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Spectral optimization problems with internal constraint

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Abstract

We consider spectral optimization problems with internal inclusion constraints, of the form

$$\min\{\lambda_k(\Omega):\ D\subset\Omega\subset\mathbb{R}^d,\ |\Omega|=m\},$$

where the set D is fixed, possibly unbounded, and λ_k is the k-th eigenvalue of the Dirichlet Laplacian on Ω. We analyze the existence of a solution and its qualitative properties, and rise some open questions.

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1. Introduction

A spectral optimization problem is a minimization problem of the form

$$\min\{J(\Omega):\ \Omega\in\mathcal{A}\}\tag{1.1}$$

where J is a cost functional depending on the spectrum of an elliptic operator defined on the (quasi) open set Ω and $\mathcal{A}$ is a class of admissible domains. A wide literature on the subject is available, dealing with existence, regularity, necessary conditions of optimality, relaxation, explicit solutions and numerical computations of the optimal shapes. We quote for instance the books [7,18,19] and the articles [2,9,17], where the reader may find a complete list of references on the field.

The simplest situation for the existence of a solution of problem (1.1) occurs when the class of admissible domains $\mathcal{A}$ satisfies an *external* inclusion constraint, i.e. consists on quasi-open sets which are supposed *a priori* contained in a given *bounded* open set D of the Euclidean space $\mathbb{R}^d$,

$$\mathcal{A}=\{\Omega:\ \Omega\subset D,\ \Omega\ \text{quasi-open}\}.$$

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In this case a general existence result, due to Buttazzo and Dal Maso (see [11]), states that problem (1.1), with the additional constraint $|\Omega| \leqslant m$ on the Lebesgue measure of the competing domains, admits a solution provided the cost functional J satisfies the following conditions:

(i) J is lower semicontinuous for the γ-convergence, suitably defined;
(ii) J is monotone decreasing for the set inclusion.

When the surrounding box D is unbounded the existence result above is no longer true, as some simple examples show. In the case $D = \mathbb{R}^d$ a quite different approach to the proof of the existence of optimal domains has been considered by Bucur in [5,6], using a refined argument related to the Lions concentration-compactness principle (see [22]), and by Mazzoleni and Pratelli in [21] using a more direct approach. However, the latter approach only works in the case $D = \mathbb{R}^d$, while the concentration-compactness approach seems more flexible for our purposes.

In this paper we consider problem (1.1) where the admissible class $\mathcal{A}$ is defined through an *internal* constraint:

$$\mathcal{A} = \{\Omega:\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open},\ |\Omega| \leqslant m\}, \tag{1.2}$$

where D is a fixed quasi-open set of finite measure, possibly unbounded. We consider mainly the cases $J(\Omega) = \lambda_k(\Omega)$; the case of general monotone decreasing functionals is at present still open (see Section 6).

In spite of its simplicity, even for cost functionals like $J(\Omega) = \lambda_1(\Omega)$, the existence proof is rather involved, and several interesting questions arise. For this functional, together with the existence of a solution, we prove some global properties for the optimal set: it has to lie at a finite distance from D (in particular the optimal set is bounded, provided D is bounded), it has finite perimeter outside $\overline{D}$, it is an open set as soon as its measure is strictly greater than the measure of (the quasi-connected) D. Local regularity properties, outside $\overline{D}$ are not discussed here, being similar to the bounding box situation, and we refer the reader for instance to [4]. We discuss as well the existence question for $J(\Omega) = \lambda_k(\Omega)$. We refer the reader to [6] and to [21] for the analysis of these functionals in the absence of any inclusion constraint in $\mathbb{R}^d$.

It is convenient for our purposes to consider also the problem

$$\min\{\lambda_k(\Omega) + \Lambda|\Omega|:\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open}\}, \tag{1.3}$$

where the measure constraint $|\Omega| \leqslant m$ is replaced by the Lagrange multiplier penalization $\Lambda|\Omega|$. The relations between the constrained problem

$$\min\{\lambda_k(\Omega):\ D \subset \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open},\ |\Omega| \leqslant m\}, \tag{1.4}$$

and the penalized version (1.3) have been analyzed for $k = 1$ in [4], while for general k only a partial result is available (see Lemma 5.10), which is enough for our purposes.

The existence of an optimal domain for problem (1.4), as well as for its penalized version (1.3), is proved in Theorem 4.7.

2. Notations and preliminaries

We introduce here the main tools we use; further details can be found for instance in [7,9].

In the sequel, we will work in the Euclidean space $\mathbb{R}^d$ with $d \geqslant 2$. Given a subset $E \subset \mathbb{R}^d$ we define the capacity of E by

$$\operatorname{cap}(E) = \inf\left\{\int |\nabla u|^2\,dx + \int u^2\,dx:\ u \in \mathcal{U}_E\right\},$$

where $\mathcal{U}_E$ is the set of all functions u of the Sobolev space $H^1(\mathbb{R}^d)$ such that $u \geqslant 1$ almost everywhere in a neighborhood of E. If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\operatorname{cap}(Z) = 0$, we say that $P(x)$ holds *quasi-everywhere* (shortly *q.e.*) on E, whereas the expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure, that we often denote by $|\cdot|$.

A subset Ω of $\mathbb{R}^d$ is said to be *quasi-open* if for every $\varepsilon > 0$ there exists an open subset Ω_ε of $\mathbb{R}^d$, with $\Omega \subset \Omega_\varepsilon$, such that $\operatorname{cap}(\Omega_\varepsilon \setminus \Omega) < \varepsilon$. Similarly, a function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *quasi-continuous* (resp. *quasi-lower* *semicontinuous*) if there exists a decreasing sequence of open sets $(\omega_n)_{n>0}$ such that $\lim_{n\to\infty}\operatorname{cap}(\omega_n)=0$ and the restriction f_n of f to the set ω_n^c is continuous (resp. lower semicontinuous). It is well known (see for instance [23]) that every function $u\in H^1(\mathbb{R}^d)$ has a quasi-continuous representative $\tilde{u}$, which is uniquely defined up to a set of capacity zero, and given by

$$\tilde{u}(x)=\lim_{\varepsilon\to 0}\frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(y)\,dy,$$

where $B_\varepsilon(x)$ denotes the ball of radius ε centered at x. We often identify the function u with its quasi-continuous representative $\tilde{u}$; in this way, we have that quasi-open sets can be characterized as the sets of strict positivity of functions in $H^1(\mathbb{R}^d)$ and that the capacity can be equivalently defined by

$$\operatorname{cap}(E)=\min\left\{\int|\nabla u|^2\,dx+\int u^2\,dx\colon u\in H^1\left(\mathbb{R}^d\right),\ u\geqslant 1 \text{ q.e. on } E\right\}.$$

The closure $\overline{D}$ of a quasi-open set D depends on the representative set of D which is only defined up to a set of capacity zero. A canonical minimal representative of $\overline{D}$ can be defined as

$$\overline{D}=\bigcap_{\operatorname{cap}(N)=0}\{C \text{ closed: } C\supseteq D\setminus N\},$$

which contains D q.e. since it can be reduced to a countable intersection.

For every quasi-open set $\Omega\subset\mathbb{R}^d$ we denote by $H_0^1(\Omega)$ the space of all functions $u\in H^1(\mathbb{R}^d)$ such that $u=0$ q.e. on $\mathbb{R}^d\setminus\Omega$, with the Hilbert space structure inherited from $H^1(\mathbb{R}^d)$,

$$\langle u,v\rangle_{H_0^1(\Omega)}=\langle u,v\rangle_{H^1(\mathbb{R}^d)}=\int\nabla u\nabla v\,dx+\int uv\,dx.$$

The usual properties of Sobolev functions on open sets extend to quasi-open sets.

Let Ω be a quasi-open set of finite measure. By R_Ω we denote the resolvent operator of the Laplace equation with Dirichlet boundary condition,

$$R_\Omega : L^2\left(\mathbb{R}^d\right)\to L^2\left(\mathbb{R}^d\right),$$

where $R_\Omega(f)$ is the weak solution of the equation

$$\begin{cases}-\Delta u=f\in L^2(\mathbb{R}^d),\\ u\in H_0^1(\Omega).\end{cases}$$

It is well known that R_Ω is a compact, self adjoint and positive operator, so its spectrum consists on a discrete decreasing sequence of eigenvalues, which we denote (counting the multiplicities) by $1/\lambda_k(\Omega)$. From now on, we call $\lambda_k(\Omega)$ the eigenvalues of the Dirichlet Laplacian.

3. The γ-convergence

We endow the admissible class of domains with the following notion of convergence.

Definition 3.1. Let $(\Omega_n)_n$ be a sequence of quasi-open sets of uniformly bounded measure. We say that Ω_n γ-converges to Ω if the resolvent operators R_{Ω_n} converge to the resolvent operator R_Ω in the operator norm of $\mathcal{L}(L^2(\mathbb{R}^d))$.

The γ-convergence is metrizable but not compact (see for instance [7, Chapter 4] and [12]). This convergence is very strong and the eigenvalues $\lambda_k(\Omega)$ turn out to be γ-continuous. Its (local) compactification has been characterized in [13] as a space of measures.

We denote by $\mathcal{M}_0$ the set of capacitary measures on $\mathbb{R}^d$, that is the set of all Borel measures, possibly taking the value $+\infty$, vanishing on all sets of zero capacity. Observe that for each Borel set S the measure

$$\infty_S(B)=\begin{cases}0 & \text{if } \operatorname{cap}(B\cap S)=0,\\ +\infty & \text{otherwise}\end{cases}$$

is a capacitary measure.

For each capacitary measure μ, we define the linear vector space

$$H^1_\mu = H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d,\mu)=\left\{u\in H^1(\mathbb{R}^d)\colon \int_{\mathbb{R}^d}|u|^2\,d\mu<\infty\right\}.$$

Taking Ω a quasi-open set, $S=\Omega^c$ and $\mu=\infty_S$ give $H^1_\mu=H^1_0(\Omega)$. In [10] it was shown that the space H^1_μ, endowed with the scalar product

$$\langle u,v\rangle=\int_{\mathbb{R}^d}\nabla u\nabla v\,dx+\int_{\mathbb{R}^d}uv\,dx+\int_{\mathbb{R}^d}uv\,d\mu,$$

is a Hilbert space. Moreover, the space H^1_μ is separable when seen as a subset of the separable metric space $H^1(\mathbb{R}^d)$. If $\{u_n\}_{n\geqslant 0}\subset H^1_\mu$ is a dense countable subset, then we define the regular set of the capacitary measure $\mu\in\mathcal{M}_0$ as

$$\Omega_\mu=\bigcup_{n\geqslant 0}\{u_n\neq 0\}.$$

Notice that if $\mu=\infty_S$, we have $\Omega_\mu=S^c$. If the set Ω_μ has finite Lebesgue measure, then

$$\|u\|^2=\int_{\mathbb{R}^d}|\nabla u|^2\,dx+\int_{\mathbb{R}^d}|u|^2\,d\mu,$$

is an equivalent norm on H^1_μ. We define the resolvent R_μ as the map

$$R_\mu: L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d),$$

which associates to each function $f\in L^2(\mathbb{R}^d)$ the solution u of the relaxed problem formally written as

$$-\Delta u+\mu u=f,\quad u\in H^1_\mu,$$

which has to be rigorously defined in the weak form

$$\begin{cases}\displaystyle\int_{\mathbb{R}^d}\nabla u\nabla\varphi\,dx+\int_{\mathbb{R}^d}u\varphi\,d\mu=\int_{\mathbb{R}^d}f\varphi\,dx\quad \forall\varphi\in H^1_\mu,\\ u\in H^1_\mu.\end{cases}$$

If μ is a capacitary measure with regular set of finite Lebesgue measure then R_μ is a compact, self adjoint, positive operator and we denote by $\lambda_k(\mu)$ the eigenvalues of R_μ^{-1}. In this case the constant function 1 is in the dual space $(H^1_\mu)'$ and R_μ can be extended to an operator from $(H^1_\mu)'$ to H^1_μ, so we can define $w_\mu:=R_\mu(1)$ and we have $\Omega_\mu=\{w_\mu>0\}$ up to zero capacity sets.

We consider the following relation of equivalence on $\mathcal{M}_0$:

$$\mu_1\sim\mu_2\quad\Longleftrightarrow\quad \mu_1(\Omega)=\mu_2(\Omega),\quad \forall\Omega \text{ quasi-open}.$$

From now on, we work with the quotient set $\mathcal{M}_0/\sim$ which we still denote by $\mathcal{M}_0$ and we call its elements capacitary measures. We introduce the following convergence on $\mathcal{M}_0$:

Definition 3.2. We say that μ_n γ-converges to μ, if the sequence of regular sets Ω_{μ_n} is of uniformly bounded Lebesgue measure and

$$R_{\mu_n}\overset{\mathcal{L}(L^2(\mathbb{R}^d))}{\longrightarrow}R_\mu.$$

Remark 3.3. With the definition above, we have the equivalence

$$\mu_n \xrightarrow{\gamma} \mu \quad \Longleftrightarrow \quad (w_{\mu_n})_{n\geqslant 0} \text{ converges in } L^2\big(\mathbb{R}^d\big) \text{ to } w_\mu.$$

Indeed, for the "$\Leftarrow$" implication, we refer to [5, Proposition 3.3]. For the direct implication, the proof is immediate. On the one hand, we have

$$R_{\mu_n}(1_{\Omega_{\mu_n}}) - R_\mu(1_{\Omega_{\mu_n}}) \to 0 \quad \text{in } L^2\big(\mathbb{R}^d\big),$$

and on the other hand

$$R_{\mu_n}(1_{\Omega_\mu}) - R_\mu(1_{\Omega_\mu}) \to 0 \quad \text{in } L^2\big(\mathbb{R}^d\big).$$

Making the difference we get that

$$\big\| R_{\mu_n}(1_{\Omega_{\mu_n}}) - R_{\mu_n}(1_{\Omega_\mu}) + R_\mu(1_{\Omega_\mu}) - R_\mu(1_{\Omega_{\mu_n}})\big\|_{L^2(\mathbb{R}^d)} \to 0$$

and using the maximum principle we conclude with

$$\big\| R_{\mu_n}(1) - R_\mu(1)\big\|_{L^2(\mathbb{R}^d)} \to 0.$$

Definition 3.4. We say that the sequence of capacitary measures μ_n γ_{loc}-converges to the capacitary measure μ, if for each bounded open set $\omega \subset \mathbb{R}^d$, we have that the sequence of capacitary measures $\mu_n \vee \infty_{\omega^c}$ γ-converges to the capacitary measure $\mu \vee \infty_{\omega_c}$.

Remark 3.5. In [3, Definition 2.7] the γ_{loc}-convergence introduced above was called γ-convergence (see also [13]) and was related to the Γ-convergence in $L^2(\mathbb{R}^d)$ of the integral functionals

$$F_\mu(u,\omega) = \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,d\mu + \chi_{H_0^1(\omega)}(u),$$

for each bounded open set $\omega \subset \mathbb{R}^d$, where

$$\chi_{H_0^1(\omega)}(u) = \begin{cases} 0 & \text{if } u \in H_0^1(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

In [13], it was also proven that with this convergence the space $\mathcal{M}_0$ is metrizable [13, Theorem 4.9] and compact [13, Theorem 4.14].

For each $t > 0$, we will denote with $\mathcal{M}_0^t$ the following set of capacitary measures

$$\mathcal{M}_0^t = \big\{\mu \in \mathcal{M}_0\colon\ |\Omega_\mu| \leqslant t\big\}.$$

Proposition 3.6. *The set $\mathcal{M}_0^t$ endowed with the metric*

$$d_\gamma(\mu_1,\mu_2) = \|w_{\mu_1} - w_{\mu_2}\|_{L^2(\mathbb{R}^d)},$$

is a complete metric space.

Proof. Let $(\mu_n)_{n\geqslant 0}$ be a sequence such that $|\Omega_{\mu_n}| \leqslant t$ and $(w_{\mu_n})_{n\geqslant 0}$ converges strongly in $L^2(\mathbb{R}^d)$ to some function $w \in H^1(\mathbb{R}^d)$. By the compactness of the γ_{loc}-convergence, each subsequence of $(\mu_n)_{n\geqslant 0}$ has a γ_{loc}-convergent subsequence, which we still denote by μ_n and whose limit is μ. By Remark 5.6 in [5], we have that $w = w_\mu$, $|\Omega_\mu| \leqslant t$ and μ_n γ-converges to μ. Since μ is uniquely determined by the relation $w = w_\mu$ (see [14, Theorem 5.1]), we have the thesis. □

The proposition below deals with the continuity of λ_k with respect to the γ-convergence.

Proposition 3.7. *Consider a sequence $(\Omega_n)_{n\geqslant 0}$ of quasi-open sets of uniformly bounded measure such that Ω_n γ-converges to the capacitary measure μ with regular set Ω_μ. Then, for every $k\geqslant 1$*

$$\lambda_k(\Omega_\mu)\leqslant\lambda_k(\mu)=\lim_{n\to\infty}\lambda_k(\Omega_n).$$

Proof. By Remark 3.3, $R_{\Omega_n}\to R_\mu$ in the operator norm of $\mathcal{L}(L^2(\mathbb{R}^d))$, and so we have

$$\lambda_k(\Omega_n)\to\lambda_k(\mu).$$

The inequality

$$\lambda_k(\Omega_\mu)\leqslant\lambda_k(\mu),$$

now follows as a consequence of the inequality of the measures $\infty_{\Omega_\mu^c}(B)\leqslant\mu(B)$, for each quasi-open set B, in the min – max definition of the eigenvalues. □

4. Existence of an optimal set

A fundamental tool allowing to understand the behavior of a minimizing sequence in $\mathbb{R}^d$ is the concentration-compactness result (see [5, Theorem 2.2]) for the resolvent operators. We adapt it below in order to manage the internal constraint. The main changes deal with the compactness situation, where translations disappear.

Theorem 4.1. *Let $(\Omega_n)_{n\geqslant 0}$ be a sequence of quasi-open sets of uniformly bounded measure, all containing a given quasi-open set D. Then, there exists a subsequence, still denoted by $(\Omega_n)_{n\geqslant 0}$, such that one of the following situations occurs.*

(i) ***Compactness.*** *The sequence $(\Omega_n)_{n\geqslant 0}$ γ-converges to a capacitary measure μ and R_{Ω_n} converges in the uniform operator topology of $L^2(\mathbb{R}^d)$ to R_μ. Moreover, we have that $D\subset\Omega_\mu$.*

(ii) ***Dichotomy.*** *There exists a sequence of subsets $\tilde{\Omega}_n\subseteq\Omega_n$, such that:*
- $\|R_{\Omega_n}-R_{\tilde{\Omega}_n}\|_{\mathcal{L}(L^2(\mathbb{R}^d),L^2(\mathbb{R}^d))}\to 0$;
- *$\tilde{\Omega}_n$ is a union of two disjoint quasi-open sets $\tilde{\Omega}_n=\Omega_n^+\cup\Omega_n^-$;*
- $d(\Omega_n^+,\Omega_n^-)\to\infty$;
- $\liminf_{n\to\infty}|\Omega_n^\pm|>0$;
- $\limsup_{n\to\infty}|\Omega_n^+\cap D|=0$ *or* $\limsup_{n\to\infty}|\Omega_n^-\cap D|=0$.

Proof. Since $(\Omega_n)_{n\geqslant 1}$ is a sequence of quasi-open sets of uniformly bounded measure we can apply [5, Theorem 2.2]. If the compactness situation holds in [5, Theorem 2.2], then one can take the sequence y_n introduced there to be convergent. In fact, suppose that y_n is divergent and notice that the solution w_{D+y_n} is just w_D translated to the left by y_n. By the maximum principle, we have that $w_{\Omega_n+y_n}\geqslant w_{D+y_n}$ and so

$$\int w_{D+y_n}w_{\Omega_n+y_n}\,dx\geqslant\int w_D^2\,dx>0.$$

Since $y_n\to\infty$, we have that $w_{D+y_n}\rightharpoonup 0$ weakly in L^2. By the strong convergence of $w_{\Omega_n+y_n}$ we have

$$\int w_{D+y_n}w_{\Omega_n+y_n}\,dx\to 0,$$

which is a contradiction and so we have that y_n is bounded and thus, we can extract a convergent subsequence. It remains to prove that we can take $y_n=0$, for every $n\in\mathbb{N}$. Let $y_n\to y$ and set $w=L^2\text{-}\lim_{n\to\infty}w_{\Omega_n+y_n}$. We have

$$\begin{aligned}\|w_{\Omega_n}-w(\cdot-y)\|_{L^2(\mathbb{R}^d)}&\leqslant\|w_{\Omega_n}-w(\cdot-y_n)\|_{L^2(\mathbb{R}^d)}+\|w(\cdot-y_n)-w(\cdot-y)\|_{L^2(\mathbb{R}^d)}\\&\leqslant\|w_{\Omega_n+y_n}-w\|_{L^2(\mathbb{R}^d)}+\|w(\cdot-y_n)-w(\cdot-y)\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

and both last terms converge to zero as $n\to\infty$. Thus, by Proposition 3.6, we have that w_{Ω_n} γ-converges to a capacitary measure μ and $w(\cdot-y)=w_\mu$. Moreover, since $w_{\Omega_n}\geqslant w_D$ for every $n\in\mathbb{N}$, we have that also $w_\mu\geqslant w_D$ and so $D\subset\Omega_\mu$ a.e.

If the dichotomy occurs in [5, Theorem 2.2], we have only to show that we can choose a subsequence such that $\limsup_{n\to\infty}|\Omega_n^+\cap D|=0$ or $\limsup_{n\to\infty}|\Omega_n^-\cap D|=0$. In fact, since $d(\Omega_n^+,\Omega_n^-)\to\infty$, we have that one of the sequences of characteristic functions $1_{\Omega_n^+}$ or $1_{\Omega_n^-}$ has a subsequence, which converges weakly in $L^2(\mathbb{R}^d)$ to zero. Taking into account that $1_D\in L^2(\mathbb{R}^d)$, we have the thesis. □

We study the existence of a solution for problem (1.4). We notice that if a solution Ω of problem (1.4) exists, then necessarily the measure of Ω is precisely equal to m. Indeed, assume by contradiction that Ω is an optimal set with measure strictly less than m. Since the map $t\mapsto|t\Omega\cup D|$ is continuous, we can choose some $t>1$ such that the set $\Omega_t=t\Omega\cup D$ is still of measure less than m. But $\lambda_k(\Omega_t)\leqslant\lambda_k(t\Omega)=\frac{1}{t^2}\lambda_k(\Omega)$ and so, the k-th eigenvalue strictly diminishes, which contradicts the optimality of Ω.

In the sequel we study problem (1.4) for any k; we will see that the most complete result is for $k=1$, while the case $k\geqslant 2$ requires some additional assumptions on the internal constraint D (see Theorem 4.7).

Theorem 4.2. *Let D be a quasi-open set and let $m\geqslant|D|$. Then, the problem*

$$\min\{\lambda_1(\Omega):\ \Omega \text{ quasi-open},\ D\subset\Omega,\ |\Omega|\leqslant m\},\tag{4.1}$$

has at least one solution.

Proof. We consider a minimizing sequence $(\Omega_n)_{n\geqslant1}$ with the property that $\liminf_{n\to\infty}|\Omega_n|$ is minimal. Clearly, this value cannot be equal to zero. According to Theorem 4.1, if we are in the compactness situation, for a subsequence (still denoted with the same indices) there exists a measure μ such that Ω_n γ-converges to μ. Moreover, the regular set Ω_μ is admissible since $|\Omega_\mu|\leqslant m$ and $D\subset\Omega_\mu$. By Proposition 3.7 we obtain that Ω_μ is a solution of (4.1).

If we are in the dichotomy situation, we get a contradiction. Since Ω_n^+ and Ω_n^- are at positive distance, we may assume that $\lambda_1(\Omega_n^+\cup\Omega_n^-)=\lambda_1(\Omega_n^+)$. Then, the sequence $\Omega_n^+\cup D$ is also minimizing since $|\lambda_1(\Omega_n^+)-\lambda_1(\Omega_n)|\to0$ (see [5, Proposition 3.7]), but either

$$\liminf_{n\to\infty}|\Omega_n^+\cup D|<\liminf_{n\to\infty}|\Omega_n|,$$

or $|\Omega_n^-\setminus D|\to0$. The first assertion is in contradiction with our assumption on the choice of a least measure minimizing sequence. The second assertion is also impossible, since it implies that $d(\Omega_n^+,\{0\})\to+\infty$, otherwise the measure of D would be infinite. Consequently, since the measure of D is finite, we get that $|\Omega_n^+\cap D|\to0$ and consider the ball B of measure equal to $\limsup|\Omega_n^+|$. Therefore, $B\cup D$ is a solution for every position of the ball B. In particular, this leads to a contradiction if the ball intersects, but not cover, a quasi-connected component of D. □

Remark 4.3. Let us notice that from every minimizing sequence we can extract a γ-convergent subsequence. The basic observation is that any minimizing sequence for which $\liminf_{n\to\infty}|\Omega_n|$ is minimal leads to an optimal set, which necessarily has the measure equal to m. Since the Lebesgue measure is lower semicontinuous for the γ-convergence, this means that any minimizing sequence should satisfy $\lim_{n\to\infty}|\Omega_n|=m$ excluding the dichotomy in the proof above.

In the sequel we show a result which gives a rather explicit behavior of a minimizing sequence for problem (1.4). For every $m>0$ we introduce the value

$$\lambda_k^*(m)=\inf\{\lambda_k(\Omega):\ \Omega \text{ quasi-open},\ |\Omega|\leqslant m\}.$$

Following [6], there exists a bounded quasi-open set Ω^m, with measure equal to m such that $\lambda_k(\Omega^m)=\lambda_k^*(m)$.

Theorem 4.4. *Let D be a quasi-open set and let $m\geqslant|D|$. For $k\in\mathbb{N}$, $k\geqslant2$, we define*

$$\alpha_k=\inf\{\lambda_k(\Omega):\ \Omega \text{ quasi-open},\ D\subset\Omega,\ |\Omega|\leqslant m\}.$$

One of the following assertions holds:

(i) *problem* (1.4) *has a solution*;
(ii) *there exist* $l \in \{1, \dots, k-1\}$ *and an admissible quasi-open set* Ω *such that* $\alpha_k = \lambda_{k-l}(\Omega) = \lambda_l^*(m - |\Omega|)$;
(iii) *there exists* $l \in \{1, \dots, k-1\}$ *such that* $\alpha_k = \lambda_l^*(m - |D|) > \lambda_{k-l}(D)$.

Proof. Let us consider a minimizing sequence $(\Omega_n)_{n\geqslant 1}$ with the property that $\liminf_{n\to\infty} |\Omega_n|$ is minimal. If compactness occurs in Theorem 4.1, then the existence of a solution follows as in Theorem 4.2.

If dichotomy occurs, as in Theorem 4.2 we may assume that

$$\left|\Omega_n^+\right| \to \alpha^+, \qquad \left|\Omega_n^-\right| \to \alpha^-, \qquad \left|\Omega_n^+ \cap D\right| \to 0.$$

Then, up to a subsequence there exists $l \in \{1, \dots, k-1\}$ such that one of the two possibilities below holds:

(A) $|\lambda_k(\Omega_n) - \lambda_{k-l}(\Omega_n^-)| \to 0$ and $\lambda_l(\Omega_n^+) \leqslant \lambda_{k-l}(\Omega_n^-) \leqslant \lambda_{l+1}(\Omega_n^+)$;
(B) $|\lambda_k(\Omega_n) - \lambda_l(\Omega_n^+)| \to 0$ and $\lambda_{k-l}(\Omega_n^-) \leqslant \lambda_l(\Omega_n^+) \leqslant \lambda_{k-l+1}(\Omega_n^-)$.

We may take the maximal l with such a property. We use now an induction argument as follows. For $k = 1$ as proved in Theorem 4.2, dichotomy does not occur, so the compactness gives (i). Assume that for $1, \dots, k-1$ Theorem 4.4 is true. We prove it for k. If compactness occurs, then (i) holds. If dichotomy occurs and we are in situation (A) we get that $(\Omega_n^- \cup D)_n$ is minimizing for the $k-l$ eigenvalue with the inclusion constraint and the corresponding measure $m - \alpha^+ \geqslant \liminf_{n\to\infty} |\Omega_n^- \cup D|$. Since l is maximal with this property, for the sequence $(\Omega_n^- \cup D)_n$ dichotomy cannot occur again, so finally (ii) holds.

If (B) occurs, then $|\Omega_n^- \setminus D| \to 0$ and we are in situation (iii). □

Remark 4.5. Theorem 4.4 gives a complete description of the behavior of a minimizing sequence for λ_k, $k \geqslant 2$. Assertion (i) implies the existence of a solution. As well if D has some suitable geometric properties (for instance if D is bounded), both alternatives (ii) and (iii) lead to the existence of a solution. In fact, if for instance (ii) occurs, the set $\Omega \cup \Omega^{m-|\Omega|}$ is a minimizer provided that $\Omega \cap \Omega^{m-|\Omega|} = \emptyset$. Similarly, for (iii) the set $D \cup \Omega^{m-|D|}$ is a minimizer, provided that $D \cap \Omega^{m-|D|} = \emptyset$.

The result below is analogous to Theorem 4.2 and Theorem 4.4, corresponding to the problem (1.3). We omit the proof, since it is the same as that of Theorem 4.2 and Theorem 4.4.

Theorem 4.6. *Let D be a quasi-open set of finite measure and let $\Lambda > 0$. For $k \in \mathbb{N}$, we define*

$$\beta_k = \inf\left\{\lambda_k(\Omega) + \Lambda|\Omega|:\ \Omega \text{ quasi-open},\ D \subset \Omega\right\}.$$

Suppose that Ω_n is a minimizing sequence for β_k such that there exists the limit $m := \lim_{n\to\infty} |\Omega_n|$. If $k = 1$, then the problem (1.3) *has solution. If $k \geqslant 2$, then one of the following assertions holds*:

(i) *problem* (1.3) *has a solution*;
(ii) *there exist* $l \in \{1, \dots, k-1\}$ *and an admissible quasi-open set* Ω *such that* $\beta_k = \lambda_{k-l}(\Omega) = \lambda_l^*(m - |\Omega|)$;
(iii) *there exists* $l \in \{1, \dots, k-1\}$ *such that* $\beta_k = \lambda_l^*(m - |D|) > \lambda_{k-l}(D)$.

We conclude summarizing the assertions above into the following existence result, whose proof will be given in Section 5.

Theorem 4.7. *Let $D \subset \mathbb{R}^d$ be a quasi-open set of finite measure such that*

$$\text{for any } R > 0, \text{ there exists } x \in \mathbb{R}^d \text{ such that } B_R(x) \cap D = \emptyset. \tag{4.2}$$

Then, for any $k > 0$ and $\Lambda > 0$, there exists a solution of (1.3). *Moreover, if D satisfies in addition also the assumption*

$$\limsup_{t\to 1^+} \frac{|D \setminus tD|}{t-1} < \infty,$$

then problem (1.4) *admits a solution, for any $k > 0$ and $m \geqslant |D|$.*

Remark 4.8. The assumption (4.2) is crucial for the proof of Theorem 4.7. We do not know whether the existence of an optimal domain occurs without it.

5. Qualitative properties of the optimal sets

A natural question that arises in the shape optimization problems with constraints like (4.1) is to understand the influence of the inclusion domain D on the optimal sets: does boundedness and/or convexity of D imply the same properties on the optimal set? As we shall see, the answer is positive for the boundedness constraint, but negative for the convexity constraint.

5.1. Regularity of the optimal set for $k = 1$

In this section we deal with the penalized version of problem (4.1)

$$\min\{\lambda_1(\Omega) + \Lambda|\Omega|:\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ quasi-open, } D \subset \Omega\}, \tag{5.1}$$

for some $\Lambda > 0$. For the local equivalence of the two problems we refer the reader to [4]. As well, we refer the reader to [4] for a complete analysis of a similar problem, in which the internal constraint $D \subset \Omega$ is replaced by an external constraint $\Omega \subset D$, with a bounded open set D.

In the case of internal constraint, new behaviors can be noticed with respect to [4]; we prove that the optimal set of (5.1) is open even if D is only quasi-open, provided that D is quasi-connected and the optimal set has a measure strictly greater than $|D|$.

Definition 5.1. We say that the quasi-open set D is quasi-connected if for every pair of non-empty quasi-open sets A_1 and A_2 having intersection of positive capacity with D and such that $D \subset A_1 \cup A_2$, we get $\mathrm{cap}(A_1 \cap A_2) > 0$.

The quasi-connectedness has a topological counterpart. Indeed, a quasi-open, quasi-connected set A has a fine interior (which differs from A by a set of zero capacity) which is finely connected (the fine topology being the coarsest topology making all superharmonic functions continuous). A non-negative superharmonic function in $H_0^1(A)$ with A finely connected, is either equal to 0 or is strictly positive (see [8,16,20]).

Example 5.2. The assumption D quasi-connected is essential to obtain that the optimal set Ω is open even when D is only quasi-open. Indeed, consider $D = B_0 \cup D_0$, where B_0 is a large ball and D_0 is a quasi-open and not open set, whose distance from B_0 is large enough and whose measure is less than $|B_0|$. It is easy to see that in this case the optimal set is of the form $B \cup D_0$, where B is a ball containing B_0.

Remark 5.3. In spite of the example above, if D is an *arbitrary* open set, then every optimal set Ω for problem (5.1) is open. Indeed, a careful inspection of the proof of Proposition 5.6 below gives that $\{u > 0\}$ is open; since $\Omega = \{u > 0\} \cup D$ we have that Ω is open.

In the following, without loss of generality we assume that $\Lambda = 1$.

Remark 5.4. The existence of a solution to (5.1) follows by the same argument we used in the proof of Theorem 4.2 and so we omit the proof.

Let D be a quasi-open, quasi-connected set of finite measure. Let Ω be a solution of problem (5.1), let $\lambda := \lambda_1(\Omega)$, and let $u := u_\Omega$ be the first normalized eigenfunction:

$$\begin{cases} -\Delta u = \lambda u, \\ u \in H_0^1(\Omega), \quad \|u\|_{L^2} = 1. \end{cases}$$

As D is quasi-connected, if Ω is optimal, then u is a solution of the minimization problem

$$\min\left\{\frac{\int |\nabla v|^2\,dx}{\int v^2\,dx} + \big|\{v > 0\}\big|:\ v \in H^1(\mathbb{R}^d),\ v \geqslant 0,\ D \subset \{v > 0\}\right\}. \tag{5.2}$$

The following lemma has a proof similar to [1, Lemma 3.2] and [4, Lemma 3.1], so we omit it.

Lemma 5.5. *Let u be a supersolution of problem (5.2), in the sense that for any $v \in H^1(\mathbb{R}^d)$ such that $v \geqslant u$, we have*

$$\frac{\int |\nabla v|^2\,dx}{\int v^2\,dx} + \big|\{v>0\}\big| \geqslant \frac{\int |\nabla u|^2\,dx}{\int u^2\,dx} + \big|\{u>0\}\big|.$$

Then there is a constant C depending only on the dimension d such that for each $r>0$, the following implication holds:

$$Cr \leqslant \frac{1}{|\partial B_r|}\int_{\partial B_r} u\,d\mathcal{H}^{d-1} \quad\Longrightarrow\quad B_r \subset \{u>0\}. \tag{5.3}$$

The next proposition follows the approach first introduced in [1]; nevertheless, we give the proof below to stress the fact that the quasi-open internal constraint does not change the argument too much.

Proposition 5.6. *Let D be a quasi-open, quasi-connected set of finite measure. Every solution Ω of problem (5.1) is an open set up to a set of capacity 0.*

Proof. Since D is quasi-connected, the optimal domain Ω is quasi-connected too. Indeed, otherwise Ω would have at least two quasi-connected components, one of which contains D. Then:

(i) either $\lambda_1(\Omega)$ is realized on a component not containing D, in which case $D \cup B$, for a suitable ball B, is a better competitor;
(ii) or $\lambda_1(\Omega)$ is realized on the component Ω_1 containing D, in which case replacing Ω by a suitable enlargement of Ω_1 gives again a better competitor.

Let u be a solution of (5.2). We prove that if $u(x)>0$, then u is positive in a small ball centered at x. Without loss of generality, we can suppose that $x=0$ and that 0 is a regular point in the sense that

$$u(0) = \lim_{r\to 0}\frac{1}{|B_r(0)|}\int_{B_r(0)} u(y)\,dy.$$

Denote by φ_r the solution of

$$\begin{cases} -\Delta\varphi_r = 1, \\ \varphi_r \in H_0^1(B_r), \end{cases} \tag{5.4}$$

where B_r denotes the ball centered in 0 of radius r. An explicit computation gives

$$\varphi_r(y) = \frac{r^2-|y|^2}{2d}.$$

Since $u \geqslant 0$ and $\Delta u + \lambda u = 0$ on $\{u>0\}$, we have that $\Delta u + \lambda u \geqslant 0$ in distributional sense on $\mathbb{R}^d$. Indeed, arguing as in [19, Lemma 7.2.5], we consider, for any non-negative $\varphi \in H^1(\mathbb{R}^d)$, the test function $p_n(u)\varphi \in H_0^1(\{u>0\})$, where $p_n(t)$ is 0, for $t \leqslant 0$, 1, for $t \geqslant 1/n$ and equals nt, for $t \in [0,1/n]$. An explicit calculation gives

$$0 = \limsup_{n\to\infty}\big\langle \Delta u + \lambda u,\, p_n(u)\varphi\big\rangle \leqslant \langle \Delta u + \lambda u, \varphi\rangle.$$

As a consequence, using the boundedness of u (see [15]), we obtain

$$\Delta\big(u - \|u\|_\infty \lambda\varphi_r\big) \geqslant -\lambda u + \lambda\|u\|_\infty \geqslant 0$$

on each ball B_r, so the function $u - \|u\|_\infty\lambda\varphi_r$ is subharmonic on B_r. By Poisson's formula, we have

$$u(0) - \|u\|_\infty \lambda \varphi_r(0) \leqslant C(d) \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, d\mathcal{H}^{d-1}(y),$$

$$u(0) - \|u\|_\infty \lambda C_1 r^2 \leqslant C(d) \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, d\mathcal{H}^{d-1}(y).$$

Suppose that $u(0) > 0$. Then, choosing r small enough, we have

$$u(0) \leqslant \frac{2C(d)}{|\partial B_r|} \int_{\partial B_r} u(y)\, d\mathcal{H}^{d-1}(y).$$

Now choose C as in Lemma 5.5 and r such that $2rCC(d) \leqslant u(0)$. Then

$$Cr \leqslant \frac{1}{|\partial B_r|} \int_{\partial B_r} u(y)\, d\mathcal{H}^{d-1}(y),$$

and so $u > 0$ on B_r. □

Remark 5.7. Alternatively, one can formulate the proposition above, requiring that the inclusion $D \subset \Omega$ holds quasi-everywhere, and in this case the optimal sets $\{u > 0\}$ in (5.2) are open and u is continuous. In fact, in [1] it is proven a stronger result on the Lipschitz continuity of u, even if this does not provide a higher regularity of the optimal set Ω.

Remark 5.8. The regularity of the free parts of the boundary is the same as in [4, Theorem 1.2], being independent of the fact that the inclusion constraint is internal or external.

Remark 5.9. If D is a quasi-open set such that there does not exist an open set containing D and having the same Lebesgue measure, then Proposition 5.6 asserts that the measure of any optimal set is strictly greater than the measure of D.

In general, this is not the case if D is an open set. Indeed, following a simple computation one can consider D to be a ball B and take a constant Λ large enough, so that the optimal set is B itself. More generally, if the partial metric derivative of the first eigenvalue on D is finite, i.e.

$$\lambda_1'(D) := \limsup_{|E \setminus D| \to 0,\ E \supset D} \frac{\lambda_1(D) - \lambda_1(E)}{|E \setminus D|} < +\infty,$$

then for every $\Lambda > \lambda_1'(D)$ there exists $\Lambda' > \Lambda$ such that the optimal solution of (5.1) with Λ' is D. Indeed, by contradiction for every $\Lambda > \lambda_1'(D)$ there exists $\varepsilon > 0$ such that for every $\Omega \supset D$ such that $|\Omega| \leqslant |D| + \varepsilon$ we have

$$\lambda_1(D) + \Lambda|D| \leqslant \lambda_1(\Omega) + \Lambda|\Omega|.$$

Then, replacing Λ with $\Lambda' > \Lambda$ such that $\Lambda' \geqslant \frac{\lambda_1(D)}{\varepsilon}$, we get that D is a global minimizer.

5.2. Bounded constraint implies bounded minimizers

In this paragraph we consider the penalized version (1.3). In fact the following lemma gives a relation between the solutions of the constrained problem (1.4) and the subsolutions of the Lagrange multiplier penalized version (1.3). Adapting a notion introduced in [6], we say that Ω^* is a shape subsolution for λ_k if there exists $c > 0$ such that

$$\lambda_k(\Omega^*) + c|\Omega^*| \leqslant \lambda_k(\Omega) + c|\Omega| \quad \forall D \subset \Omega \subset \Omega^*. \tag{5.5}$$

For simplicity, we consider the internal constraint D regular enough such that

$$\limsup_{t \to 1^+} \frac{|D \setminus tD|}{t - 1} < \infty. \tag{5.6}$$

This condition is for instance satisfied if D is bounded and Lipschitz, or if D is star shaped.

Lemma 5.10. *Suppose that the internal constraint D satisfies* (5.6) *and assume that Ω_m is a solution of*

$$\min\{\lambda_k(\Omega):\ D\subset\Omega\subset\mathbb{R}^d,\ \Omega\text{-quasi-open},\ |\Omega|\leqslant m\}. \tag{5.7}$$

Then Ω_m is a shape subsolution for λ_k.

Proof. We first notice that $|\Omega_m|=m$. Suppose by contradiction, that for each $\varepsilon>0$, there is some quasi-open set Ω_ε such that $D\subset\Omega_\varepsilon\subset\Omega_m$ and

$$\lambda_k(\Omega_\varepsilon)+\varepsilon|\Omega_\varepsilon|<\lambda_k(\Omega_m)+\varepsilon|\Omega_m|. \tag{5.8}$$

By the compactness of the inclusion $H_0^1(\Omega)\subset L^2(\Omega)$, we can suppose, up to a subsequence that Ω_ε γ-converges to some capacitary measure μ, whose regular set Ω_μ is such that

$$|\Omega_\mu|\leqslant\liminf_{\varepsilon\to0}|\Omega_\varepsilon|,$$
$$\lambda_k(\Omega_\mu)\leqslant\lambda_k(\mu)=\lim_{\varepsilon\to0}\lambda_k(\Omega_\varepsilon)\leqslant\lambda_k(\Omega_m),$$

where the last inequality is due to (5.8) and Lemma 3.7. Thus, we obtain that Ω_μ is a solution of (5.7) and so $|\Omega_\mu|=m$ and $\lambda_k(\Omega_\mu)=\lambda_k(\Omega_m)$.

Let $\Omega'_\varepsilon=t_\varepsilon\Omega_\varepsilon\cup D$, where t_ε is such that $|\Omega'_\varepsilon|=m$. Then, we have that

$$\begin{aligned}\lambda_k(\Omega_\varepsilon)+\varepsilon|\Omega_\varepsilon|&<\lambda_k(\Omega_m)+\varepsilon|\Omega_m|\\&\leqslant\lambda_k(\Omega'_\varepsilon)+\varepsilon|\Omega'_\varepsilon|\\&\leqslant\lambda_k(t_\varepsilon\Omega_\varepsilon)+\varepsilon|t_\varepsilon\Omega_\varepsilon\cup D|\\&\leqslant\frac{1}{t_\varepsilon^2}\lambda_k(\Omega_\varepsilon)+\varepsilon\big(|t_\varepsilon\Omega_\varepsilon|+|D\setminus t_\varepsilon\Omega_\varepsilon|\big)\\&\leqslant\frac{1}{t_\varepsilon^2}\lambda_k(\Omega_\varepsilon)+\varepsilon\big(|t_\varepsilon\Omega_\varepsilon|+|D\setminus t_\varepsilon D|\big),\end{aligned} \tag{5.9}$$

and so

$$\frac{t_\varepsilon^2-1}{t_\varepsilon^2}\lambda_k(\Omega_\varepsilon)\leqslant\varepsilon\big(\big(t_\varepsilon^d-1\big)|\Omega_\varepsilon|+|D\setminus t_\varepsilon D|\big). \tag{5.10}$$

By hypothesis (5.6) passing to the limit as $t_\varepsilon\to1^+$, there is some constant C such that

$$\lambda_k(\Omega_\varepsilon)\leqslant\varepsilon C, \tag{5.11}$$

for ε small enough. But, by (5.9), $\lambda_k(\Omega_\varepsilon)\to\lambda_k(\Omega_m)$ and so, we have a contradiction. □

We give the following technical result for which we refer to [1, Lemma 3.4] and to [4, Lemma 3.1] in the case $k=1$ and to [6, Lemma 2.3] for the general case.

Lemma 5.11. *Let Ω be a shape subsolution for λ_k and let w be the solution of the equation*

$$\begin{cases}-\Delta w=1\in\Omega,\\ w\in H_0^1(\Omega).\end{cases}$$

Then there exist two constants C_0 and r_0 such that for each $x\in\mathbb{R}^d$ such that $d(x,D)>r_0$ and for each $r<r_0$ the following implication holds:

$$\big(\|w\|_{L^\infty(B_r)}\leqslant C_0r\big)\quad\Rightarrow\quad(w=0\ \text{on}\ B_{\frac{r}{2}}). \tag{5.12}$$

The proof of this lemma relies on the fact that shape subsolutions for λ_k are local *shape subsolutions* for the energy problem (see [6, Definition 2.1] for more details).

Proposition 5.12. *Suppose that D is a quasi-open set of finite measure and that Ω is a shape subsolution for λ_k. Then there exists $L > 0$ such that $\Omega \subset \overline{D} + B_L(0)$. In particular if D is bounded, then Ω is bounded.*

Proof. With no loss of generality we set $c = 1$ in (5.5). Assume by contradiction that such L does not exist. Then, there is a sequence $(x_n)_{n\geqslant 1} \subset \Omega$ such that $d(x_n, \overline{D}) \to +\infty$ and $|x_n - x_m| \geqslant 2r_0$, when $n \neq m$. Since $\Omega = \{w > 0\}$, we have $w(x_n) > 0$, for every n and so, by Lemma 5.11, there are constants $C_0 > 0$ and $0 < r_0$ such that we have the bound

$$\|w\|_{L^\infty(B_r(x_n))} \geqslant C_0 r \quad \forall r < r_0.$$

For each n, consider $y_n \in B_r(x_n)$ such that

$$w(y_n) \geqslant \frac{1}{2} C_0 r.$$

Consider the function $\varphi_r(\cdot - y_n)$, as defined in (5.4). Then $w - \varphi_r(\cdot - y_n)$ is subharmonic, since

$$\Delta\big(w - \varphi_r(\cdot - y_n)\big) \geqslant 0.$$

So, we have the inequalities

$$\int_{B_r(y_n)} \big(w(x) - \varphi_r(x - y_n)\big)\,dx \geqslant |B_r|\big(w(y_n) - \varphi_r(x - y_n)\big) \geqslant |B_r|\left(\frac{C_0}{2} r - r^2 \varphi_1(0)\right);$$

$$\int_{B_r(y_n)} w(x)\,dx \geqslant r^{2+d}\|\varphi\|_{L^1} + |B_r|\left(\frac{C_0}{2} r - r^2 \varphi_1(0)\right).$$

Choose now $0 < r < r_0$ small enough such that $\frac{C_0}{2} r - r^2 \varphi_1(0) > 0$. Then there is a constant $c > 0$, such that

$$\int_{B_r(y_n)} w(x)\,dx \geqslant c \quad \forall n \in \mathbb{N}.$$

The fact that the balls $B_r(y_n)$ are all disjoint contradicts the integrability of w. □

Remark 5.13. The constant c, depends on C_0, r_0 and $\lambda_k(D)$. In fact, the proof of the proposition above gives an estimate on the number of admissible points x_n. Therefore the value of L could be estimated more explicitly.

We are now in a position to prove the existence Theorem 4.7.

Proof of Theorem 4.7. We prove the thesis by induction. In the case $k = 1$, the existence of an optimal domain follows by Theorem 4.6. Suppose that the thesis is true for $l = 1, \dots, k-1$ and let Ω_n be a minimizing sequence for the problem (1.3). Up to extracting a subsequence, we can suppose that $|\Omega_n|$ is convergent. Applying Theorem 4.6, for $m := \lim |\Omega_n|$, we have three possibilities. If case (i) occurs, then we have a solution of (1.3). Suppose that (ii) holds and let Ω and $l < k$ be as in (ii) of Theorem 4.6. By the inductive step, there is a solution Ω_{k-l} of (1.3) with $k = k - l$. By Proposition 5.12, we have that for any $R > 0$, there exists $x \in \mathbb{R}^d$ such that $B_R(x) \cap \Omega_{k-l} = \emptyset$. Let Ω_l^* be the optimal set for $\lambda_l + \Lambda|\cdot|$ in $\mathbb{R}^d$. By [6, Theorem 3.3], Ω_l^* is bounded and so, we may suppose that $\Omega_l^* \cap \Omega_{k-l} = \emptyset$. Thus, the set $\Omega_k := \Omega_l^* \cup \Omega_{k-l}$ is a solution of (1.3). If (iii) occurs, we reason as above, obtaining that the set $\Omega_l^* \cup D$ is a solution of (1.3). The proof that the problem (1.4) has a solution follows by the same argument and Theorem 4.4. Note that, in order to construct an optimal set for λ_k, in the case when (ii) occurs, we need that the optimal sets for (1.4) are bounded. This is true since, under the assumption (5.6), any solution of (1.4) is a subsolution for λ_k (see Lemma 5.10) and so, is bounded, by Proposition 5.12. □

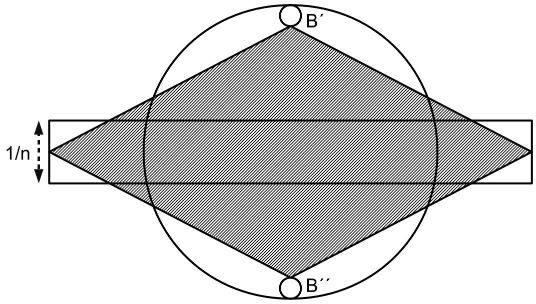

Fig. 1. Convex obstacle does not imply convex optimal set.

5.3. Convex constraint does not imply convex optimal set

In this section we will prove that the solution Ω of the optimization problem (4.1) might not be convex even if the constraint D is convex. Consider the sequence of constraints $(D_n)_{n\geqslant 1}$, where $D_n = (-\frac{1}{n}, \frac{1}{n}) \times (-1, 1)$ and consider the sequence of bounded open sets $(\Omega_n)_{n\geqslant 1}$ such that for each n big enough, Ω_n is a solution of the shape optimization problem:

$$\min\{\lambda_1(\Omega):\ D_n \subset \Omega,\ \Omega \text{ quasi-open},\ |\Omega| = m\}. \tag{5.13}$$

Proposition 5.14. *For every $m < 4/\pi$, there is $N > 0$ such that Ω_n is not convex for all $n \geqslant N$.*

Proof. We begin with some observations on the optimal sets.

1. By a Steiner symmetrization argument, all the sets Ω_n are Steiner symmetric with respect to the axes x and y (in consequence, they are also star shaped sets).
2. For n large enough, we consider the set $\Omega'_n = D_n \cup B^*(m - \frac{4}{n})$, where for any $a > 0$, $B^*(a)$ denotes the ball centered in 0 of measure a. By the optimality of Ω_n, we have

$$\lambda_1(\Omega_n) \leqslant \lambda_1(\Omega'_n) \leqslant \lambda_1\left(B^*\left(m - \frac{4}{n}\right)\right).$$

By Theorem 4.1, there is a γ-converging subsequence still denoted by $(\Omega_n)_{n\geqslant 1}$. Let Ω be the γ-limit of this subsequence. Then

- $\lambda_1(\Omega) \leqslant \liminf_{n\to\infty} \lambda_1(\Omega_n) \leqslant \liminf_{n\to\infty} \lambda_1(B^*(m - \frac{4}{n})) = \lambda_1(B^*(m))$;
- $|\Omega| \leqslant \liminf_{n\to\infty} |\Omega_n| = m$.

Using the fact that the ball is the unique minimizer of λ_1 under a measure constraint, we obtain $\Omega = B^*(m)$. Consider a ball B' of center $(0, \sqrt{\frac{m}{\pi}} - \varepsilon)$ and radius ε and a ball B'' of center $(0, -\sqrt{\frac{m}{\pi}} + \varepsilon)$ and radius ε. Then

$$\Omega_n \cap B' \xrightarrow[n\to\infty]{\gamma} \Omega \cap B' = B', \qquad \Omega_n \cap B'' \xrightarrow[n\to\infty]{\gamma} \Omega \cap B'' = B''.$$

Then there is some n large enough such that both sets $B' \cap \Omega_n$ and $B'' \cap \Omega_n$ are non-empty, and Ω_n cannot be convex (see Fig. 1).

In fact, if by contradiction Ω_n was convex, then we should have that the rhombus R with vertices $(-1, 0)$, $(0, -\sqrt{\frac{m}{\pi}} + \varepsilon)$, $(1, 0)$, $(0, \sqrt{\frac{m}{\pi}} - \varepsilon)$ is contained in Ω_n. But

$$|R| = 2\left(\sqrt{\frac{m}{\pi}} - \varepsilon\right) > m,$$

for ε small enough and $m \leqslant 4/\pi$, and this is a contradiction. □

5.4. Lack of monotonicity

We show here that in problem (4.1) the optimal solutions are not monotone with respect to m, i.e. $m_1 < m_2$ does not imply in general that $\Omega_1 \subset \Omega_2$ where Ω_i is optimal with the constraint m_i. Similarly, in the penalized problem (5.1), the same lack of monotonicity occurs with respect to Λ, i.e. $\Lambda_1 < \Lambda_2$ does not imply in general that $\Omega^1 \supset \Omega^2$ where Ω^i is optimal with the penalization Λ_i. Here we consider only the case of penalization, since the first one follows as a consequence, taking $m_1 = |\Omega^2|$ and $m_2 = |\Omega^1|$.

Let us consider in $\mathbb{R}^2$ the internal constraint D of the form $D = B_{1/2}(0) \cup R^{\varepsilon,\eta}$ where $R^{\varepsilon,\eta}$ is the rectangle $(\eta, 0) + (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \times (-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon})$. The parameters ε, η will be fixed later.

Note that $\frac{\pi}{4} = |B_{1/2}(0)| < |R^{\varepsilon,\eta}| = 1$ and that $\lambda_1(B_{1/2}(0)) < \lambda_1(R^{\varepsilon,\eta})$ for ε small enough. As well, we notice that the distance between $B_{1/2}(0)$ and $R^{\varepsilon,\eta}$ tends to $+\infty$ as $\eta \to +\infty$. Following Remark 5.13 for every Λ and $\varepsilon > 0$, there exists η large enough such that every solution Ω of (5.1) satisfies one of the following two possibilities:

(A) $\Omega = B \cup R^{\varepsilon,\eta}$, where B is a ball containing $B_{1/2}(0)$ and disjoint from $R^{\varepsilon,\eta}$;
(B) $\Omega = B_{1/2}(0) \cup A$, where A is a connected open set containing $R^{\varepsilon,\eta}$ and disjoint from $B_{1/2}(0)$.

Lemma 5.15. *Let $\Lambda > 0$ be fixed, let Ω_ε be a solution of the problem*

$$\min\{\lambda_1(\Omega) + \Lambda|\Omega|\colon\ \Omega \supset R^{\varepsilon,0}\},$$

and let B be a ball solving

$$\min\{\lambda_1(\Omega) + \Lambda|\Omega|\colon\ \Omega \subset \mathbb{R}^2\}.$$

Then we have

$$\lambda_1(B) = \lim_{\varepsilon\to 0} \lambda_1(\Omega_\varepsilon), \qquad |B| + 1 = \lim_{\varepsilon\to 0} |\Omega_\varepsilon|.$$

Proof. By Steiner symmetrization along both axes, the sets Ω_ε are Steiner symmetric, and so star shaped. Therefore the sets Ω_ε fulfill a uniform exterior segment condition which, together with the compactness result [13, Theorem 4.14], is enough (see [7, Chapter 4]) to give that Ω_ε γ_{loc}-converges to some open set Ω.

We first notice that

$$\lambda_1(\Omega_\varepsilon) + \Lambda|\Omega_\varepsilon| \leqslant \lambda_1\big(B_1(0)\big) + \Lambda\big|B_1(0)\big| + \Lambda := c, \tag{5.14}$$

which gives that both measures of Ω_ε and $\lambda_1(\Omega_\varepsilon)$ are uniformly bounded. Because of that and of the Steiner symmetrization above, all Ω_ε are contained in the set

$$\big\{(x, y)\colon\ |xy| \leqslant c\big\}. \tag{5.15}$$

From the properties of the γ_{loc}-convergence, for every ball $B_R(0)$ we have that

$$\big|\Omega \cap B_R(0)\big| \leqslant \liminf_{\varepsilon\to 0} \big|\Omega_\varepsilon \cap B_R(0)\big|.$$

Since

$$\liminf_{\varepsilon\to 0} \big|\Omega_\varepsilon \cap B_R(0)\big| \leqslant \liminf_{\varepsilon\to 0} |\Omega_\varepsilon| - 1,$$

we get

$$|\Omega| + 1 \leqslant \liminf_{\varepsilon\to 0} |\Omega_\varepsilon|. \tag{5.16}$$

We prove now that

$$\lambda_1(\Omega) \leqslant \liminf_{\varepsilon\to 0} \lambda_1(\Omega_\varepsilon). \tag{5.17}$$

Let u_ε be the first normalized eigenfunction on Ω_ε. By the concentration-compactness principle, we may have: compactness, vanishing or dichotomy. The vanishing is ruled out by the fact that in this case we would have

$\lambda_1(\Omega_\varepsilon) \to +\infty$, which contradicts (5.14). The dichotomy is ruled out too, by the following argument. Let u_ε^i, $i = 1, 2$ be the two sequences provided by the dichotomy. From the concentration-compactness principle, at least one sequence of quasi-open sets $\{u_\varepsilon^i > 0\}$ has a distance from the origin going to $+\infty$. In the same time $\lambda_1(\{u_\varepsilon^i > 0\})$ are equibounded. This is in contradiction with the inclusion (5.15). Therefore the compactness occurs, i.e. $u_\varepsilon(\cdot + y_\varepsilon)$ converges strongly in $L^2(\mathbb{R}^2)$ to some function $u \in H_0^1(\Omega)$. Again, by Steiner symmetrization the vectors y_ε can be taken equal to 0. Consequently (5.17) is achieved.

Taking test domains of the form $B \cup R^{\varepsilon,0}$ with $B \cap R^{\varepsilon,0} = \emptyset$ we have that

$$\lambda_1(B) + \Lambda|B| + \Lambda \geqslant \lambda_1(\Omega_\varepsilon) + \Lambda|\Omega_\varepsilon|,$$

and passing to the limit

$$\lambda_1(B) + \Lambda|B| + \Lambda \geqslant \lambda_1(\Omega) + \Lambda|\Omega| + \Lambda.$$

Using the optimality of the ball B we get $\Omega = B$ and inequalities (5.16)–(5.17) become equalities. □

Let us fix Λ_2 such that a global solution of

$$\min\{\lambda_1(\Omega) + \Lambda_2|\Omega|:\ \Omega \subset \mathbb{R}^2\},$$

is the ball $B_1(0)$. Then for ε small enough given by Lemma 5.15 and for η large enough given by Remark 5.13 the solution of (5.1) with Λ_2 is

$$\Omega_\varepsilon^2 = B_1(0) \cup R^{\varepsilon,\eta}.$$

Indeed, from Lemma 5.15, for ε small enough we have

$$\lambda_1(\Omega_\varepsilon) + \Lambda_2|\Omega_\varepsilon| + \Lambda_2\big|B_{1/2}(0)\big| > \lambda_1\big(B_1(0)\big) + \Lambda_2\big|B_1(0)\big| + \Lambda_2\big|R^{\varepsilon,\eta}\big|,$$

so situation (A) occurs.

For the ε fixed above, take Λ_1 small enough such that a ball B' containing $R^{\varepsilon,0}$ is a global minimizer for

$$\min\{\lambda_1(\Omega) + \Lambda_1|\Omega|:\ \Omega \subset \mathbb{R}^2\}.$$

Then we are in situation (B) since $|B_{1/2}(0)| < |R^{\varepsilon,0}|$. This concludes our argument since no monotonicity may occur.

5.5. *The optimal set for λ_k has finite perimeter*

The proof of the fact that the solution of the problem (1.3) has a finite perimeter of the free boundary (i.e. outside the closure of the constraint), is based on a shape subsolutions technique involving the energy functional $E(\Omega)$, whose definition we recall here (see [6]).

Definition 5.16. For each quasi-open set Ω of finite Lebesgue measure, we define the energy functional

$$E(\Omega) = \inf\left\{\frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega w\,dx:\ w \in H_0^1(\Omega)\right\}. \tag{5.18}$$

We denote by w_Ω the solution of

$$\begin{cases} -\Delta w_\Omega = 1, \\ w_\Omega \in H_0^1(\Omega), \end{cases} \tag{5.19}$$

which, in particular, is the minimizer in (5.18).

The proof of the following proposition uses the same argument as in Theorem 2.2. of [6]. We adapt this technique to the case of the internal constraint. Note that, when $k = 1$, the proposition below can be proved directly, working with the first eigenfunction u_1, instead of w_Ω.

Proposition 5.17. *Let Ω be an optimal domain for (1.3). Then the perimeter of Ω in $\mathbb{R}^d \setminus \overline{D}$ is finite.*

Proof. By Lemma 4.1 of [6], we have that there exists some positive constant c depending on Ω such that

$$\lambda_k(\tilde{\Omega}) - \lambda_k(\Omega) \leqslant c\big(E(\tilde{\Omega}) - E(\Omega)\big) \quad \forall D \subset \tilde{\Omega} \subset \Omega \text{ quasi-open.} \tag{5.20}$$

Using the optimality of Ω, we have

$$E(\Omega) + \Lambda|\Omega| \leqslant E(\tilde{\Omega}) + \Lambda|\Omega| \quad \forall D \subset \tilde{\Omega} \subset \Omega \text{ quasi-open,} \tag{5.21}$$

where $\Lambda = 1/c$, i.e. Ω is an energy subsolution. Let $w = w_\Omega$ be the solution of (5.19). Consider the set $\Omega_\varepsilon = D \cup \{w > \varepsilon\}$ instead of $\tilde{\Omega}$ in (5.21). By the fact that $(w-\varepsilon)^+ \in H_0^1(\Omega_\varepsilon)$, we have

$$E(\Omega) + \Lambda|\Omega| \leqslant E(\Omega_\varepsilon) + \Lambda|\Omega_\varepsilon| \leqslant \frac{1}{2}\int \big|\nabla(w-\varepsilon)^+\big|^2\,dx - \int (w-\varepsilon)^+\,dx + \Lambda|\Omega_\varepsilon|, \tag{5.22}$$

and since $E(\Omega) = \frac{1}{2}\int |\nabla w|^2\,dx - \int w\,dx$, we obtain

$$\begin{aligned}
\varepsilon|\Omega| &\geqslant \int w\,dx - \int (w-\varepsilon)^+\,dx \\
&\geqslant \frac{1}{2}\int_{\{0<w\leqslant\varepsilon\}} |\nabla w|^2\,dx + \Lambda|\Omega \setminus \Omega_\varepsilon| \\
&\geqslant \frac{1}{2}\int_{\{0<w\leqslant\varepsilon\}\setminus\overline{D}} |\nabla w|^2\,dx + \Lambda\big|\{0<w\leqslant\varepsilon\}\setminus\overline{D}\big| \\
&\geqslant \frac{1}{2|\{0<w\leqslant\varepsilon\}\setminus\overline{D}|}\bigg(\int_{\{0<w\leqslant\varepsilon\}\setminus\overline{D}} |\nabla w|\,dx\bigg)^2 + \Lambda\big|\{0<w\leqslant\varepsilon\}\setminus\overline{D}\big|.
\end{aligned} \tag{5.23}$$

Thus, we have that

$$\int_{\{0<w\leqslant\varepsilon\}\setminus\overline{D}} \big|\nabla w(x)\big|\,dx \leqslant \varepsilon|\Omega|\sqrt{\Lambda/2}. \tag{5.24}$$

By the co-area formula

$$\frac{1}{\varepsilon}\int_0^\varepsilon P\big(\{w>t\};\mathbb{R}^d\setminus\overline{D}\big)\,dt \leqslant |\Omega|\sqrt{\Lambda/2}, \tag{5.25}$$

for each $\varepsilon > 0$ small enough. Then, there is a sequence $(\varepsilon_n)_{n\geqslant 1}$ converging to 0 such that

$$P\big(\{w>\varepsilon_n\};\mathbb{R}^d\setminus\overline{D}\big) \leqslant |\Omega|\sqrt{\Lambda/2}.$$

Passing to the limit we have

$$P\big(\{w>0\};\mathbb{R}^d\setminus\overline{D}\big) \leqslant |\Omega|\sqrt{\Lambda/2},$$

as required. □

6. Open problems and complements

We give a list of some open problems that arose during the work on this article. We denote by $\Omega(D,m)$ a quasi-open set of Lebesgue measure m, which solves the shape optimization problem (4.1).

1. Let D be an open convex set such that for every $m \geqslant |D|$ there exists a convex solution to the shape optimization problem (4.1). Is it true that then D is a ball?
2. Is it true that for every quasi-open set D with finite measure, problem (1.4) has a solution?

3. If D is not a ball, is there some $\varepsilon > 0$, such that for every $0 < m < \varepsilon$, the set $\Omega(D, |D| + m)$ is unique? Note that this is certainly not true when m is large, since for a bounded D any ball of measure m and containing D is a solution.
4. If $m' > m$, is there an optimal set $\Omega(D, m')$ containing the optimal set $\Omega(D, m)$? Note that the symmetric statement (if $m' < m$, then for each optimal set $\Omega(D, m)$, there is an optimal set $\Omega(D, m') \subset \Omega(D, m)$) is false. Indeed, take for instance D the unit square in $\mathbb{R}^2$ centered in 0 and $m' = \frac{\pi}{2} < m$. Then $\Omega(D, m')$ is the ball centered at 0 with radius $\frac{1}{\sqrt{2}}$ and $\Omega(D, m)$ is any ball of radius $\sqrt{\frac{m}{\pi}}$. Clearly, there are balls $\Omega(D, m)$ which do not contain $\Omega(D, m')$.
5. An interesting problem, similar to (4.1), is given by the minimization of the energy integral functional

$$E(\Omega) = -\int w_\Omega(x)\,dx.$$

We can repeat in this case all the arguments above, obtaining similar existence, boundedness and regularity results. In particular, working with the energy functional simplifies the analysis of Proposition 5.6, obtaining that optimal sets are open, even without the quasi-connectedness assumption on D.

One can consider more general spectral optimization problems where the cost is

$$J(\Omega) = \Phi\big(\lambda(\Omega)\big),$$

for a suitable function Φ. If Φ is:

(a) monotone increasing, that is $\Phi(\lambda) \leqslant \Phi(\lambda')$ whenever $\lambda_k \leqslant \lambda'_k$, for every k,
(b) lower semicontinuous, that is $\Phi(\lambda) \leqslant \liminf_{n\to\infty} \Phi(\lambda^n)$, whenever $\lambda_k^n \to \lambda_k$ for every k,

then the optimization problem with an external bounded constraint has a solution thanks to [11]. The general case for the internal constraint problem remains, on the contrary, open.

Finally, one can consider shape optimization problems with cost functional of integral form. Given a right-hand side f we consider the PDE

$$-\Delta u = f \quad \text{in } \Omega,\ u \in H_0^1(\Omega),$$

which provides, for every admissible domain Ω, a unique solution u_Ω that we assume extended by zero outside of Ω. The cost is in this case of the form

$$J(\Omega) = \int_{\mathbb{R}^d} j\big(x, u_\Omega(x)\big)\,dx.$$

If $j(x, \cdot)$ is lower semicontinuous, decreasing and such that $j(x, s) \geqslant a(x) - c|s|^2$ for suitable $a \in L^1(\mathbb{R}^d)$ and $c > 0$, then the optimization problem with an external bounded constraint has again a solution thanks to [11] but the general case for the internal constraint problem remains open.

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