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Rolling manifolds on space forms ☆

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Abstract

In this paper, we consider the rolling problem (R) without spinning nor slipping of a smooth connected oriented complete Riemannian manifold (M,g) onto a space form $(\hat{M},\hat{g})$ of the same dimension $n\geqslant 2$. This amounts to study an n-dimensional distribution $\mathcal{D}_{\mathrm{R}}$, that we call the rolling distribution, and which is defined in terms of the Levi-Civita connections ∇^g and $\nabla^{\hat{g}}$. We then address the issue of the complete controllability of the control system associated to $\mathcal{D}_{\mathrm{R}}$. The key remark is that the state space Q carries the structure of a principal bundle compatible with $\mathcal{D}_{\mathrm{R}}$. It implies that the orbits obtained by rolling along loops of (M,g) become Lie subgroups of the structure group of $\pi_{Q,M}$. Moreover, these orbits can be realized as holonomy groups of either certain vector bundle connections ∇^{Rol}, called the rolling connections, when the curvature of the space form is non-zero, or of an affine connection (in the sense of Kobayashi and Nomizu, 1996 [14]) in the zero curvature case. As a consequence, we prove that the rolling (R) onto an Euclidean space is completely controllable if and only if the holonomy group of (M,g) is equal to $\mathrm{SO}(n)$. Moreover, when $(\hat{M},\hat{g})$ has positive (constant) curvature we prove that, if the action of the holonomy group of ∇^{Rol} is not transitive, then (M,g) admits $(\hat{M},\hat{g})$ as its universal covering. In addition, we show that, for n even and $n\geqslant 16$, the rolling problem (R) of (M,g) against the space form $(\hat{M},\hat{g})$ of positive curvature $c>0$, is completely controllable if and only if (M,g) is not of constant curvature c.

1. Introduction

In this paper, we study the rolling of a manifold over another one. Unless otherwise precised, manifolds are smooth, connected, oriented, of finite dimension $n\geqslant 2$, endowed with a Riemannian metric. The rolling is assumed to be without spinning (NS) or without spinning nor slipping (R). Here we only consider the rolling problem (R). When both manifolds are isometrically embedded into an Euclidean space, the rolling problem is classical in differential geometry (see [21]), through the notions of "development of a manifold" and "rolling maps". For instance, É. Cartan defines holonomy by rolling a manifold against its tangent space without spinning nor slipping (cf. [5,7]). The most basic issue linked to the rolling problem (R) is that of *controllability*, i.e., to determine, for two given points q_{init} and

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q_{final} in the state space Q, if there exists a curve γ so that the rolling (R) along γ steers the system from q_{init} to q_{final}. If this is the case for every points q_{init} and q_{final} in Q, then the rolling (R) is said to be *completely controllable*.

If the manifolds rolling on each other are two-dimensional, the controllability issue is well-understood thanks to the work of [2,6,8,16,1] especially. For instance, in the simply connected case, the rolling (R) is completely controllable if and only if the manifolds are not isometric. In the case where the manifolds are isometric, [2] also provides a description of the reachable sets in terms of isometries between the manifolds. In particular, these reachable sets are immersed submanifolds of Q of dimension either 2 or 5. In case the manifolds rolling on each other are isometric convex surfaces, [16] provides a beautiful description of a two-dimensional reachable set: consider the initial configuration given by two (isometric) surfaces in contact so that one is the image of the other one by the symmetry with respect to the (common) tangent plane at the contact point. Then, this symmetry property (chirality) is preserved along the rolling (R). If the (isometric) convex surfaces are not spheres nor planes, the reachable set starting at a contact point where the Gaussian curvatures are distinct, is open (and thus of dimension 5).

After [2], the state space (Q) of the rolling problem (R) is given by

$$Q = \left\{ A : T|_x M \to T|_{\hat{x}} \hat{M} \;\middle|\; A \text{ o-isometry},\ x \in M,\ \hat{x} \in \hat{M} \right\},$$

where "o-isometry" means positively oriented isometry (see [6,17,11] for an alternative description). The set of admissible controls is equal to the set of absolutely continuous (a.c.) curves on M. We next construct an n-dimensional distribution $\mathcal{D}_{\text{R}}$, that we call the rolling distribution, so that its tangent curves coincide with the admissible curves of $(\Sigma)_R$. A standard procedure in geometric control in order to address the controllability issue simply consists of studying the Lie algebra spanned by the vector fields tangent to $\mathcal{D}_{\text{R}}$. More precisely, one tries to compute the dimension of the evaluation at every point $q \in Q$ of this Lie algebra. However, this strategy turns out to be delicate for the rolling problem, even if one of the manifolds is assumed to be the Euclidean space. Indeed, in that particular case, this amounts to determine the dimension of the holonomy group associated to the Levi-Civita connection of a Riemannian manifold (M, g), only from the infinitesimal information provided by the evaluation at any point x of the curvature tensor associated to ∇^g and its covariant derivatives of arbitrary order (cf. [10] for more details).

However, when one of the manifolds, let say $(\hat{M}, \hat{g})$, is a space form, i.e., a simply connected complete Riemannian manifold of constant curvature, we prove, in Section 4, that there is a principal bundle structure on the bundle $\pi_{Q,M} : Q \to M$, which is compatible with the rolling distribution $\mathcal{D}_{\text{R}}$. From this fundamental feature, we show how to address the complete controllability of the rolling problem (R) without resorting to any Lie bracket computation. Indeed, if $\hat{M}$ has zero curvature, i.e., it is the Euclidean plane, we reduce the description of reachable sets to the study of an affine connection and its holonomy group, a subgroup of $\text{SE}(n)$, in the sense of [14]. Then, we deduce that the rolling (R) is completely controllable if and only if the (Riemannian) holonomy group of ∇^g is equal to $\text{SO}(n)$. This result is actually similar to Theorem IV.7.1, p. 193 and Theorem IV.7.2, p. 194 in [14].

In the case where $\hat{M}$ has non-zero constant curvature (up to a trivial reduction equal to 1 or -1), the description of reachable sets resumes to the study of a vector bundle connection ∇^{Rol} of the vector bundle $\pi_{TM \oplus \mathbb{R}} : TM \oplus \mathbb{R} \to M$ and its holonomy group H^{Rol}, which is a subgroup of $\text{SO}(n+1)$ or $\text{SO}_0(n,1)$ depending whether the curvature of $\hat{M}$ is equal to 1 or -1 respectively. Recall that $\text{SO}_0(n,1)$ is the identity component of $O(n,1)$. We then prove that the rolling problem (R) is completely controllable if and only if H^{Rol} is equal to $\text{SO}(n+1)$ or $\text{SO}_0(n,1)$ respectively.

The structure of H^{Rol} is further investigated for the rolling onto an n-dimensional unit sphere S^n. We prove that if the action of H^{Rol} onto S^n is not transitive, then (M, g) admits the unit sphere as Riemannian universal covering. This rigidity result can be seen as a de Rham type of result of global nature and we will provide in another paper [9] the details of the extension of de Rham decomposition theorem to the case of rolling on a space form of negative curvature.

Then by adapting to the classical argument of Simons [22] to our particular situation, we prove that for n even and $n \geqslant 16$, the rolling problem (R) of (M, g) against the space form $(\hat{M}, \hat{g})$ of positive curvature $c > 0$, is completely controllable if and only if (M, g) is not of constant curvature c. In that way, we recover some of the results of [13].

To conclude this introduction, we would like to propose some open problems. The first one deals with the rolling problem of two (locally) symmetric spaces. Indeed, the Lie algebraic structure of the rolling distribution does not involve the covariant derivatives of the curvature tensors on M and $\hat{M}$ (see [11]) and therefore its analysis turns out to be a purely algebraic question. Another question refers to the rolling onto a space of constant positive curvature, where the action of the rolling holonomy group is irreducible and transitive. One reasonably expects a list of possibilities similar to that of Berger. In addition, one may investigate the structure of the group of (local) symmetries associated to the rolling distribution, in particular when both manifolds M and $\hat{M}$ have constant curvature. Finally, what could be necessary conditions on M and $\hat{M}$ insuring that the rolling distribution is a principal bundle connection over $Q \to M$? Recall that we provide here a sufficient condition for that, namely that $\hat{M}$ has constant curvature.

2. Notations

For any sets A, B, C and $U \subset A \times B$ and any map $F : U \to C$, we write U_a and U^b for the sets defined by $\{b \in B \mid (a,b) \in U\}$ and $\{a \in A \mid (a,b) \in U\}$ respectively. Similarly, let $F_a : U_a \to C$ and $F^b : U^b \to C$ be defined by $F_a(b) := F(a,b)$ and $F^b(a) := F(a,b)$ respectively. For any sets $V_1, \dots, V_n$ the map $\mathrm{pr}_i : V_1 \times \cdots \times V_n \to V_i$ denotes the projection onto the i-th factor.

In this paper, a smooth manifold is a finite-dimensional, second countable, Hausdorff manifold (see e.g. [15]). For any smooth map $\pi : E \to M$ between smooth manifolds E and M, the set $\pi^{-1}(\{x\}) =: \pi^{-1}(x)$ is called the π-fiber over x and it is sometimes denoted by $E|_x$, when π is clear from the context. The set of smooth sections of π is denoted by $\Gamma(\pi)$. The value $s(x)$ of a section s at x is usually denoted by $s|_x$. For a smooth map $\pi : E \to M$ and $y \in E$, let $V|_y(\pi)$ be the set of all $Y \in T|_y E$ such that $\pi_*(Y) = 0$. If π is a smooth bundle, the collection of spaces $V|_y(\pi)$, $y \in E$, defines a smooth submanifold $V(\pi)$ of $T(E)$ and the restriction $\pi_{T(E)} : T(E) \to E$ to $V(\pi)$ is denoted by $\pi_{V(\pi)}$. In this case $\pi_{V(\pi)}$ is a vector subbundle of $\pi_{T(E)}$ over E.

One uses $\mathrm{VF}(M)$ to denote the set of smooth vector fields on M. The flow of a vector field $Y \in \mathrm{VF}(M)$ is a smooth onto map $\Phi_Y : D \to M$ defined on an open subset D of $\mathbb{R} \times M$ containing $\{0\} \times M$.

For any maps $\gamma : [a,b] \to M$, $\omega : [c,d] \to M$ into M such that $\gamma(b) = \omega(c)$ we define

$$\omega \sqcup \gamma : [a, b+d-c] \to M; \quad (\omega \sqcup \gamma)(t) = \begin{cases} \gamma(t), & t \in [a,b], \\ \omega(t-b+c), & t \in [b, b+d-c]. \end{cases}$$

Also we write $\gamma^{-1} : [a,b] \to M$; $\gamma^{-1}(t) := \gamma(b+a-t)$. In the space of loops $[0,1] \to M$ based at some given point x_0, one defines an operation "." of concatenation by $\omega.\gamma := (t \mapsto \omega(\frac{t}{2})) \sqcup (t \mapsto \gamma(\frac{t}{2}))$. For $y \in M$, we use $\Omega_y(M)$ to denote the set of all piecewise C^1-loops $[0,1] \to M$ of M based at y.

A continuous map $\gamma : I \to M$ from a real compact interval I into a smooth manifold M is called *absolutely continuous*, or *a.c.* for short if, for every $t_0 \in I$, there is a smooth coordinate chart (ϕ, U) of M such that $\gamma(t_0) \in U$ and $\phi \circ \gamma|_{\gamma^{-1}(U)}$ is absolutely continuous.

Given a smooth distribution $\mathcal{D}$ on M, we call an absolutely continuous curve $\gamma : I \to M$, $I \subset \mathbb{R}$, *$\mathcal{D}$-admissible* if γ is tangent to $\mathcal{D}$ almost everywhere (a.e.), i.e., if for almost all $t \in I$ it holds that $\dot{\gamma}(t) \in \mathcal{D}|_{\gamma(t)}$. For $x_0 \in M$, the endpoints of all the $\mathcal{D}$-admissible curves of M starting at x_0 form the set called *$\mathcal{D}$-orbit through* x_0 and denoted $\mathcal{O}_{\mathcal{D}}(x_0)$. More precisely,

$$\mathcal{O}_{\mathcal{D}}(x_0) = \{\gamma(1) \mid \gamma : [0,1] \to M,\ \mathcal{D}\text{-admissible},\ \gamma(0) = x_0\}. \tag{1}$$

By the Orbit Theorem (see [3]), it follows that $\mathcal{O}_{\mathcal{D}}(x_0)$ is an immersed smooth submanifold of M containing x_0. It is also known that one may restrict to piecewise smooth curves in the description of the orbit, i.e., the curves γ in (1) can be taken piecewise smooth.

Let $\pi : E \to M$ be a vector bundle and $\nabla : \mathrm{VF}(M) \times \Gamma(\pi) \to \Gamma(\pi)$ a linear connection on π. As is standard, we write for $X \in \mathrm{VF}(M)$, $s \in \Gamma(\pi)$ the value of ∇ as $\nabla_X s \in \Gamma(\pi)$. A parallel transport of $s_0 \in E|_{x_0}$ along an a.c. path $\gamma : [a,b] \to M$ from $\gamma(a) = x_0$ to $\gamma(b)$ is written as $(P^\nabla)^b_a(\gamma) s_0$. The parallel transport map

$$(P^\nabla)^b_a(\gamma) : E|_{\gamma(a)} \to E|_{\gamma(b)} \tag{2}$$

is a linear isomorphism and one also writes $(P^\nabla)^a_b(\gamma) := (P^\nabla)^b_a(\gamma^{-1}) = (P^\nabla)^b_a(\gamma)^{-1}$. The holonomy group of ∇ at x_0 is defined to be the subgroup $H^\nabla|_{x_0}$ of $\mathrm{GL}(E|_{x_0})$ given by

$$H^\nabla|_{x_0} = \{(P^\nabla)^1_0(\gamma) \mid \gamma \in \Omega_{x_0}(M)\}.$$

One writes R^∇ for the curvature tensor of ∇ and if the connection ∇ is clear from the context, one simply writes $P = P^\nabla$ and $R = R^\nabla$ for the parallel transport operator and the curvature operator, respectively. Finally, the Levi-Civita connection of a Riemannian manifold (N,h) is written as ∇^h or simply ∇ when h is clear from the context.

We use $\mathrm{Iso}(N,h)$ to denote the group of isometries of a Riemannian manifold (N,h). The isometries respect parallel transport in the sense that for any absolutely continuous $\gamma:[a,b]\to N$ and $F\in\mathrm{Iso}(N,h)$ one has (cf. [20, p. 41, Eq. (3.5)])

$$F_*|_{\gamma(t)}\circ\big(P^{\nabla^h}\big)^t_a(\gamma)=\big(P^{\nabla^h}\big)^t_a(F\circ\gamma)\circ F_*|_{\gamma(a)}. \tag{3}$$

The following result is standard.

Proposition 2.1. *(Cf. [14, Chapter IV, Theorem 4.1].) Let (N,h) be a Riemannian manifold and for any absolutely continuous $\gamma:[0,1]\to M$, $\gamma(0)=y_0$, define*

$$\Lambda^{\nabla^h}_{y_0}(\gamma)(t)=\int_0^t\big(P^{\nabla^h}\big)^0_s(\gamma)\dot\gamma(s)\,\mathrm{d}s\in T|_{y_0}N,\qquad t\in[0,1].$$

Then the map $\Lambda^{\nabla^h}_{y_0}:\gamma\mapsto\Lambda^{\nabla^h}_{y_0}(\gamma)(\cdot)$ is an injection from the set of absolutely continuous curves $[0,1]\to N$ starting at y_0 onto an open subset of the Banach space of absolutely continuous curves $[0,1]\to T|_{y_0}N$ starting at 0. Moreover, the map $\Lambda^{\nabla^h}_{y_0}$ is a bijection onto the latter Banach space if (and only if) (N,h) is a complete Riemannian manifold.

3. State space and distributions

3.1. State space

3.1.1. Definition of the state space

After [2,3] we make the following definition.

Definition 3.1. The *state space* $Q=Q(M,\hat M)$ for the rolling of two n-dimensional *connected, oriented* smooth Riemannian manifolds (M,g), $(\hat M,\hat g)$ is defined as

$$Q=\{A:T|_xM\to T|_{\hat x}\hat M\mid A\text{ o-isometry},\ x\in M,\ \hat x\in\hat M\},$$

with "o-isometry" means "orientation preserving isometry": if $(X_i)_{i=1}^n$ is a pos. oriented g-orthonormal frame of M at x then $(AX_i)_{i=1}^n$ is a pos. oriented $\hat g$-orthonormal frame of $\hat M$ at $\hat x$.

The linear space of $\mathbb{R}$-linear map $A:T|_xM\to T|_{\hat x}\hat M$ is canonically isomorphic to the tensor product $T^*|_xM\otimes T|_{\hat x}\hat M$. We write

$$T^*M\otimes T\hat M=\bigcup_{(x,\hat x)\in M\times\hat M}T^*|_xM\otimes T|_{\hat x}\hat M$$

and if a point $A\in T^*M\otimes T\hat M$ belongs to $T^*|_xM\otimes T|_{\hat x}\hat M$, we usually write it as $q=(x,\hat x;A)$. With projection $\pi_{T^*M\otimes T\hat M}:T^*M\otimes T\hat M\to M\times\hat M$; $(x,\hat x;A)\mapsto(x,\hat x)$, the space $T^*M\otimes T\hat M$ becomes a vector bundle over $M\times\hat M$ of rank n^2 and $\pi_Q:=\pi_{T^*M\otimes T\hat M}|_Q:Q\to M\times\hat M$ is a smooth subbundle of rank $n(n-1)/2$ with fibers diffeomorphic to $\mathrm{SO}(n)$.

Remark 3.2. Let $q=(x,\hat x;A)\in Q$ and $B\in(T^*M\otimes T\hat M)|_{(x,\hat x)}$. Then $\nu(B)|_q\in V|_q(\pi_{T^*M\otimes T\hat M})$ is tangent to Q (i.e., is an element of $V|_q(\pi_Q)$) if and only if $\hat g(AX,BY)+\hat g(BX,AY)=0$ for all $X,Y\in T|_xM$. This latter condition can be stated equivalently as $B\in A(\mathfrak{so}(T|_xM))$, i.e. $V|_{(x,\hat x;A)}(\pi_Q)$ is naturally $\mathbb{R}$-linearly isomorphic to $A(\mathfrak{so}(T|_xM))$.

3.2. Distribution and the control problems

3.2.1. The rolling distribution $\mathcal{D}_{\mathrm{R}}$

In this section, using the subsequent lift operation, we build a smooth distribution $\mathcal{D}_{\mathrm{R}}$ on the spaces Q and $T^*M\otimes T\hat M$ whose tangent curves are the solutions of (8). For the next definition, we use the fact that if $A\in Q$, then $P^t_0(\hat\gamma)\circ A\circ P^0_t(\gamma)\in Q$ for all t where $\gamma,\hat\gamma$ are any smooth curves in M, $\hat M$ respectively.

Definition 3.3. For $q = (x, \hat{x}; A) \in Q$ and $X \in T|_x M$ we define a vector $\mathscr{L}_R(X)|_q \in T|_q Q$ as

$$\mathscr{L}_R(X)|_q = \frac{d}{dt}\bigg|_0 \big(P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma)\big) \tag{4}$$

where $\gamma, \hat{\gamma}$ are any smooth curves in $M, \hat{M}$ respectively such that $\dot{\gamma}(0) = X$ and $\dot{\hat{\gamma}}(0) = AX$.

Remark 3.4. The definition of $\mathscr{L}_R(X)$ as given above is independent of the choice of $\gamma, \hat{\gamma}$ such that they satisfy $\dot{\gamma}(0) = X$, $\dot{\hat{\gamma}}(0) = AX$.

This map naturally induces $\mathscr{L}_R : \mathrm{VF}(M) \to \mathrm{VF}(Q)$ as follows. For $X \in \mathrm{VF}(M)$ we define $\mathscr{L}_R(X)$, the *rolling lifted* vector field associated to X, by

$$\mathscr{L}_R(X) : Q \to TQ,$$
$$q \mapsto \mathscr{L}_R(X)|_q.$$

The rolling lift map $\mathscr{L}_R$ allows one to construct a distribution on Q of rank n as follows.

Definition 3.5. The *rolling distribution* $\mathcal{D}_R$ on Q is the n-dimensional smooth distribution defined by

$$\forall q = (x, \hat{x}; A) \in Q, \quad \mathcal{D}_R|_q = \mathscr{L}_R(T|_x M)|_q. \tag{5}$$

One defines $\pi_{Q,M} := \mathrm{pr}_1 \circ \pi_Q : Q \to M$.

Remark 3.6. The map $\pi_{Q,M} : Q \to M$ is a bundle: if $F = (X_i)_{i=1}^n$ is a local oriented orthonormal frame of M defined on an open set U, the local trivialization of $\pi_{Q,M}$ induced by F as

$$\tau_F : \pi_{Q,M}^{-1}(U) \to U \times F_{\mathrm{OON}}(\hat{M}); \qquad \tau_F(x, \hat{x}; A) = \big(x, (AX_i|_x)_{i=1}^n\big),$$

is a diffeomorphism, where $F_{\mathrm{OON}}(\hat{M})$ is the bundle of all oriented orthonormal frames on $\hat{M}$.

The differential $(\pi_{Q,M})_*$ maps each $\mathcal{D}_R|_q$, $q = (x, \hat{x}; A) \in Q$, isomorphically onto $T|_x M$, implying the local existence of rolling curves described in the following proposition (cf. [10]).

Proposition 3.7.

(i) *For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and a.c. $\gamma : [0, a] \to M$, $a > 0$, such that $\gamma(0) = x_0$, there exists a unique a.c. $q : [0, a'] \to Q$, $q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$, with $0 < a' \leqslant a$ (and a' maximal with the latter property), which is tangent to $\mathcal{D}_R$ a.e. and $q(0) = q_0$. We denote this unique curve q by*

$$t \mapsto q_{\mathcal{D}_R}(\gamma, q_0)(t) = \big(\gamma(t), \hat{\gamma}_{\mathcal{D}_R}(\gamma, q_0)(t); A_{\mathcal{D}_R}(\gamma, q_0)(t)\big),$$

and refer to it as the rolling curve *with initial conditions (γ, q_0) or* along γ with initial position q_0. *In the case that $\hat{M}$ is a complete manifold one has $a' = a$.*
Conversely, any absolutely continuous curve $q : [0, a] \to Q$, which is a.e. tangent to $\mathcal{D}_R$, is a rolling curve along $\gamma := \pi_{Q,M} \circ q$, i.e., has the form $q_{\mathcal{D}_R}(\gamma, q(0))$.

(ii) *Writing $\Lambda_{x_0} = \Lambda_{x_0}^{\nabla}$ and $\hat{\Lambda}_{\hat{x}_0} = \hat{\Lambda}_{\hat{x}_0}^{\hat{\nabla}}$ (see Proposition 2.1), then, for any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and a.c. curve γ starting from x_0, the corresponding rolling curve is given by*

$$q_{\mathcal{D}_R}(\gamma, q_0)(t) = \big(\gamma(t), \hat{\Lambda}_{\hat{x}_0}^{-1}\big(A_0 \circ \Lambda_{x_0}(\gamma)\big)(t); P_0^t\big(\hat{\Lambda}_{\hat{x}_0}^{-1}\big(A_0 \circ \Lambda_{x_0}(\gamma)\big)\big) \circ A_0 \circ P_t^0(\gamma)\big). \tag{6}$$

(iii) *Let $q_0 = (x_0, \hat{x}_0; A_0) \in Q$, $X \in T|_{x_0} M$ and $\gamma : [0, a] \to M$; $\gamma(t) = \exp_{x_0}(tX)$, a geodesic of (M, g) with $\gamma(0) = x_0$, $\dot{\gamma}(0) = X$. The rolling curve $q_{\mathcal{D}_R}(\gamma, q_0) = (\gamma, \hat{\gamma}_{\mathcal{D}_R}(\gamma, q_0); A_{\mathcal{D}_R}(\gamma, q_0)) : [0, a'] \to Q$, $0 < a' \leqslant a$, along γ with initial position q_0 is given by*

$$\hat{\gamma}_{\mathcal{D}_R}(\gamma, q_0)(t) = \widehat{\exp}_{\hat{x}_0}(tA_0X), \qquad A_{\mathcal{D}_R}(\gamma, q_0)(t) = P_0^t\big(\hat{\gamma}_{\mathcal{D}_R}(\gamma, q_0)\big) \circ A_0 \circ P_t^0(\gamma),$$

where $\widehat{\exp}$ is the exponential mapping of $(\hat{M}, \hat{g})$. Moreover, $a' = a$ if $\hat{M}$ is complete.

(iv) *If $\gamma:[a,b]\to M$ and $\omega:[c,d]\to M$ are two a.c. curves with $\gamma(b)=\omega(c)$ and $q_0\in Q$, then*

$$q_{\mathcal{D}_{\rm R}}(\omega\sqcup\gamma,q_0)=q_{\mathcal{D}_{\rm R}}\big(\omega,q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(b)\big)\sqcup q_{\mathcal{D}_{\rm R}}(\gamma,q_0). \tag{7}$$

On the group $\Omega_{x_0}(M)$ of piecewise differentiable loops of M based at x_0 one has

$$q_{\mathcal{D}_{\rm R}}(\omega.\gamma,q_0)=q_{\mathcal{D}_{\rm R}}\big(\omega,q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(1)\big).q_{\mathcal{D}_{\rm R}}(\gamma,q_0),$$

where $\gamma,\omega\in\Omega_{x_0}(M)$.

Remark 3.8. The curve $t\mapsto q(t)=(\gamma(t),\hat\gamma(t);A(t))\in Q$, $t\in[a,b]$, is a rolling curve if and only if it is an admissible curve of the following driftless control affine system

$$(\Sigma)_R\qquad\begin{cases}\dot\gamma(t)=u(t),\\ \dot{\hat\gamma}(t)=A(t)u(t),\\ \overline{\nabla}_{(u(t),A(t)u(t))}A(t)=0,\end{cases}\qquad\text{for a.e. } t\in[a,b], \tag{8}$$

where $\overline{\nabla}$ is the vector bundle connection on $\pi_{T^*M\otimes T\hat M}$ canonically induced by $\nabla,\hat\nabla$ and the control u belongs to $\mathcal{U}(M)$, the set of measurable TM-valued functions u defined on some interval $I=[a,b]$ such that there exists a.c. $y:[a,b]\to M$ verifying $u=\dot y$ a.e. on $[a,b]$. We can write the above system as

$$\begin{cases}\dot{\hat\gamma}(t)=A(t)\dot\gamma(t),\\ \overline{\nabla}_{(\dot\gamma(t),\dot{\hat\gamma}(t))}A(t)=0\end{cases}$$

where γ is a.c. In the model of rolling of a Riemannian manifold (M,g) against another one $(\hat M,\hat g)$, the first (resp. second) equation models the so-called *no-slipping condition* (resp. *no-spinning condition*). A complete argument for the above remark is provided in [10].

3.3. Global properties of a $\mathcal{D}_{\rm R}$-orbit

The next proposition describes on one hand the symmetry of the rolling problem with respect to (M,g) and $(\hat M,\hat g)$ and on the other hand that each $\mathcal{D}_{\rm R}$-orbit is a smooth bundle over M. Proofs are omitted (cf. [10]).

Proposition 3.9.

(i) *Let $\widehat{\mathcal{D}_{\rm R}}$ be the rolling distribution in $\hat Q:=Q(\hat M,M)$. Then the map $\iota:Q\to\hat Q$; $\iota(x,\hat x;A)=(\hat x,x;A^{-1})$ is a diffeomorphism of Q onto $\hat Q$ and $\iota_*\mathcal{D}_{\rm R}=\widehat{\mathcal{D}_{\rm R}}$. In particular, $\iota(\mathcal{O}_{\mathcal{D}_{\rm R}}(q))=\mathcal{O}_{\widehat{\mathcal{D}_{\rm R}}}(\iota(q))$.*

(ii) *Let $q_0=(x_0,\hat x_0;A_0)\in Q$ and suppose that $\hat M$ is complete. Then $\pi_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0),M}:=\pi_{Q,M}|_{\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)}:\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\to M$ is a smooth subbundle of $\pi_{Q,M}$.*

Proposition 3.10. *For any Riemannian isometries $F\in\mathrm{Iso}(M,g)$ and $\hat F\in\mathrm{Iso}(\hat M,\hat g)$ of (M,g), $(\hat M,\hat g)$ respectively, one defines smooth free right and left actions of $\mathrm{Iso}(M,g)$, $\mathrm{Iso}(\hat M,\hat g)$ on Q by*

$$q_0\cdot F:=\big(F^{-1}(x_0),\hat x_0;A_0\circ F_*|_{F^{-1}(x_0)}\big),\qquad \hat F\cdot q_0:=\big(x_0,\hat F(\hat x_0);\hat F_*|_{\hat x_0}\circ A_0\big),$$

where $q_0=(x_0,\hat x_0;A_0)\in Q$. Set $\hat F\cdot q_0\cdot F:=(\hat F\cdot q_0)\cdot F=\hat F\cdot(q_0\cdot F)$. For any $q_0=(x_0,\hat x_0;A_0)\in Q$, a.c. $\gamma:[0,1]\to M$, $\gamma(0)=x_0$, and $F\in\mathrm{Iso}(M,g)$, $\hat F\in\mathrm{Iso}(\hat M,\hat g)$, one has

$$\hat F\cdot q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(t)\cdot F=q_{\mathcal{D}_{\rm R}}\big(F^{-1}\circ\gamma,\hat F\cdot q_0\cdot F\big)(t), \tag{9}$$

for all $t\in[0,1]$ where $q_{\mathcal{D}_{\rm R}}(\gamma,q_0)(t)$ is defined. In particular, $\hat F\cdot\mathcal{O}_{\mathcal{D}_{\rm R}}(q_0)\cdot F=\mathcal{O}_{\mathcal{D}_{\rm R}}(\hat F\cdot q_0\cdot F)$.

Proof. The fact that the group actions are well defined is clear and the smoothness of these actions can be proven by writing out the Lie group structures of the isometry groups (using e.g. Lemma III.6.4 in [20]). If $q_0\cdot F=q_0\cdot F'$ for some $F,F'\in\mathrm{Iso}(M,g)$ and $q_0\in Q$, then $F^{-1}(x_0)=F'^{-1}(x_0)$, $F_*|_{x_0}=F'_*|_{x_0}$ and hence $F=F'$ since M is connected (see [20, p. 43]). This proves the freeness of the right $\mathrm{Iso}(M,g)$-action. The same argument proves the freeness of the left $\mathrm{Iso}(\hat{M},\hat{g})$-action.

Finally, Eq. (9) follows from a simple application of Eq. (3). Indeed, we first recall the rolling curve $q_{\mathcal{D}_R}(\gamma,q_0)=(\gamma,\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0);A_{\mathcal{D}_R}(\gamma,q_0))$ satisfies

$$P_t^0\big(\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\dot{\hat{\gamma}}_{\mathcal{D}_R}(\gamma,q_0)(t)=A_0P_t^0(\gamma)\dot{\gamma}(t),$$
$$A_{\mathcal{D}_R}(\gamma,q_0)(t)=P_0^t\big(\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\circ A_0\circ P_t^0(\gamma).$$

First, by using (3), we get

$$\begin{aligned}
&P_t^0\big(\hat{F}\circ\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{F}\circ\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)(t)\\
&\quad=\hat{F}_*P_t^0\big(\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\hat{F}_*^{-1}\big(\hat{F}_*\dot{\hat{\gamma}}_{\mathcal{D}_R}(\gamma,q_0)(t)\big)\\
&\quad=\hat{F}_*A_0P_t^0(\gamma)\dot{\gamma}(t)=(\hat{F}_*A_0F_*)\big(F_*^{-1}P_t^0(\gamma)F_*\big)F_*^{-1}\dot{\gamma}(t)\\
&\quad=(\hat{F}_*A_0F_*)P_t^0\big(F^{-1}\circ\gamma\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(F^{-1}\circ\gamma\big)(t),
\end{aligned}$$

and since by definition one has

$$P_t^0\big(\hat{\gamma}_{\mathcal{D}_R}\big(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F\big)\big)\dot{\hat{\gamma}}_{\mathcal{D}_R}\big(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F\big)=(\hat{F}_*A_0F_*)P_t^0\big(F^{-1}\circ\gamma\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(F^{-1}\circ\gamma\big)(t),$$

the uniqueness of solutions of a system of ODEs gives that $\hat{F}\circ\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)=\hat{\gamma}_{\mathcal{D}_R}(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F)$. Hence (9) is a consequence of the following

$$\begin{aligned}
\hat{F}_*A_{\mathcal{D}_R}(\gamma,q_0)F_*&=\hat{F}_*\big(P_0^t\big(\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\circ A_0\circ P_t^0(\gamma)\big)F_*\\
&=P_0^t\big(\hat{F}\circ\hat{\gamma}_{\mathcal{D}_R}(\gamma,q_0)\big)\circ(\hat{F}_*A_0F_*)\circ P_t^0\big(F^{-1}\circ\gamma\big)\\
&=P_0^t\big(\hat{\gamma}_{\mathcal{D}_R}\big(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F\big)\big)\circ(\hat{F}_*A_0F_*)\circ P_t^0\big(F^{-1}\circ\gamma\big)\\
&=A_{\mathcal{D}_R}\big(F^{-1}\circ\gamma,\hat{F}\cdot q_0\cdot F\big).\qquad\square
\end{aligned}$$

The following proposition and its corollary are given without their proofs.

Proposition 3.11. *Let $\pi_1:(M_1,g_1)\to(M,g)$ and $\hat{\pi}:(\hat{M}_1,\hat{g}_1)\to(\hat{M},\hat{g})$ be Riemannian coverings. Write $Q_1=Q(M_1,\hat{M}_1)$ and $(\mathcal{D}_R)_1$ for the rolling distribution in Q_1. Then the map $\Pi:Q_1\to Q$; $\Pi(x_1,\hat{x}_1;A_1)=(\pi(x_1),\hat{\pi}(\hat{x}_1);\hat{\pi}_*|_{\hat{x}_1}\circ A_1\circ(\pi_*|_{x_1})^{-1})$ is a covering map of Q_1 over Q and $\Pi_*(\mathcal{D}_R)_1=\mathcal{D}_R$. Moreover, for every $q_1\in Q_1$ the restriction onto $\mathcal{O}_{(\mathcal{D}_R)_1}(q_1)$ of Π is a covering map $\mathcal{O}_{(\mathcal{D}_R)_1}(q_1)\to\mathcal{O}_{\mathcal{D}_R}(\Pi(q_1))$. Then, for every $q_1\in Q_1$, $\Pi(\mathcal{O}_{(\mathcal{D}_R)_1}(q_1))=\mathcal{O}_{\mathcal{D}_R}(\Pi(q_1))$ and one has $\mathcal{O}_{(\mathcal{D}_R)_1}(q_1)=Q_1$ if and only if $\mathcal{O}_{\mathcal{D}_R}(\Pi(q_1))=Q$.*

As an immediate corollary of the above proposition, we obtain the following result regarding the complete controllability of $(\mathcal{D}_R)$.

Corollary 3.12. *Let $\pi_1:(M_1,g_1)\to(M,g)$ and $\hat{\pi}:(\hat{M}_1,\hat{g}_1)\to(\hat{M},\hat{g})$ be Riemannian coverings. Write $Q=Q(M,\hat{M})$, $\mathcal{D}_R$ and $Q_1=Q(M_1,\hat{M}_1)$, $(\mathcal{D}_R)_1$ respectively for the state space and for the rolling distribution in the respective state space. Then the control system associated to $\mathcal{D}_R$ is completely controllable if and only if the control system associated to $(\mathcal{D}_R)_1$ is completely controllable. As a consequence, when one addresses the complete controllability issue for the rolling distribution $\mathcal{D}_R$, one can assume with no loss of generality that both manifolds M and $\hat{M}$ are simply connected.*

4. Rolling against a space form

For the rest of the paper we assume that $(\hat{M},\hat{g})$ is a space form, i.e., a simply connected complete Riemannian manifold of constant curvature. The possible cases are: (i) Euclidean space with Euclidean metric (zero curvature), (ii) sphere (positive curvature), and (iii) hyperbolic space (negative curvature), cf. e.g. [20].

We first reduce the original control problem to the following one: fix (any) $x_0 \in M$ and consider rolling of M along loops $\gamma \in \Omega_{x_0}(M)$; one obtains a control problem whose state space is the fiber $\pi_{Q,M}^{-1}(x_0)$ and the reachable sets are $\pi_{Q,M}^{-1} \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$, where $q_0 \in \pi_{Q,M}^{-1}(x_0)$. It is then trivial to see that complete controllability of the original problem is equivalent to the complete controllability of the reduced rolling problem. Note that this fact holds true for the general rolling problem of one Riemannian manifold against another one.

On the other hand, the rolling problem against a space form of constant curvature $c \in \mathbb{R}$ actually presents a fundamental feature which turns out to be the crucial ingredient to address the controllability issue. We next prove that on the bundle $\pi_{Q,M} : Q \to M$ one can define a principal bundle structure that preserves the rolling distribution $\mathcal{D}_{\mathrm{R}}$. As a consequence, the reachable sets $\pi_{Q,M}^{-1}(x_0) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0)$ become Lie subgroups of the structure group of $\pi_{Q,M}$. We will prove that these orbits in fact can be realized as holonomy groups of certain vector bundle connections if $c \neq 0$ and as a holonomy group of an affine connection (in the sense of [14]). Therefore the original problem of complete controllability reduces to the study of appropriate connections.

4.1. Orbit structure

We first recall standard results on space forms. Following Section V.3 of [14], we define the n-dimensional space form $\hat{M}_{n;c}$ of curvature $c \neq 0$ as a subset of $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, given by

$$\hat{M}_{n;c} := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_1^2 + \cdots + x_n^2 + c^{-1} x_{n+1}^2 = c^{-1},\ x_{n+1} + \frac{c}{|c|} \geqslant 0 \right\}.$$

Equip $\hat{M}_{n;c}$ with a Riemannian metric $\hat{g}_{n;c}$ defined as the restriction to $\hat{M}_{n;c}$ of the non-degenerate symmetric $(0,2)$-tensor $s_{n;c} := (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2 + c^{-1}(\mathrm{d}x_{n+1})^2$. The condition $x_{n+1} + \frac{c}{|c|} \geqslant 0$ in the definition $\hat{M}_{n;c}$ guarantees that $\hat{M}_{n;c}$ is connected also when $c < 0$.

Let $G_c(n)$ be the identity component of the Lie group of linear maps $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ leaving invariant the bilinear form

$$\langle x, y \rangle_{n;c} := \sum_{i=1}^{n} x_i y_i + c^{-1} x_{n+1} y_{n+1},$$

for $x = (x_1, \dots, x_{n+1})$, $y = (y_1, \dots, y_{n+1})$ and having determinant $+1$. In other words, a linear map $B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ belongs to $G_c(n)$ if and only if $\det(B) = +1$ and $\langle Bx, By \rangle_{n;c} = \langle x, y \rangle_{n;c}$, $\forall x, y \in \mathbb{R}^{n+1}$, or, equivalently, $B^T I_{n;c} B = I_{n;c}$, $\det(B) = +1$, where $I_{n;c} = \mathrm{diag}(1, 1, \dots, 1, c^{-1})$. In particular, $G_1(n) = \mathrm{SO}(n+1)$ and $G_{-1}(n) = \mathrm{SO}_0(n,1)$. The Lie algebra of the Lie group $G_c(n)$ will be denoted by $\mathfrak{g}_c(n)$. Notice that an $(n+1) \times (n+1)$ real matrix B belongs to $\mathfrak{g}_c(n)$ if and only if $B^T I_{n;c} + I_{n;c} B = 0$, where $I_{n;c}$ was introduced above.

Sometimes we identify the form $s_{n;c}$ on $\mathbb{R}^{n+1}$ with $\langle \cdot, \cdot \rangle_{n;c}$ using the canonical identification of the tangent spaces $T|_v \mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$. Notice that if $\hat{x} \in \hat{M}_{n;c}$ and $V \in T|_{\hat{x}} \mathbb{R}^{n+1}$, then $V \in T|_{\hat{x}} \hat{M}_{n;c}$ if and only if $s_{n;c}(V, \hat{x}) = 0$.

If $c = 0$, the space form $(\hat{M}_{n;0}, \hat{g}_{n;0})$ is simply equal to $\mathbb{R}^n$ with the Euclidean metric, $G_n(0)$ is set to be the group $\mathrm{SE}(n) := \mathrm{SE}(\mathbb{R}^n)$, the special Euclidean group of $(\hat{M}_{n;0}, \hat{g}_{n;0})$. Recall that $\mathrm{SE}(n)$ is equal to $\mathbb{R}^n \times \mathrm{SO}(n)$ as a set, and is equipped with the group operation $\star$ given by

$$(v, L) \star (u, K) := (Lu + v, L \circ K).$$

The natural action, also written as $\star$, of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ is given by

$$(u, K) \star v := Kv + u, \quad (u, K) \in \mathrm{SO}(V),\ v \in V.$$

Finally recall that, with this notation, the isometry group of $(\hat{M}_{n;c}, \hat{g}_{n;c})$ is equal to $G_c(n)$ for all $c \in \mathbb{R}$ (cf. [14]) as explicitly recalled in the next proof. From now on we set $(\hat{M}, \hat{g}) = (\hat{M}_{n;c}, \hat{g}_{n;c})$ for $c \in \mathbb{R}$. In the next proposition we detail the principal bundle structure of $\pi_{Q,M}$.

Proposition 4.1.

(i) *The bundle $\pi_{Q,M} : Q \to M$ is a principal $G_c(n)$-bundle with a left action $\mu : G_c(n) \times Q \to Q$ defined for every $q = (x, \hat{x}; A)$ by*

$$\mu\big((\hat{y}, C), q\big) = (x, C\hat{x} + \hat{y}; C \circ A), \quad \textit{if } c = 0,$$
$$\mu(B, q) = (x, B\hat{x}; B \circ A), \quad \textit{if } c \neq 0.$$

Moreover, the action μ preserves the distribution $\mathcal{D}_R$, i.e., for any $q \in Q$ and $B \in G_c(n)$, $(\mu_B)_*\mathcal{D}_R|_q = \mathcal{D}_R|_{\mu(B,q)}$ where $\mu_B : Q \to Q; q \mapsto \mu(B, q)$.

(ii) *For any given $q = (x, \hat{x}; A) \in Q$ there is a unique subgroup $\mathcal{H}_q$ of $G_c(n)$, called the holonomy group of $\mathcal{D}_R$, such that*

$$\mu\big(\mathcal{H}_q \times \{q\}\big) = \mathcal{O}_{\mathcal{D}_R}(q) \cap \pi^{-1}_{Q,M}(x).$$

Also, if $q' = (x, \hat{x}'; A') \in Q$ is in the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}_q$ and $\mathcal{H}_{q'}$ are conjugate in $G_c(n)$ and all conjugacy classes of $\mathcal{H}_q$ in $G_c(n)$ are of the form $\mathcal{H}_{q'}$. This conjugacy class will be denoted by $\mathcal{H}$. Moreover, $\pi_{\mathcal{O}_{\mathcal{D}_R}(q),M} : \mathcal{O}_{\mathcal{D}_R}(q) \to M$ is a principal $\mathcal{H}$-bundle over M.

Proof. (i) We begin by showing that if $B \in G_c(n)$, then $\mu(B, q) \in Q$. Let $X \in T|_x$. If $c = 0$, then $B = (\hat{y}, C) \in \mathrm{SE}(n) = \mathbb{R}^n \times \mathrm{SO}(n)$ and

$$\big\|\mu(B, q)X\big\|_{\hat{g}_{n;0}} = \|CAX\|_{\hat{g}_{n;0}} = \|AX\|_{\hat{g}_{n;0}} = \|X\|_g$$

while if $c \neq 0$,

$$\big\|\mu(B, q)X\big\|_{\hat{g}_{n;c}} = \|BAX\|_{\hat{g}_{n;c}} = \|AX\|_{\hat{g}_{n;c}} = \|X\|_g.$$

Since $G_c(n)$ is connected for every $c \in \mathbb{R}$, it follows that $\mu(B, q) = (x, \hat{z}; A')$ viewed as a map $T|_x M \to T_{\hat{z}}\hat{M}_{n;c}$ is also orientation preserving and therefore indeed $\mu(B, q) \in Q$.

Clearly μ is smooth, satisfies the group action property and the action is free. We show that μ-action is transitive and proper, implying that $\pi_{Q,M}$ endowed with $G_c(n)$ action μ becomes a principal bundle.

Let $q = (x, \hat{x}; A), q' = (x, \hat{x}', A') \in \pi^{-1}_{Q,M}(x)$ and suppose $(X_i)_{i=1}^n$ is some orthonormal frame of M at x. Since $G_c(n)$, identified as an open subgroup of $\mathrm{Iso}(M_{n;c}, \hat{g}_{n;c})$, acts transitively on the space of orthonormal frames of $\hat{M}_{n;c}$, there is an $\hat{F} \in G_c(n)$ such that $\hat{F}_*(AX_i) = A'X_i$ for all $i = 1, \ldots, n$. This implies that $\hat{F}(\hat{x}) = \hat{x}'$ and $\hat{F}_*A = A'$.

If $c = 0$ we set $B_{\hat{F}} = (\hat{x}' - \hat{F}_*|_{\hat{x}}(\hat{x}), \hat{F}_*|_{\hat{x}})$, where $\hat{F}_*|_{\hat{x}}$ is thought as a map $\mathbb{R}^n \to \mathbb{R}^n$ through canonical identifications of $T|_{\hat{x}}\hat{M}_{n;c}$ and $T|_{\hat{x}'}\hat{M}_{n;c}$ with $\mathbb{R}^n$. If $c \neq 0$ the element $B_{\hat{F}}$ of $G_c(n)$ is uniquely determined by setting it to be equal to $\hat{F}_*|_{\hat{x}}$ on $T|_{\hat{x}}\hat{M}_{n;c}$ and imposing that $B_{\hat{F}}(\hat{x}) = \hat{x}'$. Therefore, we get $\mu(B_{\hat{F}}, q) = q'$ which therefore shows the transitivity.

We first prove that if $\hat{F} \in G_c(n)$ and $B_{\hat{F}} \in G_c(n)$ as defined above, then $\mu(B_{\hat{F}}, q) = \hat{F} \cdot q$ where $q = (x, \hat{x}; A)$ and the right hand side is defined in Proposition 3.10. If $c = 0$, then

$$\begin{aligned}\mu(B_{\hat{F}}, q) &= \mu\big(\big(\hat{F}(\hat{x}) - \hat{F}_*|_{\hat{x}}(\hat{x}), \hat{F}_*|_{\hat{x}}\big), q\big) = \big(x, \hat{F}_*|_{\hat{x}}(\hat{x}) + \big(\hat{F}(\hat{x}) - \hat{F}_*|_{\hat{x}}(\hat{x})\big); \hat{F}_*|_{\hat{x}} \circ A\big)\\ &= \big(x, \hat{F}(\hat{x}); \hat{F}_*|_{\hat{x}} \circ A\big) = \hat{F} \cdot q,\end{aligned}$$

while if $c \neq 0$, $\mu(B_{\hat{F}}, q) = (x, B_{\hat{F}}(\hat{x}); B_{\hat{F}} \circ A) = (x, \hat{F}(\hat{x}); \hat{F}_*|_{\hat{x}} \circ A) = \hat{F} \cdot q$.

To prove the properness, consider a sequence B_n in $G_c(n)$ and $q_n = (x_n, \hat{x}_n; A_n)$ in Q such that $q_n \to q = (x, \hat{x}; A)$ and $\mu(B_n, q_n) \to q' = (x', \hat{x}'; A')$ as $n \to \infty$. Choose the unique $\hat{F}_n \in \mathrm{Iso}(M_{n;c}, \hat{g}_{n;c})$ such that $B_n = B_{\hat{F}_n}$ as above. Then $\mu(B_n, q_n) = \hat{F}_n \cdot q \to q'$ implies in particular that $\hat{F}_n(\hat{x}_n) \to \hat{x}'$ and we also have $\hat{x}_n \to \hat{x}$. Since the action of the isometry group of a complete connected Riemannian manifold is proper, we hence obtain a subsequence $\hat{F}_{n_i}$ of $\hat{F}_n$ converging to $\hat{F} \in G_c(n)$. Then B_{n_i} converges to $B_{\hat{F}}$ and we are done.

It remains to check the claim that the action μ preserves $\mathcal{D}_R$ in the sense stated above. Let $B \in G_c(n)$. From what precedes, there is a unique $\hat{F} \in \mathrm{Iso}(\hat{M}_{n;c}, \hat{g}_{n;c})$ such that $B = B_{\hat{F}}$. Let $q = (x, \hat{x}; A) \in Q$ and let γ be any smooth curve in M such that $\gamma(0) = x$. By what was proved above and Proposition 3.10 imply that for all t,

$$\mu\big(B, q_{\mathcal{D}_R}(\gamma, q)(t)\big) = \hat{F} \cdot q_{\mathcal{D}_R}(\gamma, q)(t) = q_{\mathcal{D}_R}(\gamma, \hat{F} \cdot q)(t) = q_{\mathcal{D}_R}\big(\gamma, \mu(B, q)\big)(t).$$

Taking derivative with respect to t at $t = 0$, we find that

$$(\mu_B)_*\mathscr{L}_{\rm R}(\dot\gamma(0))\big|_q = (\mu_B)_*\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 q_{\mathcal{D}_{\rm R}}(\gamma,q)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \mu\big(B, q_{\mathcal{D}_{\rm R}}(\gamma,q)(t)\big)$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 q_{\mathcal{D}_{\rm R}}\big(\gamma,\mu(B,q)\big)(t) = \mathscr{L}_{\rm R}\big(\dot\gamma(0)\big)\big|_{\mu(B,q)}.$$

This implies that $(\mu_B)_*\mathcal{D}_{\rm R} = \mathcal{D}_{\rm R}|_{\mu(B,q)}$ and hence allows us to conclude the proof of (i).

(ii) This follows from the general theory of principal bundle connections. See [12,14]. □

4.2. Rolling against an Euclidean space

In this section, we give a necessary and sufficient condition for the controllability of $(\Sigma)_R$ in the case that $\hat M = \mathbb{R}^n$ equipped with the Euclidean metric, i.e. $(\hat M, \hat g) = (\hat M_{n;0}, \hat g_{n;0})$.

Now fix a point q_0 of $Q = Q(M,\mathbb{R}^n)$ of the form $q_0 = (x_0, 0; A_0)$, i.e., the initial contact point on M is equal to x_0 and, on $\mathbb{R}^n$, it is the origin. Since $\mathbb{R}^n$ is flat, for any a.c. curve $t \mapsto \hat\gamma(t)$ in $\mathbb{R}^n$ and $\hat X \in \mathbb{R}^n$ we have $P_0^t(\hat\gamma(t))\hat X = \hat X$, where we understand the canonical isomorphisms $T|_{\hat\gamma(0)}\mathbb{R}^n \cong \mathbb{R}^n \cong T|_{\hat\gamma(t)}\mathbb{R}^n$. We can then parameterize the rolling curves explicitly in the form:

$$q_{\mathcal{D}_{\rm R}}\big(\gamma,(x_0,\hat x;A)\big)(t) = \left(\gamma(t), \hat x + A\int_0^t P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s;\ A P_t^0(\gamma)\right), \tag{10}$$

for any $(x_0, 0; A_0), (x_0, \hat x; A) \in Q$ and $\gamma \in \Omega_{x_0}(M)$.

We will make some standard observations for subgroups G of an Euclidean group $\mathrm{SE}(V)$, where (V,h) is a finite-dimensional inner product space. Call an element of G of the form (v, id_V) a *pure translation* of G and write $T = T(G)$ for the set that they form. Clearly T is a subgroup of G. Let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the projections $\mathrm{SE}(V) \to V$ and $\mathrm{SE}(V) \to \mathrm{SO}(V)$.

Proposition 4.2. *Let G be a Lie subgroup of* $\mathrm{SE}(V)$ *with* $\mathrm{pr}_2(G) = \mathrm{SO}(V)$. *Then either of the following cases hold*:

(i) $G = \mathrm{SE}(V)$, *or*
(ii) *there exists $v^* \in V$ which is a fixed point of G.*

Proof. Suppose first that $T = T(G)$ is non-trivial, i.e., there exists a pure translation $(v, \mathrm{id}_V) \in T$, $v \neq 0$. Then for any $(w, A) \in G$ it holds that

$$G \ni (w,A)^{-1}\star(v,\mathrm{id}_V)\star(w,A) = \big(-A^{-1}w, A^{-1}\big)\star(v+w, A)$$
$$= \big(A^{-1}(v+w) - A^{-1}w, \mathrm{id}_V\big) = \big(A^{-1}v, \mathrm{id}_V\big),$$

which implies that

$$T \supset \big\{\big(A^{-1}v,\mathrm{id}_V\big)\ \big|\ (w,A)\in G\big\} = \big\{\big(A^{-1}v,\mathrm{id}_V\big)\ \big|\ A \in \mathrm{pr}_2(G) = \mathrm{SO}(V)\big\}$$
$$= S^{n-1}\big(0,\|v\|\big)\times\{\mathrm{id}_V\}$$

where $S^{n-1}(w,r)$, $w \in \mathbb{R}^n$, $r > 0$, is the sphere of radius r centered at $w \in V$ and $\|\cdot\| = h(\cdot,\cdot)^{1/2}$. If $w \in V$ such that $\|w\| \leqslant \|v\|$ then it is clear that there are $u, u' \in S^{n-1}(0,\|v\|)$ such that $u + u' = w$ (choose $u \in S^{n-1}(0,\|v\|) \cap S^{n-1}(w,\|v\|)$ and $u' = w - u$). Therefore

$$(w,\mathrm{id}_V) = (u,\mathrm{id}_V)\star\big(u',\mathrm{id}_V\big) \in T,$$

i.e., $\overline{B}(0,\|v\|) \subset T$ where $\overline{B}(w,r)$ is the closed ball of radius r centered at w. For $k \in \mathbb{N}$,

$$\{\underbrace{\overline{B}\big(0,\|v\|\big)+\cdots+\overline{B}\big(0,\|v\|\big)}_{k\ \text{times}}\}\times\{\mathrm{id}_V\} = \underbrace{\big(\overline{B}\big(0,\|v\|\big)\times\{\mathrm{id}_V\}\big)\star\cdots\star\big(\overline{B}\big(0,\|v\|\big)\times\{\mathrm{id}_V\}\big)}_{k\ \text{times}} \subset T.$$

From this we conclude that $V \times \{\mathrm{id}_V\} = T$. Therefore we get the case (i) since

$$\begin{aligned} G = T \star G &= \big\{(u, \mathrm{id}_V) \star (w, A) \,\big|\, u \in V,\ (w, A) \in G\big\} \\ &= \big\{(u + w, A) \,\big|\, u \in V,\ (w, A) \in G\big\} = \big\{(u, A) \,\big|\, u \in V,\ A \in \mathrm{pr}_2(G) = \mathrm{SO}(V)\big\} \\ &= V \times \mathrm{SO}(V) = \mathrm{SE}(V). \end{aligned}$$

The case that is left to investigate is the one where T is trivial, i.e., $T = \{(0, \mathrm{id}_V)\}$. In this case the smooth surjective Lie group homomorphism $\mathrm{pr}_2|_G : G \to \mathrm{SO}(V)$ is also injective. In fact, if $A = \mathrm{pr}_2(v, A) = \mathrm{pr}_2(w, A)$ for $(v, A), (w, A) \in G$ and $v \neq w$, then

$$G \ni (w, A) \star (v, A)^{-1} = (w, A) \star \big(-A^{-1}v, A^{-1}\big) = (w - v, \mathrm{id}_V) \in T,$$

and since $(w - v, \mathrm{id}_V) \neq (0, \mathrm{id}_V)$, this contradicts the triviality of T. It follows that $\mathrm{pr}_2|_G$ is a Lie group isomorphism onto $\mathrm{SO}(V)$ and hence a diffeomorphism. In particular, G is compact since $\mathrm{SO}(V)$ is compact.

Take a non-zero $v \in V$ and writing μ_{H} for the (right- and) left-invariant normalized (to 1) Haar measure of the compact group G, we define $v^* := \int_G (B \star v)\, \mathrm{d}\mu_{\mathrm{H}}(B)$. Thus for $(w, A) \in G$,

$$\begin{aligned} (w, A) \star v^* = w + Av^* &= \int_G \big(w + A(B \star v)\big)\, \mathrm{d}\mu_{\mathrm{H}}(B) = \int_G \big(\big((w, A) \star B\big) \star v\big)\, \mathrm{d}\mu_{\mathrm{H}}(B) \\ &= \int_G (B \star v)\, \mathrm{d}\mu_{\mathrm{H}}(B) = v^*, \end{aligned}$$

where, in the second equality, we have used the linearity of the integral and normality of the Haar measure and in the last phase the left invariance of the Haar measure. This proves that v^* is a fixed point of G and completes the proof. □

The previous proposition allows us prove the main theorem of this section.

Theorem 4.3. *Suppose (M, g) is a complete Riemannian n-manifold and $(\hat{M}, \hat{g}) = \mathbb{R}^n$ is the Euclidean n-space. Then the rolling problem (R) is completely controllable if and only if the holonomy group of (M, g) is* $\mathrm{SO}(n)$.

Proof. We write $H|_x$ for the holonomy group $H^{\nabla}|_x$ of ∇ at $x \in M$.

Suppose first that (R) is completely controllable. We need to show that $H|_{x_0} = \mathrm{SO}(T|_{x_0}M)$ for some given $x_0 \in M$. Let $A_0 := \mathrm{id}_{T|_{x_0}M}$ and $q_0 := (x_0, 0; A_0) \in Q$ where we understand the canonical identification $T|_0(T|_{x_0}M) = T|_{x_0}M$.

Given $B \in \mathrm{SO}(T|_{x_0}M)$, set $q = (x_0, 0; AB) \in Q$. By assumption $\mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q_0) = Q$ so there exists a $\gamma \in \Omega_{x_0}(M)$ (notice that $\pi_{Q,M}(q_0) = x_0 = \pi_{Q,M}(q)$) such that $q' = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q_0)(1)$ which by (10) means that

$$(x_0, 0; AB) = \left(x_0, A\int_0^1 P_s^0(\gamma)\dot{\gamma}(s)\, \mathrm{d}s; AP_1^0(\gamma)\right)$$

and thus $B = P_1^0(\gamma) \in H|_{x_0}$. This proves the necessity of the condition.

Assume now that the holonomy group of M is $\mathrm{SO}(n)$, i.e., for any $x \in M$ we have $H|_x = \mathrm{SO}(T|_xM)$. Let $q = (x, 0; A) \in Q$ and let $\mathcal{H}_q$ be the subgroup of $\mathrm{SE}(n)$ such that $\mu(\mathcal{H}_q \times \{q\}) = \pi_{Q,M}^{-1}(x) \cap \mathcal{O}_{\mathcal{D}_{\mathrm{R}}}(q)$ as in Proposition 4.1 case (ii).

We claim that $\mathrm{pr}_2(\mathcal{H}_q) = \mathrm{SO}(n)$. Indeed, if $B \in \mathrm{SO}(n)$, then $A^{-1}BA \in \mathrm{SO}(T|_xM) = H|_x$ and hence there is a $\gamma \in \Omega_x(M)$ such that $A^{-1}BA = P_1^0(\gamma)$. Let $(\hat{y}, C) \in \mathcal{H}_q$ be such that $\mu((\hat{y}, C), q) = q_{\mathcal{D}_{\mathrm{R}}}(\gamma, q)(1)$. Then from (10) and the definition of μ in Proposition 4.1 we obtain

$$(\hat{y}, CA) = \left(A\int_0^1 P_0^s(\gamma)\dot{\gamma}(s)\, \mathrm{d}s, AP_1^0(\gamma)\right)$$

and hence $B = AP_0^1(\gamma)A^{-1} = C \in \mathrm{pr}_2(\mathcal{H}_q)$, which establishes the claim.

It follows from Proposition 4.2 that either (i) $\mathcal{H}_q = \mathrm{SE}(n)$ or (ii) there exists a fixed point $w_q^* \in \mathbb{R}^n$ of $\mathcal{H}_q$. If (i) holds for some $q_0 = (x_0, 0; A_0) \in Q$, then by Proposition 4.1 we obtain

$$\pi_{Q,M}^{-1}(x_0) \cap \mathcal{O}_{\mathcal{D}_\mathrm{R}}(q_0) = \mu\big(\mathcal{H}_q \times \{q\}\big) = \mu\big(\mathrm{SE}(n) \times \{q\}\big) = \pi_{Q,M}^{-1}(x_0)$$

and hence $\mathcal{O}_{\mathcal{D}_\mathrm{R}}(q_0) = Q$ because $\pi_{\mathcal{O}_{\mathcal{D}_\mathrm{R}}(q_0),M}$ is a subbundle of $\pi_{Q,M}$. Thus the rolling problem (R) is completely controllable if (i) holds.

Therefore suppose that (ii) holds, i.e., for every $q \in Q$ of the form $q = (x, 0; A)$ there is a fixed point $w_q^* \in \mathbb{R}^n$ of $\mathcal{H}_q$. We will prove that this implies that (M, g) is flat which is a contradiction since (M, g) does not have a trivial holonomy group.

Thus for any point of Q of the form $q = (x, 0; A)$ and all loops $\gamma \in \Omega_x(M)$ we have by (10) and Proposition 4.1,

$$\begin{aligned} w_q^* &= \big(\mu^q\big)^{-1}\big(q_{\mathcal{D}_\mathrm{R}}(\gamma, q)(1)\big) \star w_q^* = \big(\mu^q\big)^{-1}\left(x, \int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s; A P_1^0(\gamma)\right) \star w_q^* \\ &= \left(A\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s, A P_1^0(\gamma)A^{-1}\right) \star w_q^* = A P_1^0(\gamma)A^{-1} w_q^* + A\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s. \end{aligned}$$

In other words we have $(P_1^0(\gamma) - \mathrm{id})A^{-1}w_q^* + \int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s = 0$. Thus if $q = (x, 0; A)$ and $q' = (x, 0; A')$ are on the same π_Q-fiber over $(x, 0)$, then $(P_1^0(\gamma) - \mathrm{id})(A^{-1}w_q^* - A'^{-1}w_{q'}^*) = 0$ for every $\gamma \in \Omega_x(M)$. On the other hand, since M has full holonomy, i.e., $H|_x = \mathrm{SO}(T|_x M)$, and $H|_x = \{P_1^0(\gamma) \mid \gamma \in \Omega_x(M)\}$, it follows from the above equation that $A^{-1}w_q^* = A'^{-1}w_{q'}^*$. This means that for every $x \in M$ there is a unique vector $V|_x \in T|_x M$ such that $V|_x = A^{-1}w_q^*$, $\forall q \in \pi_Q^{-1}(x, 0)$. Moreover, the map $V : M \to TM$; $x \mapsto V|_x$ is a vector field on M (smoothness of V is deduced below) satisfying

$$P_1^0(\gamma)V|_x - V|_x = -\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s, \quad \forall\gamma \in \Omega_x(M). \tag{11}$$

It follows from this that, for any piecewise C^1 path $\gamma : [0, 1] \to M$, we have

$$V|_{\gamma(1)} = P_0^1(\gamma)\left(V|_{\gamma(0)} - \int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s\right). \tag{12}$$

Indeed, if $\omega \in \Omega_{\gamma(1)}(M)$, then $\gamma^{-1}.\omega.\gamma \in \Omega_{\gamma(0)}(M)$ and therefore

$$\begin{aligned} &P_1^0(\gamma)P_1^0(\omega)P_0^1(\gamma)V|_{\gamma(0)} - V|_{\gamma(0)} \\ &\quad = P_1^0\big(\gamma^{-1}.\omega.\gamma\big)V|_{\gamma(0)} - V|_{\gamma(0)} \\ &\quad = -\int_0^1 P_s^0\big(\gamma^{-1}.\omega.\gamma\big)\frac{\mathrm{d}}{\mathrm{d}s}\big(\gamma^{-1}.\omega.\gamma\big)(s)\,\mathrm{d}s \\ &\quad = -\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s - P_1^0(\gamma)\int_0^1 P_s^0(\omega)\dot\omega(s)\,\mathrm{d}s - P_1^0(\gamma)P_1^0(\omega)\int_0^1 P_s^0\big(\gamma^{-1}\big)\frac{\mathrm{d}}{\mathrm{d}s}\gamma^{-1}(s)\,\mathrm{d}s \\ &\quad = -\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s + P_1^0(\gamma)\big(P_1^0(\omega)V|_{\gamma(1)} - V|_{\gamma(1)}\big) + P_1^0(\gamma)P_1^0(\omega)P_0^1(\gamma)\int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s, \end{aligned}$$

that is $(P_1^0(\omega) - \mathrm{id})P_0^1(\gamma)(V|_{\gamma(0)} - \int_0^1 P_s^0(\gamma)\dot\gamma(s)\,\mathrm{d}s) = (P_1^0(\omega) - \mathrm{id})V|_{\gamma(1)}$. Eq. (12) then follows from this since $\{P_1^0(\omega) \mid \omega \in \Omega_{\gamma(1)}(M)\} = H|_{\gamma(1)} = \mathrm{SO}(T|_{\gamma(1)}M)$.

Since (M, g) is complete, the geodesic $\gamma_X(t) = \exp_x(tX)$ is defined for all $t \in [0, 1]$. Inserting this to Eq. (12) and noticing that $P_s^0(\gamma_X)\dot{\gamma}_X(s) = X$ in this case for all $s \in [0, 1]$, we get $V|_{\gamma_X(1)} = P_0^1(\gamma_X)(V|_x - X)$. In particular, one deduces from this formula that V is smooth on M. If $X := V|_x$ and $z := \gamma_X(1) = \exp_x(V|_x)$, we get $V|_z = 0$.

Fix $z \in M$ such that $V|_z = 0$ and fix also some $q^* \in Q$ of the form $q^* = (z, 0; A_0)$ (one may e.g. take $A_0 = \mathrm{id}_{T|_z M}$). Eq. (12) is clearly equivalent to

$$P_t^0(\gamma)V|_{\gamma(t)} = V|_{\gamma(0)} - \int_0^t P_s^0(\gamma)\dot{\gamma}(s)\,\mathrm{d}s$$

for any piecewise smooth path $\gamma : [0, T] \to M$, $T > 0$. Taking γ to be smooth and differentiating the above equation w.r.t. to t, we get $P_t^0(\gamma)\nabla_{\dot{\gamma}(t)}V = -P_t^0(\gamma)\dot{\gamma}(t)$, i.e., $\nabla_{\dot{\gamma}(t)}V = -\dot{\gamma}(t)$. Since γ was an arbitrary smooth curve, this implies that

$$\nabla_X V = -X, \quad \forall X \in \mathrm{VF}(M). \tag{13}$$

For any $X \in \mathrm{VF}(M)$, the vector $R(X, V)V$ can be seen to vanish everywhere since

$$\begin{aligned} R(X, V)V &= \nabla_X\nabla_V V - \nabla_V\nabla_X V - \nabla_{[X,V]}V = -\nabla_X V + \nabla_V X + [X, V] \\ &= [V, X] + [X, V] = 0, \end{aligned}$$

where, in the second equality, we used (13).

For any $X \in T|_z M$, we write $\gamma_X(t) = \exp_z(tX)$ for the geodesic through z in the direction of X. It follows that

$$V|_{\gamma_X(t)} = P_0^t(\gamma_X)\left(V|_z - \int_0^t P_s^0(\gamma_X)\dot{\gamma}_X(s)\,\mathrm{d}s\right) = P_0^t(\gamma_X)\left(-\int_0^t X\,\mathrm{d}s\right) = P_0^t(\gamma_X)(-tX) = -t\dot{\gamma}_X(t).$$

Now for given $X, v \in T|_z M$ let $Y(t) = \frac{\partial}{\partial s}|_0 \exp_z(t(X + sv))$ be the Jacobi field along γ_X such that $Y(0) = 0$, $\nabla_{\dot{\gamma}_X(t)}Y|_{t=0} = v$. Then one has

$$\nabla_{\dot{\gamma}_X(t)}\nabla_{\dot{\gamma}_X}Y = R\big(\dot{\gamma}_X(t), Y(t)\big)\dot{\gamma}_X(t) = \frac{1}{t^2}R\big(V|_{\gamma_X(t)}, Y(t)\big)V|_{\gamma_X(t)} = 0,$$

for $t \neq 0$ which means that $t \mapsto \nabla_{\dot{\gamma}_X(t)}Y$ is parallel along γ_X, i.e., $\nabla_{\dot{\gamma}_X(t)}Y = P_0^t(\gamma_X)\nabla_{\dot{\gamma}_X(0)}Y = P_0^t(\gamma_X)v$. This allows us to compute, for any t,

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\|Y(t)\|_g^2 &= 2\frac{\mathrm{d}}{\mathrm{d}t}g\big(\nabla_{\dot{\gamma}_X(t)}Y, Y(t)\big) = 2g\big(\underbrace{\nabla_{\dot{\gamma}_X(t)}\nabla_{\dot{\gamma}_X}Y}_{=0}, Y(t)\big) + 2g(\nabla_{\dot{\gamma}_X(t)}Y, \nabla_{\dot{\gamma}_X(t)}Y) \\ &= 2g\big(P_0^t(\gamma_X)v, P_0^t(\gamma_X)v\big) = 2\|v\|_g^2 \end{aligned}$$

and then $\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|_g^2 = 2\|v\|_g^2 t + \frac{\mathrm{d}}{\mathrm{d}t}|_0\|Y(t)\|_g^2 = 2\|v\|_g^2 t$, since $\frac{\mathrm{d}}{\mathrm{d}t}|_0\|Y(t)\|_g^2 = 2g(\nabla_{\dot{\gamma}_X(0)}Y, Y(0)) = 0$ as $Y(0) = 0$. Therefore,

$$\|Y(t)\|_g^2 = \|v\|_g^2 t^2 + \|Y(0)\|_g^2 = \|v\|_g^2 t^2,$$

which means that $\|t(\exp_z)_*|_{tX}(v)\|_g = \|tv\|_g$ and hence, when $t = 1$,

$$\big\|(\exp_z)_*|_X(v)\big\|_g = \|v\|_g, \quad \forall X, v \in T|_z M. \tag{14}$$

This proves that $\exp_z$ is a local isometry $(T|_z M, g|_z) \to (M, g)$ and hence a Riemannian covering. Thus (M, g) is flat and the proof is finished. □

Remark 4.4. For results and proofs in similar lines to those of the above proposition and theorem, see Theorem IV.7.1, p. 193 and Theorem IV.7.2, p. 194 in [14].

4.3. Rolling against a non-flat space form

4.3.1. The rolling connection

Let $\pi_{TM\oplus\mathbb{R}}: TM \oplus \mathbb{R} \to M$ be the vector bundle over M where $\pi_{TM\oplus\mathbb{R}}(X,r) = \pi_{TM}(X)$. In this section we will prove the following result.

Theorem 4.5. *There exists a vector bundle connection ∇^{Rol} of the vector bundle $\pi_{TM\oplus\mathbb{R}}$ that we call the* rolling connection*, and which we define as follows: for every $x \in M$, $Y \in T|_x M$, $X \in \mathrm{VF}(M)$, $r \in C^\infty(M)$,*

$$\nabla^{\mathsf{Rol}}_Y(X,r) = \big(\nabla_Y X + r(x)Y, Y(r) - cg(X|_x, Y)\big), \tag{15}$$

such that in the case of (M,g) rolling against the space form $(\hat{M}_{n;c}, \hat{g}_{n;c})$, $c \neq 0$, the holonomy group G of $\mathcal{D}_{\mathrm{R}}$ is isomorphic to the holonomy group $H^{\nabla^{\mathsf{Rol}}}$ of ∇^{Rol}.

Moreover, if one defines a fiber inner product h_c on $TM \oplus \mathbb{R}$ by

$$h_c\big((X,r),(Y,s)\big) = g(X,Y) + c^{-1}rs,$$

where $X, Y \in T|_x M$, $r, s \in \mathbb{R}$, then ∇^{Rol} is a metric connection in the sense that for every $X, Y, Z \in \mathrm{VF}(M)$, $r, s \in C^\infty(M)$,

$$Z\big(h_c\big((X,r),(Y,s)\big)\big) = h_c\big(\nabla^{\mathsf{Rol}}_Z(X,r),(Y,s)\big) + h_c\big((X,r),\nabla^{\mathsf{Rol}}_Z(Y,s)\big).$$

Before providing the proof of the theorem, we present the equations of parallel transport w.r.t. ∇^{Rol} along a general curve and along a geodesic of M and also the curvature of ∇^{Rol}. Let $\gamma : [0,1] \to M$ be an a.c. curve on M, $\gamma(0) = x$ and let $(X_0, r_0) \in T|_x M \oplus \mathbb{R}$. Then the parallel transport $(X(t), r(t)) = (P^{\nabla^{\mathsf{Rol}}})^t_0(\gamma)(X_0, r_0)$ of (X_0, r_0) is determined from the equations

$$\begin{cases} \nabla_{\dot\gamma(t)} X + r(t)\dot\gamma(t) = 0, \\ \dot r(t) - cg\big(\dot\gamma(t), X(t)\big) = 0, \end{cases} \tag{16}$$

for a.e. $t \in [0,1]$. In particular, if γ is a geodesic on (M,g), one may derive the following uncoupled second order differential equations for X and r, for all t,

$$\begin{cases} \nabla_{\dot\gamma(t)}\nabla_{\dot\gamma(t)} X + cg\big(X(t), \dot\gamma(t)\big)\dot\gamma(t) = 0, \\ \ddot r(t) + c\|\dot\gamma(t)\|^2_g r(t) = 0. \end{cases} \tag{17}$$

One easily checks that the connection ∇^{Rol} on $\pi_{TM\oplus\mathbb{R}}$ has the curvature,

$$R^{\nabla^{\mathsf{Rol}}}(X,Y)(Z,r) = \big(R(X,Y)Z - c\big(g(Y,Z)X - g(X,Z)Y\big), 0\big), \tag{18}$$

where $X, Y, Z \in \mathrm{VF}(M)$, $r \in C^\infty(M)$.

Proof of Theorem 4.5. We have proved in Proposition 4.1 that the rolling distribution $\mathcal{D}_{\mathrm{R}}$ is a principal bundle connection for the principal $G_c(n)$-bundle $\pi_{Q,M} : Q \to M$. By a standard procedure (cf. Definition 2.1.3 and Proposition 2.3.7 in [12]), the previous fact implies that there is a vector bundle $\xi : E \to M$ with fibers isomorphic to $\mathbb{R}^{n+1}$ and a unique linear vector bundle connection $\nabla^{\mathsf{Rol}} : \Gamma(\xi) \times \mathrm{VF}(M) \to \Gamma(\xi)$ which induces the distribution $\mathcal{D}_{\mathrm{R}}$ on Q. Then the holonomy group G of $\mathcal{D}_{\mathrm{R}}$ and $H^{\nabla^{\mathsf{Rol}}}$ of ∇^{Rol} are isomorphic. We will eventually show that ξ is further isomorphic to $\pi_{TM\oplus\mathbb{R}}$ and give the explicit expression (15) for the connection of $\pi_{TM\oplus\mathbb{R}}$ induced by this isomorphism from ∇^{Rol} on ξ.

There is a canonical non-degenerate metric $h_c : E \odot E \to M$ on the vector bundle ξ (positive definite when $c > 0$ and indefinite if $c < 0$) and the connection ∇^{Rol} is a metric connection w.r.t. to h_c, i.e., for any $Y \in \mathrm{VF}(M)$ and $s, \sigma \in \Gamma(\nu)$,

$$Y\big(h_c(s,\sigma)\big) = h_c\big(\nabla^{\mathsf{Rol}}_Y s, \sigma\big) + h_c\big(s, \nabla^{\mathsf{Rol}}_Y \sigma\big). \tag{19}$$

The construction of ξ goes as follows (see [12, Section 2.1.3]). Define a left $G_c(n)$-group action β on $Q \times \mathbb{R}^{n+1}$ by $\beta(B,(q,v)) = (\mu(B,q), Bv)$, where $q \in Q$, $v \in \mathbb{R}^{n+1}$, $B \in G_c(n)$. The action β is clearly smooth, free and proper.

Hence $E := (Q \times \mathbb{R}^{n+1})/\beta$ is a smooth manifold of dimension $n + (n+1) = 2n+1$. The β-equivalence class (i.e., β-orbit) of $(q, v) \in Q \times \mathbb{R}^{n+1}$ is denoted by $[(q, v)]$. Then one defines $\xi([(q, v)]) = \pi_{Q,M}(q)$ which is well defined since the β-action preserves the fibers of $Q \times \mathbb{R}^{n+1} \to M$; $(q, v) \mapsto \pi_{Q,M}(q)$. We prove now that ξ is isomorphic, as a vector bundle over M, to

$$\pi_{TM\oplus\mathbb{R}} : TM \oplus \mathbb{R} \to M,$$
$$(X, t) \mapsto \pi_{TM}(X).$$

Indeed, let $f \in \Gamma(\xi)$ and notice that for any $q \in Q$ there exists a unique $\overline{f}(q) \in \mathbb{R}^{n+1}$ such that $[(q, \overline{f}(q))] = f(\pi_{Q,M}(q))$ by the definition of the action β. Then $\overline{f} : Q \to \mathbb{R}^{n+1}$ is well defined and, for each $q = (x, \hat{x}; A)$, there are unique $X|_q \in T|_x M$, $r(q) \in \mathbb{R}$ such that

$$\overline{f}(q) = AX|_q + r(q)\hat{x}.$$

The maps $q \mapsto X|_q$ and $q \mapsto r(q)$ are smooth. We show that the vector $X|_q$ and the real number $r(q)$ depend only on x and hence define a vector field and a function on M. One has $[((x, \hat{x}; A), v)] = [((x, \hat{y}; B), w)]$ if and only if there is $C \in G_c(n)$ such that $C\hat{x} = \hat{y}$, $CA = B$ and $Cv = w$. This means that $C|_{\operatorname{im} A} = BA^{-1}|_{\operatorname{im} A} : T|_{\hat{x}} \hat{M}_{n;c} \to T|_{\hat{y}} \hat{M}_{n;c}$ (with $\operatorname{im} A$ denoting the image of A) and this defines C uniquely as an element of $G_c(n)$ and also, by the definition of $\overline{f}$, $C\overline{f}(x, \hat{x}, A) = \overline{f}(x, \hat{y}, B)$. Therefore,

$$BX|_{(x,\hat{y};B)} + r(x, \hat{y}; B)\hat{y} = C\big(AX|_{(x,\hat{x};A)} + r(x, \hat{x}; A)\hat{x}\big) = BX|_{(x,\hat{x};A)} + r(x, \hat{x}; A)\hat{y},$$

which shows that $X|_{(x,\hat{y};B)} = X|_{(x,\hat{x};A)}$, $r(x, \hat{y}; B) = r(x, \hat{x}; A)$ and proves the claim.

Hence for each $f \in \Gamma(\xi)$ there are unique $X_f \in \mathrm{VF}(M)$ and $r_f \in C^\infty(M)$ such that

$$f(x) = \big[\big((x, \hat{x}; A), AX_f|_x + r_f(x)\hat{x}\big)\big]$$

(here the right hand side does not depend on the choice of $(x, \hat{x}; A) \in \pi_{Q,M}^{-1}(x)$).

Conversely, given $X \in \mathrm{VF}(M)$, $r \in C^\infty(M)$ we may define $f_{(X,r)} \in \Gamma(\xi)$ by

$$f_{(X,r)}(x) = \big[\big((x, \hat{x}; A), AX|_x + r(x)\hat{x}\big)\big],$$

where the right hand side does not depend on the choice of $(x, \hat{x}; A) \in \pi_{Q,M}^{-1}(x)$.

Clearly, for $f \in \Gamma(\xi)$, one has $f_{(X_f, r_f)} = f$ and, for $(X, r) \in \mathrm{VF}(M) \times C^\infty(M)$, one has $(X_{f_{(X,r)}}, r_{f_{(X,r)}}) = (X, r)$. This proves that the map defined by

$$\Gamma(\xi) \to \mathrm{VF}(M) \times C^\infty(M),$$
$$f \mapsto (X_f, r_f)$$

is a bijection. It is easy to see that it is actually a $C^\infty(M)$-module homomorphism. Since $C^\infty(M)$-modules $\Gamma(\xi)$ and $\mathrm{VF}(M) \times C^\infty(M)$ are isomorphic and since $\mathrm{VF}(M) \times C^\infty(M)$ is obviously isomorphic, as a $C^\infty(M)$-module, to $\Gamma(\pi_{TM\oplus\mathbb{R}})$, it follows that ξ and $\pi_{TM\oplus\mathbb{R}}$ are isomorphic vector bundles over M.

We now describe the connection ∇^{Rol} and the inner product structure h_c on ξ and we determine to which objects they correspond in the isomorphic bundle $\pi_{TM\oplus\mathbb{R}}$.

By Section 2.1.3 in [12] and the above notation, one defines for $f \in \Gamma(\xi)$, $Y \in T|_x M$, $x \in M$

$$\nabla^{\mathsf{Rol}}_Y f|_x := \big[\big((x, \hat{x}; A), \mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f}\big)\big],$$

where $\overline{f} : Q \to \mathbb{R}^{n+1}$ is defined above and $\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f}$ is defined componentwise (i.e., we let $\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}$ to operate separately to each of the $n+1$ component functions of $\overline{f}$). The definition does not depend on $(x, \hat{x}; A) \in \pi_{Q,M}^{-1}(x)$ as should be evident from the above discussions. The inner product on ξ, on the other hand, is defined by

$$h_c\big(\big[\big((x, \hat{x}; A), v\big)\big], \big[\big((x, \hat{y}; B), w\big)\big]\big) = g(X, Y) + c^{-1}rt,$$

where $v = AX + r\hat{x}$, $w = BY + t\hat{y}$. It is clear that h_c is well defined.

We work out the expression for ∇^{Rol}. For clarity, we write $\iota : \hat{M}_{n;c} \to \mathbb{R}^{n+1}$ for the inclusion. Let $f \in \Gamma(\xi)$, $Y \in T|_x M$, $x \in M$. Then $\overline{f}(y, \hat{y}, B) = \iota_*(BX_f|_y) + r_f(y)\hat{y}$ where $X_f \in \mathrm{VF}(M)$, $r_f \in C^\infty(M)$ and

$$\mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\overline{f} = \mathscr{L}_{\mathrm{R}}(Y)|_{(x,\hat{x};A)}\big((y, \hat{y}; B) \mapsto \iota_*(BX_f|_y)\big) + Y(r_f)\hat{x} + r_f(x)AY.$$

Take a path γ on M such that $\dot\gamma(0)=Y$. Then $\dot q_{\mathcal{D}_{\rm R}}(\gamma,q)(0)=\mathscr{L}_{\rm R}(Y)|_q$, where $q=(x,\hat x;A)$, and $\mathscr{L}_{\rm R}(Y)|_q((y,\hat y;B)\mapsto \iota_*(BX_f|_y))=\frac{\mathrm{d}}{\mathrm{d}t}|_0\iota_*(A_{\mathcal{D}_{\rm R}}(\gamma,q)(t)X_f|_{\gamma(t)})$. Since

$$\begin{aligned}
&s_{n;c}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \iota_*\big(A_{\mathcal{D}_{\rm R}}(\gamma,q)(t)X_f|_{\gamma(t)}\big),\hat x\right)\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 s_{n;c}\big(\iota_*A_{\mathcal{D}_{\rm R}}(\gamma,q)(t)X_f|_{\gamma(t)},\hat\gamma_{\mathcal{D}_{\rm R}}(\gamma,q)(t)\big)-s_{n;c}(\iota_*AX_f|_x,\iota_*AY)\\
&\quad=-\hat g_{n;c}(AX_f|_x,AY)=-g(X_f|_x,Y)=s_{n;c}\big(-cg(X_f|_x,Y)\hat x,\hat x\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathscr{L}_{\rm R}(Y)|_q\big((y,\hat y;B)\mapsto \iota_*(BX_f|_y)\big)&=\iota_*\hat\nabla_{AY}\big(A_{\mathcal{D}_{\rm R}}(\gamma,q_0)(\cdot)X_f\big)-cg(X_f|_x,Y)\hat x\\
&=\iota_*A\nabla_YX_f-cg(X_f|_x,Y)\hat x
\end{aligned}$$

and so $\mathscr{L}_{\rm R}(Y)|_q\overline{f}=\iota_*A(\nabla_YX_f+r_f(x)Y)+(Y(r_f)-cg(X_f|_x,Y))\hat x$.

Correspondingly, using the isomorphism of ξ and $\pi_{TM\oplus\mathbb{R}}$, to the connection ∇^{Rol} and the non-degenerate metric h_c on ξ, there is a connection ∇^{Rol} and an indefinite metric h_c (with the same names as the ones on ξ) on $\pi_{TM\oplus\mathbb{R}}$ such that for $X\in\mathrm{VF}(M)$, $r\in C^\infty(M)$ and $Y\in T|_xM$,

$$\nabla^{\mathsf{Rol}}_Y(X,r)=\big(\nabla_YX+r(x)Y,Y(r)-cg(X|_x,Y)\big),\tag{20}$$

where $(x,\hat x;A)\in Q$ is any point of $\pi^{-1}x$ and $h_c((X,r),(Y,s))=g(X,Y)+c^{-1}rs$ for $X,Y\in T|_xM$, $r,s\in\mathbb{R}$. To finish the proof, we need to show that ∇^{Rol} is metric w.r.t. h_c. Indeed, if $X,Y,Z\in\mathrm{VF}(M)$, $r,s\in C^\infty(M)$, we get

$$\begin{aligned}
&h_c\big(\nabla^{\mathsf{Rol}}_Z(X,r),(Y,s)\big)+h_c\big((X,r),\nabla^{\mathsf{Rol}}_Z(Y,s)\big)\\
&\quad=h_c\big(\big(\nabla_ZX+rZ,Z(r)-cg(X,Z)\big),(Y,s)\big)+h_c\big((X,r),\big(\nabla_ZY+sZ,Z(s)-cg(Y,Z)\big)\big)\\
&\quad=g(\nabla_ZX+rZ,Y)+c^{-1}Z(r)s-g(X,Z)s+g(\nabla_ZY+sZ,X)+c^{-1}rZ(s)-rg(Y,Z)\\
&\quad=g(\nabla_ZX,Y)+g(X,\nabla_ZY)+c^{-1}Z(r)s+c^{-1}rZ(s)\\
&\quad=Z\big(g(X,Y)+c^{-1}rs\big)=Z\big(h_c\big((X,r),(Y,s)\big)\big).\qquad\square
\end{aligned}$$

4.4. Rolling holonomy for a space form of positive curvature

In this section, we assume that $c=1$, i.e. $(\hat M,\hat g)=(\hat M_{n;1},\hat g_{n;1})$ is the n-dimensional unit sphere S^n. It is now clear, thanks to Theorem 4.5, that the controllability of the rolling problem of a manifold M against the sphere S^n amounts to checking whether the connection ∇^{Rol} of $\pi_{TM\oplus\mathbb{R}}$ has full holonomy or not, i.e., whether $H^{\mathsf{Rol}}:=H^{\nabla^{\mathsf{Rol}}}$ is $\mathrm{SO}(n+1)$ or not.

The classical investigation of the holonomy group H of the Levi-Civita connection in Riemannian geometry is divided into several steps. The first one consists of studying the reducibility of the action of H^{Rol} and this issue is tackled by de Rham theorem (see [20]). The second step then deals with the question of transitivity of the irreducible action of H on the unit sphere. In particular, if this action is not transitive, the corresponding Riemannian manifold is shown to be (locally) symmetric (cf. [22,12,18,4]). Then, from the list of compact connected subgroups of $\mathrm{SO}(n)$ having a transitive action on the unit sphere, one proceeds by either excluding candidates or constructing examples of manifolds having a prescribed holonomy group.

As regards to H^{Rol} the situation turns out to be much more simple and is summarized in the following theorem.

Theorem 4.6. *Let ∇^{Rol} be the rolling connection associated to the rolling problem (R) of a complete simply connected Riemannian manifold (M,g) onto the unit sphere S^n, $n\geqslant 2$. We use H^{Rol} to denote the holonomy group of ∇^{Rol}. Assume that the action of H^{Rol} on the unit sphere is not transitive, then (M,g) admits the unit sphere as its universal covering space.*

We deduce from the list of compact connected subgroups of $\mathrm{SO}(n)$ having a transitive action on the unit sphere (cf. [12, Section 3.4.3]) an immediate corollary regarding the complete controllability of the rolling problem (R)

associated to the manifolds (M, g), $(\hat{M}, \hat{g})$ as in the previous theorem. Indeed, a transitive action on the unit sphere S^k for even dimension $k \geqslant 16$ corresponds to a unique compact connected subgroup of $\mathrm{SO}(k+1)$, namely $\mathrm{SO}(k+1)$ itself.

Corollary 4.7. *If n is even and $n \geqslant 16$, then the rolling problem (R) associated to a complete simply connected (M, g) against a space form $(\hat{M}_{n;c}, \hat{g}_{n;c})$ of positive curvature $c > 0$ is completely controllable if and only if (M, g) is not of constant curvature c.*

The proof of Theorem 4.6 is divided in two steps. We first assume that the action of H^{Rol} is reducible and then deal with the case of irreducible and non-transitive action.

4.5. Reducibility

Theorem 4.8. *Let (M, g) be a complete connected Riemannian manifold and $(\hat{M}, \hat{g}) = S^n$ be the unit sphere. If the rolling holonomy group H^{Rol} corresponding to the rolling of (M, g) against S^n acts reducibly, then S^n is a Riemannian covering of (M, g).*

Proof. We write $h = h_1$ for the inner product on $TM \oplus \mathbb{R}$. Fix once and for all a point $x_0 \in M$. Since H^{Rol} acts reducibly, then there are two non-trivial subspaces $V_1, V_2 \subset T|_{x_0}M \oplus \mathbb{R}$ and invariant by the action of $H^{\mathrm{Rol}}|_{x_0}$, the holonomy group of ∇^{Rol} at x_0. Since the holonomy group of ∇^{Rol} acts h-orthogonally on $T|_{x_0}M$, it follows that $V_1 \perp V_2$.

Define subbundles $\pi_{\mathcal{D}_j} : \mathcal{D}_j \to M$, $j = 1, 2$, of $\pi_{TM\oplus\mathbb{R}}$ such that for any $x \in M$ one chooses a piecewise C^1 curve $\gamma : [0, 1] \to M$ from x_0 to x and defines

$$\mathcal{D}_j|_x = \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1(\gamma) V_j, \quad j = 1, 2.$$

These definitions are independent of the chosen path γ: if ω is another curve, then $\omega^{-1}.\gamma \in \Omega_{x_0}(M)$ is a loop based at x_0 and by the invariance of V_j, $j = 1, 2$, under the holonomy of ∇^{Rol},

$$\big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1(\gamma) V_j = \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1(\omega) \underbrace{\big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1(\omega^{-1}.\gamma) V_j}_{=V_j} = \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1(\omega) V_j.$$

Moreover, since parallel transport $(P^{\nabla^{\mathrm{Rol}}})_0^1(\gamma)$ is an h-orthogonal map, it follows that $\mathcal{D}_1 \perp \mathcal{D}_2$ w.r.t. the vector bundle metric h.

It is a standard fact that $\mathcal{D}_j$, $j = 1, 2$, are smooth embedded submanifolds of $TM \oplus \mathbb{R}$ and that the restriction of $\pi_{TM\oplus\mathbb{R}}$ to $\mathcal{D}_j$ defines a smooth subbundle $\pi_{\mathcal{D}_j}$ as claimed. Moreover, it is clear that $\pi_{\mathcal{D}_1} \oplus \pi_{\mathcal{D}_2} = \pi_{TM\oplus\mathbb{R}}$, and this sum is h-orthogonal.

We will now assume that both $\mathcal{D}_j$, $j = 1, 2$, have dimension at least 2. The case where one of them has dimension $= 1$ can be treated in a similar fashion and will be omitted. So we let $m + 1 = \dim \mathcal{D}_1$ where $m \geqslant 1$ and then $n - m = (n + 1) - (m + 1) = \dim \mathcal{D}_2 \geqslant 2$, i.e., $1 \leqslant m \leqslant n - 2$. Define for $j = 1, 2$

$$\mathcal{D}_j^M = \mathrm{pr}_1(\mathcal{D}_j) = \big\{X \bigm| (X, r) \in \mathcal{D}_j\big\} \subset TM,$$
$$N_j = \big\{x \in M \bigm| (0, 1) \in \mathcal{D}_j|_x\big\} \subset M.$$

Trivially, $N_1 \cap N_2 = \emptyset$. Also, N_j, $j = 1, 2$, are closed subsets of M since they can be written as $N_j = \{x \in M \mid p_j^{\perp}(T|_x) = T|_x\}$ where $p_j^{\perp} : TM \oplus \mathbb{R} \to \mathcal{D}_j$ is the h-orthogonal projection onto $\mathcal{D}_j$ and T is the (smooth) constant section $x \mapsto (0, 1)$ of $\pi_{TM\oplus\mathbb{R}}$.

We next provide a sketch of the proof. We show that N_j are non-empty totally geodesic submanifolds of M and, for any $x_j \in N_j$, $j = 1, 2$, that (M, g) is locally isometric to the sphere

$$S = \big\{(X_1, X_2) \in T|_{x_1}^{\perp} N_1 \oplus T|_{x_2}^{\perp} N_2 \bigm| \|X_1\|_g^2 + \|X_2\|_g^2 = 1\big\},$$

with the metric $G := (g|_{T|_{x_1}^{\perp} N_1} \oplus g|_{T|_{x_2}^{\perp} N_2})|_S$. Here $\perp$ denotes the orthogonal complement inside $T|_x M$ w.r.t. g. Since (S, G) is isometric to the Euclidean sphere $(S^n, s_{n;1})$ this would finish the argument. The latter is rather long and we decompose it in a sequence of ten lemmas.

Lemma 4.9. *The sets N_j, $j = 1, 2$, are non-empty.*

Proof. Note that $N_1 \cup N_2 \neq M$ since otherwise $N_1 = M \backslash N_2$ would be open and closed and similarly for N_2. If (say) $N_1 \neq \emptyset$, then $N_1 = M$ by connectedness of M, i.e., the point $(0, 1) \in \mathcal{D}_1|_x$ for all $x \in M$. Then, for all $x \in M$, $X \in \mathrm{VF}(M)$, $\mathcal{D}_1|_x \ni \nabla^{\mathsf{Rol}}_{X|_x}(0, 1) = (X|_x, 0)$, by the invariance of $\mathcal{D}_1$, the holonomy of ∇^{Rol} and (15), implying that $\mathcal{D}_1 = TM \oplus \mathbb{R}$, a contradiction.

Let $x' \in M \backslash (N_1 \cup N_2)$ be arbitrary. Choose a basis $(X_0, r_0), \dots, (X_m, r_m)$ of $\mathcal{D}_1|_{x'}$. Then at least one of the numbers $r_0, \dots, r_m$ is non-zero, since otherwise one would have $(X_i, r_i) = (X_i, 0) \perp (0, 1)$ for all i and thus $\mathcal{D}_1|_{x'} \perp (0, 1)$, i.e., $(0, 1) \in \mathcal{D}_2|_{x'}$, i.e., $x' \in N_2$ which is absurd. We assume that it is r_0 which is non-zero. By taking appropriate linear combinations of (X_i, r_i), $i = 0, \dots, m$ (and by Gram–Schmidt's process), one may change the basis (X_i, r_i), $i = 0, \dots, m$, of $\mathcal{D}_1|_x$ so that $r_1, \dots, r_m = 0$, $r_0 \neq 0$ and that $(X_0, r_0), (X_1, 0), \dots, (X_m, 0)$ are h-orthonormal. Also, $X_0, \dots, X_m$ are non-zero: for $X_1, \dots, X_m$ this is evident, and for X_0 it follows from the fact that if $X_0 = 0$, then $r_0 = 1$ and hence $x' \in N_1$, which contradicts our choice of x'.

Now let $\gamma : \mathbb{R} \to M$ be the unit speed geodesic with $\gamma(0) = x'$, $\dot{\gamma}(0) = \frac{X_0}{\|X_0\|_g}$. Parallel translate (X_i, r_i) along γ by ∇^{Rol} to get $\pi_{\mathcal{D}_1}$-sections $(X_i(t), r_i(t))$ along γ. In particular, from (17) one gets $\ddot{r}_i(t) + r_i(t) = 0$, with $r_0(0) \neq 0$, $r_1(0) = \dots = r_m(0) = 0$. From the second equation in (16) one obtains $\dot{r}_i(0) = g(\dot{\gamma}(0), X_i(0)) = \|X_0\|_g^{-1} g(X_0, X_i)$ and thus $\dot{r}_i(0) = 0$ for $i = 1, \dots, m$ since $(X_i, 0)$ is h-orthogonal to (X_0, r_0). Moreover, $\dot{r}_0(0) = \|X_0\|_g$. Hence $r_i(t) = 0$ for all t and $i = 1, \dots, m$ and $r_0(t) = \|X_0\|_g \sin(t) + r_0 \cos(t)$. In particular, at $t = t_0 := \arctan(-\frac{r_0}{\|X_0\|_g})$ one has $r_i(t_0) = 0$ for all $i = 0, \dots, m$ which implies that $\mathcal{D}_1|_{\gamma(t_0)} \perp (0, 1)$, i.e., $\gamma(t_0) \in N_2$. This proves that N_2 is non-empty. The same argument with $\mathcal{D}_1$ and $\mathcal{D}_2$ interchanged shows that N_1 is non-empty. □

Lemma 4.10. *For any $x \in M$ and any unit vector $u \in T|_x M$,*

$$\big(P^{\nabla^{\mathsf{Rol}}}\big)_0^t(\gamma_u)(0, 1) = \big(-\sin(t)\dot{\gamma}_u(t), \cos(t)\big). \tag{21}$$

Proof. Here and in what follows, $\gamma_u(t) := \exp_x(tu)$. Write $(X_0(t), r_0(t)) := (P^{\nabla^{\mathsf{Rol}}})_0^t(\gamma_u)(0, 1)$. The second equation in (16) implies that $\dot{r}_0(0) = g(\dot{\gamma}_u(0), X_0(0)) = g(u, 0) = 0$ and, since $r_0(0) = 1$, the second equation in (17) gives $r_0(t) = \cos(t)$. Notice that, for all $t \in \mathbb{R}$,

$$\begin{aligned}\nabla_{\dot{\gamma}_u(t)}\big(-\sin(t)\dot{\gamma}_u(t)\big) + r_0(t)\dot{\gamma}_u(t) &= \nabla_{\dot{\gamma}_u(t)}\big(-\sin(t)\big)\dot{\gamma}_u(t) - \sin(t)\nabla_{\dot{\gamma}_u(t)}\dot{\gamma}_u(t) + \cos(t)\dot{\gamma}_u(t)\\ &= -\cos(t)\dot{\gamma}_u(t) - 0 + \cos(t)\dot{\gamma}_u(t) = 0,\end{aligned}$$

i.e., $-\sin(t)\dot{\gamma}_u(t)$ solves the same first order ODE as $X_0(t)$, $\nabla_{\dot{\gamma}_u(t)}X_0 + r_0(t)\dot{\gamma}_u(t) = 0$ by the first equation in (16). Moreover, since $(-\sin(t)\dot{\gamma}_u(t))|_{t=0} = 0 = X_0(0)$, then $X_0(t) = -\sin(t)\dot{\gamma}_u(t)$, which, combined with the fact that $r_0(t) = \cos(t)$ proven above, gives (21). □

Lemma 4.11. *The sets N_j, $j = 1, 2$, are complete, totally geodesic submanifolds of (M, g) and $\mathcal{D}_j^M|_x = T|_x N_j$, $\forall x \in N_j$, $j = 1, 2$.*

Proof. We show this for N_1. The same argument then proves the claim for N_2. Let $x \in N_1$ and $u \in \mathcal{D}_1^M|_x$ a unit vector. Since $(0, 1) \in \mathcal{D}_1|_x$, Eq. (21) implies that

$$\mathcal{D}_1|_{\gamma_u(t)} \ni \big(P^{\nabla^{\mathsf{Rol}}}\big)_0^t(\gamma_u)(0, 1) = \big(-\sin(t)\dot{\gamma}_u(t), \cos(t)\big).$$

Next notice that

$$\begin{aligned}\nabla^{\mathsf{Rol}}_{\dot{\gamma}_u(t)}\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big) &= \big(-\sin(t)\dot{\gamma}_u(t) + \sin(t)\dot{\gamma}_u(t), \cos(t) - g\big(\dot{\gamma}_u(t), \cos(t)\dot{\gamma}_u(t)\big)\big)\\ &= (0, 0),\end{aligned}$$

and hence, since $(\cos(t)\dot{\gamma}_u(t), \sin(t))|_{t=0} = (u, 0) \in \mathcal{D}_1|_x$ (because $u \in \mathcal{D}_1^M|_x$, hence there is some $r \in \mathbb{R}$ such that $(u, r) \in \mathcal{D}_1|_x$ and since $(0, 1) \in \mathcal{D}_1|_x$, then $\mathcal{D}_1|_x \ni (u, r) - r(0, 1) = (u, 0)$), we have, for all $t \in \mathbb{R}$, $(\cos(t)\dot{\gamma}_u(t), \sin(t)) = (P^{\nabla^{\mathsf{Rol}}})_0^t(u, 0) \in \mathcal{D}_1|_{\gamma_u(t)}$, and then

$$\mathcal{D}_1|_{\gamma_u(t)} \ni \sin(t)\big(\cos(t)\dot{\gamma}_u(t), \sin(t)\big) + \cos(t)\big(-\sin(t)\dot{\gamma}_u(t), \cos(t)\big) = (0, 1).$$

This proves that any geodesic starting from a point of N_1 with the initial direction from $\mathcal{D}_1^M$ stays in N_1 forever. Hence, once it has been shown that N_1 is a submanifold of M with tangent space $T|_x N_1 = \mathcal{D}_1^M|_x$ for all $x \in N_1$, then automatically N_1 is totally geodesic and complete.

Let $x \in N_1$. If U is an open neighbourhood of x and $(X_{m+1}, r_{m+1}), \dots, (X_n, r_n)$ local $\pi_{\mathcal{D}_2}$-sections forming a basis of $\mathcal{D}_2$ over U, then $N_1 \cap U = \{x \in U \mid r_{m+1}(x) = \dots = r_n(x) = 0\}$.

Thus let $(X_{m+1}, r_{m+1}), \dots, (X_n, r_n) \in \mathcal{D}_2|_x$ be a basis of $\mathcal{D}_2|_x$. Choose $\epsilon > 0$ such that $\exp_x$ is a diffeomorphism from $B_g(0, \epsilon)$ onto its image U_ϵ and define for $y \in U_\epsilon$, $j = m+1, \dots, n$,

$$(X_j, r_j)|_y = \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^1\big(\tau \mapsto \exp_x\big(\tau \exp_x^{-1}(y)\big)\big)(X_j, r_j).$$

Then (X_j, r_j) are local $\pi_{\mathcal{D}_2}$-sections and it is clear that

$$N_1 \cap U_\epsilon = \big\{y \in U_\epsilon \;\big|\; r_{m+1}(y) = \dots = r_n(y) = 0\big\}.$$

Moreover, from (16), $\nabla r_j|_x = X_j|_x$, $j = m+1, \dots, n$, which are linearly independent. Hence, by taking $\epsilon > 0$ possibly smaller, we may assume that the local vector fields ∇r_j, $j = m+1, \dots, n$, are linearly independent on U_ϵ. But this means that $N_1 \cap U_\epsilon = \{y \in U_\epsilon \mid r_{m+1}(y) = \dots = r_n(y) = 0\}$ is a smooth embedded submanifold of U_ϵ with tangent space

$$\begin{aligned} T|_x N_1 &= \big\{X \in T|_x M \;\big|\; g(\nabla r_j, X) = 0,\ j = m+1, \dots, n\big\} \\ &= \big\{X \in T|_x M \;\big|\; g(X_j, X) = 0,\ j = m+1, \dots, n\big\} = \mathcal{D}_1^M\big|_x. \end{aligned}$$

Since $x \in N_1$ was arbitrary, this proves that N_1 is indeed an embedded submanifold of M and $T|_x N_1 = \mathcal{D}_1^M|_x$ for all $x \in N_1$. □

Lemma 4.12. *Let $d_i(x) := d_g(N_i, x)$, $x \in M$. Then in the set where d_i is smooth,*

$$\big(\nabla \cos\big(d_i(\cdot)\big), \cos\big(d_i(\cdot)\big)\big) \in \mathcal{D}_i^M, \tag{22}$$

where ∇ is the gradient w.r.t. g.

Proof. Let $x \in M \backslash N_1$. Choose $y \in N_1$, $u \in (T|_y N_1)^\perp$ such that $\gamma_u : [0, d_i(x)] \to M$ is the minimal normal unit speed geodesic from N_1 to x. Since $(0, 1) \in \mathcal{D}_1|_y$ (because $y \in N_1$), the parallel translate of $(0, 1)$ along γ_u stays in $\mathcal{D}_1$ which, in view of (21), gives

$$\begin{aligned} \mathcal{D}_1|_x \ni \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^{d_1(x)}(\gamma_u)(0,1) &= \big(-\sin\big(d_1(x)\big)\dot{\gamma}_u\big(d_1(x)\big), \cos\big(d_1(x)\big)\big) \\ &= \big(-\sin\big(d_1(x)\big)\nabla\big(d_1(\cdot)\big)\big|_x, \cos\big(d_1(x)\big)\big) \\ &= \big(\nabla \cos\big(d_1(\cdot)\big)\big|_x, \cos\big(d_1(x)\big)\big), \end{aligned}$$

where the last two equalities hold true if x is not in the cut nor the conjugate locus of N_1 (nor is x in N_1, by assumption). Working in the complement of these points, which is a dense subset of M and using a continuity argument, we may assure that the result holds true everywhere where d_i is smooth. The same argument proves formula (22) for d_2. □

Lemma 4.13. *For every $Y \in \mathrm{VF}(M)$, one has, wherever $d_1(\cdot)$ is smooth, that*

$$g\big(R\big(Y, \nabla d_1(\cdot)\big)\nabla d_1(\cdot), Y\big) = g(Y, Y) - \big(\nabla_Y\big(d_1(\cdot)\big)\big)^2. \tag{23}$$

Proof.

It is known (see [19]) that for any $Y, Z \in \mathrm{VF}(M)$, $d_1(\cdot)$ satisfies a PDE

$$-g\big(R\big(Y|_y, \nabla d_1(y)\big)\nabla d_1(y), Z|_y\big) = \mathrm{Hess}^2\big(d_1(\cdot)\big)(Y|_y, Z|_y) + \big(\nabla_{\nabla d_1(y)}\mathrm{Hess}\big(d_1(\cdot)\big)\big)(Y|_y, Z|_y),$$

for every $y \in M$ such that d_1 is smooth at y (and this is true in a dense subset of M). In particular, $y \notin N_1$. Also, since the set of points $y \in M$ where $\cos(d_1(y)) = 0$ or $\sin(d_1(y)) = 0$ is clearly Lebesgue zero-measurable, we may assume that $\cos(d_1(y)) \neq 0$ and $\sin(d_1(y)) \neq 0$.

Notice that $(X_0, r_0) := (\nabla\cos(d_1(\cdot)), \cos(d_1(\cdot)))$ belongs to $\mathcal{D}_1$ and has h-norm equal to 1. We may choose in a neighbourhood U of y vector fields $X_1, \dots, X_m \in \mathrm{VF}(U)$ such that $(X_0, r_0), (X_1, 0), \dots, (X_m, 0)$ is an h-orthonormal basis of $\mathcal{D}_1$ over U. Assume also that (X_0, r_0) is smooth on U. This implies that there are smooth one-forms ω^i_j, $i, j = 0, \dots, m$, defined by (set here $r_1 = \dots = r_m = 0$) $\nabla^{\mathrm{Rol}}_Y (X_i, r_i) = \sum_{i=0}^m \omega^j_i(Y)(X_j, r_j)$, $Y \in \mathrm{VF}(M)$, or, more explicitly,

$$\begin{cases} \nabla_Y X_j + r_j Y = \sum_{i=0}^m \omega^i_j(Y) X_i, \\ Y(r_j) - g(Y, X_j) = \sum_{i=0}^m \omega^i_j(Y) r_i. \end{cases}$$

Since $(X_0, r_0), \dots, (X_m, r_m)$ are h-orthonormal, it follows that $\omega^i_j = -\omega^j_i$. The fact that $r_1 = \dots = r_m = 0$ implies that $-g(Y, X_j) = \omega^0_j(Y) r_0$, $j = 1, \dots, m$, i.e.,

$$\omega^j_0(Y) = \frac{g(Y, X_j)}{\cos(d_1(\cdot))}.$$

Since $\omega^0_0 = 0$, one has $\nabla_Y X_0 + r_0 Y = \sum_{j=1}^m \omega^j_0(Y) X_j$, which simplifies to

$$\nabla_Y \nabla d_1(\cdot) = -\cot\big(d_1(\cdot)\big)\nabla_Y\big(d_1(\cdot)\big)\nabla d_1(\cdot) + \cot\big(d_1(\cdot)\big) Y - \frac{1}{\sin(d_1(\cdot))\cos(d_1(\cdot))}\sum_{j=1}^m g(X_j, Y) X_j.$$

Writing $S(Y) := \nabla_Y \nabla d_1(\cdot) = \mathrm{Hess}(d_1(\cdot))(Y, \cdot)$, one obtains

$$\begin{aligned}
(\nabla_{\nabla d_1(\cdot)} S)(Y) &= \nabla_{\nabla d_1(\cdot)}\big(S(Y)\big) - S(\nabla_{\nabla d_1(\cdot)} Y) \\
&= \frac{1}{\sin^2(d_1(\cdot))}\nabla_Y\big(d_1(\cdot)\big)\nabla d_1(\cdot) - \cot\big(d_1(\cdot)\big) g\big(\nabla_{\nabla d_1(\cdot)} Y, \nabla d_1(\cdot)\big)\nabla d_1(\cdot) \\
&\quad - \frac{1}{\sin^2(d_1(\cdot))} Y - \left(\frac{1}{\cos^2(d_1(\cdot))} - \frac{1}{\sin^2(d_1(\cdot))}\right)\sum_{j=1}^m g(Y, X_j) X_j \\
&\quad - \frac{1}{\sin(d_1(\cdot))\cos(d_1(\cdot))}\sum_{j=1}^m \big(g(Y, \nabla_{\nabla d_1(\cdot)} X_j) X_j + g(Y, X_j)\nabla_{\nabla d_1(\cdot)} X_j\big) \\
&\quad + \cot\big(d_1(\cdot)\big) \underbrace{\nabla_{\nabla_{\nabla d_1(\cdot)} Y}\big(d_1(\cdot)\big)}_{= g(\nabla d_1(\cdot), \nabla_{\nabla d_1(\cdot)} Y)} \nabla d_1(\cdot),
\end{aligned}$$

where we used that $\nabla_{\nabla d_1(\cdot)}(d_1(\cdot)) = g(\nabla d_1(\cdot), \nabla d_1(\cdot)) = 1$. On the other hand,

$$\begin{aligned}
\mathrm{Hess}^2\big(d_1(\cdot)\big)(Y, \cdot) &= S^2(Y) = S\big(S(Y)\big) \\
&= S\left(-\cot\big(d_1(\cdot)\big)\nabla_Y\big(d_1(\cdot)\big)\nabla d_1(\cdot) + \cot\big(d_1(\cdot)\big) Y - \frac{1}{\sin(d_1(\cdot))\cos(d_1(\cdot))}\sum_{j=1}^m g(X_j, Y) X_j\right) \\
&= -\cot^2\big(d_1(\cdot)\big)\nabla_Y\big(d_1(\cdot)\big)\nabla d_1(\cdot) + \cot^2\big(d_1(\cdot)\big) Y - \frac{2}{\sin^2(d_1(\cdot))}\sum_{j=1}^m g(X_j, Y) X_j \\
&\quad + \frac{1}{\sin^2(d_1(\cdot))\cos^2(d_1(\cdot))}\sum_{j=1}^m g(X_j, Y) X_j,
\end{aligned}$$

where we used that $\nabla d_1(\cdot), X_1, \dots, X_m$ are g-orthonormal (recall that $X_0 = -\sin(d_1(\cdot))\nabla d_1(\cdot)$). Thus, for any $Y, Z \in \mathrm{VF}(M)$, one has on U that

$$\begin{aligned}
&-g\big(R\big(Y,\nabla d_1(\cdot)\big)\nabla d_1(\cdot),Z\big)\\
&\quad=-g(Y,Z)+\left(\frac{1}{\sin^2(d_1(\cdot))}-\cot^2\big(d_1(\cdot)\big)\right)\nabla_Y\big(d_1(\cdot)\big)\nabla_Z\big(d_1(\cdot)\big)\\
&\qquad-\frac{1}{\sin(d_1(\cdot))\cos(d_1(\cdot))}\sum_{j=1}^m\big(g(Y,\nabla_{\nabla d_1(\cdot)}X_j)g(X_j,Z)+g(Y,X_j)g(\nabla_{\nabla d_1(\cdot)}X_j,Z)\big).
\end{aligned}$$

We also set $Z=Y$ and hence get that $-g(R(Y,\nabla d_1(\cdot))\nabla d_1(\cdot),Y)$ is equal to

$$g(Y,Y)-\nabla_Y\big(d_1(\cdot)\big)\nabla_Y\big(d_1(\cdot)\big)+\frac{2}{\sin(d_1(\cdot))\cos(d_1(\cdot))}\sum_{j=1}^m g(Y,\nabla_{\nabla d_1(\cdot)}X_j)g(X_j,Y).$$

Here $\sum_{j=1}^m g(Y,\nabla_{\nabla d_1(\cdot)}X_j)g(X_j,Y)$ is equal to

$$\begin{aligned}
-\frac{1}{\sin(d_1(\cdot))}\sum_{j=1}^m g(Y,\nabla_{X_0}X_j)g(X_j,Y)&=-\frac{1}{\sin(d_1(\cdot))}\sum_{j=1}^m g\left(Y,\sum_{i=1}^m\omega^i_j(X_0)X_i g(X_j,Y)\right)\\
&=-\frac{1}{\sin(d_1(\cdot))}\sum_{i,j=1}^m \underbrace{\omega^i_j(X_0)}_{(\star)_1}\underbrace{g(Y,X_i)g(X_j,Y)}_{(\star)_2}=0,
\end{aligned}$$

where expression $(\star)_1$ is skew-symmetric in (i,j) while $(\star)_2$ is symmetric on (i,j). Hence the sum is zero. We finally obtain $g(R(Y,\nabla d_1(\cdot))\nabla d_1(\cdot),Y)=g(Y,Y)-(\nabla_Y(d_1(\cdot)))^2$, as claimed. It is clear that this formula now holds at every point of M where $d_1(\cdot)$ is smooth and for any $Y\in\mathrm{VF}(M)$. In particular, if Y is a unit vector g-perpendicular to $\nabla d_1(\cdot)$ at a point y of M, then $\nabla_Y d_1(\cdot)|_y=g(\nabla d_1(\cdot)|_y,Y|_y)=0$ and hence $\sec(Y,d_1(\cdot))|_y=+1$. □

Lemma 4.14. *For every $x\in N_1$, a unit vector $u\in(T|_xN_1)^\perp$ and $v\in T|_xM$ with $v\perp u$,*

$$\big\|(\exp_x)_*|_{tu}(v)\big\|_g=\left|\frac{\sin(t)}{t}\right|\|v\|_g,\quad t\in\mathbb{R}. \tag{24}$$

In particular, for all unit vectors $u_1,u_2\in(T|_xN_1)^\perp$ one has $\exp_x(\pi u_1)=\exp_x(\pi u_2)$.

Proof. Let $Y_{u,v}(t)=\frac{\partial}{\partial s}|_0\exp_x(t(u+sv))$ be the Jacobi field along $\gamma_u(t)=\exp_x(tu)$ such that $Y_{u,v}(0)=0$, $\nabla_{\dot\gamma_u(0)}Y_{u,v}=v$. Since $v\perp u$, it follows from the Gauss lemma (see [20]) that $Y_{u,v}(t)\perp\dot\gamma_u(t)$ for all t. Moreover, the assumption $u\in(T|_xN_1)^\perp$ implies that $\nabla d_1(\cdot)|_{\gamma_u(t)}=\dot\gamma_u(t)$ and thus $\nabla_{Y_{u,v}(t)}(d_1(\cdot))=g(\dot\gamma_u(t),Y_{u,v}(t))=0$. By polarization, (23) rewrites as $R(Z(t),\dot\gamma_u(t))\dot\gamma_u(t)=Z(t)-g(Z(t),\dot\gamma_u(t))\dot\gamma_u(t)$, for any vector field Z along γ_u. In particular, $\nabla_{\dot\gamma_u}\nabla_{\dot\gamma_u}Y_{u,v}=-R(Y_{u,v},\dot\gamma_u)\dot\gamma_u=-Y_{u,v}$, since $g(Y_{u,v}(t),\dot\gamma_u(t))=0$ for all t. On the other hand, the vector field $Z(t)=\sin(t)P^t_0(\gamma_u)v$ satisfies along γ_u, for all t that $\nabla_{\dot\gamma_u(t)}\nabla_{\dot\gamma_u}Z=-Z(t)$ with $Z(0)=0$ and $\nabla_{\dot\gamma_u}Z|_{t=0}=v$, i.e., the same initial value problem as $Y_{u,v}$. This implies that $Y_{u,v}(t)=\sin(t)P^t_0(\gamma_u)v$, from which we obtain (24) because $Y_{u,v}(t)=t(\exp_x)_*|_{tu}(v)$. The last claim follows from the fact that the map $\exp_x|_S:S\to M$ where $S=\{u\in(T|_xN_1)^\perp\mid\|u\|=\pi\}$ is a constant map. Indeed, if $u\in S$, $v\in T|_uS$ and we identify v as an element of $T|_xM$ as usual, then by what we have just proved (note that $u=\pi\frac{u}{\|u\|_g}$), $\|(\exp_x)_*|_u(v)\|_g=\frac{\sin(\pi)}{\pi}\|v\|_g=0$. Hence $\exp_x|_S$ has zero differential on all over S which is connected, since its dimension is $n-m-1\geqslant 1$ by assumption. Hence $\exp_x|_S$ is a constant map. □

Lemma 4.15. *For every $x\in N_1$ and unit normal vector $u\in(T|_xN_1)^\perp$, the geodesic $t\mapsto\gamma_u(t)$ meets N_2 exactly at $t\in(\mathbb{Z}+\frac{1}{2})\pi$, similarly with the roles of N_1 and N_2 interchanged.*

Proof. Let $x\in N_1$ and $u\in(T|_xN_1)^\perp$ be a unit vector normal vector to N_1. For $(X,r)\in\mathcal{D}_1|_x$ define $(X(t),r(t))=(P^{\nabla^{\mathsf{Rol}}})^t_0(\gamma_u)(X,r)$. Then by (16), (17) we have (notice that $g(u,X)=0$ since $u\in(T|_xN_1)^\perp=(\mathcal{D}^M_1|_x)^\perp$ and $X\in\mathcal{D}^M_1|_x$) $r(t)=r(0)\cos(t)$. Hence, $(X(t),r(t))$ is h-orthogonal to $(0,1)$ if and only of $r(t)=0$, i.e., $r(0)\cos(t)=0$. This proves that $(0,1)\perp\mathcal{D}_1|_{\gamma_u(t)}$, i.e., $(0,1)\in\mathcal{D}_2|_{\gamma_u(t)}$, i.e., $\gamma_u(t)\in N_2$ if and only if $t\in(\frac{1}{2}+\mathbb{Z})\pi$ (obviously, there is a vector $(X,r)\in\mathcal{D}_1|_x$ with $r\neq 0$). □

Lemma 4.16. *The submanifolds N_1, N_2 are isometrically covered by Euclidean spheres of dimensions m and $n-m$, respectively, and the fundamental groups of N_1 and N_2 are finite and have the same number of elements. More precisely, for any $x \in N_1$ define*

$$S_x = \{u \in (T|_x N_1)^\perp \mid \|u\|_g = 1\},$$

equipped with the restriction of the metric $g|_x$ of $T|_x M$. Then $S_x \to N_2$; $u \mapsto \exp_x(\frac{\pi}{2}u)$, is a Riemannian covering. The same claim holds with N_1 and N_2 interchanged.

Proof. Denote by C_1 the component of N_1 containing x. We show first that $C_1 = N_1$, i.e., N_1 is connected. Let $y_1 \in N_1$. Since C_1 is a closed subset of M, there is a minimal geodesic γ_v in M from C_1 to y_1 with $\dot{\gamma}_v(0) = v$ a unit vector, $x_1 := \gamma_v(0) \in C_1$ and $\gamma_v(d) = y_1$, with $d := d_g(y_1, C_1)$. By minimality, $v \in (T|_{x_1} C_1)^\perp = (T|_{x_1} N_1)^\perp$. Hence by Lemma 4.15, $x_2 := \exp_{x_1}(\frac{\pi}{2}v) = \gamma_v(\frac{\pi}{2})$ belongs to N_2. Since the set $S_{x_2} = \{u \in (T|_{x_2} N_2)^\perp \mid \|u\|_g = 1\}$ is connected (its dimension is $m \geqslant 1$ by assumption), Lemma 4.15 implies that $\exp_{x_2}(\frac{\pi}{2}S_{x_2})$ is contained in a single component C_1' of N_1. Writing $u := \dot{\gamma}_v(\frac{\pi}{2})$, we have $\pm u \in S_{x_2}$ so

$$C_1' \ni \exp_{x_2}\left(-\frac{\pi}{2}u\right) = \exp_{x_2}\left(-\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\frac{\pi}{2}} \exp_{x_1}(tv)\right) = \exp_{x_1}\left(\left(\frac{\pi}{2} - t\right)v\right)\bigg|_{t=\frac{\pi}{2}} = x_1,$$

and since also $x_1 \in C_1$, it follows that $C_1' = C_1$. But this implies that

$$\gamma_v(\pi) = \exp_{x_1}(\pi v) = \exp_{x_2}\left(\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\frac{\pi}{2}} \exp_{x_1}(tv)\right) = \exp_{x_2}\left(\frac{\pi}{2}u\right) \in C_1.$$

It also follows from $u \in (T|_{x_2} N_2)^\perp$ that $\dot{\gamma}_v(\pi) = \frac{\mathrm{d}}{\mathrm{d}t}|_{\frac{\pi}{2}} \exp_{x_2}(tu) \in (T|_{\gamma_v(\pi)} N_1)^\perp$. Since $\exp_{x_2}((d - \frac{\pi}{2})u) = y_1 \in N_1$, Lemma 4.15 implies that $d - \frac{\pi}{2} \in (\frac{1}{2} + \mathbb{Z})\pi$, from which, since $d \geqslant 0$, we get $d \in \mathbb{N}_0\pi$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. By taking $x_2' = \gamma_v(\frac{3}{2}\pi) \in N_2$ we may show similarly that $\gamma_v(2\pi) \in C_1$ and by induction we get $\gamma_v(k\pi) \in C_1$ for every $k \in \mathbb{N}_0$. In particular, since $d \in \mathbb{N}_0\pi$, we get $y_1 = \gamma_v(d) \in C_1$. Since $y_1 \in N_1$ was arbitrary, we get $N_1 \subset C_1$ which proves the claim. Repeating the argument with N_1 and N_2 interchanged, we see that N_2 is connected.

Eq. (24) shows that, taking $u \in S_x$ and $v \in T|_u S_x$, i.e., $v \perp u$, $v \perp T|_x N_1$,

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 \exp_x\left(\frac{\pi}{2}(u + tv)\right)\right\|_g = \left\|(\exp_x)_*|_{\frac{\pi}{2}u}\left(\frac{\pi}{2}v\right)\right\|_g = \|v\|_g.$$

This shows that $u \mapsto \exp_x(\frac{\pi}{2}u)$ is a local isometry $S_x \to N_2$. In particular, the image is open and closed in N_2, which is connected, hence $u \mapsto \exp_x(\frac{\pi}{2}u)$ is onto N_2. According to Proposition II.1.1 in [20], $u \mapsto \exp_x(\frac{\pi}{2}u)$ is a covering $S_x \to N_2$.

Similarly, for any $y \in N_2$ the map $S_y \to N_1$; $u \mapsto \exp_y(\frac{\pi}{2}u)$ is a Riemannian covering.

Finally, let us prove the statement about fundamental groups. Fix a point $x_i \in N_i$ and write $\phi_i(u) = \exp_{x_i}(\frac{\pi}{2}u)$, $i = 1, 2$, for maps $\phi_1 : S_{x_1} \to N_2$, $\phi_2 : S_{x_2} \to N_1$. The fundamental groups $\pi_1(N_1)$, $\pi_1(N_2)$ of N_1, N_2 are finite since their universal coverings are the (normal) spheres S_{x_2}, S_{x_1} which are compact. Also, $\phi_1^{-1}(x_2)$ and $\phi_2^{-1}(x_1)$ are in one-to-one correspondence with $\pi_1(N_2)$ and $\pi_1(N_1)$ respectively.

Define $\Phi_1 : \phi_1^{-1}(x_2) \to S_{x_2}$; $\Phi_1(u) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{\frac{\pi}{2}} \exp_{x_1}(tu) \in S_{x_2}$ and similarly $\Phi_2 : \phi_2^{-1}(x_1) \to S_{x_1}$; $\Phi_2(u) = -\frac{\mathrm{d}}{\mathrm{d}t}|_{\frac{\pi}{2}} \exp_{x_2}(tu) \in S_{x_1}$. Clearly, for $u \in \phi_1^{-1}(x_2)$,

$$\phi_2\big(\Phi_1(u)\big) = \exp_{x_2}\left(-\frac{\pi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\frac{\pi}{2}} \exp_{x_1}(tu)\right) = \exp_{x_1}\left(\left(\frac{\pi}{2} - t\right)u\right)\bigg|_{t=\frac{\pi}{2}} = x_1,$$

i.e., Φ_1 maps $\phi_1^{-1}(x_2) \to \phi_2^{-1}(x_1)$. Similarly Φ_2 maps $\phi_2^{-1}(x_1) \to \phi_1^{-1}(x_2)$. Finally, Φ_1 and Φ_2 are inverse maps to each other since for $u \in \phi_1^{-1}(x_2)$,

$$\Phi_2\big(\Phi_1(u)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\frac{\pi}{2}} \exp_{x_2}\left(-t\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{\frac{\pi}{2}} \exp_{x_1}(su)\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\frac{\pi}{2}} \exp_{x_1}\left(\left(\frac{\pi}{2} - t\right)u\right) = u,$$

and similarly $\Phi_1(\Phi_2(u)) = u$ for $u \in \phi_2^{-1}(x_1)$. □

For sake of simplicity, we finish the proof of Theorem 4.8 under the assumption that N_2 is simply connected and indicate in Remark 4.19 below how to handle the general case.

If N_2 is simply connected, then $S_x \to N_2$; $u \mapsto \exp_x(\frac{\pi}{2}u)$, defined in Lemma 4.16 is an isometry for some (and hence every) $x \in N_1$. From Lemma 4.16, it follows that N_1 is (simply connected and) isometric to a sphere as well. We next get the following.

Lemma 4.17. *Fix $x_i \in N_j$, $j = 1, 2$, and let*

$$S_{x_1} = \{u \in (T|_{x_1} N_1)^\perp \mid \|u\|_g = 1\}, \qquad S_{x_2} = \{u \in (T|_{x_2} N_2)^\perp \mid \|u\|_g = 1\},$$

the unit normal spheres to N_1, N_2 at x_1, x_2 respectively. Consider first the maps

$$\begin{aligned} &f_1 : S_{x_1} \to N_2, \qquad f_2 : S_{x_2} \to N_1, \\ &f_1(u) = \exp_{x_1}\left(\frac{\pi}{2}u\right), \qquad f_2(v) = \exp_{x_2}\left(\frac{\pi}{2}v\right), \end{aligned} \tag{25}$$

and the map w which associates to $(u, v) \in S_{x_1} \times S_{x_2}$ the unique element of $S_{f_2(v)}$ such that $\exp_{f_2(v)}(\frac{\pi}{2}w(u, v)) = f_1(u)$. Finally let

$$\begin{aligned} &\Psi : \left]0, \frac{\pi}{2}\right[\times S_{x_1} \times S_{x_2} \to M, \\ &\Psi(t, u, v) = \exp_{f_2(v)}\big(tw(u, v)\big). \end{aligned} \tag{26}$$

Suppose that $\tilde{S} :=]0, \frac{\pi}{2}[\times S_{x_1} \times S_{x_2}$ is endowed with the metric $\tilde{g}$ such that

$$\tilde{g}|_{(t,u,v)} = dt^2 + \sin^2(t) g|_{T|_u S_{x_1}} + \cos^2(t) g|_{T|_v S_{x_2}}.$$

Then Ψ is a local isometry.

Proof. We use G to denote the *geodesic vector field* on TM, i.e., for $u \in TM$ we have

$$G|_u := \ddot{\gamma}_u(0) = \frac{d^2}{dt^2}\bigg|_0 \exp_{\pi_{TM}(u)}(tu).$$

The projections on M by π_{TM} of its integral curves are geodesics. Indeed, first we notice that

$$G|_{\dot{\gamma} u(t)} = \frac{d^2}{ds^2}\bigg|_0 \exp_{\gamma_u(t)}\big(s\dot{\gamma}_u(t)\big) = \frac{d^2}{ds^2}\bigg|_0 \gamma_u(t+s) = \ddot{\gamma}_u(t),$$

and hence, if Γ is a curve on TM defined by $\Gamma(t) = \dot{\gamma}_u(t)$, then $\dot{\Gamma}(t) = \ddot{\gamma}_u(t) = G|_{\dot{\gamma}_u(t)} = G|_{\Gamma(t)}$, and $\Gamma(0) = u$. Hence Γ satisfies the same initial value problem as $t \mapsto \Phi_G(t, u)$, which implies that $\Phi_G(t, u) = \dot{\gamma}_u(t)$, $\forall t \in \mathbb{R}$, $u \in TM$, and in particular, $(\pi_{TM} \circ \Phi_G)(t, u) = \gamma_u(t)$, $\forall t \in \mathbb{R}$, $u \in TM$.

For every $u \in TM$ there is a direct sum decomposition $H_u \oplus V_u$ of $T|_u TM$ where $V_u = V|_u(\pi_{TM})$ is the π_{TM}-vertical fiber over u and H_u is defined as

$$H_u = \left\{\frac{d}{dt}\bigg|_0 P_0^t(\gamma_X)u \;\bigg|\; X \in T|_{\pi_{TM}(u)} M\right\}.$$

We write the elements of $T|_u TM$ w.r.t. this direct sum decomposition as (A, B) where $A \in H_u$, $B \in V_u$. It can now be shown that (see [20, Lemma 4.3, Chapter II])

$$\big((\Phi_G)_t\big)_*\big|_u(A, B) = \big(Z_{(A,B)}(t), \nabla_{\dot{\gamma}_u(t)} Z_{(A,B)}\big), \quad (A, B) \in T|_u TM,\ u \in TM,$$

with $Z_{(A,B)}$, the unique Jacobi field along geodesic γ_u such that $Z_{(A,B)}(0) = A$, $\nabla_{\dot{\gamma}_u(0)} Z_{(A,B)} = B$.

We are now ready to prove the claim. First observe that $\Psi(t, u, v) = (\pi_{TM} \circ \Phi_G)(t, w(u, v))$ and hence, for $(\frac{\partial}{\partial t}, X_1, X_2) \in T\tilde{S}$,

$$\begin{aligned}\Psi_*\left(\frac{\partial}{\partial t}, X_1, X_2\right) &= (\pi_{TM})_*\left(\frac{\partial}{\partial t}\Phi_G\big(t, w(u,v)\big) + \big((\Phi_G)_t\big)_*\big|_{w(u,v)} w_*(X_1, X_2)\right) \\ &= (\pi_{TM})_*\big(G|_{\Phi_G(t,w(u,v))} + \big(Z_{w_*(X_1,X_2)}(t), \nabla_{\frac{\partial}{\partial t}(\pi_{TM}\circ\Phi_G)(t,w(u,v))} Z_{w_*(X_1,X_2)}\big)\big) \\ &= \dot\gamma_{w(u,v)}(t) + Z_{w_*(X_1,X_2)}(t).\end{aligned}$$

On the other hand, $(\pi_{TM}\circ\Phi_G)(\frac{\pi}{2}, w(u,v)) = f_1(u)$, from where $(f_1)_*|_u(X_1) = Z_{w_*(X_1,X_2)}(\frac{\pi}{2})$. Similarly, since $(\pi_{TM}\circ\Phi_G)(0, w(u,v)) = \pi_{TM}(w(u,v)) = f_2(v)$, we get $(f_2)_*|_v(X_2) = Z_{w_*(X_1,X_2)}(0)$.

As in the proof of Lemma 4.14, we see that the Jacobi equation that $Z_{w_*(X_1,X_2)}$ satisfies is $\nabla_{\dot\gamma_{w(u,v)}(t)}\nabla_{\dot\gamma_{w(u,v)}} Z_{w_*(X_1,X_2)} = -Z_{w_*(X_1,X_2)}(t)$. It is clear that this implies that $Z_{w_*(X_1,X_2)}$ has the form $Z_{w_*(X_1,X_2)}(t) = \sin(t)P_0^t(\gamma_{w(u,v)})V_1 + \cos(t)P_0^t(\gamma_{w(u,v)})V_2$, for some $V_1, V_2 \in T|_{f_2(u)}M$. Using the boundary values of $Z_{w_*(X_1,X_2)}(t)$ at $t=0$ and $t=\frac{\pi}{2}$ as derived above, we get $V_1 = P_{\frac{\pi}{2}}^0(\gamma_{w(u,v)})((f_1)_*|_u(X_1))$ and $V_2 = (f_2)_*|_v(X_2)$. Define

$$\begin{aligned} Y_1(t) &= \sin(t)P_0^t(\gamma_{w(u,v)})V_1 = \sin(t)P_{\frac{\pi}{2}}^t(\gamma_{w(u,v)})\big((f_1)_*|_u(X_1)\big), \\ Y_2(t) &= \cos(t)P_0^t(\gamma_{w(u,v)})V_2 = \cos(t)P_0^t(\gamma_{w(u,v)})\big((f_2)_*|_v(X_2)\big),\end{aligned}$$

which means that $Z = Y_1 + Y_2$. Notice that Y_1 and Y_2 are Jacobi fields along $\gamma_{w(u,v)}$.

Since $w(u,v) \in (T|_{f_2(v)}N_1)^\perp$ and $\dot\gamma_{w(u,v)}(\frac{\pi}{2}) \in (T|_{f_1(u)}N_2)^\perp$ and

$$Y_1\left(\frac{\pi}{2}\right) = (f_1)_*|_u(X_1) \in T|_{f_1(u)}N_2, \qquad Y_2(0) = (f_2)_*|_v(X_2) \in T|_{f_2(v)}N_1,$$

it follows that $Y_1, Y_2 \perp \gamma_{w(u,v)}$. We claim that moreover $Y_1 \perp Y_2$. Indeed, since $(f_2)_*|_v(X_2) \in T|_{f_2(v)}N_1$ and $(0,1) \in \mathcal{D}_1|_{f_2(v)}$ (by definition of N_1), we have $((f_2)_*|_v(X_2), 0) \in \mathcal{D}_1|_{f_2(v)}$ and hence, for all t, $(Z_1(t), r_1(t)) := (P^{\nabla^{\mathrm{Rol}}})_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_2), 0) \in \mathcal{D}_1$. On the other hand, r_1 satisfies $\ddot r_1 + r_1 = 0$ with initial conditions $r_1(0) = 0$ and $\dot r_1(0) = g(\dot\gamma_{w(u,v)}(0), Z_1(0)) = g(w(u,v), (f_2)_*|_v(X_2)) = 0$ so $r_1(t) = 0$ for all t. Thus $Z_1(t)$ satisfies $\nabla_{\dot\gamma_{w(u,v)}(t)} Z_1 = 0$, i.e., $Z_1(t) = P_0^t(\gamma_{w(u,v)})((f_2)_*|_v(X_2))$. Similarly, if $w'(u,v) := -\frac{\mathrm{d}}{\mathrm{d}t}|_{\frac{\pi}{2}}\exp_{f_2(v)}(tw(u,v)) = -\dot\gamma_{w(u,v)}(\frac{\pi}{2})$,

$$\left(Z_2\left(\frac{\pi}{2}-t\right), r_2\left(\frac{\pi}{2}-t\right)\right) := \big(P^{\nabla^{\mathrm{Rol}}}\big)_0^t(\gamma_{w'(u,v)})\big((f_1)_*|_u(X_1), 0\big) \in \mathcal{D}_2,$$

and we have $r_2(\frac{\pi}{2}-t) = 0$ and $Z_2(\frac{\pi}{2}-t) = P_0^t(\gamma_{w'(u,v)})((f_1)_*|_v(X_1))$, i.e., $Z_2(t) = P_{\frac{\pi}{2}}^t(\gamma_{w(u,v)})((f_1)_*|_v(X_1))$. But since $\mathcal{D}_1 \perp \mathcal{D}_2$ w.r.t. h, we have that $(Z_1, r_1) \perp (Z_2, r_2)$ w.r.t. h, i.e., $g(Z_1(t), Z_2(t)) = 0$ for all t (since $r_1(t) = r_2(t) = 0$). Thus,

$$\begin{aligned} g\big(Y_1(t), Y_2(t)\big) &= \sin(t)\cos(t) g\big(P_{\frac{\pi}{2}}^t(\gamma_{w(u,v)})\big((f_1)_*|_u(X_1)\big), P_0^t(\gamma_{w(u,v)})\big((f_2)_*|_v(X_1)\big)\big) \\ &= \sin(t)\cos(t) g\big(Z_2(t), Z_1(t)\big) = 0.\end{aligned}$$

This proves the claim, i.e., $Y_1 \perp Y_2$. Since $\|w(u,v)\|_g = 1$, one has

$$\begin{aligned}\left\|\Psi_*\left(\frac{\partial}{\partial t}, X_1, X_2\right)\right\|_g^2 &= \big\|\dot\gamma_{w(u,v)}(t) + Y_1(t) + Y_2(t)\big\|_g^2 = \big\|\dot\gamma_{w(u,v)}(t)\big\|_g^2 + \big\|Y_1(t)\big\|_g^2 + \big\|Y_2(t)\big\|_g^2 \\ &= 1 + \sin^2(t)^2\big\|(f_1)_*|_u(X_1)\big\|_g^2 + \cos^2(t)\big\|(f_2)_*|_v(X_2)\big\|_g.\end{aligned}$$

Finally, since $(f_1)_*|_u(X_1) = (\exp_{x_1})_*|_{\frac{\pi}{2}u}(\frac{\pi}{2}X_1)$ and $(f_2)_*|_v(X_2) = (\exp_{x_2})_*|_{\frac{\pi}{2}v}(\frac{\pi}{2}X_2)$, Eq. (24) implies that $\|(f_1)_*|_u(X_1)\|_g = |\sin(\frac{\pi}{2})|\|X_1\|_g = \|X_1\|_g$, $\|(f_2)_*|_v(X_2)\|_g = |\sin(\frac{\pi}{2})|\|X_2\|_g = \|X_2\|_g$, and therefore $\|\Psi_*(\frac{\partial}{\partial t}, X_1, X_2)\|_g^2 = 1 + \sin^2(t)\|X_2\|_g^2 + \cos^2(t)\|X_1\|_g^2 = \tilde g|_{(t,u,v)}(\frac{\partial}{\partial t}, X_1, X_2)$, i.e., Ψ is a local isometry $\tilde S \to M$. □

Lemma 4.18. *The manifold M has constant curvature equal to* 1.

Proof. By Lemma 4.17, we know that $\Psi : \tilde S \to M$ is a local isometry. Now $(\tilde S, \tilde g)$ has constant curvature $= 1$ since it is isometric to an open subset of the unit sphere (cf. [19, Chapter 1, Section 4.2]). The image $\Psi(\tilde S)$ of Ψ is clearly a dense subset of M (indeed, $\Psi(\tilde S) = M\backslash(N_1 \cup N_2)$), which implies that M has constant curvature $= 1$. □

This completes the proof the theorem in the case $1 \leqslant m \leqslant n-2$, since a complete Riemannian manifold (M,g) with constant curvature $=1$ is covered, in a Riemannian sense, by the unit sphere, i.e., S^n. The cases $m=0$ and $m=n-1$, i.e., $\dim \mathcal{D}_1 = 1$ and $\dim \mathcal{D}_2 = 1$, respectively, are treated exactly in the same way as above, but in this case N_1 is a discrete set which might not be connected. □

Remark 4.19. The argument can easily be modified to deal with the case where N_2 (nor N_1) is not simply connected. The simplifying assumption of simply connectedness of N_1 and N_2 made previously just serves to render the map $w(\cdot,\cdot)$ globally defined on $S_{x_1} \times S_{x_2}$. Otherwise we must define w only locally and, in its definition, make a choice corresponding to different sheets (of which there is a finite number).

4.6. Non-transitive irreducible action

Following the same line of arguments that have been used in proving the classification of Riemannian holonomy groups, the next step to take after proving Theorem 4.8 consists of studying the case where $H^{\mathrm{Rol}}|_{x_0}$ acts irreducibly on $T|_{x_0}M \oplus \mathbb{R}$ and non-transitively on the h_1-unit sphere of $T|_{x_0}M \oplus \mathbb{R}$, where the latter means that there are more than one $H^{\mathrm{Rol}}|_{x_0}$-orbit on that unit sphere. We will prove that in this case of irreducible and non-transitive action of the rolling holonomy group, the manifold (M,g) has to have, again, constant curvature one. To do this, we will use the results from [22].

For the ease of reading, we first recall some definitions and the key results from [22]. Let V be a vector space. The action of $\mathrm{GL}(V)$ on V induces in a natural way an action of $\mathrm{GL}(V)$ on the tensor spaces $\bigotimes^k V^* \otimes \bigotimes^m V$ of (k,m)-tensors by

$$(gT)(X_1,\dots,X_k,\omega_1,\dots,\omega_m) := T\big(g^{-1}X_1,\dots,g^{-1}X_k,\omega_1\circ g,\dots,\omega_m\circ g\big),$$

where $T \in \bigotimes^k V^* \otimes \bigotimes^m V$, $X_1,\dots,X_k \in V$, $\omega_1,\dots,\omega_m \in V^*$.

If $\mathcal{P} \in \bigotimes^3 V^* \otimes V$, we write usually $\mathcal{P}(X,Y)Z$ for $\mathcal{P}(X,Y,Z,\cdot) \in V$, where $X,Y,Z \in V$. If $g \in \mathrm{GL}(V)$ and $\mathcal{P}$ is a $(1,3)$-tensor, then $(g\mathcal{P})(X,Y) = g \circ \mathcal{P}(g^{-1}X, g^{-1}Y) \circ g^{-1}$. This implies that $\mathfrak{gl}(V)$ acts on an element $\bigotimes^3 V^* \otimes V$ by

$$(A\mathcal{P})(X,Y) = -\mathcal{P}(AX,Y) - \mathcal{P}(X,AY) - \big[\mathcal{P}(X,Y),A\big]_{\mathfrak{so}}, \tag{27}$$

where $A \in \mathfrak{gl}(V)$. Let G be a subgroup of $O(V)$, where V is an inner product space. We recall that G acts (a) irreducibly in V if the only G-invariant subspaces of G are $\{0\}$ and V and (b) transitively on (the unit sphere of) V if for one (and hence any) unit vector $X \in V$ one has $GX = S^{n-1}(V)$, where $S^{n-1}(V)$ is the unit sphere of V. We also recall that if a connected subgroup G of $O(V)$ acts irreducibly in V, then G is compact (see [14, Appendix 5]). The concept of a curvature tensor, in abstract setting, is defined as follows.

Definition 4.20. Let V be a vector space with inner product $\langle\cdot,\cdot\rangle$. Then a $(1,3)$-tensor $\mathcal{R} \in \bigotimes^3 V^* \otimes V$ is called *a curvature* (*tensor*) *in* V if the following conditions hold for all $X,Y,Z,W \in V$

$$\mathcal{R}(X,Y) = -\mathcal{R}(Y,X), \tag{28}$$

$$\big\langle\mathcal{R}(X,Y)Z,W\big\rangle = -\big\langle\mathcal{R}(X,Y)W,Z\big\rangle, \tag{29}$$

$$\big\langle\mathcal{R}(X,Y)Z,W\big\rangle = \big\langle\mathcal{R}(Z,W)X,Y\big\rangle, \tag{30}$$

$$\mathcal{R}(X,Y)Z + \mathcal{R}(Y,Z)X + \mathcal{R}(Z,X)Y = 0. \tag{31}$$

From these one makes the following observations. Eqs. (28), (29) imply that $\mathcal{R}$ can be seen as a map $\mathcal{R}: \wedge^2 V \to \wedge^2 V$ by defining $\langle\mathcal{R}(X\wedge Y), Z\wedge W\rangle$ to be $\langle\mathcal{R}(X,Y)Z,W\rangle$, where in the former $\langle\cdot,\cdot\rangle$ is the inner product in $\wedge^2 V$ induced in the standard way by $\langle\cdot,\cdot\rangle$ in V. Then Eq. (30) means that $\mathcal{R}$ as a map $\wedge^2 V \to \wedge^2 V$ is orthogonal.

Definition 4.21. Let V be an inner product space, G a compact subgroup of $O(V)$ with Lie algebra $\mathfrak{g}$ and $\mathcal{R}$ a curvature tensor in V. The triple $(V,\mathcal{R},G)$ is a *holonomy system* if

$$\mathcal{R}(X,Y) \in \mathfrak{g}, \quad \forall X,Y \in V.$$

Notice that by (29), if $\mathcal{R}$ is a curvature in V, then $\mathcal{R}(X,Y): V \to V$ is skew-symmetric, i.e. $\mathcal{R}(X,Y) \in \mathfrak{so}(V)$ for all $X, Y \in V$. Moreover, it is easy to see that for all $g \in O(V)$ one has that $g\mathcal{R}$ is a curvature in V.

Definition 4.22. If $(V, \mathcal{R}, G)$ is a holonomy system, we write $G(\mathcal{R})$ for the linear span in $\mathfrak{gl}(V)$ of $\{(g\mathcal{R})(X,Y) \mid X, Y \in V\}$.

Clearly for all $\mathcal{Q} \in G(\mathcal{R})$, $g \in G$ one has $g\mathcal{Q} \in G(\mathcal{R})$ and hence $A\mathcal{Q} \in G(\mathcal{R})$ for all $A \in \mathfrak{g}$. Moreover, if $g \in G$ and $X, Y \in V$, then since one can write $(g\mathcal{R})(X,Y)$ as $\mathrm{Ad}(g)\mathcal{R}(X,Y)$ which belongs to $\mathfrak{g}$, because $\mathcal{R}(X,Y) \in \mathfrak{g}$, we get that $\mathcal{Q}(X,Y) \in \mathfrak{g}$ for all $\mathcal{Q} \in G(\mathcal{R})$, $X, Y \in V$. Thus we may pose the following definition.

Definition 4.23. If $(V, \mathcal{R}, G)$ is a holonomy system, we define $\mathfrak{g}^{\mathcal{R}}$ as the linear span of $\{\mathcal{Q}(X,Y) \mid \mathcal{Q} \in G(\mathcal{R}), X, Y \in V\}$ in $\mathfrak{g}$.

The subset $\mathfrak{g}^{\mathcal{R}}$ of $\mathfrak{g}$ is more than just a subspace as will be shown next.

Lemma 4.24. *The linear space $\mathfrak{g}^{\mathcal{R}}$ is an ideal in $\mathfrak{g}$.*

Proof. Let $\mathcal{Q} \in G(\mathcal{R})$, $X, Y \in V$, $A \in \mathfrak{g}$. By Eq. (27),

$$\left[\mathcal{Q}(X,Y), A\right]_{\mathfrak{so}} = -\mathcal{Q}(AX,Y) - \mathcal{Q}(X,AY) - (A\mathcal{Q})(X,Y).$$

We observed just before the previous definition that $A\mathcal{Q} \in G(\mathcal{R})$. Thus all the terms on the right belong to $\mathfrak{g}^{\mathcal{R}}$ by the very definition of it. Therefore $\mathfrak{g}^{\mathcal{R}}$ is an ideal in $\mathfrak{g}$. □

Hence the following definition makes sense.

Definition 4.25. Let $(V, \mathcal{R}, G)$ be a holonomy system. We write $G^{\mathcal{R}}$ for the Lie subgroup of G corresponding to the ideal $\mathfrak{g}^{\mathcal{R}}$ of $\mathfrak{g}$.

We need to define the concepts of an irreducible, transitive and symmetric holonomy systems.

Definition 4.26. A holonomy system $(V, \mathcal{R}, G)$ is said to be

(1) *reducible* (resp. *irreducible*) if G acts reducibly (resp. irreducibly) in V;
(2) *symmetric* if $g\mathcal{R} = \mathcal{R}$ for all $g \in G$.

If G is connected, the symmetry (2) of a holonomy system $(V, \mathcal{R}, G)$ can be written in the infinitesimal way as: $A\mathcal{R} = 0$, $\forall A \in \mathfrak{g}$. We state the main result of [22].

Proposition 4.27.

- *Let $(V, \mathcal{R}, G)$ be an irreducible holonomy system. If $G^{\mathcal{R}}$ does not act transitively on (the unit sphere of) V, then $(V, \mathcal{R}, G)$ is symmetric.*
- *If $(V, \mathcal{R}, G)$ and $(V, \mathcal{R}', G)$ are two irreducible symmetric holonomy systems with the same V and G and if both $\mathcal{R}$ and $\mathcal{R}'$ are non-zero, then there exists $c \in \mathbb{R}$ such that $\mathcal{R}' = c\mathcal{R}$.*

We next deduce from the previous proposition our main result.

Theorem 4.28. *Let (M, g) be a simply connected Riemannian manifold and (S^n, s_{n+1}) be the unit sphere. Then the rolling holonomy group $H^{\mathsf{Rol}}|_{x_0}$, for some $x_0 \in M$, cannot act both irreducibly and non-transitively on $T|_{x_0}M \oplus \mathbb{R}$.*

Proof. We argue by contradiction. Assume that $H^{\mathsf{Rol}}|_{x_0}$ acts irreducibly and non-transitively on $T|_{x_0}M \oplus \mathbb{R}$. Since M is connected, it follows that for any $x \in M$, $H^{\mathsf{Rol}}|_x$ acts irreducibly and non-transitively on $T|_x M \oplus \mathbb{R}$. We will continue using x_0 in the notations below, but we don't consider it to be fixed anymore. Notice moreover that simply connectedness of M implies that $H^{\mathsf{Rol}}|_{x_0}$ is connected. Write $M \times \mathbb{R}$. The canonical, positively directed unit vector field in the $\mathbb{R}$ gives rise to a vector field ∂_t in $M \times \mathbb{R}$ in a natural way. We equip $M \times \mathbb{R}$ with the metric h_1,

$$h_1\big((X, r\partial_t), (Y, s\partial_t)\big) = g(X, Y) + rs, \quad (X, r\partial_t), (Y, s\partial_t) \in T(M \times \mathbb{R}).$$

If $\mathrm{pr}_1 : M \times \mathbb{R} \to M$ is the projection onto the first factor, then the pull-back bundle $\mathrm{pr}_1^*(\pi_{TM\oplus\mathbb{R}})$ is canonically isomorphic to $\pi_{T(M\times\mathbb{R})}$. We define a connection ∇^{R} on the manifold as the pull-back $\mathrm{pr}_1^*(\nabla^{\mathsf{Rol}})$ determined by

$$\begin{aligned}\nabla^{\mathsf{R}}_{(X,r\partial_t)}(Y, s\partial_t) &= \nabla^{\mathsf{Rol}}_X(Y, s), \quad \forall X, Y \in \mathrm{VF}(M),\ r, s \in C^\infty(M)\\ &= \big(\nabla_X Y + sX, \big(X(s) - g(X, Y)\big)\partial_t\big).\end{aligned}$$

One has ∇^{R} is h_1-compatible (i.e. metric w.r.t. h_1) and if $T^R := T^{\nabla^{\mathsf{R}}}$, then $T^{\mathsf{R}}((X, r\partial_t), (Y, s\partial_t)) = r(Y, s\partial_t) - s(X, r\partial_t)$, so it is not the Levi-Civita connection of $(M \times \mathbb{R}, h_1)$.

Write $H^{\mathsf{R}} := H^{\nabla^{\mathsf{R}}}$ for the holonomy group(s) of ∇^{R}. Next we show that for every $(x_0, s_0) \in M \times \mathbb{R}$, one has $H^{\mathsf{R}}|_{(x_0,s_0)} = H^{\mathsf{Rol}}|_{x_0}$, where the isomorphism $T|_{(x_0,s_0)}(M \times \mathbb{R}) \cong T|_{x_0}M \oplus \mathbb{R}$ is understood. Indeed, suppose $(\gamma, \tau) : [0, 1] \to M \times \mathbb{R}$ is a piecewise smooth path, $(\gamma, \tau)(0) = (x_0, s_0)$ and $(X_0, r_0\partial_t|_{s_0}) \in T|_{(x_0,s_0)}(M \times \mathbb{R})$. Let $(X(t), r(t)\partial_t|_{\tau(t)}) := (P^{\nabla^{\mathsf{R}}})^t_0(\gamma, \tau)(X_0, r_0\partial_t|_{s_0})$ and $(\overline{X}(t), \bar{r}(t)) := (P^{\nabla^{\mathsf{Rol}}})^t_0(\gamma)(X_0, r_0)$. It is enough to show that $(\overline{X}(t), \bar{r}(t)\partial_t|_{\tau(t)}) = (X(t), r(t)\partial_t|_{\tau(t)})$ for all $t \in [0, 1]$. But this is clear since $\nabla^{\mathsf{R}}_{(\dot\gamma(t),\dot\tau(t)\partial_t|_{\tau(t)})}(\overline{X}, \bar{r}\partial_t) = \nabla^{\mathsf{Rol}}_{\dot\gamma(t)}(\overline{X}, \bar{r}) = 0$. Thus for every $(x_0, s_0) \in M \times \mathbb{R}$, the ∇^{R}-holonomy group $H^{\mathsf{R}}|_{(x_0,s_0)} \subset \mathrm{SO}(T|_{(x_0,s_0)}(M \oplus \mathbb{R}))$ acts irreducibly and non-transitively on $T|_{(x_0,s_0)}(M \oplus \mathbb{R})$. Proposition 4.27 therefore implies that for all $(x_0, s_0) \in M \times \mathbb{R}$ the holonomy system $S_{(x_0,s_0)} := (T|_{(x_0,s_0)}(M \oplus \mathbb{R}), R^{\nabla^{\mathsf{R}}}|_{(x_0,s_t)}, H^{\mathsf{R}}|_{(x_0,s_0)})$ is symmetric. Notice that the fact that $S_{(x_0,s_0)}$ is a holonomy system in the first place follows from three facts: (1) $R^{\nabla^{\mathsf{R}}}|_{(x_0,s_t)}$ satisfies Eqs. (28)–(31), (2) $H^{\mathsf{R}}|_{(x_0,s_0)}$ is compact since it is a connected subgroup of $\mathrm{SO}(T|_{(x_0,s_0)}(M \times \mathbb{R}))$ acting irreducibly (see [14, Appendix 5]) and (3) Ambrose–Singer theorem implies that $R^{\nabla^{\mathsf{R}}}|_{(x_0,s_t)}((X, r\partial_t), (Y, s\partial_t))$ always belongs to the Lie algebra of $H^{\mathsf{R}}|_{(x_0,s_0)}$. Moreover, we have explicitly

$$R^{\nabla^{\mathsf{R}}}\big((X, r\partial_t), (Y, s\partial_t)\big)(Z, u\partial_t) = R^{\nabla^{\mathsf{Rol}}}(X, Y)(Z, u) = \big(R(X, Y)Z - B(X, Y)Z, 0\big),$$

where $B(X, Y)Z := g(Y, Z)X - g(X, Z)Y$. Notice that $R^{\nabla^{\mathsf{R}}}$ cannot vanish identically on $M \times \mathbb{R}$, since in that case $H^{\mathsf{R}}|_{(x_0,s_0)}$ would be trivial by Ambrose–Singer theorem, which contradicts the irreducibility of its action.

Consider the open set $O := \{(x, s) \in M \times \mathbb{R} \mid R^{\nabla^{\mathsf{R}}} \neq 0\}$. We claim that O is actually empty, which leads us to the sought contradiction. Indeed, suppose $(x_0, s_0), (x, s) \in O$ and choose some path $(\gamma, \tau) : [0, 1] \to M$ from (x, s) to (x_0, s_0). Then if $R_0^{\nabla^{\mathsf{R}}}$ denotes the parallel translate $(P^{\nabla^{\mathsf{R}}})^1_0(\gamma, \tau)R^{\nabla^{\mathsf{R}}}|_{(x,s)}$, then $R_0^{\nabla^{\mathsf{R}}}$ is a non-zero curvature tensor in $T|_{(x_0,s_0)}(M \times \mathbb{R})$. The Ambrose–Singer theorem implies that $(T|_{(x_0,s_0)}(M \times \mathbb{R}), R_0^{\nabla^{\mathsf{R}}}, H^{\mathsf{R}}|_{(x_0,s_0)})$ and $(T|_{(x_0,s_0)}(M \times \mathbb{R}), R^{\nabla^{\mathsf{R}}}|_{(x_0,s_0)}, H^{\mathsf{R}}|_{(x_0,s_0)})$ are both holonomy systems. Also, Proposition 4.27 implies that the holonomy system $(T|_{(x_0,s_0)}(M \times \mathbb{R}), R_0^{\nabla^{\mathsf{R}}}, H^{\mathsf{R}}|_{(x_0,s_0)})$ is symmetric. Therefore, if one writes $V = T|_{(x_0,s_0)}(M \times \mathbb{R})$, $G = H^{\mathsf{R}}|_{(x_0,s_0)}$, $\mathcal{R} = R^{\nabla^{\mathsf{R}}}|_{(x_0,s_0)} \neq 0$, $\mathcal{R}' = R_0^{\nabla^{\mathsf{R}}} \neq 0$, Proposition 4.27 shows that there exists a unique $c \neq 0$ such that $(P^{\nabla^{\mathsf{R}}})^1_0(\gamma, \tau)R^{\nabla^{\mathsf{R}}}|_{(x,s)} = cR^{\nabla^{\mathsf{R}}}|_{(x_0,s_0)}$.

Let E be the exponential mapping of ∇^{R} starting at (x_0, s_0) and choose $\mathcal{U} \subset T|_{(x_0,s_0)}(M \times \mathbb{R})$ small enough such that this exponential mapping is a diffeomorphism of $\mathcal{U}$ onto an open subset $U \ni (x_0, s_0)$ of $M \times \mathbb{R}$ which is contained in the open set O. Then by the above formula, for every $(x, s) \in U$ one has a unique $f(x, s) \neq 0$ such that

$$f(x, s)\big(P^{\nabla^{\mathsf{R}}}\big)^1_0\big(t \mapsto E\big(tE^{-1}(x, s)\big)\big)R^{\nabla^{\mathsf{R}}}\big|_{(x_0,s_0)} = R^{\nabla^{\mathsf{R}}}\big|_{(x,s)}.$$

Clearly $(x, s) \mapsto f(x, s)$ is smooth. Let Γ be a ∇^{R}-geodesic through (x_0, s_0). Then since for t small, $f(\Gamma(t))R^{\nabla^{\mathsf{R}}}|_{(x_0,s_0)} = (P^{\nabla^{\mathsf{R}}})^0_t(\Gamma)R^{\nabla^{\mathsf{R}}}|_{\Gamma(t)}$, we get

$$\dot\Gamma(0)(f)R^{\nabla^{\mathsf{R}}}\big|_{(x_0,s_0)} = \nabla_{\dot\Gamma(0)}R^{\nabla^{\mathsf{R}}}. \tag{32}$$

If $\mathrm{pr}_2 : M \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second factor, one sees from the explicit expression of $R^{\nabla^{\mathsf{R}}}$ that $(\mathrm{pr}_2)_*(R^{\nabla^{\mathsf{R}}}((Y, s\partial_t), (Z, u\partial_t))(W, v\partial_t)) = 0$, for every $(Y, s\partial_t), (Z, u\partial_t), (W, v\partial_t) \in T|_{(x_0,s_0)}(M \times \mathbb{R})$. Thus (32) shows that $(\mathrm{pr}_2)_*(\nabla_{\dot\Gamma(0)}R^{\nabla^{\mathsf{R}}}((Y, s\partial_t), (Z, u\partial_t))(W, v\partial_t)) = 0$.

Write $(X, r\partial_t|_{s_0}) = \dot{\Gamma}(0)$ and take $Y, Z, W \in \mathrm{VF}(M)$, $s, u, v \in C^\infty(M)$. Then one has

$$\begin{aligned}
&\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}\big(R^{\nabla^{\mathrm{R}}}\big((Y, s\partial_t), (Z, u\partial_t)\big)(W, v\partial_t)\big) \\
&\quad = \big(\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)} R^{\nabla^{\mathrm{R}}}\big)\big((Y, s\partial_t), (Z, u\partial_t)\big)(W, v\partial_t) + R^{\nabla^{\mathrm{R}}}\big(\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}(Y, s\partial_t), (Z, u\partial_t)\big)(W, v\partial_t) \\
&\qquad + R^{\nabla^{\mathrm{R}}}\big((Y, s\partial_t), \nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}(Z, u\partial_t)\big)(W, v\partial_t) + R^{\nabla^{\mathrm{R}}}\big((Y, s\partial_t), (Z, u\partial_t)\big)\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}(W, v\partial_t),
\end{aligned}$$

and hence, $(\mathrm{pr}_2)_*(\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}(R^{\nabla^{\mathrm{R}}}((Y, s\partial_t), (Z, u\partial_t))(W, v\partial_t))) = 0$. Moreover, one also has

$$\begin{aligned}
&(\mathrm{pr}_2)_*\nabla^{\mathrm{R}}_{\dot{\Gamma}(0)}\big(R^{\nabla^{\mathrm{R}}}\big((Y, s\partial_t), (Z, u\partial_t)\big)(W, v\partial_t)\big) \\
&\quad = (\mathrm{pr}_2)_*\nabla^{\mathrm{R}}_{(X, r\partial_t|_{s_0})}\big(R(Y,Z)W - B(Y,Z)W, 0\big) = -g\big(X, R(Y,Z)W - B(Y,Z)W\big).
\end{aligned}$$

Hence $g(X, R|_{x_0}(Y,Z)W - B|_{x_0}(Y,Z)W) = 0$ and since Y, Z, W, Γ, and thus X, were arbitrary, we deduce from this that $R|_{x_0} = B|_{x_0}$. But this then implies that $R^{\nabla^{\mathrm{R}}}|_{(x_0,s_0)} = 0$, which is in contradiction with the definition of the set O containing (x_0, s_0). □

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