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A multidimensional nonlinear sixth-order quantum diffusion equation ☆

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Abstract

This paper is concerned with the analysis of a sixth-order nonlinear parabolic equation whose solutions describe the evolution of the particle density in a quantum fluid. We prove the global-in-time existence of weak nonnegative solutions in two and three space dimensions under periodic boundary conditions. Moreover, we show that these solutions are smooth and classical whenever the particle density is strictly positive, and we prove the long-time convergence to the spatial homogeneous equilibrium at a universal exponential rate. Our analysis strongly uses the Lyapunov property of the entropy functional.

MSC: 35B45; 35G25; 35K55

Keywords: Higher-order diffusion equations; Quantum diffusion model; Entropy-dissipation estimate; Gradient flow

1. Introduction

Degond et al. derived in [5] a nonlocal quantum diffusion model for charged particles in, for instance, semiconductors or cold plasmas by applying a moment method to a Wigner–BGK model. An asymptotic expansion of the nonlocal model in terms of the scaled Planck constant $\hbar^2$ leads to a family of parabolic equations for the particle densities $n(t;x)$. The first member of this family is the classical heat equation $\partial_t n = \Delta n$. The second one is the fourth-order Derrida–Lebowitz–Speer–Spohn (DLSS) equation, see (3) below, which is analyzed in [9,13]. This paper is concerned with the third family member, obtained from an expansion to order $\hbar^4$ (see [3, Appendix]), which reads as

$$\partial_t n = \operatorname{div}\left(n\nabla\left(\frac{1}{2}\left(\partial^2_{ij}\log n\right)^2 + \frac{1}{n}\partial^2_{ij}\left(n\partial^2_{ij}\log n\right)\right)\right). \qquad (1)$$

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Here and in the following, we employ the notations $\partial_i = \partial/\partial x_i$, $\partial_{ij}^2 = \partial^2/\partial x_i \partial x_j$, etc. and the summation convention over repeated indices from 1 to d. We study the initial-value problem for (1) in the d-dimensional torus $\mathbb{T}^d \cong [0,1]^d$ (imposing periodic boundary conditions) in dimensions $d = 2$ and $d = 3$. The one-dimensional problem has recently been studied in [14].

Specifically, we establish and compare two solution concepts for (1). The first concept is concerned with weak nonnegative solutions; in this framework we generalize the global existence result from [14] to the *multidimensional* situation. The second concept is that of positive classical solutions; in analogy to the results obtained by Bleher et al. for the fourth-order DLSS equation [2], we are able to establish the existence of such regular solutions for (1) locally in time. Naturally, a classical solution is also a weak solution on the time interval of its existence. Vice versa, from a given weak solution, one obtains classical solutions on all time subintervals on which the weak solution is strictly positive and has a uniformly bounded energy (see below for the definition). Since we are not able to rule out the loss of strict positivity due to the evolution, it thus might happen that the classical solution concept breaks down on certain, possibly even infinite time intervals along the globally well-defined weak solution.

We shall provide further motivations to study (1) in Section 2 below. At this point, we simply want to put Eq. (1) into the general context of higher-order parabolic equations. Mainly initiated by the research on pattern formation in Cahn–Hilliard and related models in the late 1980's, the literature on the rich mathematical structure of nonlinear fourth-order and sixth-order equations has grown rapidly over the last two decades. Particular interest has been devoted to equations that are *positivity preserving*: such equations allow for the introduction of a suitable solution concept such that a nonnegative initial datum leads to a nonnegative global solution. Clearly, this is a core feature for equations that model the evolution of particle densities, etc. On the other hand, positivity preservation is a rare property, since general parabolic equations of fourth- or higher-order do not obey comparison principles. For instance, even the linear equation $\partial_t n + (-\Delta)^m n = 0$ is *not* positivity preserving if $m > 1$.

Among the positivity preserving models, the probably most famous study object is the fourth-order thin-film equation

$$\partial_t n + \operatorname{div}\bigl(n^\beta \nabla \Delta n\bigr) = 0. \tag{2}$$

It describes the surface tension-dominated motion of thin viscous films of height $n(t;x) \geqslant 0$ under free slip ($\beta = 2$) or no-slip ($\beta = 3$) boundary conditions. The available literature on the existence, (non-)uniqueness and qualitative properties of solutions is huge and steadily growing; see [1] for a collection of references.

Other models for thin viscous films lead to *sixth-order* equations. One example is

$$\partial_t n = \operatorname{div}\bigl(n^\beta \nabla \Delta^2 n\bigr),$$

which models the spreading of a thin viscous fluid under the driving force of an elastic plate [8]. The model was first introduced in [15, Formula (A8)] in space dimension $d = 1$ with $\beta = 3$ together with a more general form of this equation arising in the isolation oxidation of silicon. Another application for such thin-film equations concerns the bonding of Silicon–Germanium films to silicon substrates [8]. Further examples of sixth-order equations can be found in [7,14,16].

The previously mentioned DLSS (or *quantum diffusion*) equation

$$\partial_t n + \partial_{ij}^2\bigl(n \partial_{ij}^2 \log n\bigr) = 0, \tag{3}$$

provides another well-studied example of a fourth-order equation. Originally, the one-dimensional version of (3) arose in the context of spin systems. Derrida et al. [6] derived it in the course of studying fluctuations of the interface between the regions of predominantly positive and negative particle spins in the Toom model. There are numerous results concerning the existence of weak solutions and their long-time behavior. We refer to [13] for some references. Eq. (1) can be seen as a sixth-order extension of (3).

In the existence analysis for equations like (2), one of the main difficulties is to establish nonnegativity of the solutions. Typically, sophisticated regularizations are constructed that lead to smooth and *strictly positive* approximative solutions. The limit of vanishing regularizations then provides a nonnegative weak solution.

For our Eq. (1), the situation is more delicate since—like for the DLSS equation (3)—the nonlinearity in the equation is not well-defined when n vanishes. This is a problem: Although nonnegativity of the solution is expected on physical grounds, the possibility that a vacuum (localized in time and space) is created from an initially strictly positive density cannot be ruled out. Thus, atop of constructing strictly positive approximations, we need to define a solution concept that works also for merely nonnegative densities with the property that the passage to the limit of vanishing regularizations is possible.

The key idea here is to rewrite the nonlinearity in (1) in a way that substitutes the logarithm by an expression that is still well-defined for $n = 0$. It turns out that the following equivalent representation of Eq. (1),

$$\partial_t n = \Delta^3 n + \partial^3_{ijk} F_1^{(ijk)}(n) + \partial^2_{ij} F_2^{(ij)}(n), \tag{4}$$

with the nonlinear operators

$$F_1^{(ijk)}(n) = 4\partial_i\sqrt{n}\big(4\partial_j\sqrt[4]{n}\partial_k\sqrt[4]{n} - 3\partial^2_{jk}\sqrt{n}\big),$$
$$F_2^{(ij)}(n) = 8\sum_{k=1}^{d}\big(\partial^2_{ik}\sqrt{n} - 4\partial_i\sqrt[4]{n}\partial_k\sqrt[4]{n}\big)\big(\partial^2_{jk}\sqrt{n} - 4\partial_j\sqrt[4]{n}\partial_k\sqrt[4]{n}\big) \tag{5}$$

is appropriate to study both concepts of solutions: weak and classical.

The construction of strictly positive approximative solutions uses yet another transformation of the nonlinearity. First, (1) is discretized in time with the implicit Euler scheme. The semi-discrete equation is regularized by an additional term of the form $\varepsilon(\Delta^3 - 1)\log n$. Each time step then requires the solution of a *strictly* elliptic problem in terms of $y = \log n$. Classical elliptic theory provides L^∞-bounds on y and thus strict positivity of $n = \exp(y)$.

The required compactness to perform the deregularization limit $\varepsilon \downarrow 0$ and later the passage to the time-continuous limit is obtained from the dissipation of a distinguished Lyapunov functional: The physical entropy

$$\mathcal{H}[u] = \int_{\mathbb{T}^d}\big(u(\log u - 1) + 1\big)\,\mathrm{d}x \tag{6}$$

is nonincreasing along the solutions. In fact, using the entropy construction method of [12], which is based on systematic integration by parts, we are able to prove that the entropy dissipation $-\mathrm{d}\mathcal{H}/\mathrm{d}t$ controls certain spatial derivatives,

$$-\frac{\mathrm{d}\mathcal{H}[n]}{\mathrm{d}t} \geqslant \kappa\int_{\mathbb{T}^d}\big(\big\|\nabla^3\sqrt{n}\big\|^2 + |\nabla\sqrt[6]{n}|^6\big)\,\mathrm{d}x, \tag{7}$$

where ∇^k denotes the tensor of all partial derivatives of order k. The resulting estimates are sufficient to pass to the limit.

Our main results about weak solutions are the following two theorems.

Theorem 1 *(Global existence of weak solutions). Let $n_0 \in L^1(\mathbb{T}^d)$ be a nonnegative function with finite entropy $\mathcal{H}[n_0] < \infty$. Then there exists a nonnegative function $n \in W^{1,4/3}_{\mathrm{loc}}(0,\infty;\ H^{-3}(\mathbb{T}^d))$, satisfying $\sqrt{n} \in L^2_{\mathrm{loc}}(0,\infty;\ H^3(\mathbb{T}^d))$ and $n(0) = n_0$, that is a solution to (4) in the following weak sense:*

$$\int_0^\infty \langle\partial_t n, \varphi\rangle\,\mathrm{d}t + \int_0^\infty\int_{\mathbb{T}^d}\big(\partial^3_{ijk}\varphi\,\partial^3_{ijk} n + \partial^3_{ijk}\varphi F_1^{(ijk)}(n) - \partial^2_{ij}\varphi F_2^{(ij)}(n)\big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{8}$$

for all test functions $\varphi \in L^4(0,T;\ H^3(\mathbb{T}^d))$.

It is not trivial at all to see that all integrals on the right-hand side of (8) are well-defined for functions n of the stated regularity. At this point, we just mention that under these hypotheses, $\sqrt[4]{n}$ is a well-defined Sobolev function; see Lemma 26 in Appendix A as well as [17] and [9, Section 3] for a discussion about the regularity of square and fourth roots of nonnegative functions. The relevant estimates on the pairings inside the integrals are established in the course of the proof; see, e.g., Lemma 10 below. Since dimension-dependent Sobolev embeddings are involved, this particular concept of weak solution does not carry over to space dimensions $d \geqslant 4$.

We recall that $H^k(\mathbb{T}^d)$ etc. are spaces of functions that are 1-periodic in each spatial coordinate direction. The derivation of the sixth-order equation (1) in [3] was performed on $\mathbb{R}^d$ and hence, it does not include the derivation of physically relevant boundary conditions. In this work, we have chosen periodic boundary conditions to simplify the analysis. In particular, integration by parts plays a pivotal role in our derivation of a priori estimates, and the boundary integrals vanish for periodic functions. We note that in [3], radially symmetric solutions for (1) satisfying no-flux-type boundary conditions have been considered instead.

Theorem 2 *(Exponential time decay). Let $n_0 \in L^1(\mathbb{T}^d)$ be a nonnegative function with finite entropy $\mathcal{H}[n_0] < \infty$ and unit mass $\int_{\mathbb{T}^d} n_0 \,\mathrm{d}x = 1$. Let n be the weak solution to (4) constructed in Theorem 1. Then there exists a constant $\lambda > 0$, depending on d, such that for all $t > 0$,*

$$\left\| n(t;\cdot) - 1 \right\|_{L^1(\mathbb{T}^d)} \leqslant \sqrt{2\mathcal{H}[n_0]}\, e^{-\lambda t}.$$

Since Eq. (4) is semi-linear parabolic, it is accessible by methods from the theory of analytic semigroups. This approach leads to the following result on classical solutions.

Theorem 3 *(Existence and uniqueness of a classical solution). Let $n_0 \in H^2(\mathbb{T}^d)$ be strictly positive. Then there exist $T_* > 0$ and precisely one smooth and strictly positive classical solution $n \in C^\infty((0,T_*); C^\infty(\mathbb{T}^d))$ to (4) with $n(t) \to n_0$ in $H^2(\mathbb{T}^d)$ as $t \downarrow 0$. Moreover, either $T_* = +\infty$, or there exists a limiting profile $n_* \in H^2(\mathbb{T}^d)$ such that $n(t) \to n_*$ in $H^2(\mathbb{T}^d)$ as $t \uparrow T_*$ and $\min_{x\in\mathbb{T}^d} n_*(x) = 0$.*

In other words, the only possibility for a classical solution to break down is the loss of strict positivity. This result parallels the one of [2] for the fourth-order DLSS equation in space dimension $d = 1$. Since stronger Sobolev embeddings are available for the sixth-order equation (4), our result holds in dimensions $d = 2$ and $d = 3$ as well. It is an open problem if loss of positivity can occur at $t > 0$ or not.

Naturally, we shall establish a connection between the concept of weak solutions, defined in (8), and classical solutions. To do so, we need to introduce the *energy*: For a positive and smooth function $u \in C^\infty(\mathbb{T}^d)$, define

$$\mathcal{E}[u] = \frac{1}{2} \int_{\mathbb{T}^d} u \left\| \nabla^2 \log u \right\|^2 \mathrm{d}x. \tag{9}$$

This functional is equivalent to the L^2-norm of $\nabla^2\sqrt{u}$ in the sense that

$$c \left\| \nabla^2 \sqrt{u} \right\|_{L^2}^2 \leqslant \mathcal{E}[u] \leqslant C \left\| \nabla^2 \sqrt{u} \right\|_{L^2}^2 \tag{10}$$

for some constants $0 < c \leqslant C$ [9,13]. For smooth and positive solutions to (4), one easily proves that $\mathcal{E}$ is a Lyapunov functional, see Lemma 6 below. The functional $\mathcal{E}[u]$ extends in a weakly lower semi-continuous manner to all non-negative functions u with $\sqrt{u} \in H^2(\mathbb{T}^d)$; see [9, Section 3] for details. Hence, if n is a weak solution in the sense of Theorem 1, then $\mathcal{E}[n(t)]$ is well-defined for almost every $t > 0$. We expect that $\mathcal{E}$ is a Lyapunov functional also for weak solutions, but currently we are not able to prove this conjecture, mainly because $\mathcal{E}$ is not a convex functional.

Theorem 4. *Assume that the weak solution n from Theorem 1 has the property that $\mathcal{E}[n(t)]$ is uniformly bounded on some interval (T_1, T_2), and that it is strictly positive at some time $t_0 \in [T_1, T_2)$; here $T_1 = 0$ and/or $T_2 = +\infty$ are admissible. Then there exists $T_* \in (T_1, T_2]$ such that n equals to the classical solution from Theorem 3 on (t_0, T_*). Moreover, either $T_* = T_2$ or $n(t)$ loses strict positivity as $t \uparrow T_*$ in the sense of Theorem 3.*

It is well known for the fourth-order equation (3), that weak solutions may not be unique [13]. We expect the same phenomenon to occur for (1). On the other hand, Theorem 4 asserts that a new weak solution n_* can branch off from a given classical solution n at some time $T > 0$ *only* if either n loses strict positivity, $\lim_{t\uparrow T} \inf_{x\in\mathbb{T}^d} n(t,x) = 0$, or if n_* has locally unbounded energy, $\limsup_{t\downarrow T} \mathcal{E}[n_*(t)] = +\infty$. This shows consistency between the notions of weak and classical solutions. In view of Theorem 2, it is reasonable to conjecture that all weak solutions become classical eventually as $t \to \infty$.

The paper is organized as follows. Section 2 provides some background information on the derivation and properties of (1). In Section 3, we derive the alternative formulation (4) of (1) and we prove the entropy inequality (7). Sections 4, 5, 6, and 7 are devoted to the proofs of Theorems 1, 2, 3, and 4, respectively. Finally, in Appendix A, we collect some technical lemmas and recall some known results which are used in the existence analysis.

Notations. All functions u defined on the torus $\mathbb{T}^d$ are assumed to be one-periodic in each coordinate. Specifically, $u : [0,1]^d \to \mathbb{R}$ is said to belong to the function space $L^p(\mathbb{T}^d)$, $W^{m,p}(\mathbb{T}^d)$ or $C^\infty(\mathbb{T}^d)$, respectively, if its periodic extension $\mathrm{E}u : \mathbb{R}^d \to \mathbb{R}$, defined by $\mathrm{E}u(x) = u(x \bmod \mathbb{T}^d)$, belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $W^{m,p}_{\mathrm{loc}}(\mathbb{R}^d)$ or $C^\infty(\mathbb{R}^d)$. Lebesgue and Sobolev norms are calculated by integrating the respective powers of $\mathrm{E}u$ and its weak derivatives (which are periodic functions on $\mathbb{R}^d$) over the unit cube $[0,1]^d$.

2. Derivation, motivation, and open problems

In this section, we indicate several motivations to study Eq. (1) by reviewing its derivation from the nonlocal quantum model, putting it in the context of gradient flows, and establishing connections to the heat and DLSS equations.

2.1. On the derivation from the nonlocal quantum model

Degond et al. derived in [5] the nonlocal and nonlinear quantum diffusion model

$$\partial_t n = \operatorname{div}(n\nabla A) \quad \text{in } \mathbb{R}^d,\ t > 0, \tag{11}$$

where the potential A is defined implicitly as the unique solution to

$$n(t;x) = \int_{\mathbb{R}^d} \operatorname{Exp}\left(A(t;x) - \frac{|p|^2}{2}\right) \mathrm{d}p.$$

The so-called quantum exponential Exp is defined as the Wigner transformed operator exponential: Denoting by W the Wigner transformation and by W^{-1} the corresponding Weyl quantization, then $\operatorname{Exp}(f) = W^{-1} \circ \exp \circ W(f)$; see [5] for details.

In the semi-classical limit $\hbar \downarrow 0$, the expression $\operatorname{Exp}(A - |p|^2/2)$ converges to e^A, so that $A = \log n$, and we recover from (11) the classical heat equation. For $\hbar > 0$, however, the quantum exponential is a complicated, genuinely nonlocal operator. An asymptotic expansion in terms of $\hbar$ has been performed in [3, Appendix], leading to the following local approximation of A in terms of n:

$$A = A_0 + \frac{\hbar^2}{12} A_1 + \frac{\hbar^4}{360} A_2 + O\big(\hbar^6\big) \tag{12}$$

with the local expressions

$$A_0 = \log n, \qquad A_1 = -2\frac{\Delta\sqrt{n}}{\sqrt{n}}, \qquad A_2 = \frac{1}{2}\big\|\nabla^2 \log n\big\|^2 + \frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big).$$

Replacing A in (11) by A_0, A_1, or A_2 yields, respectively, the heat equation, the DLSS equation (3), or the sixth-order equation (1). In this sense, (3) and (1) constitute, respectively, the primary and secondary quantum corrections to the classical diffusion equation.

2.2. Gradient-flow structure

Eq. (1) possesses—at least on a formal level—a variational structure. The divergence form implies that solutions n formally conserve the total mass, i.e., the integral $m = \int_{\mathbb{T}^d} n(t;x)\,\mathrm{d}x$ is independent of t. By homogeneity, we can assume $m = 1$ without loss of generality. Thus, any solution to (1) defines a curve $t \mapsto n(t)$ in the space of probability measures on $\mathbb{T}^d$. Provided that n is regular enough, this curve realizes a steepest descent in the energy landscape of the energy functional $\mathcal{E}$ from (9) with respect to the L^2-Wasserstein metric. Indeed, by a formal calculation, we obtain the gradient-flow representation

$$\partial_t n = \operatorname{div}\left(n\nabla\frac{\delta\mathcal{E}[n]}{\delta n}\right)$$

from (11) with $A \equiv A_2$, where $A_2 = \delta\mathcal{E}[n]/\delta n$ is the variational derivative of $\mathcal{E}$.

This variational structure is a remarkable property by itself. Atop of that, it establishes yet another connection to the heat and DLSS equations. It is well known since the seminal paper [11] that the heat equation is the gradient flow of the entropy functional $\mathcal{H}$ from (6) with respect to the L^2-Wasserstein distance. The dissipation of $\mathcal{H}$ along its own gradient flow amounts to the *Fisher information*,

$$\mathcal{F}[n] = -\frac{1}{2}\frac{\mathrm{d}\mathcal{H}[n]}{\mathrm{d}t} = \frac{1}{2}\int_{\mathbb{T}^d} n|\nabla \log n|^2\,\mathrm{d}x,$$

while the second-order time derivative produces the energy from (9),

$$\mathcal{E}[n] = \frac{1}{4}\frac{\mathrm{d}^2\mathcal{H}[n]}{\mathrm{d}t^2} = \frac{1}{2}\int_{\mathbb{T}^d} n\left\|\nabla^2 \log n\right\|^2\mathrm{d}x.$$

The Fisher information, in turn, has been proven to generate the DLSS equation (3) as a gradient flow with respect to the L^2-Wasserstein distance [9]. It is readily checked that $\mathcal{E}$ also equals the first-order time derivative of the entropy along solutions of the DLSS equation. In this sense, the sixth-order equation (1) is related to the fourth-order equation (3) in the same way as (3) itself is related to the heat equation.

We mention this point because the intimate relation between the heat and the DLSS equations (and, more generally, between second-order porous medium and fourth-order diffusion equations) has been the key tool in obtaining optimal rates for the intermediate asymptotics of solutions to (3) in [18]. It would be interesting to derive estimates on the long-time behavior of solutions to (1) by similar means.

2.3. Open problems

Finally, we propose several questions about Eq. (1) that we consider of interest:

- With our methods, we are able to prove the dissipation property (7) only in dimensions $d \leqslant 3$. Is $\mathcal{H}$ still a Lyapunov functional in higher dimensions $d \geqslant 4$?
- Is the Fisher information $\mathcal{F}$ a Lyapunov functional? Our only result in this direction so far is a formal proof of dissipation of $\mathcal{F}$ in dimension $d = 1$.
- Is the energy $\mathcal{E}$ monotone along the weak solutions constructed here? If the answer is affirmative, then the additional hypotheses on the uniform boundedness of the energy could be removed from Theorem 4.
- Does (1) admit global weak solutions in dimensions $d \geqslant 4$? Even if we assume that an inequality of the form (7) continues to hold, it is far from clear how to rewrite the weak formulation (8) in a form that does not take advantage of Sobolev embeddings in low dimensions.
- If (1) is posed on $\mathbb{R}^d$ instead of $\mathbb{T}^d$, one readily verifies that there exists a family of self-similar solutions u_s, namely

  $$u_s(t;x) = \lambda(t)^{-d}U\big(\lambda(t)^{-1}x\big) \quad \text{with } \lambda(t) = (1+6t)^{1/6},$$

  with the Gaussian profile

  $$U(z) = \exp\left(-\frac{|z|^2}{2\sqrt[3]{2}}\right).$$

  Do these "spreading Gaussians" play the same role for (1) as they do for the heat equation and for the DLSS equation? In other words, is U an attracting stationary solution of (1) after the self-similar rescaling with $x = \lambda(t)\xi$ and $t = (e^{6\tau} - 1)/6$, and do arbitrary solutions converge to U at a universal exponential rate? In dimension $d = 1$, there is numerical evidence for an affirmative answer.

3. Alternative formulations and functional inequalities

In this section, we derive two alternative formulations of the sixth-order equation (1) and prove an energy-dissipation formula and an entropy-dissipation estimate. First, we show that (1) can be written as the sum of a symmetric sixth-order term and a fourth-order remainder, and as the sum of a linear sixth-order part and a fifth-order remainder.

Lemma 5. *Eq.* (1) *can be written for smooth positive solutions equivalently as*

$$\partial_t n = \partial^3_{ijk}\big(n\partial^3_{ijk}\log n\big) + 2\partial^2_{ij}\big(n\partial^2_{ik}\log n\,\partial^2_{jk}\log n\big) \quad \text{in } \mathbb{T}^d,\ t>0, \tag{13}$$

and also equivalently as

$$\partial_t n = \Delta^3 n + \partial^3_{ijk}F_1^{(ijk)}(n) + \partial^2_{ij}F_2^{(ij)}(n) \quad \text{in } \mathbb{T}^d,\ t>0, \tag{14}$$

where the nonlinear operators $F_1^{(ijk)}$ and $F_2^{(ij)}$ are defined in (5).

We recall that we have employed the summation convention in the above formulas.

Proof. For the following formal calculations, we introduce the shorthand notations $y=\log n$, $y_i=\partial_i\log n$, $y_{ij}=\partial^2_{ij}\log n$, etc. Observing that $\partial_k n = ny_k$, $n\partial_k(1/n) = -(\partial_k n)/n = -y_k$, we calculate

$$\frac{1}{2}n\partial_k\big(\partial^2_{ij}\log n\big)^2 = n\partial^2_{ij}y\,\partial^2_{ij}y_k,$$

and

$$\begin{aligned} n\partial_k\left(\frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big)\right) &= \partial^3_{ijk}(ny_{ij}) - y_k\partial^2_{ij}(ny_{ij}) \\ &= \partial^2_{ij}\big(y_k(ny_{ij}) + ny_{ijk}\big) - y_k\partial^2_{ij}(ny_{ij}) \\ &= \partial_i\big(y_k\partial_j(ny_{ij}) + y_{jk}(ny_{ij})\big) - y_k\partial^2_{ij}(ny_{ij}) + \partial^2_{ij}(ny_{ijk}) \\ &= y_{ik}\partial_j(ny_{ij}) + y_{ijk}(ny_{ij}) + y_{jk}\partial_i(ny_{ij}) + \partial^2_{ij}\big(n\partial^2_{ij}y_k\big) \\ &= 2y_{ik}\partial_j(ny_{ij}) + n\partial^2_{ij}y\,\partial^2_{ij}y_k + \partial^2_{ij}\big(n\partial^2_{ij}y_k\big) \\ &= 2\partial_j(ny_{ij}y_{ik}) - n\partial^2_{ij}y\,\partial^2_{ij}y_k + \partial^2_{ij}\big(n\partial^2_{ij}y_k\big). \end{aligned}$$

Summing these results, we obtain

$$\frac{1}{2}n\partial_k\big(\partial^2_{ij}\log n\big)^2 + n\partial_k\left(\frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big)\right) = \partial^2_{ij}\big(n\partial^2_{ij}y_k\big) + 2\partial_j(ny_{ij}y_{ik}).$$

Differentiation with respect to x_k yields

$$\partial_k\left(\frac{1}{2}n\partial_k\big(\partial^2_{ij}\log n\big)^2 + n\partial_k\left(\frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big)\right)\right) = \partial^3_{ijk}\big(n\partial^3_{ijk}y\big) + 2\partial^2_{jk}(ny_{ij}y_{ik}),$$

which shows (13).

Similarly, introducing $u=\sqrt[4]{n}$, $u_i=\partial_i u$, $u_{ij}=\partial^2_{ij}u$, etc. and observing that $\partial_k n = 4u^3u_k$, $\partial^2_{ij}n = 12u^2u_iu_j + 4u^3u_{ij}$, and $uu_{ij} = \partial^2_{ij}(u^2)/2 - u_iu_j$, we calculate

$$\begin{aligned} n\partial^3_{ijk}y &= \partial^3_{ijk}n - \frac{3}{n}\partial^2_{ij}n\,\partial_k n + \frac{2}{n^2}\partial_i n\,\partial_j n\,\partial_k n \\ &= \partial^3_{ijk}n - 48u^2u_{ij}u_k - 16uu_iu_ju_k \\ &= \partial^3_{ijk}n - 12\partial^2_{ij}\big(u^2\big)\partial_k\big(u^2\big) + 16u_iu_j\partial_k\big(u^2\big) \\ &= \partial^3_{ijk}n + 4\partial_k\sqrt{n}\big(4\partial_i\sqrt[4]{n}\,\partial_j\sqrt[4]{n} - 3\partial^2_{ij}\sqrt{n}\big) \\ &= \partial^3_{ijk}n + F_1^{(ijk)}(n), \\ 2ny_{ik}y_{jk} &= 32u^4\left(\frac{u_{ik}}{u} - \frac{u_iu_k}{u^2}\right)\left(\frac{u_{jk}}{u} - \frac{u_ju_k}{u^2}\right) \\ &= 8\big(\partial^2_{ik}\big(u^2\big) - 4u_iu_k\big)\big(\partial^2_{jk}\big(u^2\big) - 4u_ju_k\big) \\ &= 8\big(\partial^2_{ik}\sqrt{n} - 4\partial_i\sqrt[4]{n}\,\partial_k\sqrt[4]{n}\big)\big(\partial^2_{jk}\sqrt{n} - 4\partial_j\sqrt[4]{n}\,\partial_k\sqrt[4]{n}\big) \\ &= F_2^{(ij)}(n). \end{aligned} \tag{15}$$

Differentiating both equations and summing them leads to

$$\partial^3_{ijk}\big(n\partial^3_{ijk}y\big)+2\partial^2_{ij}(ny_{ik}y_{jk})=\Delta^3 n+\partial^3_{ijk}F_1^{(ijk)}+\partial^2_{ij}F_2^{(ij)}, \tag{16}$$

which gives (14). □

In the next lemma, we make our claim about the Lyapunov property of the energy $\mathcal{E}$, defined in (9), more precise.

Lemma 6. *If $n\in C^\infty((t_1,t_2);C^\infty(\mathbb{T}^d))$ is a positive and classical solution to (1), then the energy $\mathcal{E}[n(t)]$ is a smooth and nonincreasing function on the interval (t_1,t_2). In fact, the energy is dissipated according to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big[n(t)\big]=-\int_{\mathbb{T}^d} n(t)\left|\nabla\left(\frac{1}{2}\big(\partial^2_{ij}\log n(t)\big)^2+\frac{1}{n(t)}\partial^2_{ij}\big(n(t)\partial^2_{ij}\log n(t)\big)\right)\right|^2\mathrm{d}x,\quad t>0. \tag{17}$$

Proof. The smoothness of $\mathcal{E}[n(t)]$ follows since on the set of positive functions $u\in C^\infty(\mathbb{T}^d)$, the operation $u\mapsto\log u$ is a smooth map from $C^\infty(\mathbb{T}^d)$ to itself. Dissipation formula (17) follows by using formulation (1) and integration by parts:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[n]&=\int_{\mathbb{T}^d}\left(\frac{1}{2}\partial_t n\big(\partial^2_{ij}\log n\big)^2+n\partial^2_{ij}(\log n)\partial^2_{ij}\left(\frac{\partial_t n}{n}\right)\right)\mathrm{d}x\\
&=\int_{\mathbb{T}^d}\partial_t n\left(\frac{1}{2}\big(\partial^2_{ij}\log n\big)^2+\frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big)\right)\mathrm{d}x\\
&=-\int_{\mathbb{T}^d} n\left|\nabla\left(\frac{1}{2}\big(\partial^2_{ij}\log n\big)^2+\frac{1}{n}\partial^2_{ij}\big(n\partial^2_{ij}\log n\big)\right)\right|^2\mathrm{d}x,\end{aligned}$$

which shows the claim. □

Finally, we prove the entropy-dissipation inequality (7).

Lemma 7. *Let $d\leqslant 3$ and let $u\in H^3(\mathbb{T}^d)$ be strictly positive on $\mathbb{T}^d$. Then there exists $\kappa>0$, depending only on d, such that*

$$\begin{aligned}&\int_{\mathbb{T}^d}\big(\partial^3_{ijk}(\log u)\partial^3_{ijk}u+\partial^3_{ijk}(\log u)F_1^{(ijk)}(u)-\partial^2_{ij}(\log u)F_2^{(ij)}(u)\big)\,\mathrm{d}x\\
&\qquad\geqslant\kappa\int_{\mathbb{T}^d}\big(\big|\nabla^3\sqrt{u}\big|^2+|\nabla\sqrt[6]{u}|^6\big)\,\mathrm{d}x.\end{aligned} \tag{18}$$

Proof. The proof is based on an extension of the entropy construction method developed in [12] for one-dimensional equations. A proof for $d=1$ is given in [14]. Therefore, we restrict ourselves to the cases $d=2$ and $d=3$. By (16), (18) is equivalent to, up to a factor,

$$\int_{\mathbb{T}^d} u\big(\big(\partial^3_{ijk}\log u\big)^2-2\partial^2_{ij}\log u\big(\partial^2_{ik}\log u\,\partial^2_{jk}\log u\big)\big)\,\mathrm{d}x\geqslant\frac{\kappa}{12}\int_{\mathbb{T}^d}\big(2^6\big|\nabla^3\sqrt{u}\big|^2+6^6|\nabla\sqrt[6]{u}|^6\big)\,\mathrm{d}x. \tag{19}$$

Setting $y=\log n$, $y_i=\partial_i\log n$, $y_{ij}=\partial^2_{ij}\log n$, etc., a computation shows that (19) is equivalent to

$$\int_{\mathbb{T}^d} u\big(12S[u]-\kappa R[u]\big)\,\mathrm{d}x\geqslant 0, \tag{20}$$

where $S[u] = y_{ijk}^2 - 2y_{ij}y_{jk}y_{ki}$ and

$$R[u] = 2y_i^2y_j^2y_k^2 + 12y_i^2y_jy_{jk}y_k + 8y_iy_jy_ky_{ijk} + 24y_iy_{ij}y_{jk}y_k + 12y_i^2y_{jk}^2 + 48y_iy_{ijk}y_{jk} + 16y_{ijk}^2.$$

The idea of the entropy construction method is to find the "right" integrations by parts which are necessary to write the integrand of (20) as a sum of squares. To this end, we define the vector-valued function $v = (v^1, \dots, v^d)^\top : \mathbb{T}^d \to \mathbb{R}^d$ by

$$v^k = \left(2y_i^2y_j^2 + y_{ii}y_j^2 + 5y_{ij}y_iy_j + 5y_{iij}y_j\right)y_k + \left(3y_i^2y_j + 11y_{iij} + 24y_iy_{ij}\right)y_{jk} - (5y_iy_j + 11y_{ij})y_{ijk}.$$

A straightforward computation shows that the weighted divergence

$$T[u] = \frac{1}{u}\operatorname{div}(uv) = e^{-y}\partial_k\left(e^y v^k\right)$$

can be written as

$$\begin{aligned} T[u] = {} & 2y_i^2y_j^2y_k^2 + 3y_i^2y_j^2y_{kk} + 16y_i^2y_jy_{jk}y_k + 9y_i^2y_jy_{jkk} + y_i^2y_{jj}y_{kk} + 7y_{ii}y_jy_{jk}y_k + 40y_iy_{ij}y_{jk}y_k \\ & + 3y_i^2y_{jk}^2 + 5y_iy_{ijj}y_{kk} + 40y_iy_{ij}y_{jkk} + 3y_iy_{ijk}y_{jk} + 11y_{ijj}y_{ikk} - 11y_{ijk}^2 + 24y_{ij}y_{jk}y_{ki}. \end{aligned}$$

By the divergence theorem, we have

$$\int_{\mathbb{T}^d} uT[u]\,\mathrm{d}x = 0.$$

Hence, (20) is equivalent to

$$\int_{\mathbb{T}^d} u\left(12S[u] - \kappa R[u] + T[u]\right)\mathrm{d}x \geqslant 0. \tag{21}$$

We prove that there exists $\kappa > 0$ such that the integrand is nonnegative. The expression $T[u]$ turns out to be the "right" integration by parts formula allowing us to prove the nonnegativity of the above integral. At this point, we need to distinguish the space dimension.

First, consider $d = 2$. Let $x \in \mathbb{T}^d$ be fixed. Without loss of generality, we may assume that $\nabla u(x)$ points into the first coordinate direction, i.e. $y_2 = 0$ at x. Then we compute

$$\begin{aligned} & 12S[u] - \varepsilon R[u] + T[u] \\ & = (2-2\varepsilon)y_1^6 + 3y_1^4(y_{11}+y_{22}) + 4(4-3\varepsilon)y_1^4y_{11} + 9y_1^3(y_{111}+y_{122}) - 8\varepsilon y_1^3y_{111} + y_1^2(y_{11}+y_{22})^2 \\ & \quad + 7y_1^2y_{11}(y_{11}+y_{22}) + 8(5-3\varepsilon)y_1^2\left(y_{11}^2+y_{12}^2\right) + 3(1-4\varepsilon)y_1^2\left(y_{11}^2+2y_{12}^2+y_{22}^2\right) \\ & \quad + 5y_1(y_{111}+y_{122})(y_{11}+y_{22}) + 40y_1\left(y_{11}(y_{111}+y_{122}) + y_{12}(y_{122}+y_{222})\right) \\ & \quad + 3(1-16\varepsilon)y_1(y_{11}y_{111}+2y_{12}y_{112}+y_{22}y_{122}) + 11\left((y_{111}+y_{122})^2 + (y_{112}+y_{222})^2\right) \\ & \quad + (1-16\varepsilon)\left(y_{111}^2+3y_{112}^2+3y_{122}^2+y_{222}^2\right) \\ & = \xi^\top A_\varepsilon \xi + \eta^\top B_\varepsilon \eta, \end{aligned}$$

where ξ and η are the vectors

$$\xi = \left(y_1^3, y_1y_{11}, y_1y_{22}, y_{111}, y_{122}\right)^\top, \qquad \eta = (y_1y_{12}, y_{112}, y_{222})^\top,$$

and the symmetric matrices A_ε and B_ε are defined by

$$A_\varepsilon = \frac{1}{2}\begin{pmatrix} 4-4\varepsilon & 19-12\varepsilon & 3 & 9-8\varepsilon & 9 \\ 19-12\varepsilon & 102-72\varepsilon & 9 & 48-48\varepsilon & 45 \\ 3 & 9 & 8-24\varepsilon & 5 & 8-48\varepsilon \\ 9-8\varepsilon & 48-48\varepsilon & 5 & 24-32\varepsilon & 22 \\ 9 & 45 & 8-48\varepsilon & 22 & 28-96\varepsilon \end{pmatrix},$$

$$B_\varepsilon = \begin{pmatrix} 46-48\varepsilon & 23-48\varepsilon & 20 \\ 23-48\varepsilon & 14-48\varepsilon & 11 \\ 20 & 11 & 12-16\varepsilon \end{pmatrix}.$$

Sylvester's criterion shows that the unperturbed matrices A_0 and B_0 are positive definite. Indeed, the principal minors of A_0 are 2, 47/4, 20, 13, and 149/4, and the principal minors of B_0 are 46, 115, and 334. Since the set of (strictly) positive definite matrices is open in the set of all real symmetric matrices, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the matrices A_ε and B_ε are positive definite, too. This shows that $12S[u] - \varepsilon R[u] + T[u] \geqslant 0$ for $0 < \varepsilon < \varepsilon_0$, which implies (21).

Next, let $d = 3$. This case is similar to the previous one, but technically more involved. Again, we fix some $x \in \mathbb{T}^d$ and assume that $\nabla u(x)$ is parallel to the first coordinate direction, i.e. $y_2 = y_3 = 0$. For easier presentation, we introduce the abbreviations

$$p_+ = y_{22} + y_{33}, \qquad p_- = y_{22} - y_{33},$$
$$q_{j+} = y_{j22} + y_{j33}, \qquad q_{j-} = y_{j22} - y_{j33}, \quad j = 1, 2, 3.$$

Observe that

$$2\big(y_{22}^2 + y_{33}^2\big) = p_+^2 + p_-^2,$$
$$2\big(y_{j22}^2 + y_{j33}^2\big) = q_{j+}^2 + q_{j-}^2,$$
$$2(y_{22}y_{j22} + y_{33}y_{j33}) = p_+q_{j+} + p_-q_{j-}.$$

With these notations, we find that

$$\begin{aligned}
&12S[u] - \varepsilon R[u] + T[u]\\
&\quad = (2-2\varepsilon)y_1^6 + 3y_1^4(y_{11} + p_+) + 4(4-3\varepsilon)y_1^4 y_{11} + 9y_1^3(y_{111} + q_{1+})\\
&\qquad - 8\varepsilon y_1^3 y_{111} + y_1^2(y_{11} + p_+)^2 + 7y_1^2 y_{11}(y_{11} + p_+) + 8(5-3\varepsilon)y_1^2\big(y_{11}^2 + y_{12}^2 + y_{13}^2\big)\\
&\qquad + 3(1-4\varepsilon)y_1^2\left(y_{11}^2 + \frac{1}{2}\big(p_-^2 + p_+^2\big) + 2\big(y_{12}^2 + y_{13}^2 + y_{23}^2\big)\right) + 5y_1(y_{111} + q_{1+})(y_{11} + p_+)\\
&\qquad + 40y_1\big(y_{11}(y_{111} + q_{1+}) + y_{12}(y_{112} + q_{2+}) + y_{13}(y_{113} + q_{3+})\big)\\
&\qquad + 3(1-16\varepsilon)y_1\left(y_{111}y_{11} + \frac{1}{2}(q_{1+}p_+ + q_{1-}p_-) + 2(y_{112}y_{12} + y_{113}y_{13} + y_{123}y_{23})\right)\\
&\qquad + 11\big((y_{111} + q_{1+})^2 + (y_{112} + q_{2+})^2 + (y_{113} + q_{3+})^2\big)\\
&\qquad + (1-16\varepsilon)\left(y_{111}^2 + 3\big(y_{112}^2 + y_{113}^2\big) + \frac{3}{2}\big(q_{1+}^2 + q_{1-}^2 + q_{2+}^2 + q_{2-}^2 + q_{3+}^2 + q_{-3}^2\big) + 6y_{123}^2\right)\\
&\quad = \xi^\top A_\varepsilon \xi + \sum_{j=2}^{3} \eta_j^\top B_\varepsilon \eta_j + \zeta^\top C_\varepsilon \zeta + 2\nu^\top C_\varepsilon \nu + \frac{1}{4}(1-16\varepsilon)\big(q_{2+}^2 + q_{2-}^2\big),
\end{aligned}$$

where

$$\xi = \big(y_1^3, y_1 y_{11}, y_1 p_+, y_{111}, q_{1+}\big)^\top, \qquad \eta_j = (y_1 y_{1j}, y_{11j}, q_{j+})^\top,$$
$$\zeta = (y_1 p_-, q_{1-})^\top, \qquad \nu = (y_1 y_{23}, y_{123})^\top.$$

The matrices A_ε and B_ε are almost identical to those given above, with minor modifications in the third and fifth rows and columns:

$$A_\varepsilon = \frac{1}{2}\begin{pmatrix} 4-4\varepsilon & 19-12\varepsilon & 3 & 9-8\varepsilon & 9\\ 19-12\varepsilon & 102-72\varepsilon & 9 & 48-48\varepsilon & 45\\ 3 & 9 & 5-12\varepsilon & 5 & 13/2-24\varepsilon\\ 9-8\varepsilon & 48-48\varepsilon & 5 & 24-32\varepsilon & 22\\ 9 & 45 & 13/2-24\varepsilon & 22 & 25-48\varepsilon \end{pmatrix},$$

$$B_\varepsilon = \begin{pmatrix} 46-48\varepsilon & 23-48\varepsilon & 20\\ 23-48\varepsilon & 14-48\varepsilon & 11\\ 20 & 11 & 47/4-12\varepsilon \end{pmatrix}.$$

Furthermore, the matrix C_ε is given by

$$C_\varepsilon = \begin{pmatrix} 3-12\varepsilon & 3/2-24\varepsilon \\ 3/2-24\varepsilon & 3-48\varepsilon \end{pmatrix}.$$

Again, the Sylvester criterion shows that A_0, B_0, and C_0 are positive definite. The principal minors of A_0 are 2, 47/4, 19/8, 5/8, and 453/64, while those of B_0 are 46, 115, and 1221/4, and those of C_0 are 3 and 27/4. Thus, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, also A_ε, B_ε, and C_ε are positive definite. □

4. Existence of weak solutions

The proof of Theorem 1 is divided into several steps.

4.1. Solution of the semi-discretized problem

Let $T > 0$ and $\tau > 0$ be given. We wish to solve, for a given initial datum $n_0 \in L^1(\mathbb{T}^d)$, the semi-discrete problem

$$\frac{1}{\tau}(n-n_0) = \Delta^3 n + \partial^3_{ijk} F_1^{(ijk)}(n) + \partial^2_{ij} F_2^{(ij)}(n) \quad \text{in } \mathbb{T}^d,$$

where $F_1^{(ijk)}$ and $F_2^{(ij)}$ are defined in (5).

Proposition 8. *For a nonnegative function $n_0 \in L^1(\mathbb{T}^d)$ of unit mass, $\|n_0\|_{L^1} = 1$, and of finite entropy, $\mathcal{H}[n_0] < \infty$, there exists a sequence of solutions $n_1^\tau, n_2^\tau, \ldots$ in $H^3(\mathbb{T}^d)$ to the elliptic problems*

$$\frac{1}{\tau}\int_{\mathbb{T}^d} \left(n_k^\tau - n_{k-1}^\tau\right)\phi \,\mathrm{d}x + \int_{\mathbb{T}^d} \left(\partial^3_{ijk}\phi\,\partial^3_{ijk} n_k^\tau + \partial^3_{ijk}\phi F_1^{(ijk)}\left(n_k^\tau\right) - \partial^2_{ij}\phi F_2^{(ij)}\left(n_k^\tau\right)\right) \mathrm{d}x = 0, \tag{22}$$

holding for all test functions $\phi \in H^3(\mathbb{T}^d)$, with the initial solution $n_0^\tau = n_0$. These solutions are of unit mass, and the entropy estimate

$$\mathcal{H}\left[n_k^\tau\right] + \kappa\tau \int_{\mathbb{T}^d} \left(\left\|\nabla^3 \sqrt{n_k^\tau}\right\|^2 + \left|\nabla \sqrt[6]{n_k^\tau}\right|^6\right) \mathrm{d}x \leqslant \mathcal{H}\left[n_{k-1}^\tau\right], \quad k \geqslant 1, \tag{23}$$

holds with $\kappa > 0$ given in Lemma 7.

Proof. For simplicity, we only give the argument for the construction of $n = n_1^\tau$ from n_0. The passage from n_k^τ to n_{k+1}^τ works precisely in the same way since finiteness of the entropy is inherited from one step to the next.

Regularized problem. In a first step, we are going to construct strictly positive solutions $n_\varepsilon \in H^3(\mathbb{T}^d)$ to the regularized problem

$$\frac{1}{\tau}(n-n_0) = \Delta^3 n + \partial^3_{ijk} F_1^{(ijk)}(n) + \partial^2_{ij} F_2^{(ij)}(n) + \varepsilon\left(\Delta^3 \log n - \log n\right). \tag{24}$$

Writing $n = e^y$, it follows from (15) that

$$\Delta^3 n = \partial^3_{ijk}\left(n\,\partial^3_{ijk} y\right) - \partial^3_{ijk} F_1^{(ijk)}(n).$$

Thus, assuming strict positivity and H^3-regularity of n, we can reformulate (24) as

$$\frac{1}{\tau}(n-n_0) = \partial^3_{ijk}\left((n+\varepsilon)\partial^3_{ijk} y\right) - \varepsilon y + \partial^2_{ij} F_2^{(ij)}(n), \tag{25}$$

which is an equation in $H^{-3}(\mathbb{T}^d)$.

Fixed point operator. We define the continuous map $S_\varepsilon : X \times [0,1] \to W^{2,4}(\mathbb{T}^d)$ on the set

$$X = \left\{u \in W^{2,4}(\mathbb{T}^d):\ \min_{x\in\mathbb{T}^d} u(x) > 0\right\}$$

as follows. For given $n \in X$ and $\sigma \in [0,1]$, introduce

$$a(y,z) = \int_{\mathbb{T}^d} \big((\sigma n + \varepsilon)\partial^3_{ijk} y \partial^3_{ijk} z + \varepsilon yz\big)\,dx$$

$$f(z) = -\frac{\sigma}{\tau}\int_{\mathbb{T}^d} (n - n_0)z\,dx + \sigma \int_{\mathbb{T}^d} F_2^{(ij)}(n)\partial^2_{ij} z\,dx$$

for all $y, z \in H^3(\mathbb{T}^d)$. Observe that a is a bounded and coercive bilinear form on $H^3(\mathbb{T}^d)$,

$$a(z,z) \geqslant \varepsilon \int_{\mathbb{T}^d} \big(|\nabla z|^2 + z^2\big)\,dx \geqslant c\varepsilon \|z\|^2_{H^3}$$

for some constant $c > 0$, and a varies continuously with $(n,\sigma) \in X \times [0,1]$, since the embedding $W^{2,4}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ is continuous.

Next, we claim that f is a bounded linear form on $H^3(\mathbb{T}^d)$. Indeed, due to the continuity of the Sobolev embedding $W^{2,4}(\mathbb{T}^d) \hookrightarrow W^{1,8}(\mathbb{T}^d)$ in dimensions $d \leqslant 3$ and the strict positivity and continuity of functions in $W^{2,4}(\mathbb{T}^d)$, the mapping $F_2^{(ij)}$ allows for the representation

$$F_2^{(ij)}(n) = 2\frac{\partial^2_{ik} n \partial^2_{kj} n}{n} - 4\frac{\partial^2_{ik} n \partial_k n \partial_j n}{n^2} + 2\frac{(\partial_k n)^2 \partial_i n \partial_j n}{n^3},$$

from which $F_2^{(ij)}(n) \in L^2(\mathbb{T}^d)$ follows for all $n \in W^{2,4}(\mathbb{T}^d)$. In fact, f varies continuously with $(n,\sigma) \in X \times [0,1]$.

The Lax–Milgram Lemma provides the existence and uniqueness of a solution $y \in H^3(\mathbb{T}^d)$ to the elliptic equation

$$a(y,z) = f(z) \quad \text{for all } z \in H^3\big(\mathbb{T}^d\big).$$

This solution depends H^3-continuously on $(n,\sigma) \in X \times [0,1]$. In particular, $y \equiv 0$ if $\sigma = 0$, and y solves (25) if $\sigma = 1$.

The definition of the fixed point operator S_ε is now completed by setting

$$S_\varepsilon(n,\sigma) = e^y.$$

Since $y \in H^3(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$, it is clear that $S_\varepsilon(n,\sigma) \in H^3(\mathbb{T}^d)$ is a strictly positive and bounded function. In view of the compactness of the embedding $H^3(\mathbb{T}^d) \hookrightarrow W^{2,4}(\mathbb{T}^d)$, S_ε maps bounded subsets of $X \times [0,1]$ into precompact sets in $W^{2,4}(\mathbb{T}^d)$. Finally, notice that $S_\varepsilon(n,0) \equiv 1$ for all $n \in X$ and $S_\varepsilon(n_*,1) = n_*$ for some $n_* \in X$ if and only if n_* is a solution to (24). To verify the last statement, observe that $n_* = S_\varepsilon(n_*,1)$ implies the H^3-regularity of n_*, which justifies the passage from (25) to (24), and in particular it allows us to define $\Delta^3 \log n_*$ as an element of $H^{-3}(\mathbb{T}^d)$.

A priori bound. Our goal is to obtain a fixed point of $S_\varepsilon(\cdot,1)$ by means of the Leray–Schauder theorem. Having already verified the continuity and relative compactness of S_ε as well as the condition $S_\varepsilon(\cdot,0) = 1$, it remains to find a suitable closed, bounded, convex subset $B \subset X$ such that all solutions $n_* \in X$ of $S_\varepsilon(n_*,\sigma) = n_*$ for some $\sigma \in [0,1]$ lie in the interior of B. We shall choose

$$B = \big\{u \in W^{2,4}\big(\mathbb{T}^d\big):\ \min u \geqslant \delta,\ \|u\|_{W^{2,4}} \leqslant \delta^{-1}\big\} \tag{26}$$

with a suitable $\delta > 0$ determined below.

Let $n_* \in W^{2,4}(\mathbb{T}^d)$ be a fixed point of $S(\cdot,\sigma)$ for some $\sigma \in [0,1]$. By construction, we have $n_* = e^{y_*} \in H^3(\mathbb{T}^d)$ for $y_* \in H^3(\mathbb{T}^d)$, and n_* is strictly positive. The convexity of $h(s) = s(\log s - 1) + 1$ implies that

$$\begin{aligned}
\frac{1}{\tau}\big(\mathcal{H}[n_*] - \mathcal{H}[n_0]\big) &= \frac{1}{\tau}\int_{\mathbb{T}^d} \big(h(n_*) - h(n_0)\big)\,dx \leqslant \frac{1}{\tau}\int_{\mathbb{T}^d} (n_* - n_0)h'(n_*)\,dx = \frac{1}{\tau}\int_{\mathbb{T}^d} (n_* - n_0)\log n_*\,dx \\
&= -\int_{\mathbb{T}^d} \big(\partial^3_{ijk} y_* \partial^3_{ijk} n_* + \partial^3_{ijk} y_* F_1^{(ijk)}(n_*) - \partial^2_{ij} y_* F_2^{(ij)}(n_*)\big)\,dx - \frac{\varepsilon}{\sigma}\int_{\mathbb{T}^d} \big(\big\|\nabla^3 y_*\big\|^2 + y_*^2\big)\,dx \\
&\leqslant -\kappa \int_{\mathbb{T}^d} \big(\big\|\nabla^3 \sqrt{n_*}\big\|^2 + |\nabla \sqrt[6]{n_*}|^6\big)\,dx - \frac{\varepsilon}{\sigma}\int_{\mathbb{T}^d} \big(\big\|\nabla^3 y_*\big\|^2 + y_*^2\big)\,dx.
\end{aligned}$$

For the last estimate, the functional inequality (18) has been used. Thus, we have proven

$$\mathcal{H}[n_*] + \tau\kappa \int_{\mathbb{T}^d} \left(\left\| \nabla^3 \sqrt{n_*} \right\|^2 + |\nabla \sqrt[6]{n_*}|^6 \right) \mathrm{d}x + \frac{\tau\varepsilon}{\sigma} \int_{\mathbb{T}^d} \left(\left\| \nabla^3 y_* \right\|^2 + y_*^2 \right) \mathrm{d}x \leqslant \mathcal{H}[n_0]. \tag{27}$$

A consequence of this inequality is that y_* is bounded in $H^3(\mathbb{T}^d)$,

$$\|y_*\|_{H^3} \leqslant C \int_{\mathbb{T}^d} \left(\left\| \nabla^3 y_* \right\|^2 + y_*^2 \right) \mathrm{d}x \leqslant \frac{C\mathcal{H}[n_0]}{\tau\varepsilon}$$

for some constant $C > 0$ depending on τ and ε (which are fixed positive numbers at this point), but not on $\sigma \in [0, 1]$. The continuity of the embedding $H^3(\mathbb{T}^d) \hookrightarrow W^{2,4}(\mathbb{T}^d)$ yields the σ-independent bound

$$\|n_*\|_{W^{2,4}} \leqslant \frac{C\mathcal{H}[n_0]}{\tau\varepsilon}, \tag{28}$$

maybe for another constant $C > 0$. Furthermore, the continuity of the embedding $H^3(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ provides the estimate

$$\operatorname{ess\,inf} n_* \geqslant \exp\left(-\|y_*\|_{L^\infty}\right) \geqslant \exp\left(-\frac{C\mathcal{H}[n_0]}{\tau\varepsilon}\right) > 0. \tag{29}$$

From (28) and (29) follows that there exists a set B of the form (26) which contains all potential fixed points n_*. The Leray–Schauder fixed point theorem in the version of [20] (see Theorem 28 in Appendix A) applies to our situation and yields the existence of a solution n_ε to $n = S_\varepsilon(n, 1)$.

Deregularization. From the entropy estimate, it follows that $\sqrt{n_\varepsilon}$ is ε-uniformly bounded in $H^3(\mathbb{T}^d)$, and $\sqrt[6]{n_\varepsilon}$ is ε-uniformly bounded in $W^{1,6}(\mathbb{T}^d)$. Hence, there exists a limit function $n \in H^3(\mathbb{T}^d)$, such that, as $\varepsilon \downarrow 0$, up to subsequences,

$$\sqrt{n_\varepsilon} \rightharpoonup \sqrt{n} \quad \text{in } H^3\left(\mathbb{T}^d\right), \tag{30}$$

$$\sqrt{n_\varepsilon} \to \sqrt{n} \quad \text{in } W^{2,4}\left(\mathbb{T}^d\right) \text{ and in } W^{1,\infty}\left(\mathbb{T}^d\right), \tag{31}$$

$$\sqrt[4]{n_\varepsilon} \to \sqrt[4]{n} \quad \text{in } W^{1,4}\left(\mathbb{T}^d\right), \tag{32}$$

$$\sqrt[4]{n_\varepsilon} \rightharpoonup \sqrt[4]{n} \quad \text{in } W^{1,12}\left(\mathbb{T}^d\right). \tag{33}$$

Here we take (30) for the definition of n; then (31) follows from the compactness of the embedding $H^3(\mathbb{T}^d) \hookrightarrow W^{2,4}(\mathbb{T}^d)$. The strong convergence in (32) is a direct consequence of Proposition 27, since $\sqrt[4]{n_\varepsilon}$ is "sandwiched" between $\sqrt{n_\varepsilon}$ and $\sqrt[6]{n_\varepsilon}$. Concerning (33), observe that $H^3(\mathbb{T}^d)$ embeds continuously into $W^{2,6}(\mathbb{T}^d)$, so that $\sqrt[4]{n_\varepsilon}$ is bounded in $W^{1,12}(\mathbb{T}^d)$ by Lemma 26. In particular, $\sqrt[4]{n_\varepsilon}$ converges weakly to some limit in that space—which necessarily agrees with the strong $W^{1,4}(\mathbb{T}^d)$-limit obtained in (32).

For the various terms in (24), this implies the following. The sequence

$$\partial^3_{ijk} n_\varepsilon = 2\sqrt{n_\varepsilon}\,\partial^3_{ijk}\sqrt{n_\varepsilon} + 6\partial_i\sqrt{n_\varepsilon}\,\partial^2_{jk}\sqrt{n_\varepsilon}$$

converges weakly in $L^2(\mathbb{T}^d)$ to $\partial^3_{ijk} n$, since $\sqrt{n_\varepsilon}$ converges strongly in $L^\infty(\mathbb{T}^d)$ and $\partial^3_{ijk}\sqrt{n_\varepsilon}$ converges weakly in $L^2(\mathbb{T}^d)$, while $\partial_i\sqrt{n_\varepsilon}$ and $\partial^2_{jk}\sqrt{n_\varepsilon}$ both converge strongly in $L^4(\mathbb{T}^d)$. Further, the sequence

$$F_1^{(ijk)}(n_\varepsilon) = 4\partial_i\sqrt{n_\varepsilon}\left(4\partial_j\sqrt[4]{n_\varepsilon}\,\partial_k\sqrt[4]{n_\varepsilon} - 3\partial^2_{jk}\sqrt{n_\varepsilon}\right)$$

converges strongly in $L^2(\mathbb{T}^d)$, since $\partial_i\sqrt{n_\varepsilon}$ converges strongly in $L^\infty(\mathbb{T}^d)$, $\partial^2_{jk}\sqrt{n_\varepsilon}$ converges strongly in $L^2(\mathbb{T}^d)$, and $\partial_j\sqrt[4]{n_\varepsilon}$ and $\partial_k\sqrt[4]{n_\varepsilon}$ both converge strongly in $L^4(\mathbb{T}^d)$. Finally, we consider

$$F_2^{(ij)}(n_\varepsilon) = 8\left(\partial_{ik}\sqrt{n_\varepsilon}\,\partial_{jk}\sqrt{n_\varepsilon} - 4\partial^2_{jk}\sqrt{n_\varepsilon}\,\partial_k\sqrt[4]{n_\varepsilon}\,\partial_i\sqrt[4]{n_\varepsilon} - 4\partial^2_{ik}\sqrt{n_\varepsilon}\,\partial_k\sqrt[4]{n_\varepsilon}\,\partial_j\sqrt[4]{n_\varepsilon} + 16\partial_i\sqrt[4]{n_\varepsilon}\,\partial_j\sqrt[4]{n_\varepsilon}\left(\partial_k\sqrt[4]{n_\varepsilon}\right)^2\right).$$

The first term converges strongly in $L^2(\mathbb{T}^d)$ since it is the product of two second-order derivatives of $\sqrt{n_\varepsilon}$ which converge strongly in $L^4(\mathbb{T}^d)$. The second and third expressions converge strongly in $L^{4/3}(\mathbb{T}^d)$ since each of them is the product of three strongly L^4-convergent terms. To obtain weak $L^{6/5}$-convergence of the last product, we use the strong L^4-convergence of $\partial_i \sqrt[4]{n_\varepsilon}$ to conclude strong convergence of $\partial_j \sqrt[4]{n_\varepsilon}(\partial_k \sqrt[4]{n_\varepsilon})^2$ in $L^{4/3}(\mathbb{T}^d)$, and combine this with the weak convergence of $\partial_i \sqrt[4]{n_\varepsilon}$ in $L^{12}(\mathbb{T}^d)$. Notice that weak convergence in $L^{6/5}(\mathbb{T}^d)$ suffices, since $F_2^{(ij)}(n_\varepsilon)$ is tested in (22) against $\phi \in H^3(\mathbb{T}^d)$ and hence, $\partial^2_{ij}\phi \in L^6(\mathbb{T}^d)$.

Finally, the entropy estimate (27) shows that $(\sqrt{\varepsilon} y_\varepsilon)$ is bounded in $H^3(\mathbb{T}^d)$ and hence,

$$\varepsilon y_\varepsilon \to 0 \quad \text{strongly in } H^3\big(\mathbb{T}^d\big).$$

The above convergence results allow us to perform the limit $\varepsilon \to 0$ in (24), i.e., both sides converge in $H^{-3}(\mathbb{T}^d)$. Hence, n is a nonnegative solution to (22).

Proof of auxiliary properties. It remains to verify that n has unit mass and that the dissipation inequality (23) holds. Conservation of mass follows directly from (22) by using $\phi = 1$ as a test function. The entropy estimate (27) shows that n_ε satisfies

$$\mathcal{H}[n_\varepsilon] + \tau\kappa \int_{\mathbb{T}^d} \big(\big\|\nabla^3\sqrt{n_\varepsilon}\big\|^2 + |\nabla \sqrt[6]{n_\varepsilon}|^6\big)\,\mathrm{d}x \leqslant \mathcal{H}[n_0].$$

Since $\nabla^3\sqrt{n_\varepsilon} \rightharpoonup \nabla^3\sqrt{n}$ weakly in $L^2(\mathbb{T}^d)$ and $\nabla\sqrt[6]{n_\varepsilon} \rightharpoonup \nabla\sqrt[6]{n_\varepsilon}$ weakly in $L^6(\mathbb{T}^d)$, we conclude by lower semi-continuity that

$$\begin{aligned}
&\mathcal{H}[n] + \tau\kappa \int_{\mathbb{T}^d} \big(\big\|\nabla^3\sqrt{n}\big\|^2 + |\nabla \sqrt[6]{n}|^6\big)\,\mathrm{d}x \\
&\quad \leqslant \lim_{\varepsilon\to 0} \mathcal{H}[n_\varepsilon] + \tau\kappa \liminf_{\varepsilon\to 0} \int_{\mathbb{T}^d} \big(\big\|\nabla^3\sqrt{n_\varepsilon}\big\|^2 + |\nabla \sqrt[6]{n_\varepsilon}|^6\big)\,\mathrm{d}x \leqslant \mathcal{H}[n_0].
\end{aligned}$$

This finishes the proof. □

4.2. Passage to the continuous limit

Proposition 8 guarantees the existence of a solution sequence $(n_0^\tau, n_1^\tau, n_2^\tau, \ldots)$ to the semi-discrete implicit Euler scheme (22). Define accordingly the piecewise constant interpolants $\overline{n^\tau} \in L^\infty(0,\infty; H^3(\mathbb{T}^d))$ by

$$\overline{n^\tau}(t) = n_k^\tau \quad \text{for } (k-1)\tau < t \leqslant k\tau,\ k \in \mathbb{N}, \qquad \overline{n^\tau}(0) = n_0^\tau,$$

and introduce the discrete time derivative

$$\delta_\tau \overline{n^\tau}(t) = \frac{1}{\tau}\big(n_k^\tau - n_{k-1}^\tau\big) \quad \text{for } (k-1)\tau < t \leqslant k\tau,\ k \in \mathbb{N}.$$

Corollary 9. *The interpolated function $\overline{n^\tau}$ satisfies*

$$\int_0^T\!\!\int_{\mathbb{T}^d} \delta_\tau \overline{n^\tau}\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{\mathbb{T}^d} \big(\partial^3_{ijk}\varphi\,\partial^3_{ijk}\overline{n^\tau} + \partial^3_{ijk}\varphi F_1^{(ijk)}\big(\overline{n^\tau}\big) - \partial^2_{ij}\varphi F_2^{(ij)}\big(\overline{n^\tau}\big)\big)\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{34}$$

for all test functions $\varphi \in L^4(0,T; H^3(\mathbb{T}^d))$.

Proof. Eq. (34) is a direct consequence of (22), and the definitions of $\overline{n^\tau}$ and $\delta_\tau\overline{n^\tau}$. Simply choose $\phi = \varphi(t) \in H^3(\mathbb{T}^d)$ as a test function in (22) for $(k-1)\tau < t \leqslant k\tau$ and integrate with respect to $t \in (0,T)$. Notice that at this point, the L^4-regularity of φ with respect to time is not of importance. In fact, we could replace L^4 by L^1. □

The following lemma summarizes various consequences of the discrete entropy estimate (23). Recall that we are working in spatial dimensions $d \leqslant 3$.

Lemma 10. *For any finite $T>0$,*

$$\big(\overline{n^\tau}\big) \quad \text{is bounded in } L^{4/3}\big(0,T;H^3\big(\mathbb{T}^d\big)\big), \tag{35}$$

$$\big(\nabla\sqrt{\overline{n^\tau}}\big) \quad \text{is bounded in } L^{12/5}\big(0,T;L^\infty\big(\mathbb{T}^d\big)\big), \tag{36}$$

$$\big(\nabla^2\sqrt{\overline{n^\tau}}\big) \quad \text{is bounded in } L^3\big(0,T;L^2\big(\mathbb{T}^d\big)\big) \text{ and in } L^{8/3}\big(0,T;L^{12/5}\big(\mathbb{T}^d\big)\big), \tag{37}$$

$$\big(\nabla\sqrt[4]{\overline{n^\tau}}\big) \quad \text{is bounded in } L^6\big(0,T;L^4\big(\mathbb{T}^d\big)\big) \text{ and in } L^{16/3}\big(0,T;L^{24/5}\big(\mathbb{T}^d\big)\big), \tag{38}$$

$$\big(\nabla\sqrt[6]{\overline{n^\tau}}\big) \quad \text{is bounded in } L^6\big(0,T;L^6\big(\mathbb{T}^d\big)\big), \tag{39}$$

uniformly with respect to $\tau>0$.

Proof. First notice that the boundedness of $\sqrt{\overline{n^\tau}}$ in $L^2(0,T;H^3(\mathbb{T}^d))$ follows from the entropy estimate (23). Indeed, by Lemma 23 from Appendix A and the conservation of mass, we find that

$$\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3} \leqslant C\big(\big\|\nabla^3\sqrt{\overline{n^\tau}}(t)\big\|_{L^2} + \big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^2}\big) = C\big(\big\|\nabla^3\sqrt{\overline{n^\tau}}(t)\big\|_{L^2} + 1\big),$$

where $C>0$ does not depend on τ. Therefore,

$$\big\|\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;H^3)} \leqslant C\big(\big\|\nabla^3\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;H^3)} + T^{1/2}\big) \leqslant C\big(\mathcal{H}[n_0] + T^{1/2}\big).$$

Estimate (39) follows also from the entropy estimate (23).

To prove the remaining estimates, first notice that, by the Gagliardo–Nirenberg inequality (see Lemma 25 in Appendix A), for some constants $B_i>0$,

$$\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^\infty} \leqslant B_1\big|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3}^{d/6}\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^2}^{1-d/6},$$
$$\big\|\nabla\sqrt{\overline{n^\tau}}(t)\big\|_{L^\infty} \leqslant B_2\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3}^{1/3+d/6}\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^2}^{2/3-d/6},$$
$$\big\|\nabla^2\sqrt{\overline{n^\tau}}(t)\big\|_{L^2} \leqslant B_3\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3}^{2/3}\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^2}^{1/3}.$$

Integrating over $(0,T)$, we infer that

$$\big\|\sqrt{\overline{n^\tau}}\big\|_{L^{12/d}(0,T;L^\infty)} \leqslant B_1\big\|\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;H^3)}^{d/6}\big\|\sqrt{\overline{n^\tau}}\big\|_{L^\infty(0,T;L^2)}^{1-d/6} \leqslant C, \tag{40}$$

$$\big\|\nabla\sqrt{\overline{n^\tau}}\big\|_{L^{12/(d+2)}(0,T;L^\infty)} \leqslant B_2\big\|\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;H^3)}^{(2+d)/6}\big\|\sqrt{\overline{n^\tau}}\big\|_{L^\infty(0,T;L^2)}^{(4-d)/6} \leqslant C, \tag{41}$$

$$\big\|\nabla^2\sqrt{\overline{n^\tau}}\big\|_{L^3(0,T;L^2)} \leqslant B_3\big\|\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;H^3)}^{2/3}\big\|\sqrt{\overline{n^\tau}}\big\|_{L^\infty(0,T;L^2)}^{1/3} \leqslant C, \tag{42}$$

where $C>0$ does not depend on τ. Estimate (41) implies the bound (36) since $12/(d+2) \geqslant 12/5$ for $d \leqslant 3$. Taking into account

$$\partial^3_{ijk}\overline{n^\tau} = \partial^3_{ijk}\big(\sqrt{\overline{n^\tau}}\big)^2 = 2\sqrt{\overline{n^\tau}}\partial^3_{ijk}\sqrt{\overline{n^\tau}} + 2\big(\partial_i\sqrt{\overline{n^\tau}}\partial^2_{jk}\sqrt{\overline{n^\tau}} + \partial_j\sqrt{\overline{n^\tau}}\partial^2_{ik}\sqrt{\overline{n^\tau}} + \partial_k\sqrt{\overline{n^\tau}}\partial^2_{ij}\sqrt{\overline{n^\tau}}\big),$$

Hölder's inequality and estimates (40)–(42) give

$$\begin{aligned}
\big\|\nabla^3\overline{n^\tau}\big\|_{L^{4/3}(0,T;L^2)}^{4/3} &\leqslant C\int_0^T\big(\big\|\sqrt{\overline{n^\tau}}\big\|_{L^\infty}^{4/3}\big\|\nabla^3\sqrt{\overline{n^\tau}}\big\|_{L^2}^{4/3} + \big\|\nabla\sqrt{\overline{n^\tau}}\big\|_{L^\infty}^{4/3}\big\|\nabla^2\sqrt{\overline{n^\tau}}\big\|_{L^2}^{4/3}\big)\,\mathrm{d}t \\
&\leqslant C\big\|\sqrt{\overline{n^\tau}}\big\|_{L^4(0,T;L^\infty)}^{4/3}\big\|\nabla^3\sqrt{\overline{n^\tau}}\big\|_{L^2(0,T;L^2)}^{4/3} + C\big\|\nabla\sqrt{\overline{n^\tau}}\big\|_{L^{12/5}(0,T;L^\infty)}^{4/3}\big\|\nabla^2\sqrt{\overline{n^\tau}}\big\|_{L^3(0,T;L^2)}^{4/3} \\
&\leqslant C,
\end{aligned}$$

since $12/d \geqslant 4$ for $d \leqslant 3$. This proves (35). The first bound in (37) follows from (41), while

$$\int_0^T\big\|\nabla^2\sqrt{\overline{n^\tau}}(t)\big\|_{L^{12/5}}^{8/3}\,\mathrm{d}t \leqslant B_4\int_0^T\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3}^{2(24+d)/27}\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{L^2}^{2(12-d)/27}\,\mathrm{d}t,$$

yields the second bound, since $2(24+d)/27 \leqslant 2$. Finally, (38) is a consequence of (37) in combination with the Lions–Villani estimate [17] on square roots (see Lemma 26 in Appendix A). □

Lemma 11. *For any finite $T>0$, the sequence*

$$\big(\delta_\tau \overline{n^\tau}\big) \quad \text{is bounded in } L^{4/3}\big(0,T;H^{-3}\big(\mathbb{T}^d\big)\big), \tag{43}$$

uniformly in $\tau>0$.

Proof. We need to show that there exists a constant $M>0$ such that

$$\left|\int_0^T\int_{\mathbb{T}^d} \delta_\tau \overline{n^\tau}(t;x)\varphi(t;x)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant M\|\varphi\|_{L^4(0,T;H^3)}$$

holds for every test function $\varphi\in L^4(0,T;H^3(\mathbb{T}^d))$, independently of $\tau>0$. Since, according to (34), the discrete time derivative can be decomposed as

$$\delta_\tau\overline{n^\tau} = \Delta^3\overline{n^\tau} + \partial^3_{ijk}F_1^{(ijk)}\big(\overline{n^\tau}\big) + \partial^2_{ij}F_2^{(ij)}\big(\overline{n^\tau}\big)$$

in the sense of $L^{4/3}(0,T;H^{-3}(\mathbb{T}^d))$, it suffices to discuss the three terms on the right-hand side separately. For $\Delta^3\overline{n^\tau}$, using Hölder inequality, it follows that

$$\left|\int_0^T\int_{\mathbb{T}^d} \partial^3_{ijk}\varphi(t;x)\partial^3_{ijk}\overline{n^\tau}(t;x)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \int_0^T \big\|\varphi(t)\big\|_{H^3}\big\|\overline{n^\tau}(t)\big\|_{H^3}\,\mathrm{d}t \leqslant \|\varphi\|_{L^4(0,T;H^3)}\big\|\overline{n^\tau}\big\|_{L^{4/3}(0,T;H^3)},$$

and the last expression is uniformly bounded with respect to τ in view of (35). Concerning $\partial^3_{ijk}F_1^{(ijk)}$, we find that

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^d} \partial^3_{ijk}\varphi(t;x)F_1^{(ijk)}\big(\overline{n^\tau}(t;x)\big)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leqslant 4\int_0^T \big\|\varphi(t)\big\|_{H^3}\big\|\nabla\sqrt{\overline{n^\tau}}(t)\big\|_{L^\infty}\big(3\big\|\nabla^2\sqrt{\overline{n^\tau}}(t)\big\|_{L^2} + 4\big\|\nabla\sqrt[4]{\overline{n^\tau}}(t)\big\|^2_{L^4}\big)\,\mathrm{d}t \\
&\quad\leqslant 4\|\varphi\|_{L^4(0,T;H^3)}\big\|\nabla\sqrt{\overline{n^\tau}}\big\|_{L^{12/5}(0,T;L^\infty)}\big(3\big\|\nabla^2\sqrt{\overline{n^\tau}}\big\|_{L^3(0,T;L^2)} + 4\big\|\nabla\sqrt[4]{\overline{n^\tau}}\big\|^2_{L^6(0,T;L^4)}\big),
\end{aligned}$$

which is bounded, in view of (36), (37), and (38). Finally,

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^d} \partial^2_{ij}\varphi(t;x)F_2^{(ij)}\big(\overline{n^\tau}(t;x)\big)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leqslant \int_0^T \big\|\nabla^2\varphi(t)\big\|_{L^6}\big\|F_2\big(\overline{n^\tau}(t)\big)\big\|_{L^{6/5}}\,\mathrm{d}t \\
&\quad\leqslant C\int_0^T \big\|\varphi(t)\big\|_{H^3}\big(\big\|\nabla^2\sqrt{\overline{n^\tau}}(t)\big\|_{L^{12/5}} + 8\big\|\nabla\sqrt[4]{\overline{n^\tau}}(t)\big\|^2_{L^{24/5}}\big)^2\,\mathrm{d}t \\
&\quad\leqslant 2C\|\varphi\|_{L^4(0,T;H^3)}\big(\big\|\nabla^2\sqrt{\overline{n^\tau}}\big\|^2_{L^{8/3}(0,T;L^{12/5})} + 16\big\|\nabla\sqrt[4]{\overline{n^\tau}}\big\|^4_{L^{16/3}(0,T;L^{24/5})}\big)
\end{aligned}$$

shows that also $\partial^2_{ij}F_2^{(ij)}$ is uniformly bounded with respect to τ in $L^{4/3}(0,T;H^{-3}(\mathbb{T}^d))$, see (37) and (38). $\square$

Lemma 12. *There exists a nonnegative function $n \in L^{4/3}(0,T;H^3(\mathbb{T}^d))$ such that along a suitable sequence $\tau \downarrow 0$,*

$$\overline{n^\tau} \rightharpoonup n \quad \text{in } L^{4/3}\big(0,T;H^3\big(\mathbb{T}^d\big)\big), \tag{44}$$

$$\delta_\tau \overline{n^\tau} \rightharpoonup \partial_t n \quad \text{in } L^{4/3}\big(0,T;H^{-3}\big(\mathbb{T}^d\big)\big), \tag{45}$$

$$\sqrt{\overline{n^\tau}} \to \sqrt{n} \quad \text{in } L^2\big(0,T;H^2\big(\mathbb{T}^d\big)\big), \tag{46}$$

$$\sqrt[4]{\overline{n^\tau}} \to \sqrt[4]{n} \quad \text{in } L^4\big(0,T;W^{1,4}\big(\mathbb{T}^d\big)\big). \tag{47}$$

Proof. Estimate (35) immediately implies (44), i.e., (a subsequence of) $\overline{n^\tau}$ converges weakly to some limit n in $L^{4/3}(0,T;H^3(\mathbb{T}^d))$. This convergence is even stronger: The τ-uniform bound (43) on $\delta_\tau\overline{n^\tau}$ allows us to apply Aubin's compactness lemma [21] to $\overline{n^\tau}$ (using Lemma A.2 of [4]). It follows that $\overline{n^\tau}$ converges strongly to the same limit n in $L^{4/3}(0,T;H^2(\mathbb{T}^d))$ and that $\delta_\tau\overline{n^\tau}$ converges to $\partial_t n$ weakly in $L^{4/3}(0,T;H^{-3}(\mathbb{T}^d))$, proving (45).

Of course, $\overline{n^\tau}$ also converges strongly to n in $L^1(0,T;L^1(\mathbb{T}^d))$. Therefore,

$$\int_0^T\int_{\mathbb{T}^d}\big|\sqrt{\overline{n^\tau}}(t;x)-\sqrt{n}(t;x)\big|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T\int_{\mathbb{T}^d}\big|\overline{n^\tau}(t;x)-n(t;x)\big|\,\mathrm{d}x\,\mathrm{d}t \to 0,$$

since $|\sqrt{a}-\sqrt{b}|^2 \leqslant |a-b|$ for $a,b \geqslant 0$. It follows that $\sqrt{\overline{n^\tau}}$ converges strongly to $\sqrt{n}$ in $L^2(0,T;L^2(\mathbb{T}^d))$. Invoking the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
&\int_0^T \big\|\nabla^2\sqrt{\overline{n^\tau}}(t)-\nabla^2\sqrt{n}(t)\big\|_{L^2}^2\,\mathrm{d}t \\
&\quad\leqslant B\int_0^T \big\|\sqrt{\overline{n^\tau}}(t)-\sqrt{n}(t)\big\|_{H^3}^{4/3}\big\|\sqrt{\overline{n^\tau}}(t)-\sqrt{n}(t)\big\|_{L^2}^{2/3}\,\mathrm{d}t \\
&\quad\leqslant B\bigg(\int_0^T\big(\big\|\sqrt{\overline{n^\tau}}(t)\big\|_{H^3}^2+\big\|\sqrt{n}(t)\big\|_{H^3}^2\big)\,\mathrm{d}t\bigg)^{2/3}\bigg(\int_0^T\big\|\sqrt{\overline{n^\tau}}(t)-\sqrt{n}(t)\big\|_{L^2}^2\,\mathrm{d}t\bigg)^{2/3},
\end{aligned}$$

which tends to zero since $\sqrt{\overline{n^\tau}}$ is uniformly bounded with respect to τ in $L^2(0,T;H^3(\mathbb{T}^d))$, by (35), and it converges strongly to $\sqrt{n}$ in $L^2(0,T;L^2(\mathbb{T}^d))$. This proves (46).

Finally, (47) is a consequence of Proposition 27 (see Appendix A), applied with $\alpha=1/2$, $\beta=1/6$, $\gamma=1/4$, and $p=2$, $q=6$, $r=4$. Indeed, a simple combination of the strong convergence of $\sqrt{\overline{n^\tau}}$ in $L^2(0,T;H^2(\mathbb{T}^d))$ with the boundedness of $(\sqrt[6]{\overline{n^\tau}})$ in $L^6(0,T;W^{1,6}(\mathbb{T}^d))$ (see (39)) gives the conclusion. □

Proof of Theorem 1. It remains to prove that the limit function $n \in L^{4/3}(0,T;H^3(\mathbb{T}^d))$ from Lemma 12 is the sought weak solution for (8). In other words, we need to identify the limit $\partial_t n$ with the right-hand side of (4). We recall that, by the weak convergence of $\delta_\tau\overline{n^\tau}$ to $\partial_t n$ in $L^{4/3}(0,T;H^{-3}(\mathbb{T}^d))$,

$$\int_0^T\langle\partial_t n,\varphi\rangle\,\mathrm{d}t = \lim_{\tau\downarrow 0}\int_0^T\big\langle\delta_\tau\overline{n^\tau},\varphi\big\rangle\,\mathrm{d}t$$

holds for all $\varphi \in L^4(0,T;H^3(\mathbb{T}^d))$. In view of (34), the goal is thus to prove that

$$\begin{aligned}
&\lim_{\tau\downarrow 0}\int_0^T\int_{\mathbb{T}^d}\big(\partial^3_{ijk}\varphi\,\partial^3_{ijk}\overline{n^\tau}+\partial^3_{ijk}\varphi F_1^{(ijk)}\big(\overline{n^\tau}\big)-\partial^2_{ij}\varphi F_2^{(ij)}\big(\overline{n^\tau}\big)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad=\int_0^T\int_{\mathbb{T}^d}\big(\partial^3_{ijk}\varphi\,\partial^3_{ijk}n+\partial^3_{ijk}\varphi F_1^{(ijk)}(n)-\partial^2_{ij}\varphi F_2^{(ij)}(n)\big)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for all test functions φ from some dense set of $L^4(0,T;H^3(\mathbb{T}^d))$. Since the C^∞ functions are dense in that set, it suffices to prove the weak convergence of $\partial^3_{ijk}\overline{n^\tau}$, $F_1^{(ijk)}(\overline{n^\tau})$, and $F_2^{(ij)}(\overline{n^\tau})$ to their respective limits $\partial^3_{ijk}n$, $F_1^{(ijk)}(n)$, and $F_2^{(ij)}(n)$ in $L^1(0,T;L^1(\mathbb{T}^d))$.

First term of the integrand. From (44), it follows in particular that $\partial^3_{ijk}\overline{n^\tau}$ converges weakly to $\partial^3_{ijk}n$ in $L^{4/3}(0,T;L^2(\mathbb{T}^d))$ for any combination of the indices i, j, and k, and thus, as $\tau\downarrow 0$,

$$\int_0^T\int_{\mathbb{T}^d}\partial^3_{ijk}\varphi\,\partial^3_{ijk}\overline{n^\tau}\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_{\mathbb{T}^d}\partial^3_{ijk}\varphi\,\partial^3_{ijk}n\,\mathrm{d}x\,\mathrm{d}t.$$

Second term of the integrand. We recall the definition of $F_1^{(ij)}$ from (5). As a consequence of (47), the first-order derivatives $\partial_j\sqrt[4]{\overline{n^\tau}}$ converge strongly to $\partial_j\sqrt[4]{n}$ in $L^4(0,T;L^4(\mathbb{T}^d))$ for all j. As a product of strongly convergent sequences, each $\partial_j\sqrt[4]{\overline{n^\tau}}\partial_k\sqrt[4]{\overline{n^\tau}}$ converges strongly in $L^2(0,T;L^2(\mathbb{T}^d))$ to the respective product $\partial_j\sqrt[4]{n}\partial_k\sqrt[4]{n}$. Clearly, all second-order derivatives $\partial^2_{jk}\sqrt{\overline{n^\tau}}$ tend strongly to their respective limits $\partial^2_{jk}\sqrt{n}$ in $L^2(0,T;L^2(\mathbb{T}^d))$ as well, taking into account (46). In combination with the strong convergence of $\partial_i\sqrt{\overline{n^\tau}}$ to $\partial_i\sqrt{n}$ in $L^2(0,T;L^2(\mathbb{T}^d))$, by (46), it follows that each $F_1^{(ijk)}(\overline{n^\tau})$ is the sum of products of two strongly convergent sequences in $L^2(0,T;L^2(\mathbb{T}^d))$ and consequently, the product converges strongly in $L^1(0,T;L^1(\mathbb{T}^d))$ to the product of the limits:

$$\int_0^T\int_{\mathbb{T}^d}\partial^3_{ijk}\varphi F_1^{(ijk)}\big(\overline{n^\tau}\big)\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_{\mathbb{T}^d}\partial^3_{ijk}\varphi F_1^{(ijk)}(n)\,\mathrm{d}x\,\mathrm{d}t.$$

Third term of the integrand. Arguing as above, it follows from (46) and (47) that both summands in $F_2^{(ij)}(\overline{n^\tau})$ converge strongly in $L^2(0,T;L^2(\mathbb{T}^d))$ to their respective limits, and so the sequence of the product converges strongly in $L^1(0,T;L^1(\mathbb{T}^d))$ to the product of the limit. This means that

$$\int_0^T\int_{\mathbb{T}^d}\partial^2_{ij}\varphi F_2^{(ij)}\big(\overline{n^\tau}\big)\,\mathrm{d}x\,\mathrm{d}t\to\int_0^T\int_{\mathbb{T}^d}\partial^2_{ij}\varphi F_2^{(ij)}(n)\,\mathrm{d}x\,\mathrm{d}t$$

finishing the proof. □

5. Exponential time decay of weak solutions

Proof of Theorem 2. Let $\tau>0$ and let $n_1^\tau,n_2^\tau,\ldots$ be the sequence of solutions to the semi-discretized problem constructed in Proposition 8. The discrete entropy estimate (23) implies that

$$\mathcal{H}\big[n_k^\tau\big]+\tau\kappa\int_{\mathbb{T}^d}\big|\nabla^3\sqrt{n_k^\tau}\big|^2\,\mathrm{d}x\leqslant\mathcal{H}\big[n_{k-1}^\tau\big],\quad k\in\mathbb{N},$$

with a positive constant $\kappa>0$ independent of k and τ. Employing the generalized logarithmic Sobolev inequality,

$$\int_{\mathbb{T}^d}n_k^\tau\log\bigg(\frac{n_k^\tau}{\|n_k^\tau\|_{L^1(\mathbb{T}^d)}}\bigg)\,\mathrm{d}x\leqslant\frac{1}{32\pi^6}\int_{\mathbb{T}^d}\big\|\nabla^3\sqrt{n_k^\tau}\big\|^2\,\mathrm{d}x,$$

which is proven as in [13], and observing that $\|n_k^\tau\|_{L^1(\mathbb{T}^d)}=\|n_0\|_{L^1(\mathbb{T}^d)}=1$, we infer that

$$\mathcal{H}\big[n_k^\tau\big]\leqslant\frac{1}{32\pi^6}\int_{\mathbb{T}^d}\big\|\nabla^3\sqrt{n_k^\tau}\big\|^2\,\mathrm{d}x.$$

Then the above entropy inequality yields

$$\mathcal{H}\big[n_k^\tau\big]+32\pi^6\tau\kappa\mathcal{H}\big[n_k^\tau\big]\leqslant\mathcal{H}\big[n_{k-1}^\tau\big],\quad k\in\mathbb{N},$$

which in turn implies for all $t \in ((k-1)\tau, k\tau]$ that

$$\mathcal{H}\big[\overline{n^\tau}(t)\big] \leqslant \big(1+32\pi^6\tau\kappa\big)^{-t/\tau}\mathcal{H}[n_0],$$

since $k \geqslant t/\tau$. Recall that $\overline{n^\tau}(t)$ converges a.e. to $n(t)$ as $\tau \to 0$, and observe that $(1+32\pi^6\tau\kappa)^{-t/\tau}$ converges to $\exp(-32\pi^6\kappa t)$. Thus the limit $\tau \to 0$ gives

$$\mathcal{H}\big[n(t)\big] \leqslant \mathcal{H}[n_0]e^{-32\pi^6\kappa t}, \quad t \geqslant 0.$$

An application of the Csiszár–Kullback–Pinsker inequality (see, e.g., [22, Section 2]) concludes the proof. □

6. Existence and uniqueness of classical solutions

In this section, we invoke the machinery of analytic semigroups to prove Theorem 3. Our approach follows closely the strategy developed in [2] by Bleher at al. for the fourth-order DLSS equation. However, the more complicated structure of the nonlinearities in our sixth-order equation induces a variety of additional technical difficulties.

6.1. Definitions

We collect some standard results on the operator Δ^3. By abuse of notation, we use the symbol Δ^3 for the $L^1(\mathbb{T}^d)$-closure of the operator $\Delta^3\varphi = \sum_{i,j,k=1}^d \partial_i^2\partial_j^2\partial_k^2\varphi$, defined for $\varphi \in C^\infty(\mathbb{T}^d)$. Define the auxiliary function $H \in C^\infty(\mathbb{R}^d)$ by

$$H(z) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-|\zeta|^6}e^{i\zeta\cdot z}\,\mathrm{d}\zeta,$$

and introduce for each $t > 0$ the so-called solution kernel $G(t) \in C^\infty(\mathbb{T}^d)$ by

$$G(t;y) = t^{-d/6}\sum_{\Lambda\in\mathbb{Z}^d} H\big(t^{-1/6}(y+\Lambda)\big).$$

The series converges since $H(z)$ decays exponentially for $|z| \to \infty$. Classical parabolic theory provides the following result.

Lemma 13. *Let $m \in \mathbb{N}_0$, $p \in [1,\infty)$, and $\alpha \in (0,1)$. If $u \in W^{m,p}(\mathbb{T}^d)$, then the convolution $U(t) = G(t)\star u$ defines a smooth curve, satisfying*

$$U \in C^\infty\big((0,\infty); C^\infty\big(\mathbb{T}^d\big)\big) \cap C^0\big([0,\infty); W^{m,p}\big(\mathbb{T}^d\big)\big), \quad \frac{\mathrm{d}}{\mathrm{d}t}U(t) = \Delta^3 U(t), \quad U(0) = u. \tag{48}$$

If $w \in C^\alpha([t_1,t_2]; W^{m,p}(\mathbb{T}^d))$ is a Hölder continuous curve on $[t_1,t_2]$, then the function

$$W(t) = \int_{t_1}^{t} G(t-s)\star w(s)\,\mathrm{d}s$$

defines a Hölder continuously differentiable curve, satisfying

$$W \in C^{1,\alpha}\big([t_1,t_2]; W^{m+6,p}\big(\mathbb{T}^d\big)\big), \quad \frac{\mathrm{d}}{\mathrm{d}t}W(t) = \Delta^3 W(t) + w(t), \quad W(t_1) = 0. \tag{49}$$

Proof. The proof of (48) and (49) is technical but standard. One possible approach, which would be most similar to [2], is to observe that $-\Delta^3$ is the generator of the analytic semigroup defined by $t \mapsto G(t)\star f$ for all $f \in L^1(\mathbb{T}^d)$. We refer to [10, Chapter 3] or to [19, Chapter 4] for further details on the semigroup approach. □

Apart from Lemma 13, we shall not need classical results on parabolic equations. Instead, we derive our core estimates with the help of the following lemma.

Lemma 14. *For given $\alpha \in \mathbb{N}_0^d$, $p \geqslant 1$, and $t > 0$, the kernel G satisfies the estimate*

$$\left\| \mathrm{D}^\alpha G(t) \right\|_{L^p} \leqslant \Gamma t^{-(|\alpha|+d(1-1/p))/6}, \tag{50}$$

where $\Gamma > 0$ is independent of $t > 0$.

Here and in the following, D^α denotes a partial derivative of order $|\alpha|$.

Proof. For $t > 0$, define the half-open cube $Q(t) = [0, t^{-1/6})^d \subset \mathbb{R}^d$. Using the change of variables $z(t) = t^{-1/6} y$, we obtain

$$\begin{aligned}
\left\| \mathrm{D}^\alpha G(t) \right\|_{L^p(\mathbb{T}^d)} &= t^{-d/6} \biggl(\int\limits_{[0,1)^d} \sum_{\Lambda \in \mathbb{Z}^d} \left| \mathrm{D}_y^\alpha H\bigl(t^{-1/6}(y+\Lambda)\bigr)\right|^p \mathrm{d}y \biggr)^{1/p} \\
&\leqslant t^{-d/6} \sum_{\Lambda \in \mathbb{Z}^d} \biggl(\int\limits_{Q(t)} \left| t^{-|\alpha|/6} \mathrm{D}_z^\alpha H\bigl(z + t^{-1/6}\Lambda\bigr)\right|^p t^{d/6} \, \mathrm{d}z \biggr)^{1/p} \\
&= t^{-(d+|\alpha|-d/p)/6} \biggl(\int\limits_{\mathbb{R}^d} \left| \mathrm{D}_z^\alpha H(z) \right|^p \mathrm{d}z \biggr)^{1/p}.
\end{aligned}$$

Here we used the Minkowski inequality and the fact that, for each $t > 0$, the space $\mathbb{R}^d$ is the disjoint union of the cubes $Q(t) + t^{-1/6}\Lambda$, where $\Lambda \in \mathbb{Z}^d$. So $\Gamma = \|\mathrm{D}^\alpha \widetilde{H}\|_{L^p(\mathbb{R}^d)}$ is the required constant. □

6.2. Existence and uniqueness of a mild solution

Our main result of this subsection is contained in the following proposition.

Proposition 15. *Let $n_0 \in H^2(\mathbb{T}^d)$ be strictly positive. Then there exist $T > 0$ and precisely one continuous curve $n : [0, T] \to H^2(\mathbb{T}^d)$ with $n(0) = n_0$ that satisfies the following "very mild" formulation of (4):*

$$n(t) = G(t) \star n_0 + \partial_{ijk}^3 \int_0^t G(t-s) \star F_1^{(ijk)}\bigl(n(s)\bigr) \, \mathrm{d}s + \partial_{ij}^2 \int_0^t G(t-s) \star F_2^{(ij)}\bigl(n(s)\bigr) \, \mathrm{d}s \tag{51}$$

for every $t \in (0, T)$. This solution is differentiable with respect to $t \in (0, T)$ with a Hölder continuous derivative, i.e. $n \in C^{1,1/12}([\tau, T]; H^2(\mathbb{T}^d))$ for every $\tau \in (0, T)$.

To prove Proposition 15, we adapt the proof of Theorem 4.2(a) in [2] to the situation at hand. That means, we are going to obtain the solution n to (51) as the unique fixed point of the map $u \mapsto \Phi[u]$, defined by

$$\Phi[u](t) = G(t) \star n_0 + \Psi[u](t) \tag{52}$$

on a suitable set $V_T \subset C^0([0, T]; H^2(\mathbb{T}^d))$, where $\Psi = \partial_{ijk}^3 \psi_1^{(ijk)} + \partial_{ij}^2 \psi_2^{(ij)}$ and

$$\psi_1^{(ijk)}[u](t) = \int_0^t G(s) \star F_1^{(ijk)}\bigl(u(t-s)\bigr) \, \mathrm{d}s, \qquad \psi_2^{(ij)}[u](t) = \int_0^t G(s) \star F_2^{(ij)}\bigl(u(t-s)\bigr) \, \mathrm{d}s. \tag{53}$$

The core ingredient of the proof of Proposition 15 is the following Lipschitz estimate on the nonlinearities $F_1^{(ijk)}$ and $F_2^{(ij)}$.

Lemma 16. *For any $0 < \delta < 1$, $F_1^{(ijk)}$ and $F_2^{(ij)}$ are Lipschitz continuous as mappings from any bounded subset of*

$$U_\delta = \Bigl\{ u \in H^2\bigl(\mathbb{T}^d\bigr) : \min_x u(x) \geqslant \delta, \ \|u\|_{H^2} \leqslant \delta^{-1} \Bigr\} \tag{54}$$

into $L^{3/2}(\mathbb{T}^d)$ and into $L^1(\mathbb{T}^d)$, respectively, satisfying

$$\|F_1^{(ijk)}(u)\|_{L^{3/2}} \leqslant M_1\delta^{-5}, \qquad \|F_1^{(ijk)}(u_1) - F_1^{(ijk)}(u_2)\|_{L^{3/2}} \leqslant M_1\delta^{-4}\|u_1 - u_2\|_{H^2}, \tag{55}$$

$$\|F_2^{(ij)}(u)\|_{L^1} \leqslant M_2\delta^{-7}, \qquad \|F_2^{(ij)}(u_1) - F_2^{(ij)}(u_2)\|_{L^1} \leqslant M_2\delta^{-6}\|u_1 - u_2\|_{H^2}, \tag{56}$$

for all $u, u_1, u_2 \in U_\delta$, where M_1 and M_2 are universal constants. Moreover, $F_1^{(ijk)}$ and $F_2^{(ij)}$ map

$$U'_\delta = \{u \in U_\delta \cap W^{3,3/2}(\mathbb{T}^d):\ \|u\|_{W^{3,3/2}} \leqslant \delta^{-1}\}$$

into $L^2(\mathbb{T}^d)$ and $L^{3/2}(\mathbb{T}^d)$, respectively, satisfying

$$\|F_1^{(ijk)}(u)\|_{L^2} \leqslant M_1\delta^{-5}, \qquad \|F_1^{(ijk)}(u_1) - F_1^{(ijk)}(u_2)\|_{L^2} \leqslant M_1\delta^{-4}\|u_1 - u_2\|_{H^2}, \tag{57}$$

$$\|F_2^{(ij)}(u)\|_{L^{3/2}} \leqslant M_2\delta^{-7}, \qquad \|F_2^{(ij)}(u_1) - F_2^{(ij)}(u_2)\|_{L^{3/2}} \leqslant M_2\delta^{-6}\|u_1 - u_2\|_{H^2}, \tag{58}$$

for all $u, u_1, u_2 \in U'_\delta$.

Proof. Since we are working in dimensions $d \leqslant 3$, every $u \in U_\delta$ is a strictly positive and continuous function on $\mathbb{T}^d$, with $\partial^2_{ij}u \in L^2(\mathbb{T}^d)$ and $\partial_i u \in L^6(\mathbb{T}^d)$. It follows that we can write

$$F_1^{(ijk)}(u) = 2\frac{\partial_i u\partial_j u\partial_k u}{u^2} - 3\frac{\partial_i u\partial^2_{jk}u}{u}, \tag{59}$$

$$F_2^{(ij)}(u) = 2\frac{\partial^2_{ik}u\partial^2_{kj}u}{u} - 4\frac{\partial^2_{ik}u\partial_k u\partial_j u}{u^2} + 2\frac{(\partial_k u)^2\partial_i u\partial_j u}{u^3}. \tag{60}$$

Thus, $F_1^{(ijk)}$ and $F_2^{(ij)}$ are sums of products of derivatives (of order one or two) of u, divided by a power of u. By application of Hölder's inequality and the continuity of the Sobolev embedding $H^2(\mathbb{T}^d) \hookrightarrow W^{1,6}(\mathbb{T}^d)$, one readily verifies the first inequalities in (55) and (56). The Lipschitz continuity is straightforward to verify from the representations (59) and (60) by repeated application of the triangle inequality. For proving (57) and (58), we use additionally the continuous embedding $W^{3,3/2}(\mathbb{T}^d) \hookrightarrow W^{2,3}(\mathbb{T}^d)$. □

A consequence of the above lemma is that Ψ maps bounded curves u into Hölder continuous curves.

Lemma 17. *Assume that there exists a $\delta > 0$ such that $u \in C([0,T]; H^2(\mathbb{T}^d))$ satisfies*

(1) either $u(t) \geqslant \delta$ *and* $\|u(t)\|_{H^2} \leqslant \delta^{-1}$,
(2) or $u(t) > 0$ *and* $\mathcal{E}[u(t)] \leqslant \delta^{-1}$

for all $0 \leqslant t \leqslant T$. Then $\Psi[u] \in C^{1/12}([0,T]; H^2(\mathbb{T}^d))$, i.e.,

$$\|\Psi[u](t') - \Psi[u](t)\|_{H^2} \leqslant L|t' - t|^{1/12} \quad \textit{for all } t, t' \in [0,T], \tag{61}$$

where $L > 0$ depends on δ, but not on u.

Proof. To begin with, we remark that

$$\|F_1^{(ijk)}(u(t))\|_{L^{3/2}} \leqslant Z_1 \quad \text{and} \quad \|F_2^{(ij)}(u(t))\|_{L^1} \leqslant Z_2 \tag{62}$$

hold for all $t \in [0,T]$, where the positive constants Z_1 and Z_2 depend on $\delta > 0$ only. Indeed, if the first set of assumptions on u is satisfied, then (62) is an immediate consequence of Lemma 16. If instead the second set of assumptions is satisfied, then Hölder's inequality implies

$$\|F_1^{(ijk)}(u(t))\|_{L^{3/2}} \leqslant 4\|\nabla\sqrt{u(t)}\|_{L^6}\big(4\|\nabla\sqrt[4]{u(t)}\|^2_{L^4} + 3\|\nabla^2\sqrt{u(t)}\|_{L^2}\big),$$

$$\|F_2^{(ij)}(u(t))\|_{L^1} \leqslant 8\big(\|\nabla^2\sqrt{u(t)}\|_{L^2} + 4\|\nabla\sqrt[4]{u(t)}\|^2_{L^4}\big)^2.$$

In view of (10) and Lemma 26 (see Appendix A), these right-hand sides are controlled in terms of $\mathcal{E}[u(t)] \leqslant \delta^{-1}$ only.

Now, let $t, t' \in [0,T]$ be given with $\tau = t' - t > 0$. For a given $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = 2$, introduce

$$\Theta^\alpha(t;\tau) = \big\| \mathrm{D}^\alpha\big(\Psi[u](t+\tau) - \Psi[u](t)\big)\big\|_{L^2}.$$

By definition of Ψ and a change of variables under the integrals, we find that

$$\begin{aligned}\Theta^\alpha(t;\tau) \leqslant \int_0^t \Big(& \big\|\mathrm{D}^\alpha \partial^3_{ijk}\big(G(s+\tau) - G(s)\big) \star F_1^{(ijk)}\big(u(t-s)\big)\big\|_{L^2} \\ &+ \big\|\mathrm{D}^\alpha \partial^2_{ij}\big(G(s+\tau) - G(s)\big) \star F_2^{(ij)}\big(u(t-s)\big)\big\|_{L^2}\Big)\,\mathrm{d}s \\ &+ \int_0^\tau \Big(\big\|\mathrm{D}^\alpha \partial^3_{ijk} G(s) \star F_1^{(ijk)}\big(u(t+\tau-s)\big)\big\|_{L^2} + \big\|\mathrm{D}^\alpha \partial^2_{ij} G(s) \star F_2^{(ij)}\big(u(t+\tau-s)\big)\big\|_{L^2}\Big)\,\mathrm{d}s.\end{aligned}$$

Using (62) and Young's inequality for convolutions,

$$\|\phi \star \psi\|_{L^p} \leqslant \Upsilon \|\phi\|_{L^q} \|\psi\|_{L^r},$$

for $\phi \in L^p(\mathbb{R}^d)$, $\psi \in L^r(\mathbb{R}^d)$, and $1 + 1/p = 1/q + 1/r$, where $\Upsilon > 0$, the term under the last integral above can be estimated for $0 < s < \tau$ as follows:

$$\begin{aligned}\big\|\mathrm{D}^\alpha \partial^3_{ijk} G(s) \star F_1^{(ijk)}\big(u(t+\tau-s)\big)\big\|_{L^2} &\leqslant \Upsilon_1 \big\|\nabla^5 G(s)\big\|_{L^{6/5}} \big\|F_1\big(u(t+\tau-s)\big)\big\|_{L^{3/2}} \\ &\leqslant \frac{\Upsilon_1 Z_1 \Gamma_1}{s^{\vartheta_1}}, \\ \big\|\mathrm{D}^\alpha \partial^2_{ij} G(s) \star F_2^{(ij)}\big(u(t+\tau-s)\big)\big\|_{L^2} &\leqslant \Upsilon_2 \big\|\nabla^4 G(s)\big\|_{L^2} \big\|F_2\big(u(t+\tau-s)\big)\big\|_{L^1} \\ &\leqslant \frac{\Upsilon_2 Z_2 \Gamma_2}{s^{\vartheta_2}},\end{aligned}$$

where, according to (50), the exponents are given by

$$\vartheta_1 = (5 + d/6)/6 < 1 \quad \text{and} \quad \vartheta_2 = (4 + d/2)/6 < 1.$$

We apply the analogous estimate to the expression under the first integral, and estimate further by employing relation (48). For $0 < s < t$, we have

$$\begin{aligned}\big\|\nabla^5\big(G(\tau+s) - G(s)\big)\big\|_{L^{6/5}} &\leqslant \left\|\nabla^5 \int_s^{s+\tau} \Delta^3 G(\sigma)\,\mathrm{d}\sigma\right\|_{L^{6/5}} \\ &\leqslant \int_s^{s+\tau} \big\|\nabla^5\big(\Delta^3 G(\sigma)\big)\big\|_{L^{6/5}}\,\mathrm{d}\sigma \leqslant \Gamma_1' \int_s^{s+\tau} \frac{\mathrm{d}\sigma}{\sigma^{1+\vartheta_1}} = \frac{\Gamma_1'}{\vartheta_1}\big(s^{-\vartheta_1} - (s+\tau)^{-\vartheta_1}\big).\end{aligned}$$

In a similar fashion, we obtain

$$\big\|\nabla^4\big(G(s+\tau) - G(s)\big)\big\|_{L^2} \leqslant \frac{\Gamma_2'}{\vartheta_2}\big(s^{-\vartheta_2} - (s+\tau)^{-\vartheta_2}\big).$$

In summary, this leads to

$$\begin{aligned}\Theta^\alpha(t;\tau) &\leqslant \frac{\Upsilon_1 Z_1 \Gamma_1'}{\vartheta_1} \int_0^t \big(s^{-\vartheta_1} - (s+\tau)^{-\vartheta_1}\big)\,\mathrm{d}s + \frac{\Upsilon_2 Z_2 \Gamma_2'}{\vartheta_2} \int_0^t \big(s^{-\vartheta_2} - (s+\tau)^{-\vartheta_2}\big)\,\mathrm{d}s \\ &\quad + \Upsilon_1 Z_1 \Gamma_1 \int_0^\tau s^{-\vartheta_1}\,\mathrm{d}s + \Upsilon_2 Z_2 \Gamma_2 \int_0^\tau s^{-\vartheta_2}\,\mathrm{d}s \\ &\leqslant \frac{\Upsilon_1 Z_1 \Gamma_1'}{(1-\vartheta_1)\vartheta_1}\big((t+\tau)^{1-\vartheta_1} - t^{1-\vartheta_1}\big) + \frac{\Upsilon_2 Z_2 \Gamma_2'}{(1-\vartheta_2)\vartheta_2}\big((t+\tau)^{1-\vartheta_2} - t^{1-\vartheta_2}\big) \\ &\quad + \frac{\Upsilon_1 Z_1 \Gamma_1}{1-\vartheta_1}\tau^{1-\vartheta_1} + \frac{\Upsilon_2 Z_2 \Gamma_2}{1-\vartheta_2}\tau^{1-\vartheta_2}.\end{aligned}$$

To finish the proof, we observe that, since $0 < \vartheta_i < 1$, we have $(t+\tau)^{1-\vartheta_i} \leqslant t^{1-\vartheta_i} + \tau^{1-\vartheta_i}$, and $\vartheta_i \leqslant 11/12$ in dimensions $d \leqslant 3$. This proves the Hölder continuity of $\Theta^\alpha(t;\tau)$ with exponent $1/12$ for $|\alpha| = 2$. The cases $|\alpha| = 1$ and $\alpha = 0$ are similar. □

Proof of Proposition 15. As indicated above, we are going to show that Φ, given by (52), is a well-defined contraction on a suitable subset $V_T \subset C([0,T]; H^2(\mathbb{T}^d))$ for some sufficiently small $T > 0$.

Recall the definition of U_δ from (54). Since $n_0 \in H^2(\mathbb{T}^d)$ is strictly positive by assumption, we can choose $\delta > 0$ such that $n_0 \in U_{2\delta}$. Accordingly, for a given $T > 0$, define

$$V_T = \{u \in C^0([0,T]; H^2(\mathbb{T}^d)):\ u(t) \in U_\delta \text{ for all } t \in [0,T]\}.$$

Fix a curve $u \in V_T$. In view of Lemma 16, $F_1^{(ijk)}(u)$ and $F_2^{(ij)}(u)$ are continuous curves on $[0,T]$ with values in $L^{3/2}(\mathbb{T}^d)$ and $L^1(\mathbb{T}^d)$, respectively.

Since $\Phi[u](0) = n_0$ for every $u \in V_T$, the H^2-distance of $\Phi[u](t)$ to n_0 becomes small as $t \downarrow 0$, uniformly in $u \in V_T$. Moreover, since the infimum of $\Phi[u](t)$ is controlled in terms of this distance, one may choose $T > 0$ sufficiently small to achieve $\Phi[u](t) \in U_\delta$ for all $t \in [0,T]$ and $u \in V_T$. Hence, $\Phi : V_T \to V_T$ is well-defined.

Next, we verify the contraction property of Φ. The calculations follow the same pattern as above, now using the Lipschitz estimates in (55) and (56). Let $u_1, u_2 \in V_T$ be given. Then, for $|\alpha| = 2$,

$$\begin{aligned}
&\big\|\mathrm{D}^\alpha\big(\Phi[u_1](t) - \Phi[u_2](t)\big)\big\|_{L^2} \\
&\leqslant \int_0^t \Big(\big\|\nabla^5 G(t-s) \star \big(F_1^{(ijk)}\big(u_1(s)\big) - F_1^{(ijk)}\big(u_2(s)\big)\big)\big\|_{L^2} \\
&\quad + \big\|\nabla^4 G(t-s) \star \big(F_2^{(ij)}\big(u_1(s)\big) - F_2^{(ij)}\big(u_2(s)\big)\big)\big\|_{L^2}\Big)\,\mathrm{d}s \\
&\leqslant \Upsilon_1 M_1 \delta^{-4} \int_0^t (t-s)^{-\vartheta_1}\big\|u_1(s) - u_2(s)\big\|_{H^2}\,\mathrm{d}s + \Upsilon_2 M_2 \delta^{-6} \int_0^t (t-s)^{-\vartheta_2}\big\|u_1(s) - u_2(s)\big\|_{H^2}\,\mathrm{d}s \\
&\leqslant \left(\frac{\Upsilon_1 M_1}{\delta^4(1-\vartheta_1)} + \frac{\Upsilon_2 M_2}{\delta^6(1-\vartheta_2)}\right) T^{1/12} \sup_{0 \leqslant s' \leqslant T} \big\|u_1(s') - u_2(s')\big\|_{H^2}.
\end{aligned}$$

Similar estimates are obtained for $|\alpha| \leqslant 1$. Diminishing T further if necessary, it follows that Φ is contractive on V_T. The claim about the Hölder continuity is a consequence of (61) in combination with (48). □

6.3. Bootstrapping

We prove that the very mild solution to (4) is actually smooth for $t > 0$. To this end, we need the following lemma.

Lemma 18. *Let $\delta > 0$ be given. For each $m \geqslant 1$, there exist continuous and increasing functions $Q_1^{(m)}, Q_2^{(m)}$: $\mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\big\|F_1^{(ijk)}(u_1) - F_1^{(ijk)}(u_2)\big\|_{H^m} \leqslant Q_1^{(m)}\big(\|u_1\|_{H^{m+1}} + \|u_2\|_{H^{m+1}}\big)\|u_1 - u_2\|_{H^{m+2}}, \tag{63}$$

$$\big\|F_2^{(ij)}(u_1) - F_2^{(ij)}(u_2)\big\|_{W^{m,3/2}} \leqslant Q_2^{(m)}\big(\|u_1\|_{H^{m+1}} + \|u_2\|_{H^{m+1}}\big)\|u_1 - u_2\|_{H^{m+2}} \tag{64}$$

hold (componentwise) for all $u \in U_\delta \cap H^{m+2}(\mathbb{T}^d)$.

Observe that this lemma does not apply for $m = 0$, in which case one has to resort to the estimates provided in Lemma 16.

Proof. Basically, we follow the ideas of the proof of Lemma 16, namely we apply several times the triangle inequality, the Hölder inequality, and continuous Sobolev embeddings. However, due to the higher-order derivatives, the proof is technically more involved. Representations (59) and (60) show that $F_1^{(ijk)}$ and $F_2^{(ij)}$ are sums of products of derivatives of u divided by a power of u, i.e. sums of monomials of the form

$$\frac{\mathrm{D}^{\alpha^1}u\ldots\mathrm{D}^{\alpha^k}u}{u^{k-1}}, \tag{65}$$

where $\alpha^\ell\in\mathbb{N}_0^d$, $\ell=1,\ldots,k$, $1\leqslant|\alpha^\ell|\leqslant 2$, and $\sum_{\ell=1}^k|\alpha^l|=K$ equals 3 and 4 for $F_1^{(ijk)}$ and $F_2^{(ij)}$, respectively. A partial derivative of such a monomial is again a sum of monomials of the form (65):

$$\mathrm{D}^\alpha\left(\frac{\mathrm{D}^{\alpha^1}u\ldots\mathrm{D}^{\alpha^k}u}{u^{k-1}}\right)=\sum\frac{\mathrm{D}^{\beta^1}u\,\mathrm{D}^{\beta^2}u\ldots\mathrm{D}^{\beta^r}u}{u^{r-1}},$$

for $\alpha\in\mathbb{N}_0^d$, $|\alpha|\leqslant m$, $1\leqslant|\beta^\ell|\leqslant m+2$, $k\leqslant r\leqslant k+|\alpha|$, and $\sum_{\ell=1}^r|\beta^\ell|=K+|\alpha|$. In view of the continuous Sobolev embeddings $H^{m+2}(\mathbb{T}^d)\hookrightarrow W^{m+1,6}(\mathbb{T}^d)$ and $H^{m+2}(\mathbb{T}^d)\hookrightarrow C^m(\mathbb{T}^d)$, it follows from the above representation of $\mathrm{D}^\alpha F_1^{(ijk)}(u)$ and $\mathrm{D}^\alpha F_2^{(ij)}(u)$ that for every $u\in H^{m+2}(\mathbb{T}^d)$, it holds that $\mathrm{D}^\alpha F_1^{(ijk)}(u)\in L^2(\mathbb{T}^d)$ and $\mathrm{D}^\alpha F_2^{(ij)}(u)\in L^{3/2}(\mathbb{T}^d)$ for each $|\alpha|\leqslant m$ and $m\geqslant 1$. Then, by the repeated application of the triangle and Hölder inequalities, we obtain functions $Q_1^{(m)}$ and $Q_2^{(m)}$ as well as the estimates (63) and (64). □

Proposition 19. *The very mild solution from Proposition* 15 *is a continuously differentiable curve from* $(0,T]$ *to* $C^\infty(\mathbb{T}^d)$.

Proof. Let $\tau\in(0,T)$ be fixed. We are going to prove, inductively on m, that

$$n\in C^{1,1/12}\big([\tau,T];H^{m+2}\big(\mathbb{T}^d\big)\big) \tag{66}$$

for every integer $m\in\mathbb{N}_0$. For $m=0$, the claim (66) is part of the conclusion of Proposition 15 above. The compositions of the Hölder continuous curve n with the locally Lipschitz continuous nonlinearities $F_1^{(ijk)}$ and $F_2^{(ij)}$ (see Lemma 16) are Hölder continuous with the same exponent, $F_1^{(ijk)}(n)\in C^{1/12}([\tau,T];L^{3/2}(\mathbb{T}^d))$ and $F_2^{(ij)}(n)\in C^{1/12}([\tau,T];L^1(\mathbb{T}^d))$. For $\psi_1^{(ijk)}$ and $\psi_2^{(ij)}$, defined in (53), the second part of Lemma 13 implies that $\psi_1^{(ijk)}[n]\in C^{1,1/12}([\tau,T];W^{6,3/2}(\mathbb{T}^d))$ and $\psi_2^{(ij)}[n]\in C^{1,1/12}([\tau,T];W^{6,1}(\mathbb{T}^d))$. In combination with (48), it thus follows directly from (51) that $n\in C^{1/12}([\tau,T];W^{3,3/2}(\mathbb{T}^d))$. An iteration leads, via (57), to the improved regularity $F_1^{(ijk)}(n)\in C^{1/12}([\tau,T];L^2(\mathbb{T}^d))$, and thus to $\psi_1^{(ijk)}[n]\in C^{1,1/12}([\tau,T];H^6(\mathbb{T}^d))$. Furthermore, by (58), we infer that $F_2^{(ij)}(n)\in C^{1/12}([\tau,T];L^{3/2}(\mathbb{T}^d))$ and hence, $\psi_2^{(ij)}[n]\in C^{1,1/12}([\tau,T];W^{6,3/2}(\mathbb{T}^d))$. By the continuity of the embedding $W^{6,3/2}(\mathbb{T}^d)\hookrightarrow H^5(\mathbb{T}^d)$, it follows that $\psi_2^{(ij)}[n]\in C^{1,1/12}([\tau,T];H^5(\mathbb{T}^d))$. Then the representation (51) proves (66) with $m=1$.

Assuming (66) for some $m\geqslant 1$, it follows from Lemma 18 that $F_1^{(ijk)}(n)\in C^{1/12}([\tau,T];\ H^m(\mathbb{T}^d))$ and $F_2^{(ij)}(n)\in C^{1/12}([\tau,T];W^{m,3/2}(\mathbb{T}^d))$. By property (49) of the kernel G and since the Sobolev embedding $W^{m+6,3/2}(\mathbb{T}^d)\hookrightarrow H^{m+5}(\mathbb{T}^d)$ is continuous, we infer that $\psi_1^{(ijk)}\in C^{1,1/12}([\tau,T];H^{m+6}(\mathbb{T}^d))$ and $\psi_2^{(ij)}\in C^{1/12}([\tau,T];H^{m+5}(\mathbb{T}^d))$. Using this inside the representation (51) and combining it with the smoothness property (48), we arrive at $n\in C^{1,1/12}([\tau,T];H^{m+3}(\mathbb{T}^d))$, which implies (66) with m replaced by $m+1$. □

Proof of Theorem 3. First, we extend the local solution $n\in C([0,T];H^2(\mathbb{T}^d))$ obtained from Proposition 15 to the respective *maximal* solution $n_{\max}$ by the usual procedure: Provided that $n(T)\in H^2(\mathbb{T}^d)$ is strictly positive, we can invoke Proposition 15 with the new initial datum $\tilde n_0:=n(T)$, thus obtaining another very mild solution $\tilde n:[0,\tilde T]\to H^2(\mathbb{T}^d)$ to (4). Using the semigroup property $G(\tau)\star G(\sigma)=G(\tau+\sigma)$ for arbitrary $\sigma,\tau>0$, it can be easily checked that the concatenation $n_+:[0,T+\tilde T]\to H^2(\mathbb{T}^d)$, given by

$$n_+(t)=\begin{cases}n(t) & \text{for } 0\leqslant t\leqslant T,\\ \tilde n(t-T) & \text{for } T\leqslant t\leqslant T+\tilde T,\end{cases}$$

is another continuous curve satisfying (51).

The maximal solution $n_{\max}:[0,T_*)\to H^2(\mathbb{T}^d)$ is the uniquely determined curve that satisfies (51) on every subinterval $[0,T]\subset[0,T_*)$, but it cannot be extended to a solution on $[0,T_*]$. In view of our solution concept, this means that

(1) *either* $T_* = +\infty$, i.e., the solution is global,
(2) *or* $n_{\max}(t) \to n_*$ in $H^2(\mathbb{T}^d)$ as $t \uparrow T_*$, but the limiting profile n_* is *not* strictly positive,
(3) *or* $n_{\max}(t)$ does not converge in $H^2(\mathbb{T}^d)$ as $t \uparrow T_*$.

We are going to exclude the last option. First notice that Proposition 19 guarantees that n is a classical and positive solution on every subinterval $(0, T] \subset (0, T_*)$, so $n_{\max} \in C^\infty((0, T_*); C^\infty(\mathbb{T}^d))$, as desired. This means that, in turn, the formal calculation (17) is rigorous. Combining this with the continuity of $n_{\max}(t)$ in $H^2(\mathbb{T}^d)$ at $t = 0$, it follows that $\mathcal{E}[n_{\max}(t)] \leqslant \mathcal{E}[n_0] < \infty$ is uniformly bounded on $[0, T_*)$. If $T_* < \infty$, then $n_{\max}$ satisfies hypothesis (2) of Lemma 17. Since $n_{\max}(t) = G(t) \star n_0 + \Psi[n_{\max}](t)$ by definition, it is a Hölder continuous curve with exponent $1/12$ in $H^2(\mathbb{T}^d)$ on, say, $[T_*/2, T_*)$ with a *uniform* Hölder constant L. This implies, in particular, that $n_{\max}(t)$ converges in $H^2(\mathbb{T}^d)$ to a limit n_*. □

7. From weak to classical solutions

In this brief last section, we prove Theorem 4 about the passage from weak to classical solutions. In preparation of the proof of Theorem 4, we first show that any weak solution satisfies the very mild formulation (51), but in a weaker sense.

Lemma 20. *Any weak solution n in the sense of Theorem 1 is a Hölder continuous curve in $H^{-3}(\mathbb{T}^d)$, satisfying, for $t > 0$,*

$$n(t) = G(t) \star n_0 + \partial^3_{ijk} \int_0^t G(t-s) \star F_1^{(ijk)}\big(n(s)\big)\,\mathrm{d}s + \partial^2_{ij} \int_0^t G(t-s) \star F_2^{(ij)}\big(n(s)\big)\,\mathrm{d}s. \tag{67}$$

Proof. By our definition of a weak solution, n lies in $W^{1,4/3}_{\mathrm{loc}}(0,\infty; H^{-3}(\mathbb{T}^d))$. As a consequence, n is a Hölder continuous curve with exponent $1/3$ in $H^{-3}(\mathbb{T}^d)$ and, in particular, n is absolutely continuous in $H^{-3}(\mathbb{T}^d)$. Hence, its time derivative $\partial_t n(t)$ is defined in $H^{-3}(\mathbb{T}^d)$ for almost every $t > 0$. Moreover, $n \in L^{4/3}_{\mathrm{loc}}(0,\infty; H^3(\mathbb{T}^d))$, thus, $n(t) \in H^3(\mathbb{T}^d)$ for almost every $t > 0$ and $\Delta^3 n \in L^{4/3}_{\mathrm{loc}}(0,\infty; H^{-3}(\mathbb{T}^d))$. It follows that

$$g := \partial_t n - \Delta^3 n \in L^{4/3}_{\mathrm{loc}}\big(0,\infty; H^{-3}\big(\mathbb{T}^d\big)\big).$$

For fixed $t > 0$, consider the continuous curve $u : (0,t) \to C^\infty(\mathbb{T}^d)$, defined by $u(s) = G(t-s) \star n(s)$. Recalling (48), it follows for arbitrary $0 < s < t$ that

$$\partial_s u(s) = -\Delta^3 G(t-s) \star n(s) + G(t-s) \star \partial_s n(s) = G(t-s) \star \big(\partial_s n(s) - \Delta^3 n(s)\big) = G(t-s) \star g(s).$$

Therefore, $u \in W^{1,4/3}_{\mathrm{loc}}(0,\infty; H^{-3}(\mathbb{T}^d))$, and

$$\lim_{t' \uparrow t} u\big(t'\big) = u(0) + \int_0^t G(t-s) \star g(s)\,\mathrm{d}s.$$

Since $u(0) = G(t) \star n_0$ and $u(t') \to n(t)$ in $H^{-3}(\mathbb{T}^d)$ as $t' \uparrow t$, formula (67) follows. □

Lemma 21. *Let n be a weak solution to (51) whose energy is uniformly bounded on (T_1, T_2). Then $F_1^{(ijk)}(n(t))$ is bounded in $L^{3/2}(\mathbb{T}^d)$ and $F_2^{(ij)}(n(t))$ is bounded in $L^1(\mathbb{T}^d)$, uniformly in (T_1, T_2).*

Proof. By the Hölder and the Sobolev inequalities and Lemma 26 (see Appendix A), it follows that

$$\begin{aligned}\big\|F_1^{(ijk)}\big(n(t)\big)\big\|_{L^{3/2}} &\leqslant 4\big\|\nabla\sqrt{n(t)}\big\|_{L^6}\big(4\big\|\nabla\sqrt[4]{n(t)}\big\|^2_{L^4} + 3\big\|\nabla^2\sqrt{n(t)}\big\|_{L^2}\big)\\ &\leqslant 4C\big\|\sqrt{n(t)}\big\|_{H^2}\big(4C^2_{\mathrm{LV}}\big\|\sqrt{n(t)}\big\|_{H^2} + 3\big)\big\|\sqrt{n(t)}\big\|_{H^2},\end{aligned}$$

$$\begin{aligned}\|F_2^{(ij)}(n(t))\|_{L^1} &\leqslant 8\big(\|\nabla^2\sqrt{n(t)}\|_{L^2} + 4\|\nabla\sqrt[4]{n(t)}\|_{L^4}^2\big)^2 \\ &\leqslant 8\big(1+4C_{\mathrm{LV}}^2\|\sqrt{n(t)}\|_{H^2}\big)^2\|\sqrt{n(t)}\|_{H^2}^2.\end{aligned}$$

The last terms are uniformly controlled in terms of $\mathcal{E}[n(t)]$ which concludes the proof. □

In the following, let n be a weak solution satisfying the hypotheses of Theorem 4. Without loss of generality we may take $t_0 = 0$. Then $n_0 \in H^2(\mathbb{T}^d)$ and $\min n_0(x) > 0$. Since we are working with a solution of uniformly bounded energy, it is a priori clear that $n(t)$ is bounded in $H^2(\mathbb{T}^d)$. Actually, more is true.

Lemma 22. *A weak solution n is a Hölder continuous curve in $H^2(\mathbb{T}^d)$ on any time interval on which its energy is uniformly bounded.*

Proof. Let $t > 0$ and $\tau > 0$ be fixed. Since $G(t-s) \in C^\infty(\mathbb{T}^d)$ and $f(n(s)) = \partial^3_{ijk}F_1^{(ijk)}(n(s)) + \partial^2_{ij}F_2^{(ij)}(n(s)) \in H^{-3}(\mathbb{T}^d)$ for all $s \in (0,t)$, we have $G(t-s) \star f(n(s)) \in C^\infty(\mathbb{T}^d)$. It follows that

$$\nabla^2 G(t-s) \star f\big(n(s)\big) = \nabla^2\partial^3_{ijk}G(t-s) \star F_1^{(ijk)}\big(n(s)\big) + \nabla^2\partial^2_{ij}G(t-s) \star F_2^{(ij)}\big(n(s)\big).$$

By Young's inequality, it follows further that

$$\begin{aligned}\|\nabla^2 G(t-s) \star f\big(n(s)\big)\|_{L^2} &\leqslant C\big(\|\nabla^5 G(t-s)\|_{L^{6/5}}\|F_1^{(ijk)}\big(n(s)\big)\|_{L^{3/2}} + \|\nabla^4 G(t-s)\|_{L^2}\|F_2^{(ij)}\big(n(s)\big)\|_{L^1}\big) \\ &\leqslant C\big((t-s)^{-(5+d/6)/6} + (t-s)^{-(4+d/2)/6}\big) \\ &\leqslant C(t-s)^{-11/12},\end{aligned}$$

where $C > 0$ is a generic constant and recalling that $d \leqslant 3$. This implies that, for all $t \in (0,T)$ and $\tau > 0$,

$$\left\|\nabla^2\int_t^{t+\tau} G(t+\tau-s) \star f\big(n(s)\big)\,\mathrm{d}s\right\|_{L^2} \leqslant C\big((t+\tau)^{1/12} - t^{1/12}\big) \leqslant C\tau^{1/12}.$$

Similarly, we find that

$$\begin{aligned}&\|\nabla^2\big(G(t+\tau-s) - G(t-s) \star f\big(n(s)\big)\big)\|_{L^2} \\ &\quad\leqslant C\big(\|\nabla^5\big(G(t+\tau-s) - G(t-s)\big)\|_{L^{6/5}}\|F_1^{(ijk)}\big(n(s)\big)\|_{L^{3/2}} \\ &\qquad+ \|\nabla^4\big(G(t+\tau-s) - G(t-s)\big)\|_{L^2}\|F_2^{(ij)}\big(n(s)\big)\|_{L^1}\big).\end{aligned}$$

By relation (48), for $m = 4, 5$,

$$\begin{aligned}\|\nabla^m\big(G(t+\tau-s) - G(t-s)\big)\|_{L^p} &\leqslant \left\|\nabla^m\int_{t-s}^{t+\tau-s}\frac{\mathrm{d}G}{\mathrm{d}\theta}(\vartheta)\,\mathrm{d}\vartheta\right\|_{L^p} \leqslant \int_{t-s}^{t+\tau-s}\|\nabla^m\Delta^3 G(\vartheta)\|_{L^p}\,\mathrm{d}\vartheta \\ &\leqslant \Gamma\int_{t-s}^{t+\tau-s}\vartheta^{-1-(m+d(1-1/p))/6}\,\mathrm{d}\vartheta.\end{aligned}$$

As in the proof of Lemma 17, this proves the continuity with the Hölder exponent $1/12$. □

The above results, together with Theorem 3, provide the proof of Theorem 4.

Appendix A

We provide a collection of functional inequalities used throughout the calculations.

Lemma 23. *Let $m \in \mathbb{N}$ be given. Then there exists a constant $C > 0$ such that for all $u \in H^m(\mathbb{T}^d)$,*

$$\|u\|_{H^m} \leqslant C\big(\|\nabla^m u\|_{L^2} + \|u\|_{L^2}\big).$$

Lemma 24. *Let $m, n \in \mathbb{N}$ and $1 \leqslant p, r \leqslant \infty$ be given and assume that $n - d/r < m - d/p$. Then the Sobolev space $W^{m,p}(\mathbb{T}^d)$ embeds compactly into $W^{n,r}(\mathbb{T}^d)$. In the borderline case, if $n - d/r = m - d/p$ is not an integer, the embedding is still continuous.*

The following result is from [23, p. 1034].

Lemma 25 *(Gagliardo–Nirenberg inequality). Let m, $n \in \mathbb{N}_0$ with $m > n$ and let $1 \leqslant p, q, r \leqslant \infty$. Assume that there exists $\theta \in (0,1)$ such that*

$$n - \frac{d}{r} = \theta\left(m - \frac{d}{p}\right) - (1-\theta)\frac{d}{q}.$$

There exists a constant $B > 0$ such that for all $u \in W^{m,p}(\mathbb{T}^d)$,

$$\left\|\nabla^n u\right\|_{L^r(\Omega)} \leqslant B\|u\|_{W^{m,p}}^{\theta}\|u\|_{L^q}^{1-\theta}. \tag{68}$$

Estimates on square roots play a key role in the proofs of our results. The following result is a consequence of Théorème 1(ii) in [17].

Lemma 26. *Let $1 < p \leqslant \infty$. Then there exists a constant $C_{\mathrm{LV}} > 0$ such that for all nonnegative functions $u \in W^{2,p}(\mathbb{T}^d)$,*

$$\|\sqrt{u}\|_{W^{1,2p}(\mathbb{T}^d)}^2 \leqslant C_{\mathrm{LV}}\|u\|_{W^{2,p}(\mathbb{T}^d)}. \tag{69}$$

Proof. Let $\phi \in C^2(\mathbb{R})$ be a nonnegative cut-off function satisfying $\phi(x) = 1$ for $0 \leqslant x \leqslant 1$, and $\phi(x) = 0$ for $x \geqslant 2$ and for $x \leqslant -1$. Define accordingly $\phi_d \in C^2(\mathbb{R}^d)$ by

$$\phi_d(x_1, x_2, \ldots, x_d) = \phi(x_1)\phi(x_2)\cdots\phi(x_d). \tag{70}$$

Given $u \in W^{2,p}(\mathbb{T}^d)$, consider $w \in W^{2,p}(\mathbb{R}^d)$ with $w(x) = \phi_d(x)\mathrm{E}u(x)$; recall that $\mathrm{E}u$ is the periodic extension of u to $\mathbb{R}^d$. By definition of ϕ_d, we have $w(x) = \mathrm{E}u(x)$ for $x \in [0,1]^d$ and $\operatorname{supp} w \subset [-1,2]^d$. On one hand,

$$\|\mathrm{D}\sqrt{u}\|_{L^{2p}(\mathbb{T}^d)}^{2p} = \sum_{j=1}^{d}\int_{[0,1]^d}\left|\partial_j\sqrt{\mathrm{E}u(x)}\right|^{2p}\mathrm{d}x \leqslant \sum_{j=1}^{d}\int_{\mathbb{R}^d}\left|\partial_j\sqrt{w(x)}\right|^{2p}\mathrm{d}x = \|\mathrm{D}\sqrt{w}\|_{L^{2p}(\mathbb{R}^d)}^{2p}. \tag{71}$$

On the other hand, with constants A_p, $B_d > 0$,

$$\begin{aligned}
\left\|\mathrm{D}^2 w\right\|_{L^p(\mathbb{R}^d)}^p &= \sum_{1\leqslant j\leqslant k\leqslant d}\int_{\mathbb{R}^d}\left|\partial_{jk}^2 w(x)\right|^p\mathrm{d}x \\
&= \sum_{1\leqslant j\leqslant k\leqslant d}\int_{\mathbb{R}^d}\left|\partial_{jk}^2\phi_d\mathrm{E}u + \partial_j\phi_d\partial_k\mathrm{E}u + \partial_k\phi_d\partial_j\mathrm{E}u + \phi_d\partial_{jk}^2\mathrm{E}u\right|^p\mathrm{d}x \\
&\leqslant A_p\|\phi_d\|_{C^2(\mathbb{R}^d)}^p\sum_{1\leqslant j\leqslant k\leqslant d}\int_{[-1,2]^d}\left(|\mathrm{E}u|^p + |\partial_j\mathrm{E}u|^p + |\partial_j\mathrm{E}u|^p + \left|\partial_{jk}^2\mathrm{E}u\right|^p\right)\mathrm{d}x \\
&\leqslant A_pB_d\|\phi_d\|_{C^2(\mathbb{R}^d)}^p\|u\|_{W^{2,p}(\mathbb{T}^d)}^p.
\end{aligned} \tag{72}$$

By Théorème 1(ii) in [17],

$$\|\mathrm{D}\sqrt{w}\|_{L^{2p}(\mathbb{R}^d)}^{2p} \leqslant K\left\|\mathrm{D}^2 w\right\|_{L^p(\mathbb{R}^d)}^p, \tag{73}$$

where $K > 0$ only depends on d and p. Then, combining (71) with (72) via (73), it follows that

$$\|\mathrm{D}\sqrt{u}\|_{L^{2p}(\mathbb{T}^d)}^{2p} \leqslant A_pB_dK\|\phi_d\|_{C^2(\mathbb{R}^d)}^d\|u\|_{W^{2,p}(\mathbb{T}^d)}^p.$$

Finally, observe that, trivially,

$$\|\sqrt{u}\|_{L^{2p}(\mathbb{T}^d)}^{2p} = \|u\|_{L^p(\mathbb{T}^d)}^p \leqslant \|u\|_{W^{2,p}(\mathbb{T}^d)}^p.$$

Hence, (69) holds with the constant

$$C_{\mathrm{LV}} = \left(1 + A_p B_d K \|\phi_d\|_{C^2(\mathbb{R}^d)}^p\right)^{1/p},$$

ending the proof. □

The following result is proved in [14, Appendix]. It is needed to obtain strong convergence of the sequences $(\sqrt{u_n})$ or $(\sqrt[4]{u_n})$, given strong convergence of the sequence (u_n) and a uniform bound on $(\sqrt[4]{u_n})$ or $(\sqrt[6]{u_n})$, respectively.

Proposition 27. *Let $0 < \beta < \gamma < \alpha < \infty$, $1 < p, q, r < \infty$ be given, where $\alpha p = \beta q = \gamma r$. Assume that (u_n) is a sequence of strictly positive functions on $\mathbb{T}^d$ with the following properties*:

(1) *u_n^α converges strongly to u^α in $W^{1,p}(\mathbb{T}^d)$, and*
(2) *u_n^β is bounded in $W^{1,q}(\mathbb{T}^d)$.*

Then u_n^γ converges strongly to u^γ in $W^{1,r}(\mathbb{T}^d)$.

The respective result holds for sequences of nonnegative functions $u_n : (0, T) \times \mathbb{T}^d \to \mathbb{R}$ upon replacing $W^{1,s}(\mathbb{T}^d)$ by $L^s(0, T; W^{1,s}(\mathbb{T}^d))$ for, respectively, $s = p, q, r$.

Finally, we recall a particular variant of the Leray–Schauder theorem that has been proven in [20].

Theorem 28 *(Leray–Schauder). Let X be a Banach space and let $B \subset X$ be a closed and convex set such that the zero element of X is contained in the interior of B. Furthermore, let $S : B \times [0, 1] \to X$ be a continuous map such that its range $S(B \times [0, 1])$ is relatively compact in X. Assume that $S(x, \sigma) \neq x$ for all $x \in \partial B$ and $\sigma \in [0, 1]$ and that $S(\partial B \times \{0\}) \subset B$. Then there exists $x_0 \in B$ such that $S(x_0, 1) = x_0$.*

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