Available online at www.sciencedirect.com
SciVerse ScienceDirect
ELSEVIER
Ann. I. H. Poincaré – AN 30 (2013) 157–178
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion

Youshan Tao [a], Michael Winkler [b,*]

[a] *Department of Applied Mathematics, Dong Hua University, Shanghai 200051, PR China*
[b] *Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany*

Received 4 December 2011; received in revised form 11 July 2012; accepted 17 July 2012

Available online 23 July 2012

Abstract

This paper deals with a boundary-value problem in three-dimensional smoothly bounded domains for a coupled chemotaxis-Stokes system generalizing the prototype

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t + \nabla P = \Delta u + n \nabla \phi, \\ \nabla \cdot u = 0, \end{cases}$$

which describes the motion of oxygen-driven swimming bacteria in an incompressible fluid.

It is proved that global weak solutions exist whenever $m > \frac{8}{7}$ and the initial data (n_0, c_0, u_0) are sufficiently regular satisfying $n_0 > 0$ and $c_0 > 0$. This extends a recent result by Di Francesco, Lorz and Markowich [M. Di Francesco, A. Lorz, P.A. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior, Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1437–1453] which asserts global existence of weak solutions under the constraint $m \in [\frac{7+\sqrt{217}}{12}, 2]$.

Résumé

Ce papier considère un problème aux limites dans des domaines tridimensionnels réguliers et bornés, plus précisément, un système couplé de chemotaxie-Stokes qui généralise le prototype

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n \nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nc, \\ u_t + \nabla P = \Delta u + n \nabla \phi, \\ \nabla \cdot u = 0 \end{cases}$$

et qui décrit le mouvement des bactéries nageuses conduites par l'oxygène dans un fluide incompressible.

On montre que les solutions faibles globales existent quand $m > \frac{8}{7}$ et la donnée initiale (n_0, c_0, u_0) est suffisamment régulière et vérifie $n_0 > 0$ et $c_0 > 0$. Cela étend le résultat récent de Di Francesco, Lorz et Markowich [M. Di Francesco, A. Lorz, P.A. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior,

* Corresponding author.
E-mail addresses: taoys@dhu.edu.cn (Y. Tao), michael.winkler@math.uni-paderborn.de (M. Winkler).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2012.07.002

Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1437–1453] qui affirme l'existence globale de solutions faibles sous la contrainte $m \in [\frac{7+\sqrt{217}}{12}, 2]$.

MSC: 35K55; 35Q92; 35Q35; 92C17

Keywords: Chemotaxis; Stokes; Nonlinear diffusion; Global existence; Boundedness

Mots-clés : Chemotaxie ; Stokes ; Diffusion nonlinéaire ; Existence globale ; Estimation uniforme

1. Introduction

We consider a mathematical model for the motion of oxygen-driven swimming bacteria in an incompressible viscous fluid. Such bacteria may orient their movement towards higher concentration of oxygen which they consume, and the motion of the fluid is under the influence of external forces such as gravity exerted from aggregating bacteria onto the fluid. Both bacteria and oxygen diffuse through the fluid, and they are also transported by the fluid (cf. [3] and [15]).

Taking into account all these processes, in [23] the authors proposed the model

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot\big(n\chi(c)\nabla c\big), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u - \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the unknown bacterial density n, the oxygen concentration c, the fluid velocity field u and the associated pressure P in the physical domain $\Omega\subset\mathbb{R}^N$. The function $\chi(c)$ measures the chemotactic sensitivity, $f(c)$ is the consumption rate of the oxygen by the bacteria, ϕ represents the gravitational potential, and the constant κ is related to the strength of nonlinear fluid convection.

There are only few results on the mathematical analysis of this chemotaxis-Navier–Stokes system (1.1). In [15], local-in-time weak solutions were constructed for a boundary-value problem for (1.1) in the three-dimensional setting. In [5], global classical solutions near constant states were constructed for (1.1) with $\Omega=\mathbb{R}^3$. In [14], global weak solutions to (1.1) with arbitrarily large initial data in $\Omega=\mathbb{R}^2$ were constructed. Very recently, in [27], a unique global classical solution has been constructed for (1.1) with arbitrarily large initial data in bounded convex domains $\Omega\subset\mathbb{R}^2$. The question whether solutions of (1.1) with large initial data exist globally or may blow up appears to remain an open and challenging topic in the three-dimensional case.

The chemotaxis-Stokes system. Main results. Well-established physical considerations suggest to modify (1.1) in at least two directions: Firstly, when the fluid motion is slow, a commonly employed approximation of the Navier–Stokes equations is given by the Stokes equations in which the nonlinear convective term $u\cdot\nabla u$ is ignored in the u-equation of (1.1). For this simplification of (1.1) thus obtained by setting $\kappa=0$, it is asserted in [5] that when $\Omega=\mathbb{R}^2$, appropriate smallness assumptions on either the initial data for c or $\nabla\phi$ ensure global existence of weak solutions, provided that some technical conditions on χ and f are satisfied. For instance, this set of conditions allows to cover the case when $\chi\equiv 1$ and f is strictly increasing and strictly concave on $[0,\infty)$. For bounded convex domains $\Omega\subset\mathbb{R}^2$ these assumptions could be relaxed in [27] to include the choices made in (1.2) with $D\equiv 1$, and moreover the global solutions constructed there are classical and bounded throughout $\Omega\times(0,\infty)$.

Secondly, the diffusion of bacteria (or, more generally, of cells) in a viscous fluid may be viewed like movement in a porous medium (see the discussions in [24,19,1,11], for instance). Adjusting the above model accordingly and fixing $\chi(c)\equiv 1$ and $f(c)=c$ for definiteness, we shall subsequently consider the chemotaxis-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot\big(D(n)\nabla n\big) - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t = \Delta u - \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

in a smoothly bounded convex domain $\Omega \subset \mathbb{R}^3$, with prescribed initial data

$$n(x,0) = n_0(x), \qquad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

and under the boundary conditions

$$D(n)\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Here we assume that

$$D \in C^{1+\theta}_{\mathrm{loc}}\big([0,\infty)\big) \quad \text{for some } \theta > 0, \tag{1.5}$$

as well as

$$D(s) \geqslant m s^{m-1} \quad \text{for all } s > 0 \tag{1.6}$$

with some $m > 1$, and that

$$\phi \in W^{1,\infty}(\Omega). \tag{1.7}$$

As to the initial data, for simplicity we shall require throughout this paper that

$$\begin{cases} n_0 \in C^1(\bar{\Omega}) \text{ is positive in } \bar{\Omega}, \\ c_0 \in C^1(\bar{\Omega}) \text{ is positive in } \bar{\Omega}, \quad \text{and} \\ u_0 \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \text{ is such that } \nabla \cdot u_0 = 0. \end{cases} \tag{1.8}$$

Under these assumptions, our main result is the following.

Theorem 1.1. *Suppose that (1.5)–(1.8) hold with some $m > \frac{8}{7}$. Then (1.2)–(1.4) possesses at least one global weak solution (n, c, u, P) in the sense of Definition 3.1 below. Moreover, for any fixed $T > 0$ this solution is bounded in $\Omega \times (0,T)$ in the sense that*

$$\big\|n(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant C(T) \quad \textit{for all } t \in (0,T) \tag{1.9}$$

is valid with some $C(T) > 0$.

Moreover, if in addition we assume that

$$D(s) > 0 \quad \text{for all } s \geqslant 0, \tag{1.10}$$

so that the first PDE in (1.2) becomes uniformly parabolic, then our solutions will actually be smooth and hence classical:

Theorem 1.2. *Suppose that (1.5)–(1.8) and (1.10) hold with some $m > \frac{8}{7}$. Then (1.2)–(1.4) possesses at least one global classical solution (n, c, u, P).*

A natural question that has to be left open here is whether the achieved lower bound $\frac{8}{7}$ for m is optimal. It should be noted in this context that even for $m = 1$ certain weak solutions exist globally in time [27]; however, it is neither known whether these solutions are classical, nor if they enjoy a boundedness property as in (1.9).

Porous medium-type diffusion in chemotaxis systems. Before going into details, let us briefly comment on known facts about the interplay of nonlinear diffusion and chemotactic cross-diffusion. Indeed, several rigorous results in the literature on corresponding Keller–Segel systems without fluid interaction indicate that increasing m in the porous medium-type diffusion Δn^m with $m > 1$ can enhance the balancing effect of diffusion on the tendency toward cell accumulation due to chemotaxis. For instance, let us consider the classical chemotaxis system in bounded domains $\Omega \subset \mathbb{R}^N$ with nonlinear diffusion and nonlinear cross-diffusion (cf. [8]),

$$\begin{cases} n_t = \nabla \cdot \big(D(n)\nabla n\big) - \nabla \cdot \big(S(n)\nabla c\big), \\ c_t = \Delta c - c + n, \end{cases} \tag{1.11}$$

under the assumption that $D(n)$ does not decay faster than algebraically as $n \to \infty$. Then known results say that if $\frac{S(n)}{D(n)} \leqslant Cn^{\frac{2}{N}-\varepsilon}$ holds for some $C > 0$, $\varepsilon > 0$ and all large n, then all solutions are global in time and bounded ([16,21,9], see also [10]), whereas if $\frac{S(n)}{D(n)} \geqslant Cn^{\frac{2}{N}+\varepsilon}$ for some $C > 0$, $\varepsilon > 0$ and large n, then there exist solutions which blow up either in finite or in infinite time [26].

We note that our results assert a range $m > \frac{8}{7}$ of global existence that is larger than the corresponding boundedness regime $m > \frac{4}{3}$ for (1.11). However, a comparison of these seems only partially adequate, because in (1.2) the chemoattractant is consumed, rather than produced, by the population individuals.

Some precedents also indicate a similar explosion-inhibiting effect of porous medium-type diffusion in chemotaxis systems when coupled to fluid equations. A first result of this flavor [4] addresses the chemotaxis-Stokes variant of (1.1) (with $\kappa = 0$) and asserts global existence of weak solutions in bounded domains $\Omega \subset \mathbb{R}^2$ when $m \in (\frac{3}{2}, 2]$ and f is increasing with $f(0) = 0$. This global existence result in the spatially two-dimensional setting could recently be extended in [22] so as to cover the whole range $m \in (1, \infty)$, and moreover it has been shown there that all solutions evolving from sufficiently regular initial data are uniformly bounded in $\Omega \times (0, \infty)$. The work [14] proves global weak solvability of the chemotaxis-Stokes variant of (1.1) for the precise value $m = \frac{4}{3}$ and $\Omega = \mathbb{R}^3$ under some additional assumptions on χ and f. This complements a corresponding result in [4] which asserts global weak solvability of the chemotaxis-Stokes variant of (1.1) for any $m \in [\frac{7+\sqrt{217}}{12}, 2]$ and bounded domains $\Omega \subset \mathbb{R}^3$.

Methods of proof. Plan of the paper. Whereas the proofs in the mentioned previous related works [4,5,14,27] are crucially based on a free-energy inequality, our method will be different in that it will rely on a similar energy estimate only at a first stage. Indeed, a corresponding inequality (see Lemma 2.3) will serve in Section 2.2 as the starting point for an iterative bootstrap procedure which will eventually yield bounds for $\int_\Omega n^p$ for any $p < 9(m-1)$.

The essential novelty in our approach, to be presented in Section 2.3, consists of a subtle combination of entropy-like estimates for $\int_\Omega n^p$ (Lemma 2.6) and $\int_\Omega |\nabla c|^{2k}$ (Lemma 2.9) in establishing corresponding estimates for coupled quantities of the form

$$\int_\Omega n^p + \int_\Omega |\nabla c|^{2k}$$

with any large $k > 1$ and certain $p > 1$ (see Section 2.3 and in particular Lemma 2.16). Finally, in Section 3 we complete the proofs of Theorems 1.1 and 1.2.

2. Estimates for non-degenerate problems

Throughout this section we shall assume that (1.5)–(1.8) and (1.10) are satisfied with some $m > 1$, and we emphasize that all constants appearing in the estimates in this section will only depend on Ω, m and the initial data. In particular, the value of the parameter function D at zero does not enter any of our estimates in a quantitative way. This will allow us to treat the degenerate case $D(s) = ms^{m-1}$ in a familiar approximative manner, namely by applying the results of this section to the shifted function $D_\varepsilon(s) = m(s+\varepsilon)^{m-1}$ for $\varepsilon > 0$ and letting $\varepsilon \searrow 0$ to end up with a weak solution of the degenerate problem.

2.1. Preliminary observations

Our first statement concerns local classical solvability of (1.2)–(1.4) in the case of non-degenerate diffusion. In its formulation, we shall refer to the standard fractional powers of the Stokes operator A regarded as a self-adjoint operator in the solenoidal subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0 \text{ in } \mathcal{D}'(\Omega)\}$ of $L^2(\Omega)$, in its natural domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$.

Lemma 2.1. *Assume* (1.5)–(1.8) *and* (1.10)*. Then there exists $T_{\max} > 0$ with the property that* (1.2)–(1.4) *possesses a classical solution (n, c, u, P) such that $n > 0$ and $c > 0$ in $\bar\Omega \times [0, T_{\max})$, that*

$$n \in C^0\big(\bar\Omega \times [0, T_{\max})\big) \cap C^{2,1}\big(\bar\Omega \times (0, T_{\max})\big),$$

$$c \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \quad \textit{and}$$
$$u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$

and such that either $T_{\max} = \infty$*, or*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{D(A^\alpha)} \to \infty \quad \textit{for all } \alpha \in \left(\frac{3}{4}, 1\right) \textit{ as } t \nearrow T_{\max}. \tag{2.1}$$

Proof. A proof of this can be obtained by a straightforward adaptation of the reasoning in [27, Lemma 2.1] and [20, Lemma 2.1], and so we may refrain from repeating the arguments here. □

Lemma 2.2. *Assume* (1.5)–(1.8) *and* (1.10). *If* (n, c, u, P) *is a classical solution of* (1.2)–(1.4) *in* $\Omega \times (0, T)$ *for some* $T > 0$*, then*

$$\int_\Omega n(x,t)\,dx = \int_\Omega n_0 \quad \textit{for all } t \in (0,T) \tag{2.2}$$

and

$$|c| \leqslant \|c_0\|_{L^\infty(\Omega)} \quad \textit{in } \Omega \times (0,T). \tag{2.3}$$

Proof. The identity (2.2) directly results from an integration of the first PDE in (1.2) over Ω, whereas the inequality (2.3) is a consequence of the parabolic maximum principle applied to the second equation in (1.2), because $n \geqslant 0$. □

Lemma 2.3. *Suppose that* (1.5)–(1.8) *and* (1.10) *hold. Then there exists* $C > 0$ *such that if* (n, c, u, P) *is a classical solution of* (1.2)–(1.4) *in* $\Omega \times (0, T)$ *for some* $T > 0$*, then*

$$\frac{d}{dt}\left\{\int_\Omega n \ln n + 2\int_\Omega |\nabla\sqrt{c}|^2\right\} + \int_\Omega n^{m-2}|\nabla n|^2 + \int_\Omega c\left|D^2 \ln c\right|^2 + \frac{1}{2}\int_\Omega n\frac{|\nabla c|^2}{c} \leqslant C\int_\Omega |u|^4 \tag{2.4}$$

for all $t \in (0, T)$.

Proof. (2.4) is a consequence of [27, Lemmas 3.2–3.4]. Since it is a cornerstone of subsequent *a priori* estimates in the present paper, let us recall the main ideas. We divide the proof into three steps.

Step 1. We derive an energy identity.

By straightforward computation (cf. [27, Lemma 3.2] for details), one verifies the identity

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega n \ln n + 2\int_\Omega |\nabla\sqrt{c}|^2\right\} + \int_\Omega \frac{D(n)}{n}|\nabla n|^2 + \int_\Omega c\left|D^2 \ln c\right|^2 + \frac{1}{2}\int_\Omega n\frac{|\nabla c|^2}{c} \\
&\quad = -\frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{c^2}(u\cdot\nabla c) + \int_\Omega \frac{\Delta c}{c}(u \cdot \nabla c) + \frac{1}{2}\int_{\partial\Omega} \frac{1}{c}\frac{\partial|\nabla c|^2}{\partial\nu} \quad \text{for all } t \in (0,T).
\end{aligned} \tag{2.5}$$

Step 2. We establish a useful integral inequality.

By some computation and the Hölder inequality (cf. [27, Lemma 3.3] for details), one proves the inequality

$$\int_\Omega \frac{|\nabla c|^4}{c^3} \leqslant (2+\sqrt{3})^2 \int_\Omega c\left|D^2 \ln c\right|^2. \tag{2.6}$$

Step 3. We proceed to prove (2.4).

To this end, we need to estimate the three terms in the right-hand side of (2.5). Firstly, the convexity of $\partial\Omega$ in conjunction with the boundary condition $\frac{\partial c}{\partial \nu} = 0$ on $\partial\Omega$ implies that ([13, Lemme I.1], [2] or [21, Lemma 3.2])

$$\frac{\partial|\nabla c|^2}{\partial\nu} \leqslant 0 \quad \text{on } \partial\Omega. \tag{2.7}$$

Then, by some computation, Young's inequality, (2.6) and the fact that $|\Delta z|^2 \leqslant 3|D^2 z|^2$ for $z \in C^2(\bar{\Omega})$ (cf. [27, Lemma 3.4] for details), one finds some constant $C > 0$ such that

$$\left| -\frac{1}{2} \int_\Omega \frac{|\nabla c|^2}{c^2} (u \cdot \nabla c) + \int_\Omega \frac{\Delta c}{c} (u \cdot \nabla c) \right| \leqslant \frac{3}{4} \int_\Omega c \left| D^2 \ln c \right|^2 + \frac{C}{4} \int_\Omega |u|^4. \tag{2.8}$$

Finally, collecting (2.5), (2.7) and (2.8) and using (1.6), we prove (2.4). □

Our next goal is to derive some first *a priori* estimates from the above energy inequality. As a useful preparation for this, we state the following.

Lemma 2.4. *Let $T > 0$. Then there exists $C(T) > 0$ such that if (n, c, u, P) is a classical solution of (1.2)–(1.4) in $\Omega \times (0, T)$, then*

$$\int_0^T \int_\Omega |u|^4 \leqslant C(T) \cdot \left\{ \int_0^T \int_\Omega n^{m-2} |\nabla n|^2 + 1 \right\}^{\frac{1}{3m-1}}. \tag{2.9}$$

Proof. First, according to standard results on maximal Sobolev regularity of the Stokes evolution equation [7, Theorem 2.7], there exists $C_1(T) > 0$ such that

$$\int_0^T \left\| u(\cdot, t) \right\|_{W^{2,\frac{12}{11}}(\Omega)}^4 dt \leqslant C_1(T) \cdot \left\{ \int_0^T \left\| n(\cdot, t) \nabla \phi \right\|_{L^{\frac{12}{11}}(\Omega)}^4 dt + 1 \right\}.$$

Since $\nabla \phi$ was assumed to be bounded, and since in the three-dimensional setting we have $W^{2,\frac{12}{11}}(\Omega) \hookrightarrow L^4(\Omega)$, we thus find $C_2(T) > 0$ fulfilling

$$\int_0^T \left\| u(\cdot, t) \right\|_{L^4(\Omega)}^4 dt \leqslant C_2(T) \cdot \left\{ \int_0^T \left\| n(\cdot, t) \right\|_{L^{\frac{12}{11}}(\Omega)}^4 dt + 1 \right\}. \tag{2.10}$$

We next invoke the Gagliardo–Nirenberg inequality (see [6] and e.g. [25] for a version involving L^r spaces with $r < 1$) to obtain $C_3 > 0$ such that

$$\begin{aligned} \int_0^T \left\| n(\cdot, t) \right\|_{L^{\frac{12}{11}}(\Omega)}^4 (\cdot, t)\, dt &= \int_0^T \left\| n^{\frac{m}{2}}(\cdot, t) \right\|_{L^{\frac{24}{11m}}(\Omega)}^{\frac{8}{m}} dt \\ &\leqslant C_3 \int_0^T \left\{ \left\| \nabla n^{\frac{m}{2}}(\cdot, t) \right\|_{L^2(\Omega)}^2 + \left\| n^{\frac{m}{2}}(\cdot, t) \right\|_{L^{\frac{2}{m}}(\Omega)}^2 \right\}^{\frac{1}{3m-1}} \cdot \left\| n^{\frac{m}{2}}(\cdot, t) \right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{8}{m} - \frac{2}{3m-1}} dt. \end{aligned}$$

In view of (2.2), we therefore have

$$\int_0^T \left\| n(\cdot, t) \right\|_{L^{\frac{12}{11}}(\Omega)}^4 (\cdot, t)\, dt \leqslant C_4 \int_0^T \left\{ \left\| \nabla n^{\frac{m}{2}}(\cdot, t) \right\|_{L^2(\Omega)}^2 + 1 \right\}^{\frac{1}{3m-1}} dt$$

for some $C_4 > 0$. Combined with (2.10) this easily yields (2.9). □

Lemma 2.5. *For each $T > 0$ there exists $C(T) > 0$ such that if (1.5)–(1.8) and (1.10) hold and (n, c, u, P) is a classical solution of (1.2)–(1.4) in $\Omega \times (0, T)$ then*

$$\int_0^T \int_\Omega n^{m-2} |\nabla n|^2 \leqslant C(T), \tag{2.11}$$

$$\int_\Omega |\nabla c|^2 \leqslant C(T) \quad \textit{for all } t \in (0,T), \quad \textit{and} \tag{2.12}$$

$$\int_0^T\int_\Omega |\nabla c|^4 \leqslant C(T). \tag{2.13}$$

Proof. Integrating (2.4) over $(0,t)$ we obtain

$$2\int_\Omega \left|\nabla\sqrt{c}(\cdot,t)\right|^2 + \int_0^t\int_\Omega n^{m-2}|\nabla n|^2 + \int_0^t\int_\Omega c\left|D^2\ln c\right|^2$$
$$\leqslant C_1\int_0^t\int_\Omega |u|^4 + \int_\Omega n_0\ln n_0 + 2\int_\Omega |\nabla\sqrt{c_0}|^2 - \int_\Omega n(\cdot,t)\ln n(\cdot,t)$$

for some $C_1>0$ and all $t\in(0,T)$. Since $-\xi\ln\xi \leqslant \frac{1}{e}$ for all $\xi>0$, and since $|\nabla\sqrt{c}|^2 = \frac{|\nabla c|^2}{4c}$, this in conjunction with Lemma 2.4 shows that there exists some $C_2(T)>0$ such that

$$\frac{1}{2}\cdot\sup_{t\in(0,T)}\int_\Omega \frac{|\nabla c|^2}{c} + \int_0^T\int_\Omega n^{m-2}|\nabla n|^2 + \int_0^T\int_\Omega c\left|D^2\ln c\right|^2$$
$$\leqslant C_1\cdot C_2(T)\cdot\left\{\int_0^T\int_\Omega n^{m-2}|\nabla n|^2+1\right\}^{\frac{1}{3m-1}} + \int_\Omega n_0\ln n_0 + 2\int_\Omega|\nabla\sqrt{c_0}|^2 + \frac{|\Omega|}{e}. \tag{2.14}$$

By the Young inequality and the fact that $\frac{1}{3m-1}<1$ thanks to our restriction $m>1>\frac{2}{3}$, we derive from (2.14) that

$$\frac{1}{2}\cdot\sup_{t\in(0,T)}\int_\Omega \frac{|\nabla c|^2}{c} + \frac{1}{2}\int_0^T\int_\Omega n^{m-2}|\nabla n|^2 + \int_0^T\int_\Omega c\left|D^2\ln c\right|^2 \leqslant C_3(T), \tag{2.15}$$

where

$$C_3(T) := \int_\Omega n_0\ln n_0 + 2\int_\Omega|\nabla\sqrt{c_0}|^2 + \frac{|\Omega|}{e} + \left(\frac{2}{3m-1}\right)^{\frac{1}{3m-2}}\cdot\frac{3m-2}{3m-1}\cdot\left(C_1\cdot C_2(T)\right)^{\frac{3m-1}{3m-2}}.$$

Therefore (2.11)–(2.13) result from (2.15) and (2.6) upon recalling that $c\leqslant\|c_0\|_{L^\infty(\Omega)}$ in $\Omega\times(0,T)$ by Lemma 2.2. □

Another basic observation is obtained in a standard way upon testing the first PDE in (1.2) by powers of n. Since $\nabla\cdot u=0$, the convective term does not play a role here. We thereby gain the preliminary estimate (2.16) which will be treated in two different ways in the sequel: In Section 2.2 we shall further estimate its right-hand side by primarily using (2.13), whereas in Section 2.3 we will use the information thereby achieved (cf. Lemma 2.7 below) to derive improved estimates on coupling (2.16) to a corresponding inequality for $\int_\Omega|\nabla c|^{2k}$, $k>1$, and thus use the dissipative features of the *second* PDE in (1.2) to absorb the right-hand side of (2.16) properly.

Lemma 2.6. *Assume* (1.5)–(1.8) *and* (1.10)*, and suppose that* (n,c,u,P) *is a classical solution of* (1.2)–(1.4) *in* $\Omega\times(0,T)$ *for some* $T>0$*. Then for each* $p>1$ *we have*

$$\frac{d}{dt}\int_\Omega n^p + \frac{2mp(p-1)}{(m+p-1)^2}\int_\Omega\left|\nabla n^{\frac{m+p-1}{2}}\right|^2 \leqslant \frac{p(p-1)}{2m}\int_\Omega n^{-m+p+1}|\nabla c|^2 \quad \textit{for all } t\in(0,T). \tag{2.16}$$

Proof. We multiply the first equation in (1.2) by n^{p-1} and integrate by parts over Ω to obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + (p-1)\int_\Omega n^{p-2}D(n)|\nabla n|^2 = (p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c \quad \text{for all } t\in(0,T),$$

where we have used that $\nabla\cdot u = 0$. Since $D(n)\geqslant mn^{m-1}$ by (1.6), this yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + m(p-1)\int_\Omega n^{m+p-3}|\nabla n|^2 \leqslant (p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c \quad \text{for all } t\in(0,T). \tag{2.17}$$

Since by Young's inequality

$$(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c \leqslant \frac{m(p-1)}{2}\int_\Omega n^{m+p-3}|\nabla n|^2 + \frac{p-1}{2m}\int_\Omega n^{-m+p+1}|\nabla c|^2,$$

(2.16) results from (2.17) upon obvious rearrangements. □

2.2. *A bound for $\int_\Omega n^p$ for $p<9(m-1)$ by iteration*

Throughout the remainder of this paper we assume that $m>10/9$, that is, $9(m-1)>1$. Building on Lemmas 2.6, 2.5 and an iteration argument, we first establish a bound for $\int_\Omega n^p$ for $p<9(m-1)$. More precisely, the main result of this subsection reads as follows.

Lemma 2.7. *Let $p_0\in(0,9(m-1))$ and $T>0$. Then there exists $C(p_0,T)>0$ such that whenever (1.5)–(1.8) and (1.10) hold and (n,c,u,P) is a classical solution of (1.2)–(1.4), we have*

$$\int_\Omega n^{p_0}(x,t)\,dx \leqslant C(p_0,T) \quad \textit{for all } t\in(0,T). \tag{2.18}$$

Proof. We divide the proof into two steps.

Step 1. We first make sure that if for some $\hat p\geqslant 1$ there exists $c_1(\hat p,T)>0$ such that

$$\int_\Omega n^{\hat p}(x,t)\,dx \leqslant c_1(\hat p,T) \quad \text{for all } t\in(0,T), \tag{2.19}$$

and if $p>1$ is such that

$$p<3(m-1)+\frac{2}{3}\hat p, \tag{2.20}$$

then we even have

$$\int_\Omega n^p(x,t)\,dx \leqslant c_2(p,T) \quad \text{for all } t\in(0,T) \tag{2.21}$$

with some $c_2(p,T)>0$.

To achieve this, we use the Hölder inequality to estimate the right-hand side in (2.16) according to

$$\int_\Omega n^{p-m+1}|\nabla c|^2 \leqslant \left(\int_\Omega n^{2(p-m+1)}\right)^{\frac{1}{2}}\cdot\left(\int_\Omega |\nabla c|^4\right)^{\frac{1}{2}}. \tag{2.22}$$

Here the Gagliardo–Nirenberg inequality provides $c_3>0$ such that

$$\begin{aligned}\left(\int_\Omega n^{2(p-m+1)}\right)^{\frac{1}{2}} &= \left\|n^{\frac{p+m-1}{2}}\right\|_{L^{\frac{4(p-m+1)}{p+m-1}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}} \\ &\leqslant c_4\Big(\left\|\nabla n^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}^{b}\cdot\left\|n^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2\hat p}{p+m-1}}(\Omega)}^{1-b} + \left\|n^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2\hat p}{p+m-1}}(\Omega)}\Big)^{\frac{2(p-m+1)}{p+m-1}},\end{aligned}$$

where

$$b = \frac{\frac{p+m-1}{2\hat{p}} - \frac{p+m-1}{4(p-m+1)}}{\frac{p+m-1}{2\hat{p}} - \frac{1}{6}} = \frac{1 - \frac{\hat{p}}{2(p-m+1)}}{1 - \frac{\hat{p}}{3(p+m-1)}} \in (0,1).$$

Now thanks to (2.20) we find that

$$b \cdot \frac{2(p-m+1)}{p+m-1} < 1,$$

so that in view of (2.22), (2.19) and Young's inequality we can thus pick $c_4 > 0$ such that

$$\frac{p(p-1)}{2m} \int_\Omega n^{p-m+1} |\nabla c|^2 \leqslant \frac{mp(p-1)}{(m+p-1)^2} \int_\Omega \left|\nabla n^{\frac{m+p-1}{2}}\right|^2 + c_4 \left(\int_\Omega |\nabla c|^4 + 1 \right).$$

Hence, from (2.16) we obtain that $y(t) := \int_\Omega n^p(x,t)\,dx$, $t \in [0,T)$, satisfies the differential inequality

$$y'(t) \leqslant c_5 \left(\int_\Omega |\nabla c|^4 + 1 \right) \quad \text{for all } t \in (0,T)$$

with some $c_5 > 0$. On integration we infer that

$$y(t) \leqslant y(0) + c_5 \left(\int_0^T \int_\Omega |\nabla c|^4 + T \right) \quad \text{for all } t \in (0,T),$$

whereupon an application of (2.13) yields (2.21).

Step 2. We proceed to prove the statement of the lemma.

To this end, given $p_0 \in (1, 9(m-1))$ we fix $\varepsilon > 0$ small enough such that still $p_0 < 9(m-1-\varepsilon)$. We now define $(p_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ by letting

$$p_1 := 1 \quad \text{and} \quad p_{k+1} := 3(m-1-\varepsilon) + \frac{2}{3} p_k, \quad k \geqslant 1.$$

Then from (2.2) we know that (2.19) is valid for $\hat{p} := p_1$, so that since evidently $p_{k+1} < 3(m-1) + \frac{2}{3} p_k$ for all $k \in \mathbb{N}$, we infer from a recurrent application of Step 1 that for each $k \in \mathbb{N}$ there exists $C(k,T) > 0$ fulfilling

$$\int_\Omega n^{p_k}(x,t)\,dx \leqslant C(k,T) \quad \text{for all } t \in (0,T).$$

Since it can easily be checked that p_k increases with k and satisfies $p_k \nearrow p_\infty := 9(m-1-\varepsilon)$ as $k \to \infty$, this implies (2.18) due to the fact that $p_0 < p_\infty$. □

2.3. A bound for $\int_\Omega n^p$ with any $p > 1$ by a coupled entropy estimate

As announced above, we shall now treat the integral on the right of (2.16) in a different way. In fact, we shall allow our estimate to depend on a certain higher norm of $|\nabla c|$ which will finally be controlled using the diffusive properties of the equation for c in (1.2). To be more precise:

Lemma 2.8. *Let $k > 1$, $T > 0$ and $\eta > 0$. Then for any $p > 2(m-1)$ fulfilling*

$$p < (8k-1)(m-1) \tag{2.23}$$

there exists $C(k,p,T,\eta) > 0$ such that if (1.5)–(1.8) and (1.10) hold and (n,c,u,P) solves (1.2)–(1.4) classically in $\Omega \times (0,T)$, then

$$\int_\Omega n^{-m+p+1} |\nabla c|^2 \leqslant \eta \int_\Omega \left|\nabla n^{\frac{m+p-1}{2}}\right|^2 + \eta \int_\Omega \left|\nabla |\nabla c|^k\right|^2 + C(k,p,T,\eta) \quad \text{for all } t \in (0,T). \tag{2.24}$$

Proof. We let

$$\varphi(s) := (2k-1)(m-1) + \frac{2}{3}ks, \quad s > 0, \tag{2.25}$$

and observe that

$$\varphi\big(9(m-1)\big) = (8k-1)(m-1).$$

Hence, by (2.23) we can find $p_0 \in (0, 9(m-1))$ such that

$$p < \varphi(p_0), \tag{2.26}$$

and since $p_0 < 9(m-1)$, Lemma 2.7 provides $C_1(T) > 0$ such that

$$\int_\Omega n^{p_0} \leqslant C_1(T) \quad \text{for all } t \in (0,T). \tag{2.27}$$

According to Lemma 2.5, we can furthermore fix $C_2(T) > 0$ satisfying

$$\int_\Omega |\nabla c|^2 \leqslant C_2(T) \quad \text{for all } t \in (0,T). \tag{2.28}$$

Now by the Hölder inequality applied with exponents $\frac{3k}{3k-1}$ and $3k$,

$$\begin{aligned}\int_\Omega n^{-m+p+1}|\nabla c|^2 &\leqslant \left(\int_\Omega n^{\frac{3k}{3k-1}(-m+p+1)}\right)^{\frac{3k-1}{3k}} \cdot \left(\int_\Omega |\nabla c|^{6k}\right)^{\frac{1}{3k}} \\ &= \big\|n^{\frac{m+p-1}{2}}\big\|_{L^{\frac{6k(-m+p+1)}{(3k-1)(m+p-1)}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}} \cdot \big\||\nabla c|^k\big\|_{L^6(\Omega)}^{\frac{2}{k}} \quad \text{for all } t \in (0,T).\end{aligned} \tag{2.29}$$

Here, from the Gagliardo–Nirenberg inequality and (2.27) we obtain $C_3(k,p) > 0$ and $C_4(k,p,T) > 0$ such that

$$\begin{aligned}&\big\|n^{\frac{m+p-1}{2}}\big\|_{L^{\frac{6k(-m+p+1)}{(3k-1)(m+p-1)}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}} \\ &\quad\leqslant C_3(k,p)\cdot\Big\{\big\|\nabla n^{\frac{m+p-1}{2}}\big\|_{L^2(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}a} \cdot \big\|n^{\frac{m+p-1}{2}}\big\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}(1-a)} + \big\|n^{\frac{m+p-1}{2}}\big\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}}\Big\} \\ &\quad\leqslant C_4(k,p,T)\cdot\Big\{\big\|\nabla n^{\frac{m+p-1}{2}}\big\|_{L^2(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}a} + 1\Big\} \quad \text{for all } t \in (0,T)\end{aligned} \tag{2.30}$$

with $a \in (0,1)$ determined by

$$-\frac{3(3k-1)(m+p-1)}{6k(-m+p+1)} = -\frac{1}{2}a - \frac{3(m+p-1)}{2p_0}(1-a),$$

that is, with

$$a = \frac{(m+p-1)\cdot[3k(-m+p+1)-(3k-1)p_0]}{(-m+p+1)\cdot k\cdot[3(m+p-1)-p_0]}. \tag{2.31}$$

We note here that according to our restriction $p > 2(m-1)$ and the fact that $p_0 < 9(m-1)$, the expression $3(m+p-1)-p_0 > 0$ indeed is positive. As for the rightmost term in (2.29), we invoke the Sobolev inequality to find $C_5(k) > 0$ such that

$$\big\||\nabla c|^k\big\|_{L^6(\Omega)}^{\frac{2}{k}} \leqslant C_5(k)\cdot\Big\{\big\|\nabla|\nabla c|^k\big\|_{L^2(\Omega)}^{\frac{2}{k}} + \big\||\nabla c|^k\big\|_{L^1(\Omega)}^{\frac{2}{k}}\Big\} \quad \text{for all } t \in (0,T), \tag{2.32}$$

where in the case $k \leqslant 2$, the last term in brackets can clearly be controlled using (2.28). However, if $k > 2$ then by Hölder's and Young's we can further estimate

$$\big\||\nabla c|^k\big\|_{L^1(\Omega)}^{\frac{2}{k}} \leqslant \frac{1}{2C_5(k)}\big\||\nabla c|^k\big\|_{L^6(\Omega)}^{\frac{2}{k}} + C_6(k)\big\||\nabla c|^k\big\|_{L^{\frac{2}{k}}(\Omega)}^{\frac{2}{k}} \quad \text{for all } t \in (0,T)$$

with some $C_6(k) > 0$, whence from (2.32) and (2.28) we all in all obtain $C_7(k,T)$ such that

$$\big\| |\nabla c|^k \big\|_{L^6(\Omega)}^{\frac{2}{k}} \leqslant C_7(k,T) \cdot \Big\{ \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^{\frac{2}{k}} + 1 \Big\} \quad \text{for all } t \in (0,T).$$

Combined with (2.30), in view of (2.29) this shows that for some $C_8(k,p,T) > 0$ we have

$$\int_\Omega n^{-m+p+1} |\nabla c|^2 \leqslant C_8(k,p,T) \cdot \Big\{ \big\| \nabla n^{\frac{m+p-1}{2}} \big\|_{L^2(\Omega)}^{\frac{2(-m+p+1)}{m+p-1}a} \cdot \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^{\frac{2}{k}} + 1 \Big\} \tag{2.33}$$

for all $t \in (0,T)$. Now by (2.31) and (2.26) we see that

$$\begin{aligned}
\frac{2(-m+p+1)}{m+p-1}a + \frac{2}{k} - 2 &= \frac{2}{k} \cdot \left\{ \frac{3k(-m+p+1) - (3k-1)p_0}{3(m+p-1) - p_0} + 1 - k \right\} \\
&= \frac{2}{k} \cdot \frac{-6(m-1)k - 2kp_0 + 3(m+p-1)}{3(m+p-1) - p_0} \\
&= \frac{6}{k} \cdot \frac{-(2k-1)(m-1) - \frac{2}{3}kp_0 + p}{3(m+p-1) - p_0} \\
&= \frac{6}{k} \cdot \frac{-\varphi(p_0) + p}{3(m+p-1) - p_0} \\
&< 0
\end{aligned}$$

and hence

$$\frac{2(-m+p+1)}{m+p-1}a + \frac{2}{k} < 2.$$

Therefore, given any $\delta > 0$, upon twice applying Young's inequality we can find $C_9(k,p,\delta) > 0$ such that

$$X^{\frac{2(-m+p+1)}{m+p-1}a} \cdot Y^{\frac{2}{k}} \leqslant \delta \cdot \big(A^2 + B^2\big) + C_9(k,p,\delta) \quad \text{for all } X \geqslant 0 \text{ and } Y \geqslant 0.$$

Applied to (2.33), this yields

$$\int_\Omega n^{-m+p+1} |\nabla c|^2 \leqslant \delta C_8(k,p,T) \cdot \Big\{ \big\| \nabla n^{\frac{m+p-1}{2}} \big\|_{L^2(\Omega)}^2 + \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^2 \Big\} + C_8(k,p,T) \cdot \big(C_9(k,p,\delta) + 1\big)$$

for all $t \in (0,T)$, and thereby proves (2.24) on choosing $\delta := \frac{\eta}{C_8(k,p,T)}$. □

The first term on the right of (2.24) may clearly be absorbed by the dissipative integral in (2.16). Accordingly, our next goal is to cope with the second appropriately. This is prepared by the following inequality in which once more the convexity of Ω is essential.

Lemma 2.9. *Let (1.5)–(1.8) and (1.10) be satisfied, and suppose that (n,c,u,P) is a classical solution of (1.2)–(1.4) for some $T > 0$. Then for all $k > 1$ we have*

$$\begin{aligned}
&\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla c|^{2k} + \frac{k-1}{2}\int_\Omega |\nabla c|^{2k-4} \big|\nabla |\nabla c|^2\big|^2 + \int_\Omega |\nabla c|^{2k-2} \big|D^2 c\big|^2 \\
&\quad \leqslant \int_\Omega nc \nabla \cdot \big(|\nabla c|^{2k-2} \nabla c\big) + \int_\Omega (u \cdot \nabla c) \nabla \cdot \big(|\nabla c|^{2k-2} \nabla c\big) \quad \textit{for all } t \in (0,T).
\end{aligned} \tag{2.34}$$

Proof. By direct computation using the second equation in (1.2) we obtain

$$\begin{aligned}
\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla c|^{2k} &= \int_\Omega |\nabla c|^{2k-2} \nabla c \cdot \nabla c_t \\
&= \int_\Omega |\nabla c|^{2k-2} \nabla c \cdot \nabla \Delta c - \int_\Omega |\nabla c|^{2k-2} \nabla c \cdot \nabla (nc + u \cdot \nabla c) \quad \text{for all } t \in (0,T).
\end{aligned} \tag{2.35}$$

Here an integration by parts shows that the second integral equals the sum on the right of (2.34), because $\frac{\partial c}{\partial \nu}=0$ on $\partial\Omega$. Moreover, in view of the pointwise identity

$$\nabla c\cdot\nabla\Delta c=\frac{1}{2}\Delta|\nabla c|^2-\left|D^2c\right|^2,$$

upon another integration by parts the first term on the right becomes

$$\begin{aligned}\int_\Omega |\nabla c|^{2k-2}\nabla c\cdot\nabla\Delta c&=\frac{1}{2}\int_\Omega |\nabla c|^{2k-2}\Delta|\nabla c|^2-\int_\Omega |\nabla c|^{2k-2}\left|D^2c\right|^2\\&=-\frac{1}{2}\int_\Omega \nabla|\nabla c|^{2k-2}\cdot\nabla|\nabla c|^2+\frac{1}{2}\int_{\partial\Omega}|\nabla c|^{2k-2}\frac{\partial|\nabla c|^2}{\partial\nu}\\&\quad-\int_\Omega |\nabla c|^{2k-2}\left|D^2c\right|^2\quad\text{for all } t\in(0,T).\end{aligned}$$

Since

$$\frac{1}{2}\nabla|\nabla c|^{2k-2}\cdot\nabla|\nabla c|^2=\frac{k-1}{2}|\nabla c|^{2k-4}\left|\nabla|\nabla c|^2\right|^2,$$

and since $\frac{\partial|\nabla c|^2}{\partial\nu}\leqslant 0$ on $\partial\Omega$ thanks to the convexity of Ω and the fact that $\frac{\partial c}{\partial\nu}=0$ on $\partial\Omega$ [2], this directly gives (2.34). □

We proceed to estimate both integrals on the right of (2.34) in a straightforward manner.

Lemma 2.10. *Let $k>1$. Then there exists $C>0$ such that if (1.5)–(1.8) and (1.10) hold and (n,c,u,P) is a classical solution of (1.2)–(1.4) in $\Omega\times(0,T)$ for some $T>0$, then*

$$\begin{aligned}&\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla c|^{2k}+\frac{k-1}{k^2}\int_\Omega\left|\nabla|\nabla c|^k\right|^2\\&\quad\leqslant C\cdot\left\{\int_\Omega n^2|\nabla c|^{2k-2}+\int_\Omega |u|^2|\nabla c|^{2k}\right\}\quad\textit{for all } t\in(0,T).\end{aligned}\tag{2.36}$$

Proof. Starting from (2.34), we rewrite

$$\int_\Omega nc\nabla\cdot\left(|\nabla c|^{2k-2}\nabla c\right)=\int_\Omega nc|\nabla c|^{2k-2}\Delta c+(k-1)\int_\Omega nc|\nabla c|^{2k-4}\nabla|\nabla c|^2\cdot\nabla c.$$

Here we use Young's inequality along with the fact that (2.3) guarantees that $|c|\leqslant\|c_0\|_{L^\infty(\Omega)}$ to estimate

$$\begin{aligned}\int_\Omega nc|\nabla c|^{2k-2}\Delta c&\leqslant\frac{\delta}{3}\int_\Omega |\nabla c|^{2k-2}|\Delta c|^2+\frac{3}{4\delta}\int_\Omega n^2c^2|\nabla c|^{2k-2}\\&\leqslant\delta\int_\Omega |\nabla c|^{2k-2}\left|D^2c\right|^2+\frac{3\|c_0\|^2_{L^\infty(\Omega)}}{4\delta}\int_\Omega n^2|\nabla c|^{2k-2}\end{aligned}\tag{2.37}$$

for any $\delta>0$, because $|\Delta c|^2\leqslant 3|D^2c|^2$ by the Cauchy–Schwarz inequality for sums. Proceeding similarly, we find that

$$(k-1)\int_\Omega nc|\nabla c|^{2k-4}\nabla|\nabla c|^2\cdot\nabla c\leqslant\delta\int_\Omega |\nabla c|^{2k-4}\left|\nabla|\nabla c|^2\right|^2+\frac{(k-1)^2\|c_0\|^2_{L^\infty(\Omega)}}{4\delta}\int_\Omega n^2|\nabla c|^{2k-2}.$$

Combined with (2.37), this easily yields (2.36) upon choosing $\delta := \min\{\frac{k-1}{4}, \frac{1}{2}\}$ and noticing the identity $|\nabla c|^{2k-4}|\nabla|\nabla c|^2|^2 = \frac{4}{k^2}|\nabla|\nabla c|^k|^2$. □

By means of an embedding argument, the first term on the right of (2.36) can be related to a higher Lebesgue norm of n as follows.

Lemma 2.11. *For all $k > 1$, $T > 0$ and $\eta > 0$ there exists $C(k, T, \eta) > 0$ such that whenever (1.5)–(1.8) and (1.10) are valid and (n, c, u, P) solves (1.2)–(1.4) classically in $\Omega \times (0, T)$, we have*

$$\int_\Omega n^2|\nabla c|^{2k-2} \leqslant \eta \int_\Omega \bigl|\nabla|\nabla c|^k\bigr|^2 + C(k, p, T, \eta) \cdot \bigl\{\|n\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)} + 1\bigr\} \quad \textit{for all } t \in (0, T). \tag{2.38}$$

Proof. Abbreviating $w := |\nabla c|^2$, from Lemma 2.5 we know that there exists $C_1(T) > 0$ such that

$$\bigl\|w^{\frac{k}{2}}\bigr\|^{\frac{2}{k}}_{L^{\frac{2}{k}}(\Omega)} \equiv \int_\Omega |\nabla c|^2 \leqslant C_1(T) \quad \text{for all } t \in (0, T). \tag{2.39}$$

Now by the Hölder inequality applied with exponents $\frac{3k}{2k+1} > 1$ and $\frac{3k}{k-1}$, we estimate

$$\int_\Omega n^2|\nabla c|^{2k-2} = \int_\Omega n^2 w^{k-1} \leqslant \biggl(\int_\Omega n^{\frac{6k}{2k+1}}\biggr)^{\frac{2k+1}{3k}} \cdot \biggl(\int_\Omega w^{3k}\biggr)^{\frac{k-1}{3k}} = \|n\|^2_{L^{\frac{6k}{2k+1}}(\Omega)} \cdot \bigl\|w^{\frac{k}{2}}\bigr\|^{\frac{2(k-1)}{k}}_{L^6(\Omega)} \tag{2.40}$$

for all $t \in (0, T)$. Here, using the Sobolev inequality and (2.39) we find $C_2(k) > 0$ and $C_3(k, T) > 0$ such that

$$\begin{aligned}\bigl\|w^{\frac{k}{2}}\bigr\|^{\frac{2(k-1)}{k}}_{L^6(\Omega)} &\leqslant C_2(k) \cdot \bigl\{\bigl\|\nabla w^{\frac{k}{2}}\bigr\|^{\frac{2(k-1)}{k}}_{L^2(\Omega)} + \bigl\|w^{\frac{k}{2}}\bigr\|^{\frac{2(k-1)}{k}}_{L^{\frac{2}{k}}(\Omega)}\bigr\} \\ &\leqslant C_3(k, T) \cdot \bigl\{\bigl\|\nabla w^{\frac{k}{2}}\bigr\|^{\frac{2(k-1)}{k}}_{L^2(\Omega)} + 1\bigr\} \quad \text{for all } t \in (0, T).\end{aligned}$$

Thus, according to Young's inequality applied to (2.40) with exponents $\frac{k}{k-1}$ and k and recalling that $w = |\nabla c|^2$, we can achieve that given any $\eta > 0$ we can find $C_4(k, T, \eta) > 0$ such that

$$\int_\Omega n^2|\nabla c|^{2k-2} \leqslant \eta \cdot \bigl\{\bigl\|\nabla|\nabla c|^k\bigr\|^2_{L^2(\Omega)} + 1\bigr\} + C_4(k, T, \eta) \cdot \|n\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)} \quad \text{for all } t \in (0, T),$$

which easily yields (2.38). □

Since the integrability powers in the norm of n appearing in (2.38) do not depend on p, it is evident that this term can be controlled by the dissipative integral in (2.16) provided that p is large enough. Our reasoning in the sequel (cf. Lemma 2.16) shall crucially rely on a precise condition for p under which this conclusion is valid. Using an interpolation argument involving Lemma 2.7 we shall derive (2.41) as such a sufficient condition, which in the case $m > \frac{8}{7}$ will turn out to be mild enough so as to be achievable together with (2.23) and another restriction on p arising below (cf. (2.52)).

Lemma 2.12. *Let $k > 1$ and $p > 1$ be such that*

$$p > 2k - (6k + 1)(m - 1). \tag{2.41}$$

Then given any $T > 0$ and $\eta > 0$ we can find $C(k, p, T, \eta) > 0$ such that if (1.5)–(1.8) and (1.10) hold, then each classical solution (n, c, u, P) of (1.2)–(1.4) in $\Omega \times (0, T)$ satisfies

$$\int_0^T \bigl\|n(\cdot, t)\bigr\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)}\,dt \leqslant \eta \int_0^T\!\!\int_\Omega \bigl|\nabla n^{\frac{m+p-1}{2}}\bigr|^2 + C(k, p, T, \eta). \tag{2.42}$$

Proof. We let

$$\chi(s) := 2k - \frac{2k}{3}s - (m-1), \quad s > 0,$$

and conclude from

$$\chi\big(9(m-1)\big) = 2k - 6k(m-1) - (m-1) = 2k - (6k+1)(m-1)$$

and (2.41) that there exists $p_0 \in (0, 9(m-1))$ such that

$$\chi(p_0) < p. \tag{2.43}$$

Applying Lemma 2.7 to this value of p_0, we find $C_1(T) > 0$ fulfilling

$$\int_\Omega n^{p_0}(x,t)\,dx \leqslant C_1(T) \quad \text{for all } t \in (0,T). \tag{2.44}$$

Now since $p > 1$ implies that $\frac{12k}{(2k+1)(m+p-1)} < 6$, we may invoke the Gagliardo–Nirenberg inequality to find $C_2(k,p) > 0$ such that

$$\begin{aligned}\|n\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)} &= \big\|n^{\frac{m+p-1}{2}}\big\|^{\frac{4k}{m+p-1}}_{L^{\frac{12k}{(2k+1)(m+p-1)}}(\Omega)} \\ &\leqslant C_2(k,p)\cdot\Big\{\big\|\nabla n^{\frac{m+p-1}{2}}\big\|^{\frac{4k}{m+p-1}a}_{L^2(\Omega)} \cdot \big\|n^{\frac{m+p-1}{2}}\big\|^{\frac{4k}{m+p-1}(1-a)}_{L^{\frac{2p_0}{m+p-1}}(\Omega)} + \big\|n^{\frac{m+p-1}{2}}\big\|^{\frac{4k}{m+p-1}}_{L^{\frac{2p_0}{m+p-1}}(\Omega)}\Big\}\end{aligned}$$

is valid for all $t \in (0,T)$ with $a \in (0,1)$ determined by

$$-\frac{3(2k+1)(m+p-1)}{12k} = -\frac{1}{2}a - \frac{3(m+p-1)}{2p_0}(1-a),$$

so that

$$a = \frac{[6k-(2k+1)p_0]\cdot(m+p-1)}{2k\cdot[3(m+p-1)-p_0]}.$$

In view of (2.44), we therefore obtain $C_3(k,p,T) > 0$ such that

$$\|n\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)} \leqslant C_3(k,p,T)\cdot\Big\{\big\|\nabla n^{\frac{m+p-1}{2}}\big\|^{\frac{2[6k-(2k+1)p_0]}{3(m+p-1)-p_0}}_{L^2(\Omega)} + 1\Big\} \quad \text{for all } t \in (0,T). \tag{2.45}$$

Here, since

$$\begin{aligned}\frac{2[6k-(2k+1)p_0]}{3(m+p-1)-p_0} - 2 &= \frac{12k-(4k+2)p_0 - 6(m+p-1) + 2p_0}{3(m+p-1)-p_0} \\ &= \frac{6[2k - \frac{2}{3}kp_0 - (m-1) - p]}{3(m+p-1)-p_0} \\ &= \frac{6[\chi(p_0)-p]}{3(m+p-1)-p_0} \\ &< 0\end{aligned}$$

according to (2.43), for any $\delta > 0$ Young's inequality provides $C_4(k,p,\delta) > 0$ such that

$$\big\|\nabla n^{\frac{m+p-1}{2}}\big\|^{\frac{2[6k-(2k+1)p_0]}{3(m+p-1)-p_0}}_{L^2(\Omega)} \leqslant \delta\big\|\nabla n^{\frac{m+p-1}{2}}\big\|^2_{L^2(\Omega)} + C_4(k,p,\delta).$$

Applied to (2.45) for sufficiently small $\delta = \delta(\eta, C_3(k,p,T)) > 0$, this immediately leads to (2.42). □

In a way similar to that in Lemma 2.11, the second integral on the right of (2.36) can be estimated in terms of appropriate norms of u.

Lemma 2.13. *Given $k > 1$, $T > 0$ and $\eta > 0$, one can find $C(k, T, \eta) > 0$ such that if under the assumptions (1.5)–(1.8) and (1.10), (n, c, u, P) solves (1.2)–(1.4) classically in $\Omega \times (0, T)$, then*

$$\int_\Omega |u|^2 |\nabla c|^{2k} \leqslant \eta \int_\Omega \big|\nabla |\nabla c|^k\big|^2 + C(k, T, \eta) \cdot \big\{ \|u\|_{L^\infty(\Omega)}^{3k-1} + 1 \big\} \quad \text{for all } t \in (0, T). \tag{2.46}$$

Proof. According to the Hölder and the Gagliardo–Nirenberg inequalities we can find $C_1(k) > 0$ such that

$$\int_\Omega |u|^2 |\nabla c|^{2k} \leqslant \|u\|_{L^\infty(\Omega)}^2 \cdot \big\| |\nabla c|^k \big\|_{L^2(\Omega)}^2$$
$$\leqslant C_1(k) \cdot \|u\|_{L^\infty(\Omega)}^2 \cdot \Big\{ \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^{\frac{6(k-1)}{3k-1}} \cdot \big\| |\nabla c|^k \big\|_{L^{\frac{2}{k}}(\Omega)}^{\frac{4}{3k-1}} + \big\| |\nabla c|^k \big\|_{L^{\frac{2}{k}}(\Omega)}^2 \Big\}$$

for all $t \in (0, T)$. Thanks to Lemma 2.5, there exists $C_2(T) > 0$ such that $\| |\nabla c|^k \|_{L^{\frac{2}{k}}(\Omega)}^{\frac{2}{k}} \leqslant C_2(T)$ for all $t \in (0, T)$, whence

$$\int_\Omega |u|^2 |\nabla c|^{2k} \leqslant C_3(k, T) \|u\|_{L^\infty(\Omega)}^2 \cdot \Big\{ \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^{\frac{6(k-1)}{3k-1}} + 1 \Big\} \quad \text{for all } t \in (0, T) \tag{2.47}$$

is valid with some $C_3(k, T) > 0$. Now given $\delta > 0$, from an application of Young's inequality with exponents $\frac{3k-1}{3(k-1)}$ and $\frac{3k-1}{2}$ we obtain $C_4(k, \delta) > 0$ satisfying

$$\|u\|_{L^\infty(\Omega)}^2 \cdot \Big\{ \big\| |\nabla c|^k \big\|_{L^2(\Omega)}^{\frac{6(k-1)}{3k-1}} + 1 \Big\} \leqslant \delta \cdot \big\| \nabla |\nabla c|^k \big\|_{L^2(\Omega)}^2 + C_4(k, \delta) \cdot \big\{ \|u\|_{L^\infty(\Omega)}^{3k-1} + 1 \big\} \quad \text{for all } t \in (0, T),$$

and therefore (2.46) readily follows from (2.47) upon an appropriate choice of δ. □

Now standard regularity estimates for the instationary Stokes equation [7] allow us to replace, after a time integration, the norm of u appearing above by a certain norm of the cell density n which, as we recall, appears as a source term in the third PDE in (1.2).

Lemma 2.14. *Let $k > 1$, $r > \frac{3}{2}$ and $T > 0$. Then there exists $C(k, r, T) > 0$ such that if (1.5)–(1.8) and (1.10) hold and (n, c, u, P) is a classical solution of (1.2)–(1.4) in $\Omega \times (0, T)$, we have*

$$\int_0^T \big\| u(\cdot, t) \big\|_{L^\infty(\Omega)}^{3k-1} dt \leqslant C(k, q, T) \cdot \left\{ \int_0^T \big\| n(\cdot, t) \big\|_{L^r(\Omega)}^{3k-1} dt + 1 \right\}. \tag{2.48}$$

Proof. From the well-known results on maximal Sobolev regularity properties of the Stokes evolution equation [7, Theorem 2.3] we obtain $C_1(k, r, T) > 0$ such that

$$\int_0^T \big\| u(\cdot, t) \big\|_{L^\infty(\Omega)}^{3k-1} dt \leqslant C_1(k, r, T) \cdot \left\{ \int_0^T \big\| n(\cdot, t) \nabla\phi \big\|_{L^r(\Omega)}^{3k-1} + 1 \right\}.$$

Since $\nabla\phi \in L^\infty(\Omega)$ and $W^{2,r}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to our restriction $r > \frac{3}{2}$, this establishes (2.48). □

Let us summarize what we obtain from Lemma 2.10 in light of Lemmas 2.11–2.14.

Lemma 2.15. *Let $k > 1$, $r > \frac{3}{2}$ and $p > 1$ be such that*

$$p > 2k - (6k + 1)(m - 1).$$

Then given any $T > 0$ and $\eta > 0$ we can find $C(k, p, r, T, \eta) > 0$ with the property that whenever (1.5)–(1.8) and (1.10) are valid and (n, c, u, P) solves (1.2)–(1.4) classically in $\Omega \times (0, T)$, we have

$$\sup_{t\in(0,T)}\int_\Omega |\nabla c|^{2k} + \int_0^T\int_\Omega \big|\nabla|\nabla c|^k\big|^2$$
$$\leqslant \eta\int_0^T\int_\Omega \big|\nabla n^{\frac{m+p-1}{2}}\big|^2 + C(k,p,r,T,\eta)\cdot\Bigg\{\int_0^T \big\|n(\cdot,t)\big\|_{L^r(\Omega)}^{3k-1}\,dt+1\Bigg\}. \tag{2.49}$$

Proof. We only need to apply Lemmas 2.11 and 2.13 with suitably small $\eta > 0$ in estimating the right-hand side in (2.36) to achieve

$$\frac{1}{2k}\frac{d}{dt}\int_\Omega |\nabla c|^{2k} + \frac{k-1}{2k^2}\int_\Omega \big|\nabla|\nabla c|^k\big|^2 \leqslant C_1(k,T)\cdot\big\{\|n\|^{2k}_{L^{\frac{6k}{2k+1}}(\Omega)} + \|u\|^{3k-1}_{L^\infty(\Omega)}+1\big\}\quad \text{for all } t\in(0,T)$$

with some $C_1(k,T) > 0$. Integrating with respect to $t \in (0,T)$ and using Lemmas 2.14 and 2.12 we readily end up with (2.49). □

By combining the above inequalities we can now derive bounds, uniformly with respect to $t \in (0,T)$, for arbitrarily high Lebesgue norms of n and ∇c. This will be achieved by deriving entropy-type estimates for coupled quantities of the form

$$\int_\Omega n^p + \int_\Omega |\nabla c|^{2k}$$

with any large $k > 1$ and certain $p > 1$ lying in some range suggested by the requirements (2.23), (2.41) and (2.52) below. The main step toward this is done in the following lemma.

Lemma 2.16. *Suppose that $k > 1$ and $p > \max\{\frac{3}{2}, 2(m-1)\}$ are such that*

$$p < (8k-1)(m-1) \tag{2.50}$$

and

$$p > 2k-(6k+1)(m-1) \tag{2.51}$$

as well as

$$p > (21-18m)k + 15m - 16. \tag{2.52}$$

Then given $T > 0$, one can find $C(k,p,T) > 0$ such that if under the hypotheses (1.5)–(1.8) and (1.10), (n,c,u,P) is any classical solution of (1.2)–(1.4) in $\Omega\times(0,T)$, then

$$\int_\Omega n^p + \int_\Omega |\nabla c|^{2k} \leqslant C(k,p,T)\quad \textit{for all } t\in(0,T). \tag{2.53}$$

Proof. We define

$$\psi(r,s) := (3k-1)\left(1-\frac{s}{r}\right)+s,\qquad r>0,\ s>0,$$

and claim that there exist $r > \frac{3}{2}$ and $p_0 \in (0, 9(m-1))$ such that

$$p_0 \leqslant r < p \tag{2.54}$$

and

$$\psi(r,p_0) < p. \tag{2.55}$$

Indeed, in the case $9(m-1) \leqslant \frac{3}{2}$ we observe that

$$\psi\left(\frac{3}{2}, 9(m-1)\right) = (3k-1)\cdot\big(1-6(m-1)\big)+9(m-1) = (21-18m)k+15m-16 < p$$

is valid in view of (2.52), so that by a continuity argument we can pick $r \in (\frac{3}{2}, p)$ close enough to $\frac{3}{2}$ and $p_0 \in (0, 9(m-1))$ sufficiently close to $9(m-1)$ such that still (2.55) holds, and we note that then (2.54) is obvious.

Conversely, if $9(m-1) > \frac{3}{2}$ then we simply pick any $r \in (\frac{3}{2}, p)$ such that $r < 9(m-1)$, and set $p_0 := r$. Then trivially $p_0 < 9(m-1)$ and also (2.54) is evident, whereas now

$$\psi(r, p_0) = \psi(r, r) = r < p$$

according to our choice of r.

Now the validity of (2.54) enables us to apply the Hölder inequality in estimating the second integral on the right of (2.49),

$$\int_0^T \|n(\cdot,t)\|_{L^r(\Omega)}^{3k-1}\,dt \leqslant \int_0^T \|n(\cdot,t)\|_{L^p(\Omega)}^{\frac{(3k-1)p(r-p_0)}{r(p-p_0)}} \cdot \|n(\cdot,t)\|_{L^{p_0}(\Omega)}^{\frac{(3k-1)p_0(p-r)}{r(p-p_0)}}\,dt. \tag{2.56}$$

Here we note that

$$\kappa := \frac{\frac{(3k-1)p(r-p_0)}{r(p-p_0)}}{p}$$

satisfies

$$\kappa - 1 = \frac{(3k-1)(r-p_0) - r(p-p_0)}{r(p-p_0)} = \frac{(3k-1)(1-\frac{p_0}{r}) + p_0 - p}{p-p_0} = \frac{\psi(r,p_0)-p}{p-p_0} < 0$$

by (2.55). Thus, from (2.56) and Lemma 2.7 we obtain $C_1(k,p,T) > 0$ and $C_2(k,p,T) > 0$ such that

$$\int_0^T \|n(\cdot,t)\|_{L^r(\Omega)}^{3k-1}\,dt \leqslant C_1(k,p,T)\int_0^T \|n(\cdot,t)\|_{L^p(\Omega)}^{p\kappa}\,dt \leqslant C_2(k,p,T)\cdot T\cdot\left(\sup_{t\in(0,T)}\int_\Omega n^p(x,t)\,dx\right)^\kappa$$

with $\kappa < 1$. Lemma 2.15, which is applicable because of (2.51), therefore yields $C_3(k,p,T) > 0$ such that

$$\sup_{t\in(0,T)}\int_\Omega |\nabla c|^{2k} + \int_0^T\int_\Omega \big|\nabla|\nabla c|^k\big|^2 \leqslant \int_0^T\int_\Omega \big|\nabla n^{\frac{m+p-1}{2}}\big|^2 + C_3(k,p,T)\cdot\left\{\left(\sup_{t\in(0,T)}\int_\Omega n^p(x,t)\,dx\right)^\kappa + 1\right\}. \tag{2.57}$$

On the other hand, from Lemmas 2.6 and 2.8 we know that since (2.50) holds, we have

$$\sup_{t\in(0,T)}\int_\Omega n^p + \int_0^T\int_\Omega \big|\nabla n^{\frac{m+p-1}{2}}\big|^2 \leqslant \int_0^T\int_\Omega \big|\nabla|\nabla c|^k\big|^2 + C_4(k,p,T)$$

with some $C_4 > 0$. Adding this to (2.57) shows that

$$\sup_{t\in(0,T)}\left\{\int_\Omega n^p + \int_\Omega |\nabla c|^{2k}\right\} \leqslant C_3(k,p,T)\cdot\left(\sup_{t\in(0,T)}\int_\Omega n^p(x,t)\,dx\right)^\kappa + C_3(k,p,T) + C_4(k,p,T).$$

Since $\kappa < 1$, in view of Young's inequality this directly leads to (2.53). □

Now it is straightforward to check that in the case $m > \frac{8}{7}$ the above lemma will indeed be applicable for arbitrarily large k and p.

Corollary 2.17. *Let $m > \frac{8}{7}$. Then for all $k > 1$, $p > 1$ and $T > 0$ there exists $C(k, p, T) > 0$ such that whenever (1.5)–(1.8) and (1.10) hold and (n, c, u, P) is a classical solution of (1.2)–(1.4) in $\Omega \times (0, T)$, we have*

$$\int_\Omega n^p + \int_\Omega |\nabla c|^{2k} \leqslant C(k, p, T) \quad \text{for all } t \in (0, T). \tag{2.58}$$

Proof. In view of Lemma 2.16, we evidently only need to check that (2.50)–(2.52) may be fulfilled simultaneously for all sufficiently large $k > 1$ and some $p_k > \frac{3}{2}$ satisfying $p_k \to \infty$ as $k \to \infty$.

To this end, we observe that

$$I_1(k) := (8k - 1)(m - 1) \to \infty \quad \text{as } k \to \infty,$$

and that since $m > \frac{8}{7}$, we have

$$I_2(k) := \frac{2k - (6k + 1)(m - 1)}{(8k - 1)(m - 1)} \to \frac{2 - 6(m - 1)}{8(m - 1)} = \frac{4 - 3m}{4m - 4} < 1 \quad \text{as } k \to \infty$$

as well as

$$I_3(k) := \frac{(21 - 18m)k + 15m - 16}{(8k - 1)(m - 1)} \to \frac{21 - 18m}{8(m - 1)} < \frac{21 - 18 \cdot \frac{8}{7}}{8 \cdot \frac{1}{7}} = \frac{3}{8} < 1 \quad \text{as } k \to \infty.$$

It is therefore possible to pick $k_0 > 1$ large enough such that $I_1(k) > \frac{3}{2}$, $I_2(k) < 1$ and $I_3(k) < 1$ for all $k > k_0$, and then, given any $k > k_0$, fix $p_k \in (\frac{3}{2}, I_1(k))$ close enough to $I_1(k)$ such that

$$\frac{p_k}{I_1(k)} > \max\{I_2(k), I_3(k)\},$$

which asserts that both (2.51) and (2.52) will be valid. □

Upon a straightforward adaptation of the well-known Moser–Alikakos iteration procedure we end up with the following main result of this section.

Corollary 2.18. *Let $m > \frac{8}{7}$. Then for all $T > 0$ there exists $C(T) > 0$ such that if (1.5)–(1.8) and (1.10) hold and (n, c, u, P) solves (1.2)–(1.4) classically in $\Omega \times (0, T)$, then*

$$\|n(\cdot, t)\|_{L^\infty(\Omega)} \leqslant C(T) \quad \text{for all } t \in (0, T). \tag{2.59}$$

Proof. For a complete proof of a corresponding statement in a more general setting, also covering the present case, we refer to [21, Lemma 4.1]. □

3. Global existence. Proof of Theorems 1.1 and 1.2

We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Let (n, c, u, P) be a classical solution of (1.2)–(1.4) in $\Omega \times (0, T)$ for some $T > 0$. Then (2.59) holds by Corollary 2.18, and we proceed to estimate u and c. It is well known (cf. [18], for instance) that the Stokes operator $A = -\mathcal{P}\Delta$, with $\mathcal{P}$ denoting the Helmholtz projection in $L^2(\Omega)$, is sectorial and generates a contraction semigroup $(e^{-tA})_{t \geqslant 0}$ in $L^2(\Omega)$ with its operator norm bounded according to

$$\|e^{-tA}\| \leqslant c_1 e^{-\mu t} \quad \text{for all } t \geqslant 0$$

with some $c_1 > 0$ and $\mu > 0$. We now pick any $\alpha \in (\frac{3}{4}, 1)$ and apply the fractional power A^α to the variation-of-constants formula

$$u(\cdot,t)=e^{-tA}u_0+\int_0^t e^{-(t-s)A}\mathcal{P}\big(n(\cdot,s)\nabla\phi\big)\,ds,\quad t\in(0,T),$$

to find $c_2>0$ and $c_3(T)>0$ such that

$$\begin{aligned}\big\|u(\cdot,t)\big\|_{D(A^\alpha)}=\big\|A^\alpha u(\cdot,t)\big\|_{L^2(\Omega)}&\leqslant c_2\cdot\left(1+\int_0^t (t-s)^{-\alpha}e^{-\mu(t-s)}\big\|n(\cdot,s)\big\|_{L^2(\Omega)}\,ds\right)\\&\leqslant c_2\cdot\left(1+\sup_{s\in(0,T)}\big\|n(\cdot,s)\big\|_{L^2(\Omega)}\cdot\int_0^\infty \sigma^{-\alpha}e^{-\mu\sigma}\,d\sigma\right)\\&\leqslant c_3(T)\quad\text{for all } t\in(0,T).\end{aligned}\tag{3.1}$$

Since in the three-dimensional setting we have $D(A^\alpha)\hookrightarrow W^{1,q}(\Omega)$ for any $q<\frac{6}{5-4\alpha}$ (cf. e.g. [18, Lemma 2.4.3]), it follows from the fact that $\alpha>\frac{3}{4}$ that $D(A^\alpha)$ is continuously embedded into both $W^{1,2}(\Omega)$ and $L^\infty(\Omega)$. Therefore, there exists some $c_4(T)>0$ such that

$$\big\|u(\cdot,t)\big\|_{W^{1,2}(\Omega)}\leqslant c_4(T)\quad\text{and}\quad\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}\leqslant c_4(T)\quad\text{for all } t\in(0,T).\tag{3.2}$$

Therefore, by (2.59) and standard parabolic regularity theory applied to the second equation in (1.2), we obtain some $c_5(T)>0$ such that

$$\big\|c(\cdot,t)\big\|_{W^{1,\infty}(\Omega)}\leqslant c_5(T)\quad\text{for all } t\in(0,T).\tag{3.3}$$

Finally, the global classical solvability statement is a straightforward consequence of Lemma 2.1, Corollary 2.18, (3.1) and (3.3). □

Let us now turn our attention to weak solutions as addressed in Theorem 1.1. We shall pursue the following natural solution concept introduced in [4].

Definition 3.1 *(Weak solution).* Let $T\in(0,\infty)$. A quadruple (n,c,u,P) is said to be a weak solution to problem (1.2)–(1.4) in $\Omega\times(0,T)$ if

(1) $n\in L^\infty(\Omega\times(0,T))$, $D(n)\nabla n\in L^2((0,T);L^2(\Omega))$ and $n_t\in L^2((0,T);(W^{1,2}(\Omega))^\star)$,
(2) $c\in L^\infty(\Omega\times(0,T))\cap L^2((0,T);W^{2,2}(\Omega))\cap W^{1,2}((0,T);L^2(\Omega))$,
(3) $u\in L^2((0,T);W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))$,
(4) the identities

$$\int_0^T\!\!\int_\Omega n_t\psi-\int_0^T\!\!\int_\Omega\nabla\psi\cdot un+\int_0^T\!\!\int_\Omega D(n)\nabla n\cdot\nabla\psi=\int_0^T\!\!\int_\Omega n\nabla c\cdot\nabla\psi,$$
$$\int_0^T\!\!\int_\Omega c_t\psi-\int_0^T\!\!\int_\Omega\nabla\psi\cdot uc+\int_0^T\!\!\int_\Omega\nabla c\cdot\nabla\psi=-\int_0^T\!\!\int_\Omega nc\psi,$$
$$\int_0^T\!\!\int_\Omega\tilde\psi_t\cdot u+\int_\Omega\tilde\psi\cdot u_0+\int_0^T\!\!\int_\Omega u\cdot\Delta\tilde\psi+\int_0^T\!\!\int_\Omega n\nabla\phi\cdot\tilde\psi=0$$

hold for all $\psi\in L^2((0,T);W^{1,2}(\Omega))$ and any $\tilde\psi\in L^2((0,T);W^{2,2}(\Omega))\cap W^{1,2}((0,T);L^2(\Omega))$ with values in $\mathbb{R}^2$, $\nabla\cdot\tilde\psi=0$ and $\tilde\psi|_{\partial\Omega}=0$. If (n,c,u,P) is a weak solution of (1.2)–(1.4) in $\Omega\times(0,T)$ for any $T\in(0,\infty)$, then we call (n,c,u,P) a global weak solution.

We are now in the position to prove our main result on global weak solvability.

Proof of Theorem 1.1. Proceeding as in [22, Theorem 1.1], for $\varepsilon \in (0,1)$ we consider the approximate problems given by

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot \big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\big) - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon), \quad x \in \Omega,\ t>0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - n_\varepsilon c_\varepsilon, \quad x \in \Omega,\ t>0, \\ u_{\varepsilon t} + \nabla P_\varepsilon - \Delta u_\varepsilon - n_\varepsilon \nabla \phi = 0, \quad x \in \Omega,\ t>0, \\ \nabla \cdot u_\varepsilon = 0, \quad x \in \Omega,\ t>0, \\ \partial_\nu n_\varepsilon(x,t) = \partial_\nu c_\varepsilon(x,t) = 0 \quad \text{and} \quad u_\varepsilon(x,t)=0, \quad x \in \partial\Omega,\ t>0, \\ n_\varepsilon(x,0)=n_0(x), \quad c_\varepsilon(x,0)=c_0(x) \quad \text{and} \quad u_\varepsilon(x,0)=u_0(x), \quad x \in \Omega, \end{cases} \tag{3.4}$$

where $D_\varepsilon(s) := D(s+\varepsilon)$ for $s \geqslant 0$. Indeed, Theorem 1.2 asserts that for each $\varepsilon \in (0,1)$ this problem admits a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ which is defined for all $t>0$. Moreover, Corollary 2.18, Lemmas 2.5, 2.2 and (3.2) say that for each $T>0$ we can find $c_1(T)>0$ such that

$$\begin{cases} \big\|n_\varepsilon(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant c_1(T) \quad \text{for all } t \in (0,T) \quad \text{and} \\ \int_0^t \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 \leqslant c_1(T) \quad \text{for all } t \in (0,T), \\ \big\|c_\varepsilon(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant c_1(T) \quad \text{and} \quad \big\|\nabla c_\varepsilon(\cdot,t)\big\|_{L^2(\Omega)} \leqslant c_1(T) \quad \text{for all } t \in (0,T), \\ \big\|u_\varepsilon(\cdot,t)\big\|_{L^\infty(\Omega)} \leqslant c_1(T) \quad \text{and} \quad \big\|\nabla u_\varepsilon(\cdot,t)\big\|_{L^2(\Omega)} \leqslant c_1(T) \quad \text{for all } t \in (0,T) \end{cases} \tag{3.5}$$

are valid, and from (1.6) and the proofs of Lemmas 2.3–2.5 we also infer that there exists some $c_2(T)>0$ such that

$$\int_0^t \int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \leqslant c_2(T) \quad \text{for all } t \in (0,T). \tag{3.6}$$

In order to achieve a strong precompactness property of $(n_\varepsilon)_{\varepsilon \in (0,1)}$, let us fix $\theta \geqslant \max(1, \frac{m}{2})$ and multiply the first equation in (3.4) by $n_\varepsilon^{\theta-1}\zeta(x)$, where $\zeta \in C_0^\infty(\Omega)$. On integrating by parts, we thereby obtain

$$\begin{aligned} \frac{1}{\theta} \int_\Omega \big(n_\varepsilon^\theta\big)_t \cdot \zeta = {} & -(\theta-1) \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\theta-2} |\nabla n_\varepsilon|^2 \zeta - \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\theta-1} \nabla n_\varepsilon \cdot \nabla \zeta \\ & + (\theta-1) \int_\Omega n_\varepsilon^{\theta-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \zeta + \int_\Omega n_\varepsilon^\theta \nabla c_\varepsilon \cdot \nabla \zeta + \frac{1}{\theta} \int_\Omega n_\varepsilon^\theta u_\varepsilon \cdot \nabla \zeta \end{aligned} \tag{3.7}$$

for $t>0$. Here we estimate

$$\left| \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\theta-2} |\nabla n_\varepsilon|^2 \zeta \right| \leqslant \|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}^{\theta-1} \cdot \left(\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \right) \cdot \|\zeta\|_{L^\infty(\Omega)},$$

$$\left| \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\theta-1} \nabla n_\varepsilon \cdot \nabla \zeta \right| \leqslant \left(\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 + \big\| D_\varepsilon(n_\varepsilon) n_\varepsilon^{2\theta-1} \big\|_{L^\infty(\Omega\times(0,T))} \cdot |\Omega| \right) \cdot \|\nabla\zeta\|_{L^\infty(\Omega)},$$

$$\begin{aligned} \left| \int_\Omega n_\varepsilon^{\theta-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon \zeta \right| & \leqslant \left(\int_\Omega n_\varepsilon^{2\theta-2} |\nabla n_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \right) \cdot \|\zeta\|_{L^\infty(\Omega)} \\ & \leqslant \left(\|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}^{2\theta-m} \cdot \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2 \right) \cdot \|\zeta\|_{L^\infty(\Omega)}, \end{aligned}$$

$$\left| \int_\Omega n_\varepsilon^\theta \nabla c_\varepsilon \cdot \nabla \zeta \right| \leqslant \left(\|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}^{2\theta} \cdot |\Omega| + \int_\Omega |\nabla c_\varepsilon|^2 \right) \cdot \|\nabla\zeta\|_{L^\infty(\Omega)}$$

and

$$\left| \int_\Omega n_\varepsilon^\theta u_\varepsilon \cdot \nabla\zeta \right| \leqslant \|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}^\theta \cdot \|u_\varepsilon\|_{L^\infty(\Omega\times(0,T))} \cdot \|\nabla\zeta\|_{L^\infty(\Omega)} \cdot |\Omega|.$$

According to our restriction on θ, we deduce from (3.5), (3.6) and

$$\left\| D_\varepsilon(n_\varepsilon)\right\|_{L^\infty(\Omega\times(0,T))} = \left\| D(n_\varepsilon+\varepsilon)\right\|_{L^\infty(\Omega\times(0,T))} \leqslant \max_{s\in[0,1+\|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}]} D(s)$$

that for such θ there exists $c_3(T) > 0$ such that

$$\left\| \left(n_\varepsilon^\theta\right)_t \right\|_{L^1((0,t);(W_0^{2,2}(\Omega))^\star)} = \int_0^t \sup_{\zeta\in C_0^\infty(\Omega),\, \|\zeta\|_{W^{2,2}(\Omega)}\leqslant 1} \left| \int_\Omega \left(n_\varepsilon^\theta\right)_t \cdot \zeta \right| \leqslant c_3(T) \quad \text{for all } t\in(0,T).$$

Similarly, we can find $c_4(T) > 0$ fulfilling

$$\|c_{\varepsilon t}\|_{L^2((0,t);(W_0^{1,2}(\Omega))^\star)} \leqslant c_4(T) \quad \text{for all } t > 0.$$

In conjunction with (3.5), (1.6), (1.5) and the Aubin–Lions compactness lemma ([12, Chapter IV] and [17]), we thus infer the existence of a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ along which

$$\begin{cases} n_\varepsilon \to n & \text{a.e. in } \Omega\times(0,\infty), \\ n_\varepsilon \overset{\star}{\rightharpoonup} n & \text{in } L^\infty\big(\Omega\times(0,\infty)\big), \\ D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \rightharpoonup D(n)\nabla n & \text{in } L^2_{\mathrm{loc}}\big(\bar\Omega\times[0,\infty)\big), \\ c_\varepsilon \to c & \text{a.e. in } \Omega\times(0,\infty), \\ \nabla c_\varepsilon \rightharpoonup \nabla c & \text{in } L^2_{\mathrm{loc}}\big(\bar\Omega\times[0,\infty)\big), \\ u_\varepsilon \rightharpoonup u & \text{in } L^2_{\mathrm{loc}}\big([0,\infty); W_0^{1,2}(\Omega)\big) \end{cases}$$

holds for some limit $(n, c, u) \in (L^\infty(\Omega\times(0,\infty)))^5$ with nonnegative n and c. Due to these convergence properties, applying standard arguments we may take $\varepsilon = \varepsilon_j \searrow 0$ in each term of the natural weak formulation of (3.4) separately to verify that in fact (n, c, u) can be complemented by some pressure function P in such a way that (n, c, u, P) is a weak solution of (1.2)–(1.4).

Finally, the boundedness statement is a straightforward consequence of Corollary 2.18, Lemma 2.2 and (3.2). □

Acknowledgement

Y. Tao is supported by the National Natural Science Foundation of China (No. 11171061).

References

[1] V. Calvez, J.A. Carrillo, Volume effects in the Keller–Segel model: Energy estimates preventing blow-up, J. Math. Pures Appl. 86 (9) (2006) 155–175.

[2] R. Dal Passo, H. Garcke, G. Grün, On a fourth-order degenerate parabolic equation: Global entropy estimates, existence, and qualitative behavior of solutions, SIAM J. Math. Anal. 29 (2) (1998) 321–342.

[3] C. Dombrowski, L. Cisneros, S. Chatkaew, R.E. Goldstein, J.O. Kessler, Self-concentration and large-scale coherence in bacterial dynamics, Phys. Rev. Lett. 93 (2004), 098103, 4 pp.

[4] M. Di Francesco, A. Lorz, P.A. Markowich, Chemotaxis–fluid coupled model for swimming bacteria with nonlinear diffusion: Global existence and asymptotic behavior, Discrete Contin. Dyn. Syst. Ser. A 28 (2010) 1437–1453.

[5] R.J. Duan, A. Lorz, P.A. Markowich, Global solutions to the coupled chemotaxis–fluid equations, Comm. Partial Differential Equations 35 (2010) 1635–1673.

[6] A. Friedman, Partial Differential Equations, Holt, Rinehart & Winston, New York, 1969.

[7] Y. Giga, H. Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains, J. Funct. Anal. 102 (1991) 72–94.

[8] T. Hillen, K. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.

[9] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (1) (2005) 52–107.

[10] S. Ishida, T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type with small data, J. Differential Equations 252 (3) (2011) 2469–2491.

[11] R. Kowalczyk, Preventing blow-up in a chemotaxis model, J. Math. Anal. Appl. 305 (2005) 566–585.

[12] J.L. Lions, Équations différentielles opérationnelles et problémes aux limites, Die Grundlehren der mathematischen Wissenschaften, Springer, 1961.
[13] P.L. Lions, Résolution de problèmes elliptiques quasilinéaires, Arch. Ration. Mech. Anal. 74 (4) (1980) 335–353.
[14] J.-G. Liu, A. Lorz, A coupled chemotaxis–fluid model: Global existence, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011) 643–652.
[15] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci. 20 (2010) 987–1004.
[16] T. Senba, T. Suzuki, A quasi-linear system of chemotaxis, Abstr. Appl. Anal. 2006 (2006) 1–21.
[17] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.
[18] H. Sohr, The Navier–Stokes Equations. An Elementary Functional Analytic Approach, Birkhäuser, Basel, 2001.
[19] Z. Szymańska, C. Morales-Rodrigo, M. Lachowicz, M. Chaplain, Mathematical modelling of cancer invasion of tissue: The role and effect of nonlocal interactions, Math. Models Methods Appl. Sci. 19 (2009) 257–281.
[20] Y. Tao, M. Winkler, A chemotaxis–haptotaxis model: The roles of nonlinear diffusion and logistic source, SIAM J. Math. Anal. 43 (2011) 685–704.
[21] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations 252 (2012) 692–715.
[22] Y. Tao, M. Winkler, Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion, Discrete Contin. Dyn. Syst. Ser. A 32 (5) (2012).
[23] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.
[24] J.L. Vázquez, The Porous Medium Equations, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.
[25] M. Winkler, A critical exponent in a degenerate parabolic equation, Math. Methods Appl. Sci. 25 (2002) 911–925.
[26] M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse?, Math. Methods Appl. Sci. 33 (2010) 12–24.
[27] M. Winkler, Global large-data solutions in a chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Differential Equations 37 (2012) 319–352.