x_0^{\lambda}} \right) \left\| F(s, \cdot) \right\|_{L^\infty(\frac{R_x}{2}, 2R_x)}$$

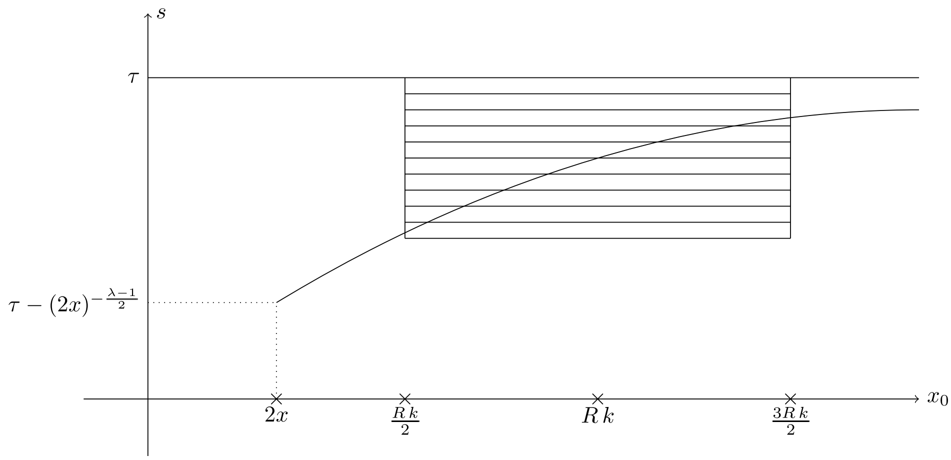

Fig. 3. The sub-family $\mathcal{R}_k$ of cubes of the covering $\mathcal{C}_{x,\tau}^{(2)}$.

$$\leqslant CR_x^{-\frac{\lambda-1}{2}} \sum_{\mathcal{C}_{x,\tau}^{(1)}} \fint_{(\tau - CR_x^{-\frac{\lambda-1}{2}})_+}^{\tau} \left\| F(s,\cdot)\right\|_{L^\infty(\frac{R_x}{2},2R_x)}\,ds$$
$$\leqslant CR_x^{-\frac{\lambda-1}{2}} \sup_{0\leqslant\tau\leqslant T} N_\infty(F;\tau,R) \leqslant Cx^{-(\frac{3+\lambda}{2}+\bar\delta)}\mathcal{H}(F). \tag{7.30}$$

We now estimate the term $J_{1,2}$ in (7.25). Let us denote as $\mathcal{C}_{x,\tau}^{(2)}$ a covering of the set $\{(x_0,s)\colon\ x_0 \geqslant 2x,\ (\tau - x_0^{-\frac{\lambda-1}{2}})_+ \leqslant s \leqslant \tau\}$ by means of boxes with the form $[\frac{R}{2},2R]\times[\tau_0,\tau_0+(R)^{-\frac{\lambda-1}{2}}]$, with $R\geqslant x$, $\tau_0\in[0,\tau]$ in which each point is covered at most by a finite number of boxes (independent of x, τ), and where the sequence of sizes R increases exponentially. (See Fig. 3.)

Then:

$$\frac{1}{x^{3/2}}\int_{2x}^{\infty} dx_0 \int_{\tau - x_0^{-\frac{\lambda-1}{2}}}^{\tau} ds\,(\tau-s)x_0^{\frac{\lambda}{2}}\left|F(s,x_0)\right|$$
$$\leqslant \frac{C}{x^{3/2}}\sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{\frac{\lambda}{2}+1}\int_{\tau_0 - R^{-\frac{\lambda-1}{2}}}^{\tau_0} ds\,(\tau-s)\left\|F(s,\cdot)\right\|_{L^\infty(\frac{R}{2},2R)}$$
$$\leqslant \frac{C}{x^{3/2}}\sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}}\left(\ \fint_{\tau_0 - R^{-\frac{\lambda-1}{2}}}^{\tau_0} ds\,\left\|F(s,\cdot)\right\|^2_{L^\infty(\frac{R}{2},2R)}\right)^{\frac{1}{2}}$$
$$\leqslant \frac{C}{x^{3/2}}\sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}} N_\infty(F;\tau,R) \leqslant \frac{C\mathcal{H}(F)}{x^{3/2}}\sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}}R^{-(2+\bar\delta)} \leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\bar\delta}}$$

where we use the fact that the series $\sum_{\mathcal{C}_{x,\tau}^{(2)}}[\cdots]$ is a geometric series, due to our choice of the sizes of the boxes. Therefore:

$$|J_1| \leqslant Cx^{-(\frac{3+\lambda}{2}+\bar\delta)}\mathcal{H}(F). \tag{7.31}$$

We now estimate J_3 (cf. (7.26)). To this end we introduce a new covering $\mathcal{C}_{x,\tau}^{(3)}$ of the set $\mathcal{D}_{x,\tau} = \{(x_0,s)\colon\ x_0\geqslant 2x,\ 0\leqslant s\leqslant (\tau - x_0^{-\frac{\lambda-1}{2}})_+\}$ by means of boxes of the form $[\frac{R}{2},2R]\times[\tau_0,\tau_0+(R)^{-\frac{\lambda-1}{2}}]$ with $R\geqslant x$, $\tau_0\in[0,\tau]$.

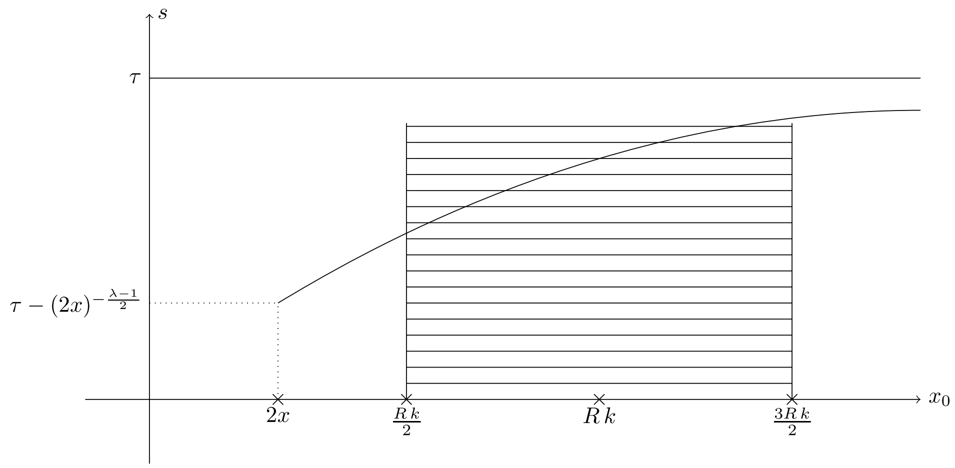

Fig. 4. The sub-family $\mathcal{R}_k$ of cubes of the covering $\mathcal{C}^{(3)}_{x,\tau}$.

We will assume that the rectangles in the set $\mathcal{C}^{(3)}_{x,\tau}$ have the following properties. (i) Each point in the set $\mathcal{D}_{x,\tau}$ is contained in at most a finite number of boxes independent of x, τ for $\tau \in [0,T]$. (ii) There exists a sequence of sizes $\{R_k\}_{k\in N}$ that increases exponentially on k, such that the number of cubes with R comparable to a given R_k (i.e. $R \in (\frac{R_k}{2}, 2R_k)$) is bounded by $CR_k^{\frac{\lambda-1}{2}}$, with C independent of x, τ for $\tau \in [0,T]$. Notice that the construction of the covering implies that, for $(x_0,s) \in [\frac{R}{2}, 2R] \times [\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}}]$ we have $\frac{1}{2}(\tau-\tau_0) \leqslant (\tau - s) \leqslant 2(\tau-\tau_0)$. (See Fig. 4.)

Then, using $\min\{1, \frac{1}{(\tau-s)^{\frac{2}{\lambda-1}}x}\} \leqslant 1$:

$$|J_3| \leqslant \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}^{(3)}_{x,\tau}} \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} ds \int_{\frac{R}{2}}^{2R} dx_0 \frac{\|F(s,\cdot)\|_{L^\infty(\frac{R}{2},2R)}}{(\tau-\tau_0)^{\frac{1}{\lambda-1}}}$$

$$\leqslant \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}^{(3)}_{x,\tau}} \left[\frac{R}{(\tau-\tau_0)^{\frac{1}{\lambda-1}}} R^{-\frac{\lambda-1}{2}}\right] \fint_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \left\|F(s,\cdot)\right\|_{L^\infty(\frac{R}{2},2R)} ds$$

$$\leqslant \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}^{(3)}_{x,\tau}} \left[R \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau-s)^{\frac{1}{\lambda-1}}}\right] N_\infty(F;\tau,R)$$

where we have used the fact that the length of the time integration in each box is of order $R^{-\frac{\lambda-1}{2}}$ as well as Cauchy–Schwartz inequality in the last step. Therefore, using the definition of $\mathcal{H}(F)$ as well as the properties of the covering $\mathcal{C}^{(3)}_{x,\tau}$ and in particular the properties of the sequence $\{R_k\}$:

$$|J_3| \leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\mathcal{C}^{(3)}_{x,\tau}} \left[R^{-(1+\bar{\delta})} \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau-s)^{\frac{1}{\lambda-1}}}\right]$$

$$\leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\{R_k\}} \left[R_k^{-(1+\bar{\delta})} \int_0^{\tau - CR_k^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau-s)^{\frac{1}{\lambda-1}}}\right] \leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\{R_k\}} R_k^{-\frac{\lambda}{2}-\bar{\delta}} \leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\bar{\delta}}}.$$

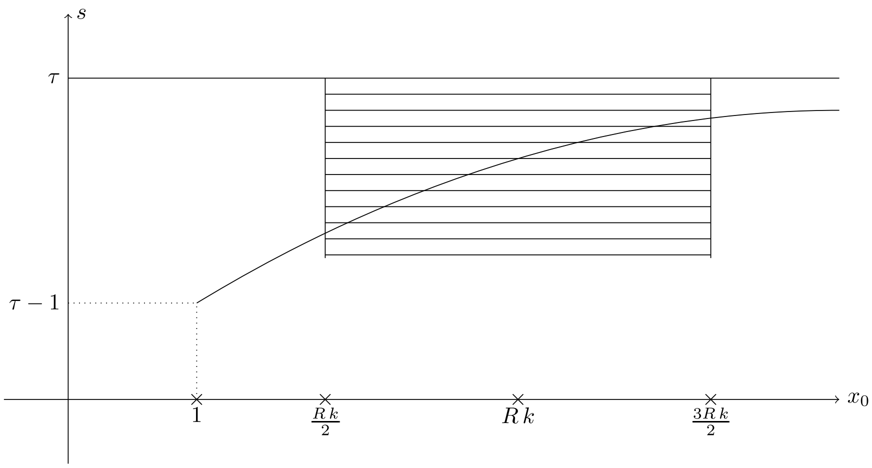

Fig. 5. The sub-family $\mathcal{R}_k$ of cubes of the covering $\mathcal{C}^{(4)}_{x,\tau}$.

Then:

$$|J_3| \leqslant \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\bar{\delta}}}. \tag{7.32}$$

We now estimate the term J_2. Using (4.11), (7.27) and (7.28) we obtain $J_2 \leqslant K_2$ where:

$$K_2 \leqslant \int_0^\infty dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, \big|F(s,x_0)\big| x_0^{\frac{\lambda+1}{2}}.$$

We introduce a covering $\mathcal{C}^{(4)}_{x,\tau}$ such that each point of the set $\{(x_0,s):\ x_0 \geqslant 1,\ (\tau - x_0^{-\frac{\lambda-1}{2}})_+ \leqslant s \leqslant \tau\}$ is contained in a bounded number of boxes having the form $[\frac{R}{2}, 2R] \times [\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}}]$, and where the values of the radii R increase exponentially. (See Fig. 5.)

Then:

$$K_2 \leqslant \int_0^\tau ds \int_0^1 dx_0 \big|F(s,x_0)\big| x_0^{\frac{\lambda+1}{2}} + \sum_{\mathcal{C}^{(4)}_{x,\tau}} \int_{\frac{R}{2}}^{2R} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, \big|F(s,x_0)\big| x_0^{\frac{\lambda+1}{2}} = K_{2,1} + K_{2,2}.$$

Using Cauchy–Schwartz we then estimate $K_{2,2}$ as:

$$K_{2,2} \leqslant C \sum_{\mathcal{C}^{(4)}_{x,\tau}} R^2 N_\infty(F;\tau,R) \leqslant C\mathcal{H}(F) \sum_{\mathcal{C}^{(4)}_{x,\tau}} R^{-\bar{\delta}} \leqslant C\mathcal{H}(F).$$

On the other hand, we estimate $K_{2,1}$ decomposing the interval $[0,1]$ as:

$$[0,1] = \bigcup_{n=0}^{\infty} [R_{n+1}, R_n], \quad R_n = 2^{-n},\ n = 0,1,2,\ldots. \tag{7.33}$$

Then, using again Cauchy–Schwartz:

$$\begin{aligned} K_{2,1} &\leqslant C \sum_{n=0}^{\infty} \int_0^\tau ds \int_{R_{n+1}}^{R_n} dx_0 \, \big\|F(s,\cdot)\big\|_{L^\infty(\frac{R_n}{2},R_n)} R_n^{\frac{\lambda+1}{2}} \\ &\leqslant C\mathcal{H}(F) \sum_{n=0}^{\infty} R_n^{\frac{\lambda+3}{2}+1} R_n^{-\frac{3}{2}} \leqslant C\mathcal{H}(F). \end{aligned}$$

Therefore:

$$J_2 \leqslant K_2 \leqslant C\mathcal{H}(F).$$

Actually we can derive a more precise approximation for J_2 rewriting it as $J_2 = I_2(\tau)x^{-\frac{3+\lambda}{2}} + J_{2;R,1} + J_{2;R}$ where:

$$I_2(\tau) = \int_0^\infty \frac{dx_0}{x_0} \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, F(s,x_0)\Theta\bigl((\tau-s)x_0^{\frac{\lambda-1}{2}}\bigr)x_0^{\frac{3+\lambda}{2}},$$

$$J_{2;R,1} = -x^{-\frac{3+\lambda}{2}} \int_{\frac{2x}{3}}^\infty \frac{dx_0}{x_0} \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, F(s,x_0)\Theta\bigl((\tau-s)x_0^{\frac{\lambda-1}{2}}\bigr)x_0^{\frac{3+\lambda}{2}},$$

$$J_{2,R} = \int_0^\tau ds \int_0^\infty \left[g\left((\tau-s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) - \Theta\bigl((\tau-s)x_0^{\frac{\lambda-1}{2}}\bigr)\left(\frac{x_0}{x}\right)^{\frac{3+\lambda}{2}}\right] F(s,x_0)\frac{dx_0}{x_0}.$$

The terms $J_{2;R}$ can be estimated using Proposition 13 (cf. (5.7), (5.8)) arguing as in the estimate of K_2 exactly as the previous estimate for J_2 since $r > \bar{\delta}$. Therefore, we obtain the estimate:

$$|J_{2,R}| \leqslant Cx^{-\frac{3+\lambda}{2}-r} \int_0^{\frac{2x}{3}} \frac{dx_0}{(x_0)^{1+\bar{\delta}-r}} \leqslant Cx^{-\frac{3+\lambda}{2}-\bar{\delta}}$$

where the constants C could be very large for small $\bar{\delta}$. On the other hand:

$$|J_{2;R,1}| \leqslant Cx^{-\frac{3+\lambda}{2}} \int_{\frac{2x}{3}}^\infty \frac{dx_0}{x_0} \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} ds\, (\tau-s)x_0^{\frac{\lambda-1}{2}}x_0^{\frac{3+\lambda}{2}}(x_0)^{-2-\bar{\delta}} \leqslant Cx^{-\frac{3+\lambda}{2}-\bar{\delta}}$$

whence:

$$\bigl|J_2 - I_2(\tau)x^{-\frac{3+\lambda}{2}}\bigr| \leqslant Cx^{-\frac{3+\lambda}{2}-\bar{\delta}}. \tag{7.34}$$

We now estimate the integrand in J_4 in (7.29). In order to apply Lebesgue's Theorem we need to prove that:

$$K_4 = \int_0^\infty dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} \frac{ds}{(\tau-s)^{\frac{\lambda+1}{\lambda-1}}}\bigl|F(s,x_0)\bigr| < \infty.$$

We define a new covering $\mathcal{C}_{x,\tau}^{(5)}$ of the set $\{(x_0,s)\colon\ x_0 \geqslant 1,\ 0 \leqslant s \leqslant (\tau - x_0^{-\frac{\lambda-1}{2}})_+\}$ having the same properties as the covering $\mathcal{C}_{x,\tau}^{(3)}$. (See Fig. 6.)

Then, using that the integrand is empty for $x_0 \leqslant 1$ due to the fact that $\tau \leqslant T \leqslant 1$, as well as Cauchy–Schwartz and the properties of the covering $\mathcal{C}_{x,\tau}^{(5)}$ we obtain:

$$\begin{aligned} K_4 &\leqslant C\sum_{\mathcal{C}_{x,\tau}^{(5)}} \frac{R^{1-\frac{\lambda-1}{2}}}{(\tau-\tau_0)^{\frac{\lambda+1}{\lambda-1}}} \int_{\tau_0}^{(\tau_0 - CR^{-\frac{\lambda-1}{2}})_+} \bigl\|F(s,\cdot)\bigr\|_{L^\infty(\frac{R}{2},2R)}\, ds \\ &\leqslant C\mathcal{H}(F)\sum_{\mathcal{C}_{x,\tau}^{(5)}} \frac{R^{-1-\frac{\lambda-1}{2}-\bar{\delta}}}{(\tau-\tau_0)^{\frac{\lambda+1}{\lambda-1}}} \end{aligned}$$

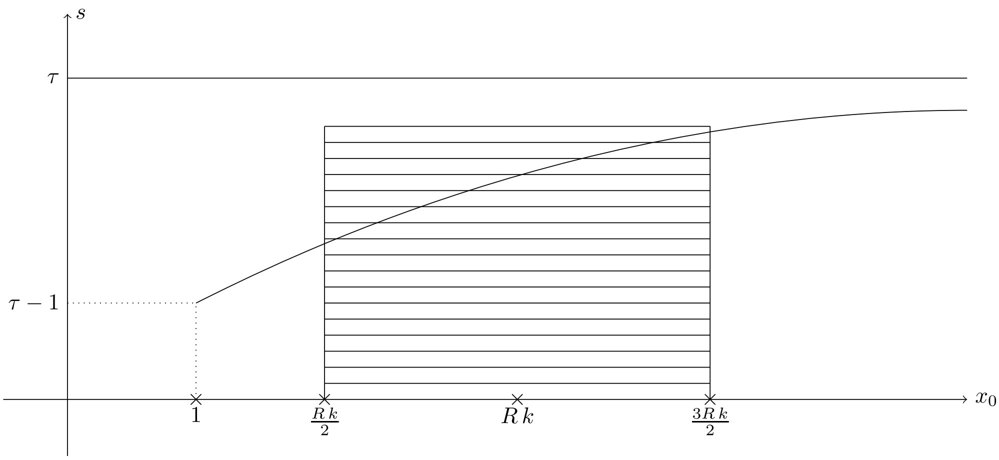

Fig. 6. The sub-family $\mathcal{R}_k$ of cubes of the covering $\mathcal{C}_{x,\tau}^{(5)}$.

$$\leqslant C\mathcal{H}(F)\sum_{\mathcal{C}_{x,\tau}^{(5)}} R_k^{-(1+\bar{\delta})} \int_{\tau_0}^{(\tau_0 - CR_k^{-\frac{\lambda-1}{2}})_+} (\tau - s)^{-\frac{\lambda+1}{\lambda-1}}\, ds$$

$$\leqslant C\mathcal{H}(F)\sum_{\{R_k\}} R_k^{-\bar{\delta}} \leqslant C\mathcal{H}(F)$$

whence:

$$K_4 \leqslant C\mathcal{H}(F). \tag{7.35}$$

In order to obtain (7.18) we need to substract from J_4 the leading contribution and to estimate the remainder. Using (4.10) we obtain:

$$J_4 = \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, g\left((\tau - s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) F(s, x_0)$$

$$= \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \Theta\big((\tau - s)x_0^{\frac{\lambda-1}{2}}\big)(x)^{-\frac{3+\lambda}{2}} F(s, x_0) + J_{4,R}$$

where:

$$|J_{4,R}| \leqslant C \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \big((\tau - s)x_0^{\frac{\lambda-1}{2}}\big)^{-\frac{2\varepsilon}{\lambda-1} - \frac{\lambda+1}{\lambda-1}} \left(\frac{x}{x_0}\right)^{-\frac{3+\lambda}{2} - \varepsilon} \big|F(s, x_0)\big|.$$

The argument yielding (7.35) implies the existence of the integral:

$$I_4(\tau) = \int_0^{\infty} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \Theta\big((\tau - s)x_0^{\frac{\lambda-1}{2}}\big)(\tau - s)^{-\frac{3+\lambda}{\lambda-1}} F(s, x_0).$$

Therefore:

$$J_4 = I_4(\tau)x^{-\frac{3+\lambda}{2}} + \int_{\frac{2x}{3}}^{\infty} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \Theta\big((\tau-s)x_0^{\frac{\lambda-1}{2}}\big)(\tau-s)^{-\frac{3+\lambda}{\lambda-1}} F(s,x_0) + J_{4,R}. \tag{7.36}$$

We then estimate the remainders in (7.36). Notice that the bounds for Θ and F yield:

$$\begin{aligned}
&\left| \int_{\frac{2x}{3}}^{\infty} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \Theta\big((\tau-s)x_0^{\frac{\lambda-1}{2}}\big)(\tau-s)^{-\frac{3+\lambda}{\lambda-1}} F(s,x_0)\right| \\
&\quad\leqslant C\left| \int_{\frac{2x}{3}}^{\infty} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, (\tau-s)^{\frac{2}{\lambda-1}} x_0 (\tau-s)^{-\frac{3+\lambda}{\lambda-1}} (x_0)^{-(2+\bar\delta)}\right| \\
&\quad\leqslant Cx^{-\bar\delta}.
\end{aligned}$$

We estimate the term $J_{4,R}$ in (7.36) as:

$$|J_{4,R}| \leqslant Cx^{-\frac{3+\lambda}{2}-\varepsilon} \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds\, \big((\tau-s)x_0^{\frac{\lambda-1}{2}}\big)^{-\frac{2\varepsilon+\lambda+1}{\lambda-1}} (x_0)^{\frac{\lambda-1}{2}+\varepsilon-\bar\delta} \leqslant Cx^{-\frac{3+\lambda}{2}-\bar\delta}$$

using $\varepsilon > \bar\delta$ whence:

$$\left|J_4 - I_4(\tau)x^{-\frac{3+\lambda}{2}}\right| \leqslant Cx^{-\frac{3+\lambda}{2}-\bar\delta}. \tag{7.37}$$

Combining (7.31), (7.32), (7.34), (7.37) we obtain that the function $J_R(\tau,x)$ defined in (7.18) satisfies

$$\left|J_R(\tau,x)\right| \leqslant \frac{C\|F\|_{X_{\frac{3}{2},2+\bar\delta}(T)}}{x^{\frac{3+\lambda}{2}+\bar\delta}} \quad \text{for } x \geqslant 1. \tag{7.38}$$

On the other hand, in order to estimate $J_R = J$ for $x \leqslant 1$ we argue as follows. We recall that $|J_1| \leqslant J_{1,1} + J_{1,2}$ with $J_{1,1}$, $J_{1,2}$ as in (7.24), (7.25). We have for $R \leqslant 1$ and $x \in (\frac{R}{2}, 2R)$:

$$J_{1,1}(\tau,x) \leqslant C\int_0^{\tau} ds\, \left\|F(s,\cdot)\right\|_{L^\infty(\frac{R}{4},4R)} \leqslant M_\infty(F;R) \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar\delta}(T)}.$$

We estimate $J_{1,2}$ covering the set $(s,x_0) \in (0,T)\times(0,1)$ by means of union of the rectangles $(0,T)\times[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, $n = 0,1,2,\dots$. Let us denote this covering as $\mathcal{C}^{(6)}$. Then:

$$\begin{aligned}
J_{1,2}(\tau,x) &\leqslant \frac{C}{x^{3/2}} \int_{2x}^{1} dx_0 \int_0^{\tau} (\tau-s)\, ds\, x_0^{\frac{\lambda}{2}} \left|F(s,x_0)\right| + \frac{C}{x^{3/2}} \int_1^{\infty} dx_0 \int_{(\tau - x_0^{-\frac{\lambda-1}{2}})_+}^{\tau} (\tau-s)\, ds\, x_0^{\frac{\lambda}{2}} \left|F(s,x_0)\right| \\
&= J_{1,2,1}(\tau,x) + J_{1,2,2}(\tau,x),
\end{aligned}$$

$$J_{1,2,1}(\tau,x) \leqslant \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \int_0^{T} (\tau-s)\, ds\, x_0^{\frac{\lambda}{2}} \left\|F(s,\cdot)\right\|_{L^\infty(\frac{1}{2^{n+1}},\frac{1}{2^n})}$$

$$\leqslant \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \left(\frac{1}{2^n}\right)^{\frac{\lambda}{2}} M_\infty\left(F; \frac{1}{2^n}\right)$$
$$\leqslant \frac{C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \frac{1}{2^{n(\frac{\lambda-1}{2})}} \leqslant \frac{C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}}{x^{3/2}}.$$

The integral term in $J_{1,2,2}(\tau,x)$ can be estimated using the covering $\mathcal{C}^{(2)}_{x,\tau}$ exactly in the same way as in the estimate of $J_{1,2}$ for $x \geqslant 1$. It then follows that $J_{1,2,2}(\tau,x) \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} x^{-3/2}$ whence:

$$J_{1,2}(\tau,x) \leqslant \frac{C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}}{x^{3/2}}. \tag{7.39}$$

In order to estimate J_3 for $x \leqslant 1$ we use (7.26). Notice that for $s < \tau \leqslant T < 1$ and $x \leqslant 1$ we have $(\tau-s)^{-\frac{1}{\lambda-1}} x^{-\frac{3}{2}} \leqslant (\tau-s)^{-\frac{1+\lambda}{\lambda-1}} x^{-\frac{3+\lambda}{2}}$ whence (7.26) yields:

$$|J_3| \leqslant \frac{C}{x^{\frac{3}{2}}} \int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds \, \frac{|F(s,x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}}$$
$$\leqslant \frac{C}{x^{\frac{3}{2}}} \int_{\frac{2x}{3}}^{1} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds \, \frac{|F(s,x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}} + \frac{C}{x^{\frac{3}{2}}} \int_1^{\infty} dx_0 \int_0^{(\tau - x_0^{-\frac{\lambda-1}{2}})_+} ds \, \frac{|F(s,x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}}$$
$$\equiv J_{3,1} + J_{3,2}.$$

The term $J_{3,2}$ can be estimated in the same manner as J_3 for $x \geqslant 1$. We just use the covering $\mathcal{C}^{(3)}_{x,\tau}$ to obtain $J_{3,2} \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} x^{-3/2}$. Since $J_{3,1} = 0$ for $0 \leqslant \tau \leqslant T < 1$ we then obtain:

$$J_3 \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} x^{-3/2}. \tag{7.40}$$

In order to estimate J_2 we use (7.27), (7.28) as well as (4.11)

$$J_2 \leqslant C x^{-\frac{3+\lambda}{2}} \int_0^{\frac{2x}{3}} dx_0 \int_0^{\tau} ds \, F(s,x_0)(\tau-s) x_0^{\lambda}$$
$$\leqslant C x^{-\frac{3}{2}} \int_0^{1} dx_0 \int_0^{\tau} ds \, F(s,x_0)(\tau-s)(x_0)^{\frac{\lambda}{2}}$$

and this integral can be estimated exactly as $J_{1,2,1}(\tau,x)$, using the covering $\mathcal{C}^{(6)}$:

$$J_2 \leqslant \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \int_0^{T} (\tau-s) \, ds \, x_0^{\frac{\lambda}{2}} \left\| F(s,\cdot) \right\|_{L^\infty(\frac{1}{2^{n+1}}, \frac{1}{2^n})}$$

whence:

$$J_2 \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} x^{-3/2}. \tag{7.41}$$

Finally we notice that (7.29) implies that $J_4 = 0$ for $x \leqslant 1$, $0 \leqslant \tau \leqslant T < 1$. Combining (7.39), (7.40), (7.41) we obtain:

$$\left|J(\tau, x)\right| \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} x^{-3/2}$$

for $0 < x \leqslant 1$. Using then also (7.38) the result follows.

To conclude the proof of Lemma 23 it only remains to show (7.21). To this end we use similar covering arguments. First we decompose the expression of $I(\tau; F)$ as:

$$I(\tau; F) = I_1 + I_2 + I_3 + I_4,$$

$$I_1 = \int_0^1 \frac{dx_0}{x_0} \int_0^\tau ds\,[\ldots], \qquad I_2 = \int_1^{\tau^{-\frac{2}{\lambda-1}}} \frac{dx_0}{x_0} \int_0^\tau ds\,[\ldots],$$

$$I_3 = \int_{\tau^{-\frac{2}{\lambda-1}}}^\infty \frac{dx_0}{x_0} \int_{\tau - x_0^{-\frac{\lambda-1}{2}}}^\tau ds\,[\ldots], \qquad I_4 = \int_{\tau^{-\frac{2}{\lambda-1}}}^\infty \frac{dx_0}{x_0} \int_0^{\tau - x_0^{-\frac{\lambda-1}{2}}} ds\,[\ldots].$$

Using (4.11) as well as the definition of $\|\cdot\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}$ we obtain, using Cauchy–Schwartz:

$$|I_1| \leqslant C\tau^{\frac{3}{2}}\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} \sum_{n=0}^\infty \left(\frac{1}{2^n}\right)^{\lambda-\frac{1}{2}} \leqslant C\tau^{\frac{3}{2}}\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}. \tag{7.42}$$

Using the splitting of the domains of integrations in rectangles as above as well as (4.11) we also obtain the following estimates:

$$|I_2| \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} \int_1^{\tau^{-\frac{2}{\lambda-1}}} dx_0\,(x_0)^{\lambda-(2+\bar{\delta})} \int_0^\tau (\tau - s)\,ds,$$

$$|I_3| \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} \int_{\tau^{-\frac{2}{\lambda-1}}}^\infty dx_0\,(x_0)^{\lambda-(2+\bar{\delta})} \int_{\tau - x_0^{-\frac{\lambda-1}{2}}}^\tau (\tau - s)\,ds,$$

$$|I_4| \leqslant C\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} \int_{\tau^{-\frac{2}{\lambda-1}}}^\infty dx_0\,(x_0)^{-(2+\bar{\delta})} \int_0^{\tau - x_0^{-\frac{\lambda-1}{2}}} (\tau - s)^{-\frac{\lambda+1}{\lambda-1}}\,ds.$$

Therefore, since the three integrals on the right-hand side are bounded by $\leqslant C\tau^{\frac{2\bar{\delta}}{\lambda-1}}\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}$:

$$|I_2| + |I_3| + |I_4| \leqslant C\tau^{\frac{2\bar{\delta}}{\lambda-1}}\|F\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)}.$$

Combining this with (7.42) we obtain (7.21). □

We now derive the asymptotics (7.3) for the solutions of (7.1) in a pointwise sense. More precise estimates for the regularity of the error terms will be derived later.

Lemma 24. *Suppose that $\varphi \in \mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}(T)$ solves (7.1) with $F \in Y^{\sigma}_{\frac{3}{2},2+\bar{\delta}}(T)$ and $\bar{\delta} < r$. Then (7.2) holds with $\mathcal{W}(\tau)$ as in (7.3) and*:

$$|||\varphi_R|||_{\frac{3}{2},\frac{3+\lambda}{2}+\bar{\delta}} \leqslant C\left(\|F\|_{X^{\sigma}_{\frac{3}{2},2+\bar{\delta}}(T)} + \|\varphi\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}(T)}\right). \tag{7.43}$$

Proof. Lemma 21 as well as the fact that $r > \bar{\delta}$ implies that $\|(\mathcal{L}_{f_0} - L)[\varphi]\|_{X_{\frac{3}{2},2+\bar{\delta}}(T)} < \infty$. Lemma 24 then follows applying Lemma 23 with F replaced by

$$\bar{F} = F + (\mathcal{L}_{f_0} - L)[\varphi]. \qquad \square \tag{7.44}$$

In order to prove suitable regularity properties of the remainder φ_R in (7.2) we will need to obtain some regularity properties for the function $I(\cdot;\bar{F})$, where the functional I is as in (7.19) and $\bar{F}$ as in (7.44). The rationale behind the argument is that an equation for φ_R contains terms that can be estimated only if some regularity for $I(\tau;\bar{F})$ is available.

We have:

Lemma 25. *Suppose that* $\sup_{\tau\in[0,T]} |||F(\tau,\cdot)|||_{\frac{3}{2},2+\bar{\delta}} < \infty$. *Let* $\Delta = \tau_1 - \tau_0$, *with* $0 \leqslant \tau_0 \leqslant \tau_1 \leqslant T$. *Then*:

$$\left|I(\tau_1;F) - I(\tau_0;F)\right| \leqslant C(\Delta)^{\frac{2\bar{\delta}}{\lambda-1}}$$

with $C > 0$ *depending on* $\sup_{\tau\in[0,T]} |||F(\tau,\cdot)|||_{\frac{3}{2},2+\bar{\delta}}, \lambda$.

Proof. We write, using (7.19):

$$\begin{aligned} I(\tau_1;F) - I(\tau_0;F) &= \int_0^\infty \frac{dx_0}{x_0}\int_{\tau_0}^{\tau_1} ds\, \Theta\big((\tau_1 - s)x_0^{\frac{\lambda-1}{2}}\big)x_0^{\frac{3+\lambda}{2}} F(s,x_0) \\ &\quad + \int_0^\infty \frac{dx_0}{x_0}\int_0^{\tau_0} ds\left[\Theta\big((\tau_1 - s)x_0^{\frac{\lambda-1}{2}}\big) - \Theta\big((\tau_0 - s)x_0^{\frac{\lambda-1}{2}}\big)\right]x_0^{\frac{3+\lambda}{2}} F(s,x_0) \\ &= L_1 + L_2. \end{aligned}$$

We have the global estimate:

$$\left|F(s,x_0)\right| \leqslant C x_0^{-(2+\bar{\delta})}.$$

Then:

$$\begin{aligned} |L_1| &\leqslant C\int_0^\infty \frac{dx_0}{x_0}\int_{\tau_0}^{\tau_1} ds\left|\Theta\big((\tau_1 - s)x_0^{\frac{\lambda-1}{2}}\big)\right|x_0^{\frac{3+\lambda}{2}} x_0^{-(2+\bar{\delta})} \\ &= C\int_0^\infty \frac{dx_0}{(x_0)^{1+\bar{\delta}}}\int_0^{\Delta x_0^{\frac{\lambda-1}{2}}}\left|\Theta(s)\right| ds = C(\Delta)^{\frac{2\delta}{\lambda-1}}\int_0^\infty \frac{dy}{(y)^{1+\bar{\delta}}}\int_0^{y^{\frac{\lambda-1}{2}}}\left|\Theta(s)\right| ds. \end{aligned}$$

Using the fact that $\Psi_1(y) = \int_0^{y^{\frac{\lambda-1}{2}}} |\Theta(s)|\, ds$ satisfies $|\Psi_1(y)| \leqslant C\min\{y^{\lambda-1}, 1\}$ (cf. (4.11)) we obtain:

$$|L_1| \leqslant C(\Delta)^{\frac{2\bar{\delta}}{\lambda-1}}. \tag{7.45}$$

We now estimate $|L_2|$. We have:

$$\begin{aligned} |L_2| &\leqslant \int_0^\infty \frac{dx_0}{x_0}\int_0^{\tau_0} ds\left|\Theta\big((\tau_1 - s)x_0^{\frac{\lambda-1}{2}}\big) - \Theta\big((\tau_0 - s)x_0^{\frac{\lambda-1}{2}}\big)\right|x_0^{\frac{3+\lambda}{2}} x_0^{-(2+\bar{\delta})} \\ &= \int_0^\infty \frac{dx_0}{(x_0)^{1+\bar{\delta}}}\int_0^{\frac{\tau_0}{\Delta}} d\rho\left|\Theta\big((1+\rho)\Delta x_0^{\frac{\lambda-1}{2}}\big) - \Theta\big(\rho\Delta x_0^{\frac{\lambda-1}{2}}\big)\right|x_0^{\frac{\lambda-1}{2}}\Delta \end{aligned}$$

$$\leqslant (\Delta)^{\frac{2\bar{\delta}}{\lambda-1}} \int_0^\infty \frac{\Psi_2(y)\,dy}{(y)^{1+\bar{\delta}}} \int_0^\infty dZ\,\big|\Theta\big(y^{\frac{\lambda-1}{2}}+Z\big)-\Theta(Z)\big|$$

where $\Psi_2(y)=\int_0^\infty dZ\,|\Theta(y^{\frac{\lambda-1}{2}}+Z)-\Theta(Z)|$. We have used the change of variables $y=(\Delta)^{\frac{2}{\lambda-1}}x_0$, and $Z=y^{\frac{\lambda-1}{2}}\rho$. Using (4.11) we obtain $\Psi_2(y)\leqslant C\min\{y^{\frac{\lambda-1}{2}},1\}$. Then:

$$|L_2|\leqslant C(\Delta)^{\frac{2\bar{\delta}}{\lambda-1}}. \tag{7.46}$$

Lemma 25 follows combining (7.45), (7.46). □

We can now prove Proposition 20 deriving suitable regularity estimates for the function φ_R.

Proof of Proposition 20. Given the function φ_R defined in (7.2) our goal is to prove (7.4). To this end, given $I(\tau;F)$ in (7.3) defined in $0\leqslant\tau\leqslant T$ we extend it to $\mathbb{R}$ defining $I(\tau;\bar{F})=I(0;\bar{F})$ for $\tau\leqslant 0$ and $I(\tau;\bar{F})=I(T;\bar{F})$ for $\tau\geqslant T$. We define a function $\tilde{I}_R(\tau)$ for $0\leqslant\tau\leqslant T$ by means of:

$$\tilde{I}_R(\tau;\bar{F})=\big(I(\cdot;\bar{F})*\chi_R\big)(\tau) \tag{7.47}$$

with $\chi_R(\tau)=R^{\frac{\lambda-1}{2}}\chi_1(R^{\frac{\lambda-1}{2}}\tau)$ where the nonnegative function $\chi_1\in C^\infty(\mathbb{R})$ is compactly supported in $[-1,1]$ and it satisfies $\int_{\mathbb{R}}\chi_1(\tau)\,d\tau=1$. We extend $I(\tau;\bar{F})$ in (7.47) as it can be easily seen, using Lemma 25 with $\bar{\delta}<r$ with r as in (2.1) that:

$$\big|\tilde{I}_R(\tau;\bar{F})-I(\tau;\bar{F})\big|\leqslant CR^{-\bar{\delta}},\quad \tau\in[0,T],\qquad \left|\frac{d\tilde{I}_R(\tau;\bar{F})}{d\tau}\right|\leqslant R^{\frac{\lambda-1}{2}-\bar{\delta}},\quad \tau\in[0,T]. \tag{7.48}$$

We define $\tilde{\varphi}_R(\tau,x)=\varphi(\tau,x)-\tilde{I}_R(\tau;\bar{F})x^{-\frac{3+\lambda}{2}}\xi(x)$ where ξ is as in (2.4). Then, using (7.2) we obtain $|\tilde{\varphi}_R(\tau,x)|\leqslant |\varphi_R(\tau,x)|+|\tilde{I}_R(\tau;\bar{F})-I(\tau;\bar{F})||\xi(x)x^{-\frac{3+\lambda}{2}}$ whence (7.43), (7.48) yield:

$$\big|\tilde{\varphi}_R(\tau,x)\big|\leqslant CR^{-(\frac{3+\lambda}{2}+\bar{\delta})},\quad x\in\left[\frac{R}{2},R\right],\ \tau\in[0,T],$$

where $\tilde{\varphi}_R$ satisfies:

$$\begin{aligned}
&(\tilde{\varphi}_R)_\tau=\mathcal{L}_{f_0}[\tilde{\varphi}_R]+\tilde{F}(\tau,x),\\
&\tilde{F}(\tau,x)=\bar{F}(\tau,x)-\frac{d\tilde{I}_R(\tau;\bar{F})}{d\tau}x^{-\frac{3+\lambda}{2}}+\tilde{I}_R(\tau;\bar{F})\mathcal{L}_{f_0}\big[x^{-\frac{3+\lambda}{2}}\xi(x)\big]
\end{aligned} \tag{7.49}$$

where $\bar{F}$ is as in (7.44). Using (7.48) as well as the fact that $\bar{F}\in Y^\sigma_{\frac{3}{2},2+\bar{\delta}}$ and the fact that $\mathcal{L}_{f_0}[x^{-\frac{3+\lambda}{2}}\xi(x)]$ decreases like $x^{-(2+\bar{\delta})}$ as $x\to\infty$, including derivatives, it follows that $\tilde{F}\in Y^\sigma_{\frac{3}{2},2+\bar{\delta}}$. We now use the rescaling $\tilde{\varphi}_R(\tau,x)=R^{-(\frac{3+\lambda}{2}+\bar{\delta})}\Phi_R(\rho,X)$, $x=RX$, $\tau=\frac{\rho}{R^{\frac{\lambda-1}{2}}}$. The function $\Phi_R(\rho,X)$ then satisfies an equation of the form $\frac{\partial\Phi_R}{\partial\rho}=\mathcal{L}^R_{f_0}[\Phi_R]+\tilde{F}_R$ where $\mathcal{L}^R_{f_0}$ is the operator obtained rescaling $\mathcal{L}_{f_0}$ as well as $\tilde{F}_R$ that is obtained rescaling $\tilde{F}$ (cf. (5.82) in the proof of Lemma 5.8 of [9]).

We can now argue as in the proof of Lemma 5.8 in [9], applying Theorem 11 in Section 4 to prove:

$$\|\tilde{\varphi}_R\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)}\leqslant C\|F\|_{Y^\sigma_{\frac{3}{2},2+\bar{\delta}}}.$$

This concludes the proof of Proposition 20. □

8. Fixed point argument

In this section we prove Theorem 1 by means of a fixed point argument. Our strategy is to define a function $\tilde{h}$ by means of (3.5) with initial data $\tilde{h}(0,x)=0$. To this end, we define two auxiliary functions $\tilde{h}_1$, $\tilde{h}_2$ that will be due respectively to the source terms $(\frac{Q[h]}{\Lambda(\tau)}+\Lambda(\tau)Q[f_0])$ and $(-\Lambda_\tau f_0(x))$. The goal of this splitting is to treat in a separate way the term containing Λ_τ since this term will require some careful analysis of the time regularity of the solutions. The function h will be then defined as $\tilde{h}(t,x)=\tilde{h}_1(t,x)+\tilde{h}_2(t,x)$.

8.1. Construction of the function $\tilde{h}_1$

We define $\tilde{h}_1$ as the solution of the problem:

$$\tilde{h}_{1,\tau} = \mathcal{L}_{f_0}[\tilde{h}_1] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0], \tag{8.1}$$

$$\tilde{h}_1(0,x) = 0, \quad x > 0. \tag{8.2}$$

The existence, uniqueness and main regularity properties of the function $\tilde{h}_1$ are given in the following result:

Proposition 26. *Given $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ with $\bar{\delta} < r$, f_0 as in (2.2)–(2.5), $0 < T \leqslant 1$ and $\Lambda \in C[0,T]$ satisfying $|\Lambda(\tau) - 1| \leqslant \frac{1}{4}$ there exists a unique function $\tilde{h}_1 \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma;\frac{1}{2}}(T)$ solution of (8.1), (8.2) if T is sufficiently small. Moreover, there exists a function $\mathcal{G}[\cdot;h,\Lambda] \in C[0,T]$ such that the function r_1 defined by means of*:

$$\tilde{h}_1(\tau,x) - \mathcal{G}[\tau;h,\Lambda]\xi(x)x^{-\frac{3+\lambda}{2}} = r_1(\tau,x;h,\Lambda) \tag{8.3}$$

satisfies $r_1(\cdot,\cdot;h,\Lambda) \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$. The function $\xi(\cdot)$ is as in (2.4). The mappings

$$\begin{aligned} &\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T) \times C[0,T] \to C[0,T], \\ &(h,\Lambda) \mapsto \mathcal{G}[\cdot;h,\Lambda], \end{aligned} \tag{8.4}$$

$$\begin{aligned} &\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T) \times C[0,T] \to \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T), \\ &(h,\Lambda) \mapsto r_1(\cdot,\cdot;h,\Lambda) \end{aligned} \tag{8.5}$$

are Lipschitz continuous assuming that $C[0,T]$ is endowed with the uniform topology and T is sufficiently small. Moreover there exist ρ_0, T_0 such that the Lipschitz constant for the maps can be made arbitrarily small if $T < T_0$ and $\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \rho_0$.

We will prove Proposition 26 with the help of some auxiliary lemmas. Estimates for $Q[f_0](x_0)$ are obtained in the next result.

Lemma 27. *Let f_0 be as in (2.2)–(2.5). We assume that this function is defined in $[0,T] \times \mathbb{R}^+$ as a function independent of τ. We then have*:

$$\big\| Q[f_0] \big\|_{Y^{\sigma}_{\frac{3}{2},(2+r)}(T)} \leqslant C, \tag{8.6}$$

$$\big\| Q[f_0] \big\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta}_1)}(T)} \leqslant C \max\big\{\sqrt{T}, T^{\frac{2(r-\bar{\delta})}{\lambda-1}}\big\}, \quad 0 < \bar{\delta}_1 \leqslant r, \tag{8.7}$$

$$\big|Q[f_0](x_0)\big| \leqslant \frac{C}{1+x_0^{2+r}}, \quad x_0 > 0. \tag{8.8}$$

Proof. We use the decomposition (2.6)

$$f_0 = \bar{f}_0(x) + h_0(x), \quad \bar{f}_0(x) = \frac{\xi(x)}{x^{\frac{3+\lambda}{2}}}$$

with $\xi(\cdot)$ as in (2.4). Then:

$$\big|h_0(x)\big| + x\big|h_0'(x)\big| \leqslant \frac{CB}{x^{\frac{3+\lambda}{2}+r}}, \quad x \geqslant 1. \tag{8.9}$$

Then:

$$Q[f_0](x_0) = Q[\bar{f}_0](x_0) + \mathcal{L}_{\bar{f}_0}[h_0](x_0) + Q[h_0](x_0),$$

$$Q[\bar{f}_0](x) = -\int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} \bar{f}_0(x)\bar{f}_0(y)\,dy + \int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} \bar{f}_0(y)\big[(x-y)^{\frac{\lambda}{2}} \bar{f}_0(x-y) - x^{\frac{\lambda}{2}} \bar{f}_0(x)\big]\,dy. \tag{8.10}$$

Using the fact that $\bar{f}_0(y) = y^{-\frac{3+\lambda}{2}}$ for large y we obtain:

$$Q[\bar{f}_0](x) = -\int_{\frac{x}{2}}^{\infty} \frac{dy}{x^{\frac{3}{2}} y^{\frac{3}{2}}} + \int_0^{\frac{x}{2}} \left[\frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}}\right]\frac{dy}{y^{\frac{3}{2}}} + \int_0^2 \big[\xi(y) - 1\big]\left[\frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}}\right]\frac{dy}{y^{\frac{3}{2}}}.$$

The first two terms on the right-hand side cancel out. The third one can be estimated, using Taylor's expansion, as:

$$\int_0^2 \frac{1}{y^{\frac{3}{2}}}\left|\frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}}\right| dy \leqslant \frac{C}{x^{\frac{5}{2}}}, \quad x > 1,$$

whence:

$$\big|Q[\bar{f}_0](x)\big| \leqslant \frac{C}{x^{\frac{5}{2}}}, \quad x > 1. \tag{8.11}$$

We can estimate $Q[h_0](x_0)$ as:

$$\big|Q[h_0](x)\big| \leqslant \frac{CB^2}{x^{2+2r}} + \left|\int_0^{\frac{x}{2}} \frac{dq}{dy}(y) S(x,y;h_0)\,dy\right|$$

where:

$$q(y;h_0) = \int_y^{\infty} \xi^{\frac{\lambda}{2}} h_0(\xi)\,d\xi, \qquad S(x,y;h_0) = \big[(x-y)^{\frac{\lambda}{2}} h_0(x-y) - x^{\frac{\lambda}{2}} h_0(x)\big].$$

Integrating by parts we obtain:

$$\left|\int_0^{\frac{x}{2}} \frac{dq}{dy}(y;h_0) S(x,y;h_0)\,dy\right| \leqslant \frac{CB^2}{x^{2+2r}} + CB^2\left|\int_0^{\frac{x}{2}} \frac{1}{y^{\frac{1}{2}+r}} \frac{1}{(x-y)^{\frac{5}{2}+r}}\,dy\right| \leqslant \frac{CB^2}{x^{2+2r}}.$$

Then:

$$\big|Q[h_0](x)\big| \leqslant \frac{CB^2}{x^{2+2r}}, \quad x > 1. \tag{8.12}$$

Finally we write $\mathcal{L}_{f_0}[h_0]$ as:

$$\mathcal{L}_{f_0}[h_0] = -\int_0^{\frac{x}{2}} S(x,y;f_0) q'(y;h_0)\,dy - x^{\lambda/2} f_0(x) q\left(\frac{x}{2};h_0\right)$$
$$- \int_0^{\frac{x}{2}} S(x,y;h_0) q'(y;f_0)\,dy - x^{\lambda/2} h_0(x) q\left(\frac{x}{2};f_0\right).$$

An immediate computation shows that the second and fourth terms on the right can be estimated as $CBx^{-(2+r)}$. The other two terms can be estimated integrating by parts. Then:

$$\big|\mathcal{L}_{\bar{f}_0}[h_0](x)\big| \leqslant \frac{CB}{x^{2+r}}, \quad x > 1. \tag{8.13}$$

In the region $x \leqslant 1$ we have trivially boundedness of $Q[f_0](x_0)$. Combining this with (8.10)–(8.13) we obtain (8.8). Estimate (8.6) follows similarly using the differentiability properties assumed for f_0.

It only remains to prove (8.7). To this end, notice that due to the definition of $\|\cdot\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}(T)}$ we need to estimate L^2 norms in time for $R \leqslant 1$. Since we have estimates in L^∞ for $Q[f_0]$ we then obtain a dependence on T like $\sqrt{T}$. Similar estimates can be obtained, using also the definition of the norms $\|\cdot\|_{Y^{\sigma}_{\frac{3}{2},(2+r)}(T)}$ for all the values of $R \leqslant T^{-\frac{2}{\lambda-1}}$, since for these values there is not splitting of the domain of integration in the t variable. In the region where $R > T^{-\frac{2}{\lambda-1}}$ we use the fact that $Q[f_0]$ is pointwise estimated by $R^{-(2+r)}$. Therefore:

$$R^{(2+\bar{\delta})}\big(N_\infty\big(Q[f_0]; t_0, R\big) + N_{2;\sigma}\big(Q[f_0]; t_0, R\big)\big) \leqslant C R^{\bar{\delta}-r} \leqslant C T^{\frac{2(r-\bar{\delta})}{\lambda-1}}$$

for $R > T^{-\frac{2}{\lambda-1}}$. Therefore (8.7) follows. $\square$

As a next step we estimate the quadratic terms in (8.1).

Lemma 28. *Given $h \in \mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)$ and Λ as in Proposition 26. Then:*

$$\left\| \frac{Q[h]}{\Lambda(\cdot)} \right\|_{Y^{\sigma}_{\frac{3+\lambda}{2},(2+\bar{\delta})}} \leqslant C\|h\|^2_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}. \tag{8.14}$$

Moreover:

$$\left\| \frac{Q[h_1]}{\Lambda_1(\cdot)} - \frac{Q[h_2]}{\Lambda_2(\cdot)} \right\|_{Y^{\sigma}_{\frac{3+\lambda}{2},(2+\bar{\delta})}} \leqslant C\left(\sum_{k=1}^{2} \|h_k\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}\right)\big(\|h_1 - h_2\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)} + \|\Lambda_1 - \Lambda_2\|_{C[0,T]}\big). \tag{8.15}$$

Proof. Estimates (8.14), (8.15) are just a consequence of Propositions 14, 19 as well as the fact that $\frac{1}{2} \leqslant \Lambda(\cdot) \leqslant \frac{3}{2}$. $\square$

Proof of Proposition 26. Existence and uniqueness of the function $\tilde{h}_1$ follow from the results in [9] (cf. 9) combined with Lemmas 27, 28.

On the other hand, the decay and regularity properties of the function r_1 defined in (8.3) are a consequence of Proposition 20. In order to apply this proposition some regularity and decay for the source terms $\frac{Q[h]}{\Lambda(\tau)}$ and $\Lambda(\tau)Q[f_0]$ are needed. In the case of $\frac{Q[h]}{\Lambda(\tau)}$ such properties are a consequence of Proposition 14. The corresponding properties for $\Lambda(\tau)Q[f_0]$ follow from Lemma 27 and the fact that $r > \bar{\delta}$. The function $\mathcal{G}[\tau; h, \Lambda]$ is given by the function $\mathcal{W}(\tau)$ in (7.3) with source F given by $\frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0]$. Notice that the linearity of the equation satisfied by $\tilde{h}_1$ as well as the Lipschitz property for $Q[h]$ in Proposition 19 implies that the map $h \to \tilde{h}_1$ is Lipschitz in h in the space $\mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}(T)$. Moreover, due to Lemma 21, the map $h \to (\mathcal{L}_{f_0} - L)(\tilde{h}_1)$ from $\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)$ to $Y^{\sigma}_{\frac{3}{2},\frac{3+\lambda}{2}+\bar{\delta}}$ is Lipschitz. Therefore the map in (8.4) has the Lipschitz dependence stated in Proposition 26. The Lipschitz property for the map in (8.5) is again a consequence of Proposition 19, the linearity of the problem under consideration and Proposition 20.

It only remains to check that the Lipschitz constant of the maps (8.4), (8.5) can be made small if $T \leqslant T_0$ and T_0 is small enough. Indeed, given two couples $(h^{(1)}, \Lambda^{(1)})$, $(h^{(2)}, \Lambda^{(2)})$ satisfying the hypothesis of the proposition, let us denote as $F^{(1)}, F^{(2)}, \tilde{h}_1^{(1)}, \tilde{h}_1^{(2)}$ and $\mathcal{W}^{(1)}, \mathcal{W}^{(2)}$ the corresponding functions $F, \tilde{h}_1, \mathcal{W}$ respectively. The stated Lipschitz properties yield:

$$\big\|F^{(1)} - F^{(2)}\big\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}} + \big\|(\mathcal{L}_{f_0} - L)\big(\tilde{h}_1^{(1)} - \tilde{h}_1^{(2)}\big)\big\|_{Y^{\sigma}_{\frac{3}{2},(2+\bar{\delta})}} \leqslant C\big(\big\|\Lambda^{(1)} - \Lambda^{(2)}\big\|_{C[0,T]} + \big\|h^{(1)} - h^{(2)}\big\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)}\big). \tag{8.16}$$

Using the inequality (8.16) combined with (7.3) and (7.21) we then obtain that the difference $\mathcal{W}^{(1)} - \mathcal{W}^{(2)}$ can be estimated as:

$$\left|\mathcal{W}^{(1)}(\tau) - \mathcal{W}^{(2)}(\tau)\right| \leqslant CT^{\frac{2\bar{\delta}}{\lambda-1}}\left(\left\|\Lambda^{(1)} - \Lambda^{(2)}\right\|_{C[0,T]} + \left\|h^{(1)} - h^{(2)}\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}\right)$$

for $0 \leqslant \tau \leqslant T$.

The fact that the Lipschitz constant for the map (8.5) can be made small for small T is just a consequence of (8.7) in Lemma 27 and (8.15) in Lemma 28 if ρ_0 in the statement of Proposition 26 is sufficiently small. $\square$

8.2. *Construction of the function* $\tilde{h}_2$

It would be natural to construct $\tilde{h}_2$ as the solution of the problem:

$$\tilde{h}_{2,\tau} = \mathcal{L}_{f_0}[\tilde{h}_2] - \Lambda_\tau f_0(x), \quad \tilde{h}_2(0,x) = 0, \; x > 0. \tag{8.17}$$

However, since it is more convenient from the technical point of view to avoid using the derivative Λ_τ we will use an alternative procedure that we describe shortly here. More precisely, we will obtain a solution of the initial value problem:

$$\psi_\tau = \mathcal{L}_{f_0}[\psi], \quad \psi(0,x) = f_0(x), \; x > 0. \tag{8.18}$$

In order to solve this problem we define the change of variables:

$$\psi(\tau,x) = f_0(x) + \zeta(\tau,x). \tag{8.19}$$

The function ζ then solves:

$$\zeta_\tau = \mathcal{L}_{f_0}[\zeta] + \mathcal{L}_{f_0}[f_0], \quad \zeta(0,x) = 0, \; x > 0. \tag{8.20}$$

This equation can be solved, assuming (2.2)–(2.5) using Theorem 9.

Variation of constants formula then suggests that $\tilde{h}_2$, solution of (8.17) is given by:

$$\tilde{h}_2(\tau,x) = -\int_0^\tau \psi(\tau-s,x)\Lambda_\tau(s)\,ds \tag{8.21}$$

and assuming that ψ is differentiable in time we would obtain:

$$\tilde{h}_2(\tau,x) = -f_0(x)\Lambda(\tau) + \psi(\tau,x) - \int_0^\tau \frac{\partial\psi}{\partial\tau}(\tau-s,x)\Lambda(s)\,ds. \tag{8.22}$$

This representation formula avoids using Λ_τ. However, it requires to prove that $\frac{\partial\psi}{\partial\tau}$ is well defined. We now prove the properties of ψ required to give a precise meaning to (8.22).

Proposition 29. *Suppose that f_0 satisfies (2.2)–(2.5). There exists a function $\psi \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma;\frac{1}{2}}(T)$ defined by means of (8.18). We have:*

$$\psi(\tau,x) = a(\tau)\xi(x)x^{-\frac{3+\lambda}{2}} + r_2(\tau,x) \tag{8.23}$$

where $\xi(\cdot)$ is the cutoff in (2.4) and where:

$$a(0) = 1, \qquad \|r_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant C. \tag{8.24}$$

Moreover:

$$\left|\frac{da}{d\tau}\right| \leqslant C, \qquad \left\|\frac{\partial r_2}{\partial\tau}\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant C. \tag{8.25}$$

The proof of this result is based on the following lemma:

Lemma 30. *Suppose that f_0 satisfies (2.2)–(2.5). There exists a constant C such that, for any $0 < T \leqslant 1$:*

$$\left\|\mathcal{L}_{f_0}[f_0]\right\|_{Y^{\sigma}_{\frac{3}{2},2+\delta}(T)} + \left\|\mathcal{L}_{f_0}\left[\mathcal{L}_{f_0}[f_0]\right]\right\|_{Y^{\sigma}_{\frac{3}{2},2+\delta}(T)} \leqslant C \tag{8.26}$$

where $\delta > 0$ is as in (2.2)–(2.5).

Proof. Using (2.2)–(2.5) we obtain the asymptotics:

$$\mathcal{L}_{f_0}[f_0] = K x^{-\frac{3+\lambda}{2}} \xi(x) + w_{0,R}(x) \tag{8.27}$$

with a remainder $w_{0,R}(x)$ that can be estimated, together with its derivatives as $x^{-(\frac{3+\lambda}{2}+\delta)}$ as $x \to \infty$. The main idea to keep in mind is that the operator $\mathcal{L}_{f_0}$ acting on power laws x^{-p} amounts to multiply then by $C_p x^r$. The constant C_p vanishes if $p = \frac{3+\lambda}{2}$. The estimate (8.26) will be then proved multiplying by the cutoff $\eta(\tau)$ and taking the operator $\mathcal{L}_{f_0}$. Since the leading power law in (8.27) is $x^{-\frac{3+\lambda}{2}}$ the action of the operator $\mathcal{L}_{f_0}$ will cancel the first order and only a remainder behaving like $x^{-(2+\delta)}$ will be left, with δ as in (2.2)–(2.5).

We now describe the details. The operator $\mathcal{L}_{f_0}$ is defined in (3.3). We then have, rearranging the integral terms:

$$\frac{1}{2}\mathcal{L}_{f_0}[f_0](x) = Q[f_0](x). \tag{8.28}$$

Using (2.2), (2.3):

$$\mathcal{L}_{f_0}[f_0](x) = \mathcal{L}_{f_{1;2}}[f_{1;2}](x) + 2\mathcal{L}_{f_{1;2}}[f_3] + \mathcal{L}_{f_3}[f_3](x). \tag{8.29}$$

We can estimate $\mathcal{L}_{f_{1;2}}[f_3]$, as well as its derivative using (2.2), (2.3), (2.5) as well as the fact that $1 < \lambda < 2$ and $0 < \delta < r$:

$$\left|\mathcal{L}_{f_{1;2}}[f_3]\right| + (1+x)\left|\frac{d}{dx}\left(\mathcal{L}_{f_{1;2}}[f_3]\right)\right| + (1+x)^2\left|\frac{d^2}{dx^2}\left(\mathcal{L}_{f_{1;2}}[f_3]\right)\right| \leqslant \frac{C}{(1+x)^{2+r+\delta}}, \quad x > 0. \tag{8.30}$$

The term $\mathcal{L}_{f_3}[f_3](x)$ can be estimated similarly:

$$\begin{aligned} &\left|\frac{1}{2}\mathcal{L}_{f_3}[f_3]\right| + (1+x)\left|\frac{1}{2}\frac{d}{dx}\left(\mathcal{L}_{f_3}[f_3]\right)\right| + (1+x)^2\left|\frac{d^2}{dx^2}\left(\mathcal{L}_{f_3}[f_3]\right)\right| \\ &\quad \leqslant \frac{C}{(1+x)^{\frac{3+\lambda}{2}+r}} \leqslant \frac{C}{(1+x)^{2+r+\delta}} \end{aligned} \tag{8.31}$$

for $x > 0$. Therefore, it only remains to estimate $\mathcal{L}_{f_{1;2}}[f_{1;2}]$ in (8.29).

We then only need to approximate the term $\mathcal{L}_{f_{1;2}}[f_{1;2}]$ that might be rewritten as:

$$\mathcal{L}_{f_{1;2}}[f_{1;2}](x) = \mathcal{L}_{f_1}[f_1](x) + 2\mathcal{L}_{f_1}[f_2](x) + \mathcal{L}_{f_2}[f_2](x). \tag{8.32}$$

We then have, using $r > \delta$:

$$\left|\mathcal{L}_{f_2}[f_2]\right| + (1+x)\left|\frac{d}{dx}\left(\mathcal{L}_{f_2}[f_2]\right)\right| + (1+x)^2\left|\frac{d^2}{dx^2}\left(\mathcal{L}_{f_2}[f_2]\right)\right| \leqslant \frac{C}{(1+x)^{2+r+\delta}}, \quad x > 0. \tag{8.33}$$

We need to obtain precise asymptotics of the terms $\mathcal{L}_{f_1}[f_1](x)$, $2\mathcal{L}_{f_1}[f_2](x)$ in order to obtain the leading order term in (8.27). Let us write $\bar{f}_1(x) = x^{-\frac{3+\lambda}{2}}$. Notice that:

$$\mathcal{L}_{\bar{f}_1}[\bar{f}_1] = 0. \tag{8.34}$$

Moreover, we have the following identity:

$$x^{\lambda/2} f_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_1(y)\, dy = x^{\lambda/2} \bar{f}_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} \bar{f}_1(y)\, dy, \quad x \geqslant 1.$$

Using (8.34) we then obtain for $x > 2$:

$$\mathcal{L}_{f_1}[f_1] = \int_0^2 y^{\lambda/2} \bar{f}_1(y)\big[\xi(y) - 1\big]\big[(x-y)^{\lambda/2} \bar{f}_1(x-y) - x^{\lambda/2} \bar{f}_1(x)\big]\,dy.$$

Taylor's expansion, as well as the fact that $\delta < \frac{\lambda-2}{2}$ implies:

$$\big|\mathcal{L}_{f_1}[f_1]\big| + (1+x)\left|\frac{d}{dx}\big(\mathcal{L}_{f_1}[f_1]\big)\right| + (1+x)^2\left|\frac{d^2}{dx^2}\big(\mathcal{L}_{f_1}[f_1]\big)\right| \leqslant \frac{C}{(1+x)^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0, \tag{8.35}$$

where we use the fact that $\mathcal{L}_{f_1}[f_1]$ and its derivatives are trivially bounded for x bounded as it might be seen using directly using the definition of $\mathcal{L}_{f_1}[f_1]$.

It remains to estimate the term $2\mathcal{L}_{f_1}[f_2](x)$ in (8.32). Using the definition of $\mathcal{L}_{f_1}[f_2](x)$ we obtain

$$2\mathcal{L}_{f_1}[f_2](x) = \mathcal{H}_1(f_1, f_2)(x) + \mathcal{H}_2(f_1, f_2)(x) \tag{8.36}$$

where:

$$\mathcal{H}_1(f_1, f_2)(x) = \int_0^{\frac{x}{2}} y^{\lambda/2} f_1(y)\big[(x-y)^{\lambda/2} f_2(x-y) - x^{\lambda/2} f_2(x)\big]\,dy$$
$$+ \int_0^{\frac{x}{2}} y^{\lambda/2} f_2(y)\big[(x-y)^{\lambda/2} f_1(x-y) - x^{\lambda/2} f_1(x)\big]\,dy,$$

$$\mathcal{H}_2(f_1, f_2)(x) = x^{\lambda/2} f_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_2(y)\,dy + x^{\lambda/2} f_2(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_1(y)\,dy.$$

The term $\mathcal{H}_2(f_1, f_2)(x)$ can be explicitly computed for large values of x:

$$\mathcal{H}_2(f_1, f_2)(x) = K_1 x^{-\frac{3+\lambda}{2}}, \quad K_1 \in \mathbb{R} \text{ and } x > 1. \tag{8.37}$$

In order to approximate $\mathcal{H}_1(f_1, f_2)(x)$ we define a function $\bar{f}_2(x) = \frac{a_1}{x^{\frac{3+\lambda}{2}+r}}$. We then have:

$$\mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x) = K_2 x^{-\frac{3+\lambda}{2}}, \quad K_1 \in \mathbb{R} \text{ and } x > 0. \tag{8.38}$$

On the other hand:

$$\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x)$$
$$= \int_0^2 y^{\lambda/2} \bar{f}_1(y)\big[\xi(y) - 1\big]\big[(x-y)^{\lambda/2} \bar{f}_2(x-y) - x^{\lambda/2} \bar{f}_2(x)\big]\,dy$$
$$+ \int_0^2 y^{\lambda/2} \bar{f}_2(y)\big[\xi(y) - 1\big]\big[(x-y)^{\lambda/2} \bar{f}_1(x-y) - x^{\lambda/2} \bar{f}_1(x)\big]\,dy.$$

Taylor's expansion, as well as the fact that $\delta < \frac{2-\lambda}{2}$, yields:

$$\big|\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x)\big| + (1+x)\left|\frac{d}{dx}\big(\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x)\big)\right|$$
$$\leqslant \frac{C}{x^{\frac{5}{2}}} \leqslant \frac{C}{x^{\frac{3+\lambda}{2}+\delta}}, \quad x \geqslant 1. \tag{8.39}$$

The boundedness of $\mathcal{L}_{f_1}[f_2](x)$ and its derivatives combined with (8.36)–(8.39) yields:

$$\left|2\mathcal{L}_{f_1}[f_2](x) - Kx^{-\frac{3+\lambda}{2}}\right| \leqslant \frac{C}{1+x^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0. \tag{8.40}$$

Combining (8.29)–(8.35), (8.40) we obtain (8.27) where:

$$\sum_{k=0}^{3}(1+x)^k\left|\frac{d^k w_{0,R}}{dx^k}\right| \leqslant \frac{C}{1+x^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0. \tag{8.41}$$

Applying $\mathcal{L}_{f_0}[\cdot]$ on both sides of (8.27), using $\mathcal{L}_{f_0}[x^{-\frac{3+\lambda}{2}}] = 0$, and arguing as in the proof of (8.35) we obtain $\sum_{k=0}^{2}(1+x)^k|\frac{d^k}{dx^k}(\mathcal{L}_{f_0}[Kx^{-\frac{3+\lambda}{2}}\xi(x)])| \leqslant \frac{C}{1+x^{2+\delta}}$. On the other hand, the action of the operator $\mathcal{L}_{f_0}$ over functions satisfying (8.41) amounts to multiplying by x^r for large values of x. Therefore $\sum_{k=0}^{2}(1+x)^k|\frac{d^k}{dx^k}(\mathcal{L}_{f_0}[w_{0,R}])| \leqslant \frac{C}{1+x^{2+\delta}}$ and the result follows. □

Proof of Proposition 29. Due to (8.26) in Lemma 30 $\mathcal{L}_{f_0}[f_0]$ is bounded in the space $Y^{\sigma}_{3/2,2+\delta}(T)$. Therefore (8.20) can be solved using the results in [9]. We obtain in this way a solution $\zeta \in \mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}(T)$. Then, ψ can be obtained by means of (8.19). Therefore expansion (8.23), (8.24) are just a consequence of Proposition 20.

It only remains to obtain estimates for the derivatives on time of the functions a, r_2. Formal differentiation of (8.20) suggests that $w = \frac{\partial \zeta}{\partial \tau}$ satisfies the following initial value problem:

$$(w)_\tau = \mathcal{L}_{f_0}[w], \quad w(0,x) = \mathcal{L}_{f_0}[f_0]. \tag{8.42}$$

Actually we can use the results in [9] to construct a solution of (8.42) as follows. We define a function $W(\tau,x)$ by means of:

$$w(\tau,x) = \mathcal{L}_{f_0}[f_0] + W(\tau,x). \tag{8.43}$$

Then, w solves (8.42) iff W solves:

$$W_\tau = \mathcal{L}_{f_0}[W] + \mathcal{L}_{f_0}\big[\mathcal{L}_{f_0}[f_0]\big], \quad W(0,x) = 0. \tag{8.44}$$

In order to be able to solve the problem (8.42) we use the hypothesis (2.2)–(2.5). Due to Lemma 30 we have that $\|\mathcal{L}_{f_0}[\mathcal{L}_{f_0}[f_0]]\|_{Y^{\sigma}_{\frac{3}{2},2+\delta}(T)}$ is bounded. Therefore, we can apply the results in [9] (cf. Theorem 9) to obtain a unique solution W of (8.44) satisfying $\|W\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\frac{3+\lambda}{2}}(T)} \leqslant C$. The function ψ, solution of (8.18) can be obtained, using also (8.19) as:

$$\psi(\tau,x) = f_0(x) + \mathcal{L}_{f_0}[f_0]\tau + \int_0^\tau W(s,x)\,ds. \tag{8.45}$$

Using (7.2), (7.4) in Proposition 20 we obtain:

$$W(\tau,x) = \mathcal{W}(\tau)x^{-\frac{3+\lambda}{2}}\xi(x) + W_R(\tau,x) \tag{8.46}$$

with

$$\left|\mathcal{W}(\tau)\right| \leqslant C, \quad 0 \leqslant \tau \leqslant T, \qquad \left\|W_R(\tau,x)\right\|_{\mathcal{Z}^{\sigma;\frac{1}{2}}_{\bar{p}}(T)} \leqslant C. \tag{8.47}$$

Then $\psi(\tau,x) = a(\tau)x^{-\frac{3+\lambda}{2}}\xi(x) + r_2(\tau,x)$ where:

$$a(\tau) = 1 + K\tau + \int_0^\tau \mathcal{W}(s)\,ds, \qquad r_2(\tau,x) = \big[f_0(x) - x^{-\frac{3+\lambda}{2}}\xi(x)\big] + w_{0,R}(x)\tau + \int_0^\tau W_R(s,x)\,ds.$$

Using (8.47) we obtain (8.25) and the proposition follows. □

Remark 31. As indicated in Section 2 the assumption (2.2)–(2.5) is very strong. However, the argument proving Proposition 29 shows that the main reason for assuming (2.2)–(2.5) is to show that $|\frac{da}{d\tau}|$, $\frac{\partial r_2}{\partial \tau}$ are bounded in a suitable sense. It would be possible to weaken (2.2), (2.3) to some assumption with the form $f_0(x) = D_1 x^{-\frac{3+\lambda}{2}} + O(x^{-\frac{3+\lambda}{2}-\delta})$ as $x \to \infty$ for some $\delta > 0$. Making such an assumption the only difference in the argument proving Proposition 29 would be that the term $\mathcal{L}_{f_0}[f_0]$ in (8.42) would behave like $O(x^{-(2+\delta)})$ instead of $O(x^{-\frac{3+\lambda}{2}})$ as $x \to \infty$. Unfortunately the well-posedness theory developed in [9] cannot cover such weakest rate of decay at infinity. The expected asymptotics for $w(\tau, x)$ as $x \to \infty$ for small τ would have the form $w(\tau, x) \sim \frac{C}{\tau^{1-\frac{2\delta}{\lambda-1}}} x^{-\frac{3+\lambda}{2}}$ as $x \to \infty, \tau \to 0$. This type of asymptotics has been obtained in [6,7] for a different equation, namely the Uehling–Uhlenbeck equation. Unfortunately since the well-posedness theory of classical solutions for the coagulation equation is more difficult, we have preferred not to consider such a case, at the price of assuming stronger regularity assumptions near the singular point. Nevertheless it would be an interesting question to prove analogous regularizing results in time.

With the previous construction we can define the function $\tilde{h}_2$ as follows.

Definition 32. For any f_0, Λ satisfying the assumptions in Proposition 26 we define $\tilde{h}_2$ by means of

$$\tilde{h}_2(\tau, x) = -f_0(x)\Lambda(\tau) + \psi(\tau, x) - \int_0^\tau w(\tau - s, x)\Lambda(s)\, ds \tag{8.48}$$

where ψ is as in (8.45) and w is as in (8.43).

Remark 33. The rationale behind Definition 32 is the following. Assuming smoothness we obtain, differentiating (8.48):

$$(\tilde{h}_2)_\tau = -f_0(x)\Lambda_\tau(\tau) + \frac{\partial \psi}{\partial \tau}(\tau, x) - \Lambda(\tau)\mathcal{L}_{f_0}[f_0] - \int_0^\tau \mathcal{L}_{f_0}[w](\tau - s, x)\Lambda(s)\, ds$$

where we have used (8.42). Exchanging the order of the integral in time and $\mathcal{L}_{f_0}$ and using again (8.48) we obtain, after some cancellations $\tilde{h}_{2,\tau} = -f_0\Lambda_\tau(\tau) + \frac{\partial\psi}{\partial\tau} + \mathcal{L}_{f_0}(\tilde{h}_2) - \mathcal{L}_{f_0}[\psi]$. Using then (8.18) we obtain that $\tilde{h}_2$ would solve (8.17).

The asymptotics of the function $\tilde{h}_2$ as $x \to \infty$ can be derived using the corresponding results for the functions ψ, w in Proposition 29.

Lemma 34. *For any $\Lambda \in C[0, T]$ satisfying the assumptions in Proposition 26 we have*:

$$\tilde{h}_2(\tau, x) = \mathcal{K}[\Lambda](\tau)\xi(x)x^{-\frac{3+\lambda}{2}} + \tilde{h}_{2,R}(\tau, x; \Lambda)$$

where

$$\mathcal{K}[\Lambda](\tau) = -\Lambda(\tau) + a(\tau) - \int_0^\tau \mathcal{W}(\tau - s)\Lambda(s)\, ds$$

with a as in Proposition 29, $\frac{da}{d\tau} = \mathcal{W}$ and $\tilde{h}_{2,R} \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$. Moreover, the map:

$$C[0, T] \to \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T) : \Lambda \to \tilde{h}_2 \tag{8.49}$$

is Lipschitz if $T \leqslant T_0$, with T_0 sufficiently small.

Proof. It is just a consequence of the definition of $\tilde{h}_2$ in (8.48) and Proposition 29. $\square$

8.3. Setting of the fixed point argument. Solution of an integral equation

Given $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ and $\Lambda \in C[0,T]$ as in Proposition 26 we can define a map $(h,\Lambda) \to \tilde{h}$ with $\tilde{h}(t,x) = \tilde{h}_1(t,x) + \tilde{h}_2(t,x)$ where $\tilde{h}_1$ is as in Proposition 26 and $\tilde{h}_2$ as in (8.48).

We now select, for any given h, the function Λ in order to have:

$$\lim_{x\to\infty}\bigl(x^{\frac{3+\lambda}{2}}\tilde{h}(t,x)\bigr) = 0. \tag{8.50}$$

Due to (8.3) and since $r_1(\cdot,\cdot;h) \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$, as well as Lemma 34 it follows that (8.50) holds if Λ solves:

$$\Lambda(\tau) = \mathcal{G}[\tau;h,\Lambda] + a(\tau) - \int_0^\tau \mathcal{W}(\tau-s,x)\Lambda(s)\,ds. \tag{8.51}$$

We first show that for $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ we can find $\Lambda = \Lambda(\cdot;h)$ such that (8.51) is satisfied.

Lemma 35. *There exists $T_0 > 0$ sufficiently small, such that, for any $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ satisfying $\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \rho_0$ Eq.* (8.51) *has a unique solution for $0 \leqslant \tau \leqslant T$, assuming that $T \leqslant T_0$. Moreover, this solution defines a mapping*:

$$\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T) \to C[0,T]: \quad h \mapsto \Lambda(\cdot;h) \tag{8.52}$$

that is contractive.

Proof. The function $\mathcal{W}$ in the integral term (8.51) is uniformly bounded due to Proposition 29. On the other hand, the function $\mathcal{G}[\tau;h,\Lambda]$ is Lipschitz contractive in Λ if T_0 is sufficiently small and $\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \rho_0$ due to Proposition 26. It then follows from (8.51) that the mapping (8.52) is contractive. □

Lemma 36. *Let us denote as B_{ρ_0} the ball of radius ρ_0 in $\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$, with ρ_0 as in Proposition 26 and let us consider the mapping from B_{ρ_0} to B_{ρ_0} given by $h \to \mathcal{T}[h]$ where $\mathcal{T}[h] = \tilde{h}_1 + \tilde{h}_2$, with $\tilde{h}_1$ as in Proposition 26 and $\tilde{h}_2$ as in* (8.48) *and with $\Lambda = \Lambda(\cdot;h)$ in* (8.1) *where $\Lambda(\cdot;h)$ is chosen as in Lemma 35. Then, there exists T_0 such that the mapping $\mathcal{T}$ is contractive in B_{ρ_0} if $T \leqslant T_0$. In such a case there exists a unique fixed point of $\mathcal{T}$ in B_{ρ_0}.*

Proof. The definitions of $\tilde{h}, \tilde{h}_1, \tilde{h}_2$ combined with (8.51) imply:

$$\mathcal{T}[h](\tau,x) = \tilde{h}(\tau,x) = r_1\bigl(\tau,x;h,\Lambda(\cdot;h)\bigr) + \tilde{h}_{2,R}\bigl(\tau,x;\Lambda(\cdot;h)\bigr). \tag{8.53}$$

Notice that $\mathcal{T}$ transforms B_{ρ_0} into B_{ρ_0} for $T \leqslant T_0$ small. Indeed, r_1 consists of two pieces that are due to the contributions of the source terms $\frac{Q[h]}{\Lambda(\tau)}$ and $\Lambda(\tau)Q[f_0]$ in (8.1) respectively. The norm $\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ of the solution due to the source term $\frac{Q[h]}{\Lambda(\tau)}$ can be bounded as $C\rho_0^2$ due to Proposition 14. On the other hand in order to estimate the contribution due to the term $\Lambda(\tau)Q[f_0]$ we use (8.7) in Lemma 27. Using then Proposition 20 it follows that the contribution due to the source $\Lambda(\tau)Q[f_0]$ is smaller than $\frac{\rho_0}{8}$ if T_0 is small enough.

On the other hand, in order to see that the contribution of the term $\tilde{h}_{2,R}$ is small for small times, we use the formulas for $\psi, \tilde{h}_2$. Using (8.45) and (8.48) we obtain:

$$\tilde{h}_2(\tau,x) = f_0(x)\bigl[1-\Lambda(\tau)\bigr] + \mathcal{L}_{f_0}[f_0]\tau + \int_0^\tau W(s,x)\,ds - \int_0^\tau w(\tau-s,x)\Lambda(s)\,ds.$$

We substract the terms behaving like $\xi(x)x^{-\frac{3+\lambda}{2}}$ in all the pieces. We then obtain:

$$\tilde{h}_{2,R}(\tau,x) = \left[f_0(x) - \xi(x)x^{-\frac{3+\lambda}{2}}\right]\left[1-\Lambda(\tau)\right] + w_{0,R}(x)\tau + \int_0^\tau W_R(s,x)\,ds - \int_0^\tau \left[w_{0,R}(x) + W_R(\tau-s,x)\right]\Lambda(s)\,ds \equiv \tilde{h}_{2,R,1}(\tau,x) + \tilde{h}_{2,R,2}(\tau,x) + \tilde{h}_{2,R,3}(\tau,x) + \tilde{h}_{2,R,4}(\tau,x)$$

(cf. (8.27), (8.43), (8.46)). Using (8.27) we obtain $\|\tilde{h}_{2,R,2}\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant CT$. We can estimate $\tilde{h}_{2,R,3}$ and $\tilde{h}_{2,R,4}$ in the space $\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ using the fact that these functions are integrals on time of functions bounded in $\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$. Using Lemma 12 we obtain $\|\tilde{h}_{2,R,3}\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} + \|\tilde{h}_{2,R,4}\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant C\sqrt{T}$. It only remains to control the term $\tilde{h}_{2,R,1}$. To this end, we use here the integral equation (8.51) that yields:

$$\left[1-\Lambda(\tau)\right] = -\mathcal{G}[\tau;h,\Lambda] + \left(1-a(\tau)\right) + \int_0^\tau \mathcal{W}(\tau-s,x)\Lambda(s)\,ds. \tag{8.54}$$

Due to Proposition 26 and Lemmas 27, 28 we can estimate the contributions to $\mathcal{G}[\tau;h,\Lambda]$ that are due to $\Lambda(\tau)$, $Q[f_0]$ and $\frac{Q[h]}{\Lambda(\tau)}$ respectively as $C\max\{\sqrt{T}, T^{\frac{2(r-\bar{\delta})}{\lambda-1}}\}$ and $C\rho_0^2$. Therefore this contribution can be estimated by $\frac{\rho_0}{8}$. The second term on the right-hand side of (8.54) can be estimated using the differentiability of a (cf. (8.25)). Therefore this term can be estimated as CT. On the other hand, the boundedness of $\mathcal{W}$ (cf. (8.47)) provides a similar estimate for the last term in (8.54). Therefore, using the regularity of f_0 we obtain $\|\tilde{h}_{2,R,1}\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \frac{\rho_0}{4}$. It then follows that $\mathcal{T}$ transforms B_{ρ_0} into B_{ρ_0} if T_0 is sufficiently small. Combining the contractivity of the map (8.52) with the Lipschitz properties of the maps (8.5), (8.49) we obtain the contractivity of $\mathcal{T}$ if $T \leqslant T_0$ sufficiently small. □

Proof of Theorem 1. We define $\tilde{f}$ by means of:

$$\tilde{f}(\tau,x) = \Lambda(\tau)f_0(x) + h(\tau,x) \tag{8.55}$$

where $h(\tau,x)$ is the fixed point associated to the operator $\mathcal{T}$ obtained in Lemma 36. Notice that:

$$h(\tau,x) = \mathcal{T}(h)(\tau,x) = r_1(\tau,x;h,\Lambda) + \tilde{h}_{2,R}(\tau,x) = \tilde{h}_1(\tau,x) + \tilde{h}_2(\tau,x)$$

where $\tilde{h}_1$, $\tilde{h}_2$ are as in Proposition 26 and (8.48) respectively. Using (8.55) and (8.48) we obtain:

$$\tilde{f}(\tau,x) = \tilde{h}_1(\tau,x) + \psi(\tau,x) - \int_0^\tau w(\tau-s,x)\Lambda(s)\,ds \tag{8.56}$$

where ψ, w are as in (8.45), (8.43). The function $\tilde{f}$ is differentiable with respect to τ due to Propositions 26, 29 and the continuity of Λ, and differentiability of w (cf. Lemma 35 and (8.43), (8.44) respectively). Therefore $\tilde{f}$ solves:

$$\tilde{f}_\tau = \frac{Q[\tilde{f}]}{\Lambda(\tau)} \tag{8.57}$$

as it can be checked as follows. Differentiating (8.56), using the fact that w solves (8.42) and exchanging the integration in time with the operator $\mathcal{L}_{f_0}[\cdot]$ we obtain:

$$\tilde{f}_\tau = (\tilde{h}_1)_\tau + \psi_\tau - \Lambda(\tau)\mathcal{L}_{f_0}[f_0] - \mathcal{L}_{f_0}\left[\int_0^\tau w(\tau-s,x)\Lambda(s)\,ds\right].$$

Eliminating the integral in the last term by means of (8.56):

$$\tilde{f}_\tau = \left[(\tilde{h}_1)_\tau - \mathcal{L}_{f_0}[\tilde{h}_1]\right] + \left[\psi_\tau - \mathcal{L}_{f_0}[\psi]\right] - \Lambda(\tau)\mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}].$$

Proposition 29 yields $\psi_\tau - \mathcal{L}_{f_0}[\psi] = 0$. Using also (8.1) we obtain:

$$\tilde{f}_\tau = \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau)Q[f_0] - \Lambda(\tau)\mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}]$$

where due to (8.55) $h = \tilde{f}(\tau, x) - \Lambda(\tau) f_0(x)$. Then:

$$\tilde{f}_\tau = \frac{Q[\tilde{f}]}{\Lambda(\tau)} - \mathcal{L}_{f_0}[\tilde{f}] + \Lambda(\tau)Q[f_0] + \Lambda(\tau)Q[f_0] - \Lambda(\tau)\mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}].$$

Using that $\mathcal{L}_{f_0}[f_0] = 2Q[f_0]$ (cf. (8.28)) we obtain that $\tilde{f}$ solves (8.57). Using the time scale t given by means of (3.4) we deduce that $f(t,x) = \tilde{f}(\tau, x)$ solves (1.1), (1.2). Using (8.2), (8.18) and (8.56) we have that f satisfies (1.3). This concludes the proof of the existence of the sought-for solution.

We prove uniqueness in the class of solutions stated in Theorem 1 as follows. Suppose that we have two solutions f^α, f^β of (1.1)–(1.3) such that $f^\alpha = \lambda^\alpha(t) f_0(x) + h^\alpha$, $f^\beta = \lambda^\beta(t) f_0(x) + h^\beta$ with $\lambda^\alpha, \lambda^\beta \in C[0,T]$, $h^\alpha, h^\beta \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$.

Using the change of variables (3.4), for both solutions and denoting as τ the new time scale in both cases we obtain functions $\tilde{f}^\alpha$, $\tilde{f}^\beta$ satisfying (8.57) with $\Lambda = \Lambda^\alpha = \lambda^\alpha$ and $\Lambda = \Lambda^\beta = \lambda^\beta$ respectively. We will write, with a bit abuse of notation $\tilde{f}^\alpha = \Lambda^\alpha(\tau) f_0(x) + h^\alpha$, $\tilde{f}^\beta = \Lambda^\beta(\tau) f_0(x) + h^\beta$. We define functions $\tilde{h}_2^\alpha, \tilde{h}_2^\beta \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ by means of (8.48) with the corresponding functions Λ^α, Λ^β. We define also the functions $\tilde{h}_1^\alpha, \tilde{h}_1^\beta \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$ by means of $\tilde{h}_1^\alpha = h^\alpha - \tilde{h}_2^\alpha$, $\tilde{h}_1^\beta = h^\beta - \tilde{h}_2^\beta$.

Using arguments analogous to the ones used in the derivation of (8.57) we obtain:

$$\left(\tilde{h}_1^k\right)_\tau - \mathcal{L}_{f_0}\left[\tilde{h}_1^k\right] = \frac{Q[h^k]}{\Lambda^k(\tau)} + \Lambda^k(\tau)Q[f_0], \quad \tilde{h}_1^k(0) = 0,\ k = \alpha, \beta.$$

Using Proposition 26 we obtain that $\tilde{h}_1^\alpha, \tilde{h}_1^\beta$ have the asymptotics (8.3). Moreover, for $T \leqslant T_0$ small enough we have that the operator $\mathcal{G}$ is contractive in h and Λ. Therefore:

$$\left\|\mathcal{G}\left[\cdot; h^\alpha, \Lambda^\alpha\right] - \mathcal{G}\left[\cdot; h^\beta, \Lambda^\beta\right]\right\|_{C[0,T]} \leqslant \theta \left\|\Lambda^\alpha - \Lambda^\beta\right\|_{C[0,T]} + \theta \left\|h^\alpha - h^\beta\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$$

where $0 < \theta < 1$ can be made arbitrarily small for T_0 sufficiently small. Moreover, since $h^\alpha, h^\beta \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$, the functions Λ^α, Λ^β solve the integral equation (8.51). The function Λ depends in a Lipschitz manner on the function $\mathcal{G}$ with a constant smaller than two if $T \leqslant T_0$. Therefore $\|\Lambda^\alpha - \Lambda^\beta\|_{C[0,T]} \leqslant 2\theta \|\Lambda^\alpha - \Lambda^\beta\|_{C[0,T]} + 2\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$ whence $\|\Lambda^\alpha - \Lambda^\beta\|_{C[0,T]} \leqslant 4\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$. Using then the contractivity of the mapping $(h, \Lambda) \to \tilde{h}_1$ (cf. Proposition 26) we then obtain $\|\tilde{h}_1^\alpha - \tilde{h}_1^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \frac{1}{4} \|h^\alpha - h^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$. On the other hand, (8.48) yields $\|\tilde{h}_2^\alpha - \tilde{h}_2^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant C\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$. Therefore, choosing θ small enough:

$$\left\|h^\alpha - h^\beta\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \left\|\tilde{h}_1^\alpha - \tilde{h}_1^\beta\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} + \left\|\tilde{h}_2^\alpha - \tilde{h}_2^\beta\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \leqslant \frac{1}{2}\left\|h^\alpha - h^\beta\right\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$$

whence $h^\alpha = h^\beta$. Then $\Lambda^\alpha = \Lambda^\beta$ and the uniqueness follows. □

8.4. Computing the fluxes of particles towards $x = \infty$

Proof of Theorem 4. The system (1.1)–(1.2) can be rewritten as follows (cf. [23]):

$$\frac{\partial}{\partial t}(xf) = -\frac{\partial}{\partial x}\left(j(f)\right), \quad j(f)(t,x) = \int_0^x \int_{x-y}^\infty yK(y,z) f(t,y) f(t,z)\, dy\, dz.$$

Integrating in $[0, R]$, $R > 0$, we obtain:

$$\frac{d}{dt}\left(\int_0^R xf(t,x)\,dx\right) = -j(f)(t,R).$$

Using the asymptotics (2.16) we obtain:

$$\lim_{R\to\infty} j(f)(t,R) = 2\pi\big(\lambda(t)\big)^2$$

whence the result follows. $\square$

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