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On an incompressible Navier–Stokes/Cahn–Hilliard system with degenerate mobility

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Abstract

We prove existence of weak solutions for a diffuse interface model for the flow of two viscous incompressible Newtonian fluids in a bounded domain by allowing for a degenerate mobility. The model has been developed by Abels, Garcke and Grün for fluids with different densities and leads to a solenoidal velocity field. It is given by a non-homogeneous Navier–Stokes system with a modified convective term coupled to a Cahn–Hilliard system, such that an energy estimate is fulfilled which follows from the fact that the model is thermodynamically consistent.

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1. Introduction

Classically the interface between two immiscible, viscous fluids has been modelled in the context of sharp interface approaches, see e.g. [21]. But in the context of sharp interface models it is difficult to describe topological changes, as e.g. pinch off and situations where different interfaces or different parts of an interface connect. In the last 20 years phase field approaches have been a promising new approach to model interfacial evolution in situations where interfacial energy effects are important, see e.g. [10]. In phase field approaches a phase field or order parameter is introduced which rapidly changes its value in the interfacial region and attains two prescribed values away from the interface.

For two-phase flow of immiscible, viscous fluids a phase-field approach first has been introduced by Hohenberg and Halperin [18], the so-called "Model H". In their work the Cahn–Hilliard equation was coupled to the Navier–Stokes system in such a way that capillary forces on the interface are modelled with the help of the phase field. The approach of Hohenberg and Halperin [18] was restricted to the case where the densities of the two fluids are the same

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or at least are very close ("matched densities"). It has been later shown by Gurtin, Polignone, Viñals [15] that the model can be derived in the context of rational thermodynamics. In particular global and local energy inequalities are true. These global energy estimates can be used to derive a priori estimates and this has been used by Boyer [7] and by Abels [2] for proofs of existence results.

Often the densities in two-phase flow are quite different. Therefore, there have been several attempts to derive phase field models for two-phase flow with non-matched densities. Lowengrub and Truskinovsky [20] derived a first thermodynamically consistent phase field model for the case of different densities. The model of Lowengrub and Truskinovsky is based on a barycentric velocity and hence the overall velocity field turns out to be not divergence free in general. In addition, the pressure enters the Cahn–Hilliard equation and as a result the coupling between the Cahn–Hilliard equation and the Navier–Stokes equations is quite strong. This and the fact that the velocity field is not divergence free make numerical and analytical approaches quite difficult. To the authors knowledge there have been so far no numerical simulations for the full Lowengrub–Truskinovsky model. With respect to analytical results we refer to the works of Abels [1,3] for existence results.

In a paper by Ding, Spelt and Shu [12] a generalization of Model H for non-matched densities and a divergence free velocity field has been derived. However it is not known whether this model is thermodynamically consistent. A first phase field model for non-matched densities and a divergence free velocity field which in addition fulfills local and hence global free energy inequalities has been derived by Abels, Garcke and Grün [5]. The model in [5] is given by the following system of Navier–Stokes/Cahn–Hilliard equations:

$$\begin{aligned}
&\partial_t\big(\rho(\varphi)\mathbf{v}\big)+\operatorname{div}\big(\mathbf{v}\otimes\big(\rho(\varphi)\mathbf{v}+\widetilde{\mathbf{J}}\big)\big)-\operatorname{div}\big(2\eta(\varphi)D\mathbf{v}\big)+\nabla p=-\operatorname{div}\big(a(\varphi)\nabla\varphi\otimes\nabla\varphi\big) && \text{in } Q_T,\\
&\operatorname{div}\mathbf{v}=0 && \text{in } Q_T,\\
&\partial_t\varphi+\mathbf{v}\cdot\nabla\varphi=\operatorname{div}\big(m(\varphi)\nabla\mu\big) && \text{in } Q_T,\\
&\mu=\Psi'(\varphi)+a'(\varphi)\frac{|\nabla\varphi|^2}{2}-\operatorname{div}\big(a(\varphi)\nabla\varphi\big) && \text{in } Q_T,
\end{aligned}$$

where $\widetilde{\mathbf{J}}=-\frac{\tilde\rho_2-\tilde\rho_1}{2}m(\varphi)\nabla\mu$, $Q_T=\Omega\times(0,T)$ for $0<T<\infty$, and $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a sufficiently smooth bounded domain. We close the system with the boundary and initial conditions

$$\begin{aligned}
&\mathbf{v}|_{\partial\Omega}=0 && \text{on } \partial\Omega\times(0,T),\\
&\partial_n\varphi|_{\partial\Omega}=\partial_n\mu|_{\partial\Omega}=0 && \text{on } \partial\Omega\times(0,T),\\
&(\mathbf{v},\varphi)|_{t=0}=(\mathbf{v}_0,\varphi_0) && \text{in } \Omega,
\end{aligned}$$

where $\partial_n\varphi=n\cdot\nabla\varphi$ and n denotes the exterior normal at $\partial\Omega$. Here $\mathbf{v}$ is the volume averaged velocity, $\rho=\rho(\varphi)$ is the density of the mixture of the two fluids, φ is the difference of the volume fractions of the two fluids and we assume a constitutive relation between ρ and the order parameter φ given by $\rho(\varphi)=\frac{1}{2}(\tilde\rho_1+\tilde\rho_2)+\frac{1}{2}(\tilde\rho_2-\tilde\rho_1)\varphi$, see [4] for details. In addition, p is the pressure, μ is the chemical potential associated to φ and $\tilde\rho_1$, $\tilde\rho_2$ are the specific constant mass densities of the unmixed fluids. Moreover, $D\mathbf{v}=\frac{1}{2}(\nabla\mathbf{v}+\nabla\mathbf{v}^T)$, $\eta(\varphi)>0$ is a viscosity coefficient, and $m(\varphi)\geqslant 0$ is a degenerate mobility coefficient. Furthermore, $\Psi(\varphi)$ is the homogeneous free energy density for the mixture and the (total) free energy of the system is given by

$$E_{\text{free}}(\varphi)=\int_\Omega\left(\Psi(\varphi)+a(\varphi)\frac{|\nabla\varphi|^2}{2}\right)dx$$

for some positive coefficient $a(\varphi)$. The kinetic energy is given by $E_{\text{kin}}(\varphi,\mathbf{v})=\int_\Omega\rho(\varphi)\frac{|\mathbf{v}|^2}{2}\,dx$ and the total energy as the sum of the kinetic and free energy is

$$\begin{aligned}
E_{\text{tot}}(\varphi,\mathbf{v})&=E_{\text{kin}}(\varphi,\mathbf{v})+E_{\text{free}}(\varphi)\\
&=\int_\Omega\rho(\varphi)\frac{|\mathbf{v}|^2}{2}\,dx+\int_\Omega\left(\Psi(\varphi)+a(\varphi)\frac{|\nabla\varphi|^2}{2}\right)dx.
\end{aligned}\tag{1.1}$$

In addition there have been further modelling attempts for two-phase flow with different densities. We refer to Boyer [8] and the recent work of Aki et al. [6]. We remark that for the model of Boyer no energy inequalities are known and the model of Aki et al. does not lead to velocity fields which are divergence free.

In [4] an existence result for the above Navier–Stokes/Cahn–Hilliard model has been shown in the case of a non-degenerate mobility $m(\varphi)$. As is discussed in [5] the case with non-degenerate mobility can lead to Ostwald ripening effects, i.e., in particular larger drops can grow to the expense of smaller ones. In many applications this is not reasonable and as pointed out in [5] degenerate mobilities avoid Ostwald ripening and hence the case of degenerate mobilities is very important in applications. In what follows we assume that $m(\varphi) = 1 - \varphi^2$ for $|\varphi| \leqslant 1$ and extend this by zero to all of $\mathbb{R}$. In this way we do not allow for diffusion through the bulk, i.e., the region where $\varphi = 1$ resp. $\varphi = -1$, but only in the interfacial region, where $|\varphi| < 1$. The degenerate mobility leads to the physically reasonable bound $|\varphi| \leqslant 1$ for the order parameter φ, which is the difference of volume fractions and therefore we can consider in this work a smooth homogeneous free energy density Ψ in contrast to the previous work [4].

For the Cahn–Hilliard equations without the coupling to the Navier–Stokes equations Elliott and Garcke [13] considered the case of a degenerate mobility, see also Grün [14]. We will use a suitable testing procedure from the work [13] to get a bound for the second derivatives of a function of φ in the energy estimates of Lemma 3.7. We point out that our result is also new for the case of Model H with degenerate mobility, i.e., $\tilde{\rho}_1 = \tilde{\rho}_2$, which implies $\widetilde{\mathbf{J}} = 0$ in the above Navier–Stokes/Cahn–Hilliard system.

The structure of the article is as follows: In Section 2 we summarize some notation and preliminary results. Then, in Section 3, we reformulate the Navier–Stokes/Cahn–Hilliard system suitably, define weak solutions and state our main result on existence of weak solutions. For the proof of the existence theorem in Sections 3.2 and 3.3 we approximate the equations by a problem with positive mobility m_ε and singular homogeneous free energy density Ψ_ε. For the solution $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mathbf{J}_\varepsilon)$ of the approximation (with $\mathbf{J}_\varepsilon = -m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon$) we derive suitable energy estimates to get weak limits. Then we extend the weak convergences to strong ones by using methods similar to the previous work of the authors [4], careful estimates of the additional singular free energy density and by an additional subtle argument with the help of time differences and a theorem of Simon [23]. We remark that this last point would be easier in the case of a constant coefficient $a(\varphi)$ in the free energy. Finally we can pass to the limit $\varepsilon \to 0$ in the equations for the weak solutions $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mathbf{J}_\varepsilon)$ and recover the identities for the weak solution of the main problem.

2. Preliminaries and notation

We denote $a \otimes b = (a_i b_j)_{i,j=1}^d$ for $a, b \in \mathbb{R}^d$ and $A_{\mathrm{sym}} = \frac{1}{2}(A + A^T)$ for a matrix $A \in \mathbb{R}^{d\times d}$. If X is a Banach space and X' is its dual, then

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g), \quad f \in X',\ g \in X,$$

denotes the duality product. We write $X \hookrightarrow\hookrightarrow Y$ if X is compactly embedded into Y. Moreover, if H is a Hilbert space, $(\cdot,\cdot)_H$ denotes its inner product. Moreover, we use the abbreviation $(.,.)_M = (.,.)_{L^2(M)}$.

Function spaces. If $M \subseteq \mathbb{R}^d$ is measurable, $L^q(M)$, $1 \leqslant q \leqslant \infty$, denotes the usual Lebesgue space and $\|.\|_q$ its norm. Moreover, $L^q(M; X)$ denotes the set of all strongly measurable q-integrable functions if $q \in [1, \infty)$ and essentially bounded strongly measurable functions, if $q = \infty$, where X is a Banach space.

Recall that, if X is a Banach space with the Radon–Nikodym property, then

$$L^q(M; X)' = L^{q'}\big(M; X'\big) \quad \text{for every } 1 \leqslant q < \infty$$

by means of the duality product $\langle f, g\rangle = \int_M \langle f(x), g(x)\rangle_{X',X}\,dx$ for $f \in L^{q'}(M; X')$, $g \in L^q(M; X)$. If X is reflexive or X' is separable, then X has the Radon–Nikodym property, cf. Diestel and Uhl [11].

Moreover, we recall the lemma of Aubin–Lions: If $X_0 \hookrightarrow\hookrightarrow X_1 \hookrightarrow X_2$ are Banach spaces, $1 < p < \infty$, $1 \leqslant q < \infty$, and $I \subset \mathbb{R}$ is a bounded interval, then

$$\left\{v \in L^p(I; X_0)\colon \frac{dv}{dt} \in L^q(I; X_2)\right\} \hookrightarrow\hookrightarrow L^p(I; X_1). \tag{2.1}$$

See J.-L. Lions [19] for the case $q > 1$ and Simon [23] or Roubíček [22] for $q = 1$.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Then $W_q^k(\Omega)$, $k \in \mathbb{N}_0$, $1 \leqslant q \leqslant \infty$, denotes the usual L^q-Sobolev space, $W_{q,0}^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_q^k(\Omega)$, $W_q^{-k}(\Omega) = (W_{q',0}^k(\Omega))'$, and $W_{q,0}^{-k}(\Omega) = (W_{q'}^k(\Omega))'$. We also use the abbreviation $H^k(\Omega) = W_2^k(\Omega)$.

Given $f \in L^1(\Omega)$, we denote by $f_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$ its mean value. Moreover, for $m \in \mathbb{R}$ we set

$$L^q_{(m)}(\Omega) := \{ f \in L^q(\Omega)\colon\ f_\Omega = m \}, \quad 1 \leqslant q \leqslant \infty.$$

Then for $f \in L^2(\Omega)$ we observe that

$$P_0 f := f - f_\Omega = f - \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$$

is the orthogonal projection onto $L^2_{(0)}(\Omega)$. Furthermore, we define

$$H^1_{(0)} \equiv H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \qquad (c,d)_{H^1_{(0)}(\Omega)} := (\nabla c, \nabla d)_{L^2(\Omega)}.$$

Then $H^1_{(0)}(\Omega)$ is a Hilbert space due to Poincaré's inequality.

Spaces of solenoidal vector-fields. For a bounded domain $\Omega \subset \mathbb{R}^d$ we denote by $C^\infty_{0,\sigma}(\Omega)$ in the following the space of all divergence free vector fields in $C^\infty_0(\Omega)^d$ and $L^2_\sigma(\Omega)$ is its closure in the L^2-norm. The corresponding Helmholtz projection is denoted by P_σ, cf. e.g. Sohr [24]. We note that $P_\sigma f = f - \nabla p$, where $p \in W^1_2(\Omega) \cap L^2_{(0)}(\Omega)$ is the solution of the weak Neumann problem

$$(\nabla p, \nabla \varphi)_\Omega = (f, \nabla \varphi) \quad \text{for all } \varphi \in C^\infty(\overline{\Omega}). \tag{2.2}$$

Spaces of continuous vector-fields. In the following let $I = [0,T]$ with $0 < T < \infty$ or let $I = [0,\infty)$ if $T = \infty$ and let X be a Banach space. Then $BC(I;X)$ is the Banach space of all bounded and continuous $f : I \to X$ equipped with the supremum norm and $BUC(I;X)$ is the subspace of all bounded and uniformly continuous functions. Moreover, we define $BC_w(I;X)$ as the topological vector space of all bounded and weakly continuous functions $f : I \to X$. By $C^\infty_0(0,T;X)$ we denote the vector space of all smooth functions $f : (0,T) \to X$ with $\operatorname{supp} f \Subset (0,T)$. We say that $f \in W^1_p(0,T;X)$ for $1 \leqslant p < \infty$, if and only if $f, \frac{df}{dt} \in L^p(0,T;X)$, where $\frac{df}{dt}$ denotes the vector-valued distributional derivative of f. Finally, we note:

Lemma 2.1. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$ and $X' \hookrightarrow Y'$ densely. Then $L^\infty(I;Y) \cap BUC(I;X) \hookrightarrow BC_w(I;Y)$.*

For a proof, see e.g. Abels [1].

3. Existence of weak solutions

In this section we prove an existence result for the Navier–Stokes/Cahn–Hilliard system from the introduction for a situation with degenerate mobility. Since in this case we will not have a control of the gradient of the chemical potential, we reformulate the equations by introducing a flux $\mathbf{J} = -m(\varphi)\nabla\mu$ consisting of the product of the mobility and the gradient of the chemical potential. In this way, the complete system is given by:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\big(2\eta(\varphi)D\mathbf{v}\big) + \nabla p + \operatorname{div}(\mathbf{v}\otimes\beta\mathbf{J}) = -\operatorname{div}\big(a(\varphi)\nabla\varphi\otimes\nabla\varphi\big) \quad \text{in } Q_T, \tag{3.1a}$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \tag{3.1b}$$

$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = -\operatorname{div}\mathbf{J} \quad \text{in } Q_T, \tag{3.1c}$$

$$\mathbf{J} = -m(\varphi)\nabla\bigg(\Psi'(\varphi) + a'(\varphi)\frac{|\nabla\varphi|^2}{2} - \operatorname{div}\big(a(\varphi)\nabla\varphi\big)\bigg) \quad \text{in } Q_T, \tag{3.1d}$$

$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } S_T, \tag{3.1e}$$

$$\partial_n\varphi|_{\partial\Omega} = (\mathbf{J}\cdot n)|_{\partial\Omega} = 0 \quad \text{on } S_T, \tag{3.1f}$$

$$(\mathbf{v},\varphi)|_{t=0} = (\mathbf{v}_0,\varphi_0) \quad \text{in } \Omega, \tag{3.1g}$$

where we set $\beta = \frac{\tilde\rho_2 - \tilde\rho_1}{2}$ and $\mathbf{J} = -m(\varphi)\nabla\mu$ as indicated above. The constitutive relation between density and phase field is given by $\rho(\varphi) = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\varphi$ as derived in Abels, Garcke and Grün [5], where $\tilde\rho_i > 0$ are the specific constant mass densities of the unmixed fluids and φ is the difference of the volume fractions of the fluids. By introducing $\mathbf{J}$, we omitted the chemical potential μ in our equations and we search from now on for unknowns $(\mathbf{v}, \varphi, \boldsymbol{J})$. In the above formulation and in the following, we use the abbreviations for space–time cylinders $Q_{(s,t)} = \Omega \times (s,t)$ and $Q_t = Q_{(0,t)}$ and analogously for the boundary $S_{(s,t)} = \partial\Omega \times (s,t)$ and $S_t = S_{(0,t)}$. Eq. (3.1e) is the no-slip boundary condition for viscous fluids, $(\mathbf{J}\cdot n)|_{\partial\Omega} = 0$ resulting from $\partial_n\mu|_{\partial\Omega} = 0$ means that there is no mass flux of the components through the boundary, and $\partial_n\varphi|_{\partial\Omega} = 0$ describes a contact angle of $\pi/2$ of the diffused interface and the boundary of the domain.

3.1. Assumptions and existence theorem for weak solutions

In the following we summarize the assumptions needed to formulate the notion of a weak solution of (3.1a)–(3.1g) and an existence result.

Assumption 3.1. We assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with smooth boundary and additionally we impose the following conditions.

(i) We assume $a, \Psi \in C^1(\mathbb{R})$, $\eta \in C^0(\mathbb{R})$ and $0 < c_0 \leqslant a(s), \eta(s) \leqslant K$ for given constants $c_0, K > 0$.
(ii) For the mobility m we assume that

$$m(s) = \begin{cases} 1 - s^2, & \text{if } |s| \leqslant 1, \\ 0, & \text{else.} \end{cases} \tag{3.2}$$

We remark that other mobilities which degenerate linearly at $s = \pm 1$ are possible. The choice (3.2) typically appears in applications, see Cahn and Taylor [9] and Hilliard [17]. Other degeneracies can be handled as well but some would need additional assumptions, see Elliott and Garcke [13].

We reformulate the model suitably due to the positive coefficient $a(\varphi)$ in the free energy, so that we can replace the two terms with $a(\varphi)$ in Eq. (3.1d) by a single one. To this end, we introduce the function $A(s) := \int_0^s \sqrt{a(\tau)}\,d\tau$. Then $A'(s) = \sqrt{a(s)}$ and

$$-\sqrt{a(\varphi)}\Delta A(\varphi) = a'(\varphi)\frac{|\nabla\varphi|^2}{2} - \operatorname{div}\bigl(a(\varphi)\nabla\varphi\bigr)$$

resulting from a straightforward calculation. By reparametrizing the potential Ψ through $\widetilde\Psi : \mathbb{R} \to \mathbb{R}$, $\widetilde\Psi(r) := \Psi(A^{-1}(r))$ we see $\Psi'(s) = \sqrt{a(s)}\widetilde\Psi'(A(s))$ and therefore we can replace line (3.1d) with the following one:

$$\mathbf{J} = -m(\varphi)\nabla\Bigl(\sqrt{a(\varphi)}\bigl(\widetilde\Psi'\bigl(A(\varphi)\bigr) - \Delta A(\varphi)\bigr)\Bigr). \tag{3.3}$$

We also rewrite the free energy with the help of A to

$$E_{\text{free}}(\varphi) = \int_\Omega \left(\widetilde\Psi\bigl(A(\varphi)\bigr) + \frac{|\nabla A(\varphi)|^2}{2}\right) dx.$$

Remark 3.2. With the above notation and with the calculation

$$\begin{aligned} -\operatorname{div}\bigl(a(\varphi)\nabla\varphi\otimes\nabla\varphi\bigr) &= -\operatorname{div}\bigl(a(\varphi)\nabla\varphi\bigr)\nabla\varphi - a(\varphi)\nabla\left(\frac{|\nabla\varphi|^2}{2}\right) \\ &= -\operatorname{div}\bigl(a(\varphi)\nabla\varphi\bigr)\nabla\varphi + \nabla\bigl(a(\varphi)\bigr)\frac{|\nabla\varphi|^2}{2} - \nabla\left(a(\varphi)\frac{|\nabla\varphi|^2}{2}\right) \\ &= \left(-\operatorname{div}\bigl(a(\varphi)\nabla\varphi\bigr) + a'(\varphi)\frac{|\nabla\varphi|^2}{2}\right)\nabla\varphi - \nabla\left(a(\varphi)\frac{|\nabla\varphi|^2}{2}\right) \\ &= -\sqrt{a(\varphi)}\Delta A(\varphi)\nabla\varphi - \nabla\left(a(\varphi)\frac{|\nabla\varphi|^2}{2}\right) \end{aligned}$$

we rewrite line (3.1a) with a new pressure $g = p + a(\varphi)\frac{|\nabla\varphi|^2}{2}$ into:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\big(2\eta(\varphi)D\mathbf{v}\big) + \nabla g + \operatorname{div}(\mathbf{v}\otimes\beta\mathbf{J}) = -\sqrt{a(\varphi)}\Delta A(\varphi)\nabla\varphi. \tag{3.4}$$

We remark that in contrast to the formulation in [4] we do not use the equation for the chemical potential here.

Now we can define a weak solution of problem (3.1a)–(3.1g).

Definition 3.3. Let $T\in(0,\infty)$, $\mathbf{v}_0\in L^2_\sigma(\Omega)$ and $\varphi_0\in H^1(\Omega)$ with $|\varphi_0|\leqslant 1$ almost everywhere in Ω. If in addition Assumption 3.1 holds, we call the triple $(\mathbf{v},\varphi,\mathbf{J})$ with the properties

$$\begin{aligned}
&\mathbf{v}\in BC_w\big([0,T];L^2_\sigma(\Omega)\big)\cap L^2\big(0,T;H^1_0(\Omega)^d\big),\\
&\varphi\in BC_w\big([0,T];H^1(\Omega)\big)\cap L^2\big(0,T;H^2(\Omega)\big)\quad\text{with } |\varphi|\leqslant 1 \text{ a.e. in } Q_T,\\
&\mathbf{J}\in L^2(Q_T)^d\quad\text{and}\\
&(\mathbf{v},\varphi)|_{t=0}=(\mathbf{v}_0,\varphi_0)
\end{aligned}$$

a weak solution of (3.1a)–(3.1g) if the following conditions are satisfied:

$$\begin{aligned}
&-(\rho\mathbf{v},\partial_t\boldsymbol{\psi})_{Q_T} + \big(\operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}),\boldsymbol{\psi}\big)_{Q_T} + \big(2\eta(\varphi)D\mathbf{v},D\boldsymbol{\psi}\big)_{Q_T}\\
&\quad-\big((\mathbf{v}\otimes\beta\mathbf{J}),\nabla\boldsymbol{\psi}\big)_{Q_T} = -\big(\sqrt{a(\varphi)}\Delta A(\varphi)\nabla\varphi,\boldsymbol{\psi}\big)_{Q_T}
\end{aligned}\tag{3.5}$$

for all $\boldsymbol{\psi}\in C_0^\infty(Q_T)^d$ with $\operatorname{div}\boldsymbol{\psi}=0$,

$$-\int_{Q_T}\varphi\partial_t\zeta\,dx\,dt + \int_{Q_T}(\mathbf{v}\cdot\nabla\varphi)\zeta\,dx\,dt = \int_{Q_T}\mathbf{J}\cdot\nabla\zeta\,dx\,dt \tag{3.6}$$

for all $\zeta\in C_0^\infty((0,T);C^1(\overline{\Omega}))$ and

$$\int_{Q_T}\mathbf{J}\cdot\boldsymbol{\eta}\,dx\,dt = -\int_{Q_T}\big(\sqrt{a(\varphi)}\big(\widetilde{\Psi}'\big(A(\varphi)\big)-\Delta A(\varphi)\big)\big)\operatorname{div}\big(m(\varphi)\boldsymbol{\eta}\big)\,dx\,dt \tag{3.7}$$

for all $\boldsymbol{\eta}\in L^2(0,T;H^1(\Omega)^d)\cap L^\infty(Q_T)^d$ which fulfill $\boldsymbol{\eta}\cdot n=0$ on S_T.

Remark 3.4. The identity (3.7) is a weak version of

$$\mathbf{J} = -m(\varphi)\nabla\big(\sqrt{a(\varphi)}\big(\widetilde{\Psi}'\big(A(\varphi)\big)-\Delta A(\varphi)\big)\big).$$

Our main result of this work is the following existence theorem for weak solutions on an arbitrary time interval $[0,T]$, where $T>0$.

Theorem 3.5. *Let Assumption 3.1 hold, $\mathbf{v}_0\in L^2_\sigma(\Omega)$ and $\varphi_0\in H^1(\Omega)$ with $|\varphi_0|\leqslant 1$ almost everywhere in Ω. Then there exists a weak solution $(\mathbf{v},\varphi,\mathbf{J})$ of (3.1a)–(3.1g) in the sense of Definition 3.3. Moreover for some $\widehat{\mathbf{J}}\in L^2(Q_T)$ it holds that $\mathbf{J}=\sqrt{m(\varphi)}\,\widehat{\mathbf{J}}$ and*

$$E_{\mathrm{tot}}\big(\varphi(t),\mathbf{v}(t)\big) + \int_{Q_{(s,t)}}2\eta(\varphi)|D\mathbf{v}|^2\,dx\,d\tau + \int_{Q_{(s,t)}}|\widehat{\mathbf{J}}|^2\,dx\,d\tau \leqslant E_{\mathrm{tot}}\big(\varphi(s),\mathbf{v}(s)\big) \tag{3.8}$$

for all $t\in[s,T)$ and almost all $s\in[0,T)$ including $s=0$. The total energy E_{tot} is the sum of the kinetic and the free energy, cf. (1.1). In particular, $\mathbf{J}=0$ a.e. on the set $\{|\varphi|=1\}$.

The proof of the theorem will be done in the next two subsections. But first of all we consider a special case which can then be excluded in the following proof. Due to $|\varphi_0|\leqslant 1$ a.e. in Ω we note that $\fint_\Omega\varphi_0\,dx\in[-1,1]$. In the situation where $\fint_\Omega\varphi_0\,dx=1$ we can then conclude that $\varphi_0\equiv 1$ a.e. in Ω and can give the solution at once. In fact, here we set $\varphi \equiv 1$, $\mathbf{J} \equiv 0$ and let $\mathbf{v}$ be the weak solution of the incompressible Navier–Stokes equations without coupling to the Cahn–Hilliard equation, where ρ and η are constants. The situation where $\fint_\Omega \varphi_0\,dx = -1$ can be handled analogously.

With this observation we can assume in the following that

$$\fint_\Omega \varphi_0\,dx \in (-1,1),$$

which will be needed for the reference to the previous existence result of the authors [4] and for the proof of Lemma 3.7, (iii).

3.2. Approximation and energy estimates

In the following we substitute problem (3.1a)–(3.1g) by an approximation with positive mobility and a singular homogeneous free energy density, which can be solved with the result from the authors in [4]. For the weak solutions of the approximation we then derive energy estimates.

First we approximate the degenerate mobility m by a strictly positive m_ε as

$$m_\varepsilon(s) := \begin{cases} m(-1+\varepsilon) & \text{for } s \leqslant -1+\varepsilon, \\ m(s) & \text{for } |s| < 1-\varepsilon, \\ m(1-\varepsilon) & \text{for } s \geqslant 1-\varepsilon. \end{cases}$$

In addition we use a singular homogeneous free energy density Ψ_ε given by

$$\Psi_\varepsilon(s) := \Psi(s) + \varepsilon\Psi_{\ln}(s), \quad \text{where}$$
$$\Psi_{\ln}(s) := (1+s)\ln(1+s) + (1-s)\ln(1-s).$$

Then $\Psi_\varepsilon \in C([-1,1]) \cap C^2((-1,1))$ fulfills the assumptions on the homogeneous free energy as in Abels, Depner and Garcke [4], which were given by

$$\lim_{s\to\pm1} \Psi_\varepsilon'(s) = \pm\infty, \qquad \Psi_\varepsilon''(s) \geqslant \kappa \quad \text{for some } \kappa \in \mathbb{R} \quad \text{and} \quad \lim_{s\to\pm1} \frac{\Psi_\varepsilon''(s)}{\Psi_\varepsilon'(s)} = +\infty.$$

To deal with the positive coefficient $a(\varphi)$, we set similarly as above $\widetilde{\Psi}_{\ln}(r) := \Psi_{\ln}(A^{-1}(r))$ and $\widetilde{\Psi}_\varepsilon(r) := \Psi_\varepsilon(A^{-1}(r))$ for $r \in [a,b] := A([-1,1])$.

Now we replace m by m_ε and Ψ by Ψ_ε and consider the following approximate problem, this time for unknowns $(\mathbf{v},\varphi,\mu)$:

$$\partial_t(\rho\mathbf{v}) + \operatorname{div}(\rho\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\big(2\eta(\varphi)D\mathbf{v}\big) + \nabla g + \operatorname{div}\big(\mathbf{v}\otimes\beta m_\varepsilon(\varphi)\nabla\mu\big) = -\sqrt{a(\varphi)}\Delta A(\varphi)\nabla\varphi \quad \text{in } Q_T, \tag{3.9a}$$
$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \tag{3.9b}$$
$$\partial_t\varphi + \mathbf{v}\cdot\nabla\varphi = \operatorname{div}\big(m_\varepsilon(\varphi)\nabla\mu\big) \quad \text{in } Q_T, \tag{3.9c}$$
$$\mu = \sqrt{a(\varphi)}\big(\widetilde{\Psi}_\varepsilon'\big(A(\varphi)\big) - \Delta A(\varphi)\big) \quad \text{in } Q_T, \tag{3.9d}$$
$$\mathbf{v}|_{\partial\Omega} = 0 \quad \text{on } S_T, \tag{3.9e}$$
$$\partial_n\varphi|_{\partial\Omega} = \partial_n\mu|_{\partial\Omega} = 0 \quad \text{on } S_T, \tag{3.9f}$$
$$(\mathbf{v},\varphi)|_{t=0} = (\mathbf{v}_0,\varphi_0) \quad \text{in } \Omega. \tag{3.9g}$$

From [4] we get the existence of a weak solution $(\mathbf{v}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon)$ with the properties

$$\mathbf{v}_\varepsilon \in BC_w\big([0,T];L^2_\sigma(\Omega)\big) \cap L^2\big(0,T;H^1_0(\Omega)^d\big),$$
$$\varphi_\varepsilon \in BC_w\big([0,T];H^1(\Omega)\big) \cap L^2\big(0,T;H^2(\Omega)\big), \qquad \Psi_\varepsilon'(\varphi_\varepsilon) \in L^2(Q_T),$$
$$\mu_\varepsilon \in L^2\big(0,T;H^1(\Omega)\big) \quad \text{and}$$
$$(\mathbf{v}_\varepsilon,\varphi_\varepsilon)|_{t=0} = (\mathbf{v}_0,\varphi_0)$$

in the following sense:

$$-(\rho_\varepsilon \mathbf{v}_\varepsilon, \partial_t \boldsymbol{\psi})_{Q_T} + \big(\operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{v}_\varepsilon), \boldsymbol{\psi}\big)_{Q_T} + \big(2\eta(\varphi_\varepsilon) D\mathbf{v}_\varepsilon, D\boldsymbol{\psi}\big)_{Q_T} - \big(\big(\mathbf{v}_\varepsilon \otimes \beta m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon\big), \nabla\boldsymbol{\psi}\big)_{Q_T} = (\mu_\varepsilon \nabla\varphi_\varepsilon, \boldsymbol{\psi})_{Q_T} \tag{3.10}$$

for all $\boldsymbol{\psi} \in C_0^\infty(Q_T)^d$ with $\operatorname{div} \boldsymbol{\psi} = 0$,

$$-(\varphi_\varepsilon, \partial_t \zeta)_{Q_T} + (\mathbf{v}_\varepsilon \cdot \nabla\varphi_\varepsilon, \zeta)_{Q_T} = -\big(m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon, \nabla\zeta\big)_{Q_T} \tag{3.11}$$

for all $\zeta \in C_0^\infty((0,T); C^1(\overline{\Omega}))$ and

$$\mu_\varepsilon = \sqrt{a(\varphi_\varepsilon)}\big(\widetilde{\Psi}_\varepsilon'\big(A(\varphi_\varepsilon)\big) - \Delta A(\varphi_\varepsilon)\big) \quad \text{almost everywhere in } Q_T. \tag{3.12}$$

Moreover,

$$E_{\mathrm{tot}}\big(\varphi_\varepsilon(t), \mathbf{v}_\varepsilon(t)\big) + \int_{Q_{(s,t)}} 2\eta(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2\,dx\,d\tau + \int_{Q_{(s,t)}} m_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2\,dx\,d\tau \leqslant E_{\mathrm{tot}}\big(\varphi_\varepsilon(s), \mathbf{v}_\varepsilon(s)\big) \tag{3.13}$$

for all $t \in [s,T)$ and almost all $s \in [0,T)$ has to hold (including $s = 0$).

Herein ρ_ε is given as $\rho_\varepsilon = \frac{1}{2}(\tilde\rho_1 + \tilde\rho_2) + \frac{1}{2}(\tilde\rho_2 - \tilde\rho_1)\varphi_\varepsilon$. Note that due to the singular homogeneous potential Ψ_ε we have $|\varphi_\varepsilon| < 1$ almost everywhere.

Remark 3.6. Note that Eq. (3.10) can be rewritten with the help of the identity

$$(\mu_\varepsilon \nabla\varphi_\varepsilon, \boldsymbol{\psi})_{Q_T} = -\big(\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\nabla\varphi_\varepsilon, \boldsymbol{\psi}\big)_{Q_T}.$$

This can be seen by testing (3.12) with $\nabla\varphi_\varepsilon \cdot \boldsymbol{\psi}$ and noting that $\boldsymbol{\psi}$ is divergence free.

For the weak solution $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ we get the following energy estimates:

Lemma 3.7. *For a weak solution* $(\mathbf{v}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ *of problem* (3.9a)–(3.9g) *we have the following energy estimates*:

(i)
$$\sup_{0\leqslant t\leqslant T} \int_\Omega \left(\rho_\varepsilon(t)\frac{|\mathbf{v}_\varepsilon(t)|^2}{2} + \frac{1}{2}\big|\nabla\varphi_\varepsilon(t)\big|^2 + \Psi_\varepsilon\big(\varphi_\varepsilon(t)\big)\right)dx + \int_{Q_T} 2\eta(\varphi_\varepsilon)|D\mathbf{v}_\varepsilon|^2\,dx\,dt + \int_{Q_T} m_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2\,dx\,dt \leqslant C,$$

(ii)
$$\sup_{0\leqslant t\leqslant T} \int_\Omega G_\varepsilon\big(\varphi_\varepsilon(t)\big)\,dx + \int_{Q_T} \big|\Delta A(\varphi_\varepsilon)\big|^2\,dx\,dt \leqslant C,$$

(iii)
$$\varepsilon^3 \int_{Q_T} \big|\Psi_{\ln}'(\varphi_\varepsilon)\big|^2\,dx\,dt \leqslant C,$$

(iv)
$$\int_{Q_T} |\widehat{\mathbf{J}}_\varepsilon|^2\,dx\,dt \leqslant C, \quad \text{where } \widehat{\mathbf{J}}_\varepsilon = -\sqrt{m_\varepsilon(\varphi_\varepsilon)}\nabla\mu_\varepsilon.$$

Here G_ε *is a non-negative function defined by* $G_\varepsilon(0) = G_\varepsilon'(0) = 0$ *and* $G_\varepsilon''(s) = \frac{1}{m_\varepsilon(s)}\sqrt{a(s)}$ *for* $s \in [-1,1]$.

Proof. ad (i): This follows directly from the estimate (3.13) derived in the work of Abels, Depner and Garcke [4]. We just note that for the estimate of $\nabla\varphi_\varepsilon$ we use $\nabla A(\varphi_\varepsilon) = \sqrt{a(\varphi_\varepsilon)}\nabla\varphi_\varepsilon$ and the fact that a is bounded from below by a positive constant due to Assumption 3.1.

ad (ii): From line (3.11) we get that $\partial_t\varphi_\varepsilon \in L^2(0,T;(H^1(\Omega))')$, since $\nabla\mu_\varepsilon \in L^2(Q_T)$ and $\mathbf{v}\cdot\nabla\varphi = \operatorname{div}(\mathbf{v}\varphi)$ with $\mathbf{v}\varphi \in L^2(Q_T)$. Then we derive for a function $\zeta \in L^2(0,T;H^2(\Omega))$ the weak formulation

$$\begin{aligned}\int_0^t \langle \partial_t\varphi_\varepsilon, \zeta\rangle\,d\tau + \int_{Q_t} \mathbf{v}_\varepsilon\cdot\nabla\varphi_\varepsilon\zeta\,dx\,d\tau &= -\int_{Q_t} m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon\cdot\nabla\zeta\,dx\,d\tau \\ &= \int_{Q_t} \sqrt{a(\varphi_\varepsilon)}\big(\widetilde{\Psi}'_\varepsilon\big(A(\varphi_\varepsilon)\big) - \Delta A(\varphi_\varepsilon)\big)\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\nabla\zeta\big)\,dx\,d\tau, \end{aligned} \tag{3.14}$$

where we additionally used (3.12) to express μ_ε. Now we set as test function $\zeta = G'_\varepsilon(\varphi_\varepsilon)$, where G_ε is defined by $G_\varepsilon(0) = G'_\varepsilon(0) = 0$ and $G''_\varepsilon(s) = \frac{1}{m_\varepsilon(s)}A'(s)$ for $s\in[-1,1]$. Note that G_ε is a non-negative function, which can be seen from the representation $G_\varepsilon(s) = \int_0^s(\int_0^r \frac{1}{m_\varepsilon(\tau)}A'(\tau)\,d\tau)\,dr$. With $\zeta = G'_\varepsilon(\varphi_\varepsilon)$ it holds that

$$\begin{aligned}&\nabla\zeta = G''_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon = \frac{1}{m_\varepsilon(\varphi_\varepsilon)}\nabla\big(A(\varphi_\varepsilon)\big) \quad \text{and therefore}\\ &\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\nabla\zeta\big) = \Delta\big(A(\varphi_\varepsilon)\big).\end{aligned}$$

Hence we derive

$$\begin{aligned}&\int_0^t \big\langle\partial_t\varphi_\varepsilon, G'_\varepsilon(\varphi_\varepsilon)\big\rangle d\tau + \int_{Q_t}\mathbf{v}_\varepsilon\cdot\nabla\varphi_\varepsilon G'_\varepsilon(\varphi_\varepsilon)\,dx\,d\tau\\ &\quad= \int_{Q_t}\sqrt{a(\varphi_\varepsilon)}\big(\widetilde{\Psi}'_\varepsilon\big(A(\varphi_\varepsilon)\big) - \Delta A(\varphi_\varepsilon)\big)\Delta A(\varphi_\varepsilon)\,dx\,d\tau\\ &\quad= \int_{Q_t}\Psi'_\varepsilon(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau - \int_{Q_t}\sqrt{a(\varphi_\varepsilon)}\big|\Delta A(\varphi_\varepsilon)\big|^2\,dx\,d\tau. \end{aligned}\tag{3.15}$$

With this notation we deduce

$$\begin{aligned}&\int_0^t \big\langle\partial_t\varphi_\varepsilon, G'_\varepsilon(\varphi_\varepsilon)\big\rangle dt = \int_\Omega G_\varepsilon\big(\varphi(t)\big)\,dx - \int_\Omega G_\varepsilon(\varphi_0)\,dx \quad \text{and}\\ &\int_{Q_t}\mathbf{v}_\varepsilon\cdot\nabla\varphi_\varepsilon G'_\varepsilon(\varphi_\varepsilon)\,dx\,dt = \int_{Q_t}\mathbf{v}_\varepsilon\cdot\nabla\big(G_\varepsilon(\varphi_\varepsilon)\big)\,dx\,dt = -\int_{Q_t}\operatorname{div}\mathbf{v}_\varepsilon G_\varepsilon(\varphi_\varepsilon)\,dx\,dt = 0.\end{aligned}$$

For the first term on the right side of (3.15) we observe

$$\begin{aligned}\int_{Q_t}\Psi'_\varepsilon(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau &= \int_{Q_t}\Psi'(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau + \varepsilon\int_{Q_t}\Psi'_{\ln}(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau\\ &\leqslant -\int_{Q_t}\Psi''(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\nabla A(\varphi_\varepsilon)\,dx\,dt\\ &= -\int_{Q_t}\Psi''(\varphi_\varepsilon)\sqrt{a(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2\,dx\,dt.\end{aligned}$$

Herein the estimate

$$\int_{Q_t}\Psi'_{\ln}(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau \leqslant 0$$

for the logarithmic part of the homogeneous free energy density is derived as follows. With an approximation of φ_ε by $\varphi_\varepsilon^\alpha = \alpha\varphi_\varepsilon$ for $0<\alpha<1$ we have that $|\varphi_\varepsilon^\alpha| < \alpha < 1$ and therefore

$$\int_{Q_t} \Psi_{\ln}'(\varphi_\varepsilon^\alpha)\Delta A(\varphi_\varepsilon^\alpha)\,dx\,d\tau = -\int_{Q_t} \Psi_{\ln}''(\varphi_\varepsilon^\alpha)\nabla\varphi_\varepsilon^\alpha \cdot \nabla A(\varphi_\varepsilon^\alpha)\,dx\,d\tau \leqslant 0,$$

where we used integration by parts. To pass to the limit for $\alpha \nearrow 1$ in the left side we observe that $\varphi_\varepsilon^\alpha \to \varphi_\varepsilon$ in $L^2(0,T;H^2(\Omega))$. Hence together with the bound $|\Psi_{\ln}'(\varphi_\varepsilon^\alpha)| \leqslant |\Psi_{\ln}'(\varphi_\varepsilon)|$ we can use Lebesgue's dominated convergence theorem to conclude

$$\int_{Q_t} \Psi_{\ln}'(\varphi_\varepsilon^\alpha)\Delta A(\varphi_\varepsilon^\alpha)\,dx\,d\tau \to \int_{Q_t} \Psi_{\ln}'(\varphi_\varepsilon)\Delta A(\varphi_\varepsilon)\,dx\,d\tau \quad \text{for } \alpha \nearrow 1.$$

With the bound from below $a(s) \geqslant c_0 > 0$ from Assumption 3.1 we derived therefore

$$\int_\Omega G_\varepsilon(\varphi(t))\,dx + \int_{Q_t} |\Delta A(\varphi_\varepsilon)|^2\,dx\,d\tau \leqslant C\left(\int_\Omega G_\varepsilon(\varphi_0)\,dx + \int_{Q_t} \Psi''(\varphi_\varepsilon)\sqrt{a(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2\,dx\,d\tau\right).$$

Now we use $m_\varepsilon(\tau) \geqslant m(\tau)$ to observe the inequality

$$\begin{aligned} G_\varepsilon(s) &= \int_0^s \left(\int_0^r \frac{1}{m_\varepsilon(\tau)} \underbrace{A'(\tau)}_{=\sqrt{a(\tau)}}\,d\tau\right) dr \\ &\leqslant \int_0^s \left(\int_0^r \frac{1}{m(\tau)}\sqrt{a(\tau)}\,d\tau\right) dr =: G(s) \quad \text{for } s \in (-1,1). \end{aligned}$$

Due to the special choice of the degenerate mobility m in (3.2) we conclude that G can be extended continuously to the closed interval $[-1,1]$ and that therefore the integral $\int_\Omega G(\varphi_0)\,dx$ and in particular the integral $\int_\Omega G_\varepsilon(\varphi_0)\,dx$ is bounded.

Moreover, since $\Psi''(s)$ is bounded in $|s| \leqslant 1$ and since we estimated $\int_\Omega |\nabla\varphi_\varepsilon(t)|^2\,dx$ in (i), we proved (ii).

ad (iii): To show this estimate we will argue similarly as in the time-discrete situation of Lemma 4.2 in Abels, Depner and Garcke [4]. We multiply Eq. (3.12) with $P_0\varphi_\varepsilon$, integrate over Ω and get almost everywhere in t the identity

$$\int_\Omega \mu_\varepsilon P_0\varphi_\varepsilon\,dx = \int_\Omega \Psi'(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx + \varepsilon\int_\Omega \Psi_{\ln}'(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx - \int_\Omega \sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx. \tag{3.16}$$

By using in identity (3.11) a test function which depends only on time t and not on $x \in \Omega$, we derive the fact that $(\varphi_\varepsilon)_\Omega = (\varphi_0)_\Omega$ and by assumption this number lies in $(-1+\alpha, 1-\alpha)$ for a small $\alpha > 0$. In addition with the property $\lim_{s\to\pm1}\Psi_{\ln}'(s) = \pm\infty$ we can show the inequality $\Psi_{\ln}'(s)(s-(\varphi_0)_\Omega) \geqslant C_\alpha|\Psi_{\ln}'(s)| - c_\alpha$ in three steps in the intervals $[-1,-1+\frac{\alpha}{2}]$, $[-1+\frac{\alpha}{2}, 1-\frac{\alpha}{2}]$ and $[1-\frac{\alpha}{2}, 1]$ successively. Altogether this leads to the following estimate:

$$\varepsilon\int_\Omega |\Psi_{\ln}'(\varphi_\varepsilon)|\,dx \leqslant C\left(\varepsilon\int_\Omega \Psi_{\ln}'(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx + 1\right). \tag{3.17}$$

We observe the fact that $\int_\Omega \mu_\varepsilon P_0\varphi_\varepsilon\,dx = \int_\Omega (P_0\mu_\varepsilon)\varphi_\varepsilon\,dx$ and due to integration by parts

$$\begin{aligned} -\int_\Omega \sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx &= \int_\Omega \sqrt{a(\varphi_\varepsilon)}\nabla A(\varphi_\varepsilon)\cdot\nabla\varphi_\varepsilon\,dx + \int_\Omega \frac{1}{2}a(\varphi_\varepsilon)^{-\frac{1}{2}}\nabla\varphi_\varepsilon\cdot\nabla A(\varphi_\varepsilon)P_0\varphi_\varepsilon\,dx \\ &= \int_\Omega a(\varphi_\varepsilon)|\nabla\varphi_\varepsilon|^2\,dx + \int_\Omega \frac{1}{2}P_0\varphi_\varepsilon|\nabla\varphi_\varepsilon|^2\,dx. \end{aligned}$$

Combining estimate (3.17) with identity (3.16) we are led to

$$\begin{aligned}\varepsilon\int_\Omega\left|\Psi_{\ln}'(\varphi_\varepsilon)\right|dx &\leqslant C\biggl(\int_\Omega\left|(P_0\mu_\varepsilon)\varphi_\varepsilon\right|dx+\int_\Omega\left|\Psi'(\varphi_\varepsilon)P_0\varphi_\varepsilon\right|dx+\int_\Omega\left|\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)P_0\varphi_\varepsilon\right|dx+1\biggr)\\ &\leqslant C\bigl(\|P_0\mu_\varepsilon\|_{L^2(\Omega)}+\|\nabla\varphi_\varepsilon\|_{L^2(\Omega)}+1\bigr)\\ &\leqslant C\bigl(\|\nabla\mu_\varepsilon\|_{L^2(\Omega)}+1\bigr).\end{aligned}$$

In the last two lines we have used in particular the facts that φ_ε is bounded between -1 and 1, that Ψ' is continuous, the energy estimate from (ii) for $\sup_{0\leqslant t\leqslant T}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega)}$ and the Poincaré inequality for functions with mean value zero.

With the last inequality we can estimate the integral of μ_ε by simply integrating identity (3.12) over Ω:

$$\begin{aligned}\left|\int_\Omega\mu_\varepsilon\,dx\right| &\leqslant\int_\Omega\left|\Psi'(\varphi_\varepsilon)\right|dx+\varepsilon\int_\Omega\left|\Psi_{\ln}'(\varphi_\varepsilon)\right|dx+\left|\int_\Omega\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\,dx\right|\\ &\leqslant C\bigl(\|\nabla\mu_\varepsilon\|_{L^2(\Omega)}+1\bigr),\end{aligned}$$

where we used similarly as above integration by parts for the integral over $\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)$. By the splitting of μ_ε into $\mu_\varepsilon=P_0\mu_\varepsilon+(\mu_\varepsilon)_\Omega$ we arrive at

$$\|\mu_\varepsilon\|^2_{L^2(\Omega)}\leqslant C\bigl(\|\nabla\mu_\varepsilon\|^2_{L^2(\Omega)}+1\bigr).$$

Then, again from identity (3.12), we derive

$$\varepsilon^2\left|\Psi_{\ln}'(\varphi_\varepsilon)\right|^2\leqslant C\bigl(|\mu_\varepsilon|^2+\left|\Delta A(\varphi_\varepsilon)\right|^2+1\bigr)$$

and together with the last estimates and an additional integration over time t this leads to

$$\varepsilon^2\left\|\Psi_{\ln}'(\varphi_\varepsilon)\right\|^2_{L^2(Q_T)}\leqslant C\bigl(\|\nabla\mu_\varepsilon\|^2_{L^2(Q_T)}+1\bigr).$$

Note that we used the bound $\|\Delta A(\varphi_\varepsilon)\|_{L^2(Q_T)}\leqslant C$ from (ii). Furthermore, due to the bounds in (i), we see $\varepsilon\|\nabla\mu_\varepsilon\|^2_{L^2(Q_T)}\leqslant C$ since $m_\varepsilon(s)\geqslant\varepsilon$ for $|s|\leqslant 1$ and therefore we arrive at

$$\varepsilon^3\left\|\Psi_{\ln}'(\varphi_\varepsilon)\right\|^2_{L^2(Q_T)}\leqslant C.$$

ad (iv): This follows directly from (i). □

3.3. *Passing to the limit in the approximation*

In this subsection we use the energy estimates to get weak limits for the sequences $(\mathbf{v}_\varepsilon,\varphi_\varepsilon,\mathbf{J}_\varepsilon)$, where $\mathbf{J}_\varepsilon=\sqrt{m_\varepsilon(\varphi_\varepsilon)}\widehat{\mathbf{J}}_\varepsilon$ $(=-m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon)$. With some subtle arguments we extend the weak convergences to strong ones, so that we are able to pass to the limit for $\varepsilon\to 0$ in Eqs. (3.10)–(3.12) to recover the identities (3.5)–(3.7) in the definition of the weak solution for the main problem (3.1a)–(3.1g).

Using the energy estimates in Lemma 3.7, we can pass to a subsequence to get

$$\begin{aligned}&\mathbf{v}_\varepsilon\rightharpoonup\mathbf{v}\quad\text{in }L^2\bigl(0,T;H^1(\Omega)^d\bigr),\\ &\varphi_\varepsilon\rightharpoonup\varphi\quad\text{in }L^2\bigl(0,T;H^1(\Omega)\bigr),\\ &\widehat{\mathbf{J}}_\varepsilon\rightharpoonup\widehat{\mathbf{J}}\quad\text{in }L^2(Q_T)^d\quad\text{and}\\ &\mathbf{J}_\varepsilon\rightharpoonup\mathbf{J}\quad\text{in }L^2(Q_T)^d\end{aligned}$$

for $\mathbf{v}\in L^2(0,T;H^1(\Omega)^d)\cap L^\infty(0,T;L^2_\sigma(\Omega))$, $\varphi\in L^\infty(0,T;H^1(\Omega))$ and $\widehat{\mathbf{J}},\mathbf{J}\in L^2(Q_T)^d$. Here and in the following all limits are meant to be for suitable subsequences $\varepsilon_k\to 0$ for $k\to\infty$.

With the notation $\mathbf{J}_\varepsilon=-m_\varepsilon(\varphi_\varepsilon)\nabla\mu_\varepsilon$ the weak solution of problem (3.9a)–(3.9g) fulfills the following equations:

$$\begin{aligned}&-(\rho_\varepsilon\mathbf{v}_\varepsilon,\partial_t\boldsymbol{\psi})_{Q_T}+\bigl(\operatorname{div}(\rho_\varepsilon\mathbf{v}_\varepsilon\otimes\mathbf{v}_\varepsilon),\boldsymbol{\psi}\bigr)_{Q_T}+\bigl(2\eta(\varphi_\varepsilon)D\mathbf{v}_\varepsilon,D\boldsymbol{\psi}\bigr)_{Q_T}\\ &\quad-\bigl((\mathbf{v}_\varepsilon\otimes\beta\mathbf{J}_\varepsilon),\nabla\boldsymbol{\psi}\bigr)_{Q_T}=-\bigl(\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\nabla\varphi_\varepsilon,\boldsymbol{\psi}\bigr)_{Q_T}\end{aligned}\tag{3.18}$$

for all $\boldsymbol{\psi} \in C_0^\infty(Q_T)^d$ with $\operatorname{div}\boldsymbol{\psi} = 0$,

$$-\int_{Q_T} \varphi_\varepsilon \partial_t \zeta\, dx\, dt + \int_{Q_T} (\mathbf{v}_\varepsilon \cdot \nabla\varphi_\varepsilon)\zeta\, dx\, dt = \int_{Q_T} \mathbf{J}_\varepsilon \cdot \nabla\zeta\, dx\, dt \tag{3.19}$$

for all $\zeta \in C_0^\infty((0,T); C^1(\overline{\Omega}))$ and

$$\int_{Q_T} \mathbf{J}_\varepsilon \cdot \boldsymbol{\eta}\, dx\, dt = -\int_{Q_T} \big(\Psi_\varepsilon'(\varphi_\varepsilon) - \sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\big) \operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\, dx\, dt \tag{3.20}$$

for all $\boldsymbol{\eta} \in L^2(0,T; H^1(\Omega)^d) \cap L^\infty(Q_T)^d$ with $\boldsymbol{\eta} \cdot n = 0$ on S_T. For the last line we used that for functions $\boldsymbol{\eta}$ with $\boldsymbol{\eta} \cdot n = 0$ on S_T it holds

$$\int_{Q_T} \mathbf{J}_\varepsilon \cdot \boldsymbol{\eta}\, dx\, dt = \int_{Q_T} \nabla\mu_\varepsilon \cdot m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\, dx\, dt = -\int_{Q_T} \mu_\varepsilon \operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\, dx\, dt$$
$$= -\int_{Q_T} \big(\Psi_\varepsilon'(\varphi_\varepsilon) - \sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\big) \operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\, dx\, dt.$$

Now we want to pass to the limit $\varepsilon \to 0$ in the above equations to achieve finally the weak formulation (3.5)–(3.7).

For the convergence in identity (3.18) we first note that

$$\partial_t\varphi_\varepsilon \quad \text{is bounded in } L^2\big(0,T; \big(H^1(\Omega)\big)'\big) \quad \text{and}$$
$$\varphi_\varepsilon \quad \text{is bounded in } L^\infty\big(0,T; H^1(\Omega)\big).$$

Therefore we can deduce from the lemma of Aubin–Lions (2.1) the strong convergence

$$\varphi_\varepsilon \to \varphi \quad \text{in } L^2\big(0,T; L^2(\Omega)\big)$$

and $\varphi_\varepsilon \to \varphi$ pointwise almost everywhere in Q_T.

From the bound of $\Delta A(\varphi_\varepsilon)$ in $L^2(Q_T)$ and from

$$\nabla A(\varphi_\varepsilon) \cdot n = \sqrt{a(\varphi_\varepsilon)}\nabla\varphi_\varepsilon \cdot n = 0 \quad \text{on } S_T,$$

we get from elliptic regularity theory the bound

$$\big\|A(\varphi_\varepsilon)\big\|_{L^2(0,T;H^2(\Omega))} \leqslant C.$$

This yields

$$A(\varphi_\varepsilon) \rightharpoonup g \quad \text{in } L^2\big(0,T; H^2(\Omega)\big)$$

at first for some $g \in L^2(0,T; H^2(\Omega))$, but then, due to the weak convergence $\nabla\varphi_\varepsilon \rightharpoonup \nabla\varphi$ in $L^2(0,T; L^2(\Omega))$ and due to the pointwise almost everywhere convergence $a(\varphi_\varepsilon) \to a(\varphi)$ in Q_T we can identify g with $A(\varphi)$ to get

$$A(\varphi_\varepsilon) \rightharpoonup A(\varphi) \quad \text{in } L^2\big(0,T; H^2(\Omega)\big).$$

The next step is to strengthen the convergence of $\nabla\varphi_\varepsilon$ in $L^2(Q_T)$. To this end, we remark that by definition A is Lipschitz-continuous with

$$\big|A(r) - A(s)\big| \leqslant \left|\int_s^r \sqrt{a(\tau)}\, d\tau\right| \leqslant C|r-s|.$$

Furthermore from the bound of $\partial_t\varphi_\varepsilon$ in $L^2(0,T; (H^1(\Omega))')$ we get with the notation $\varphi_\varepsilon(.+h)$ for a shift in time

$$\big\|\varphi_\varepsilon(.+h) - \varphi_\varepsilon\big\|_{L^2(0,T-h;(H^1(\Omega))')} \leqslant Ch,$$

which leads to the estimate

$$\begin{aligned}\big\|A\big(\varphi_\varepsilon(.+h)\big)-A(\varphi_\varepsilon)\big\|_{L^2(0,T-h;(H^1(\Omega))')} &\leqslant C\big\|\varphi_\varepsilon(.+h)-\varphi_\varepsilon\big\|_{L^2(0,T-h;(H^1(\Omega))')}\\ &\leqslant Ch\to 0 \quad \text{as } h\to 0.\end{aligned}$$

Together with the bound of $A(\varphi_\varepsilon)$ in $L^2(0,T;H^2(\Omega))$ we can use a theorem of Simon [23, Th. 5] to conclude the strong convergence

$$A(\varphi_\varepsilon)\to A(\varphi) \quad \text{in } L^2\big(0,T;H^1(\Omega)\big).$$

From $\nabla A(\varphi_\varepsilon)=\sqrt{a(\varphi_\varepsilon)}\nabla\varphi_\varepsilon$ we get then in particular the strong convergence

$$\nabla\varphi_\varepsilon\to\nabla\varphi \quad \text{in } L^2\big(0,T;L^2(\Omega)\big).$$

In addition we want to use an argument of Abels, Depner and Garcke from [4, Sec. 5.1] which shows that due to the a priori estimate in Lemma 3.7 and the structure of Eq. (3.18) we can deduce the strong convergence $\mathbf{v}_\varepsilon\to\mathbf{v}$ in $L^2(Q_T)^d$. In few words we show with the help of some interpolation inequalities the bound of $\partial_t(P_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon))$ in the space $L^{\frac{8}{7}}(W^1_\infty(\Omega)')$ and together with the bound of $P_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon)$ in $L^2(0,T;H^1(\Omega)^d)$ this is enough to conclude with the lemma of Aubin–Lions the strong convergence

$$P_\sigma(\rho_\varepsilon\mathbf{v}_\varepsilon)\to P_\sigma(\rho\mathbf{v}) \quad \text{in } L^2(Q_T)^d.$$

From this we can derive $\mathbf{v}_\varepsilon\to\mathbf{v}$ in $L^2(Q_T)^d$. For the details we refer to [4, Sec. 5.1 and Appendix].

With the last convergences and the weak convergence $\mathbf{J}_\varepsilon\rightharpoonup\mathbf{J}$ in $L^2(Q_T)$ we can pass to the limit $\varepsilon\to 0$ in line (3.18) to achieve (3.5).

The convergence in line (3.19) follows from the above weak limits of φ_ε and $\mathbf{J}_\varepsilon$ in $L^2(Q_T)$ and the strong ones of $\mathbf{v}_\varepsilon$ and $\nabla\varphi_\varepsilon$ in $L^2(Q_T)$.

Finally, the convergence in line (3.20) can be seen as follows: The left side converges due to the weak convergence of $\mathbf{J}_\varepsilon$ and for the right side we calculate

$$\begin{aligned}&\int_{Q_T}\big(\Psi'_\varepsilon(\varphi_\varepsilon)-\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\big)\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt\\ &\quad=\int_{Q_T}\Psi'(\varphi_\varepsilon)\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt+\varepsilon\int_{Q_T}\Psi'_{\ln}(\varphi_\varepsilon)\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt\\ &\qquad-\int_{Q_T}\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)\operatorname{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt.\end{aligned}\tag{3.21}$$

The first and the third term can be treated similarly as in Elliott and Garcke [13]. For the convenience of the reader we give the details.

First we observe the fact that $m_\varepsilon\to m$ uniformly since for all $s\in\mathbb{R}$ it holds:

$$\big|m_\varepsilon(s)-m(s)\big|\leqslant m(1-\varepsilon)\to 0 \quad \text{for } \varepsilon\to 0.$$

Hence we conclude with the pointwise convergence $\varphi_\varepsilon\to\varphi$ a.e. in Q_T that

$$m_\varepsilon(\varphi_\varepsilon)\to m(\varphi) \quad \text{a.e. in } Q_T.$$

In addition with the convergences $\Psi'(\varphi_\varepsilon)\to\Psi'(\varphi)$, $a(\varphi_\varepsilon)\to a(\varphi)$ a.e. in Q_T and with the weak convergence $\Delta A(\varphi_\varepsilon)\to\Delta A(\varphi)$ in $L^2(Q_T)$ we are led to

$$\int_{Q_T}\Psi'(\varphi_\varepsilon)m_\varepsilon(\varphi_\varepsilon)\operatorname{div}\boldsymbol{\eta}\,dx\,dt\to\int_{Q_T}\Psi'(\varphi)m(\varphi)\operatorname{div}\boldsymbol{\eta}\,dx\,dt \quad \text{and}$$

$$\int_{Q_T}\sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)m_\varepsilon(\varphi_\varepsilon)\operatorname{div}\boldsymbol{\eta}\,dx\,dt\to\int_{Q_T}\sqrt{a(\varphi)}\Delta A(\varphi)m(\varphi)\operatorname{div}\boldsymbol{\eta}\,dx\,dt.$$

The next step is to show that $m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon \to m'(\varphi)\nabla\varphi$ in $L^2(Q_T)$. To this end we split the integral in the following way:

$$\int_{Q_T} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - m'(\varphi)\nabla\varphi\right|^2 dx\,dt$$
$$= \int_{Q_T\cap\{|\varphi|<1\}} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - m'(\varphi)\nabla\varphi\right|^2 dx\,dt + \int_{Q_T\cap\{|\varphi|=1\}} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - m'(\varphi)\nabla\varphi\right|^2 dx\,dt.$$

Since $\nabla\varphi = 0$ a.e. on the set $\{|\varphi| = 1\}$, see for example Gilbarg and Trudinger [16, Lem. 7.7], we obtain

$$\int_{Q_T\cap\{|\varphi|=1\}} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - m'(\varphi)\nabla\varphi\right|^2 dx\,dt = \int_{Q_T\cap\{|\varphi|=1\}} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon\right|^2 dx\,dt$$
$$\leqslant C \int_{Q_T\cap\{|\varphi|=1\}} |\nabla\varphi_\varepsilon|^2\,dx\,dt \to C \int_{Q_T\cap\{|\varphi|=1\}} |\nabla\varphi|^2\,dx\,dt = 0.$$

Although m'_ε is not continuous, we can conclude on the set $\{|\varphi_\varepsilon| < 1\}$ the convergence $m'_\varepsilon(\varphi_\varepsilon) \to m'(\varphi)$ a.e. in Q_T. Indeed, for a point $(x,t) \in Q_T$ with $|\varphi(x,t)| < 1$ and $\varphi_\varepsilon(x,t) \to \varphi(x,t)$, it holds that $|\varphi_\varepsilon(x,t)| < 1-\delta$ for some $\delta > 0$ and ε small enough and in that region m'_ε and m' are continuous. Hence we have

$$m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon \to m'(\varphi)\nabla\varphi \quad \text{a.e. in } Q_T \tag{3.22}$$

and the generalized Lebesgue convergence theorem now gives

$$\int_{Q_T\cap\{|\varphi|<1\}} \left|m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon - m'(\varphi)\nabla\varphi\right|^2 dx\,dt \to 0,$$

which proves finally $m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon \to m'(\varphi)\nabla\varphi$ in $L^2(Q_T)$. Similarly as above, together with the convergences $\Psi'(\varphi_\varepsilon) \to \Psi'(\varphi)$, $a(\varphi_\varepsilon) \to a(\varphi)$ a.e. in Q_T and with the weak convergence $\Delta A(\varphi_\varepsilon) \to \Delta A(\varphi)$ in $L^2(Q_T)$ we are led to

$$\int_{Q_T} \Psi'(\varphi_\varepsilon)m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\boldsymbol{\eta}\,dx\,dt \to \int_{Q_T} \Psi'(\varphi)m'(\varphi)\nabla\varphi\cdot\boldsymbol{\eta}\,dx\,dt \quad \text{and}$$
$$\int_{Q_T} \sqrt{a(\varphi_\varepsilon)}\Delta A(\varphi_\varepsilon)m'_\varepsilon(\varphi_\varepsilon)\nabla\varphi_\varepsilon\cdot\boldsymbol{\eta}\,dx\,dt \to \int_{Q_T} \sqrt{a(\varphi)}\Delta A(\varphi)m'(\varphi)\nabla\varphi\cdot\boldsymbol{\eta}\,dx\,dt.$$

Now we are left to show that the second term of the right side in (3.21) converges to zero. To this end, we split it in the following way:

$$\varepsilon\int_{\Omega_T} \Psi'_{\ln}(\varphi_\varepsilon)\,\mathrm{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt$$
$$= \varepsilon\int_{\{|\varphi_\varepsilon|\leqslant 1-\varepsilon\}} \Psi'_{\ln}(\varphi_\varepsilon)\,\mathrm{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt + \varepsilon\int_{\{|\varphi_\varepsilon|>1-\varepsilon\}} \Psi'_{\ln}(\varphi_\varepsilon)\,\mathrm{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\,dx\,dt$$
$$=: (I)_\varepsilon + (II)_\varepsilon.$$

On the set $\{|\varphi_\varepsilon| \leqslant 1-\varepsilon\}$ we use that $\Psi'_{\ln}(\varphi_\varepsilon) = \ln(1+\varphi_\varepsilon) - \ln(1-\varphi_\varepsilon) + 2$ and therefore $|\Psi'_{\ln}(\varphi_\varepsilon)| \leqslant |\ln\varepsilon| + C$ to deduce that

$$\left|(I)_\varepsilon\right| \leqslant \varepsilon\big(|\ln\varepsilon| + C\big)\int_{Q_T} \left|\mathrm{div}\big(m_\varepsilon(\varphi_\varepsilon)\boldsymbol{\eta}\big)\right|dx\,dt \to 0.$$

On the set $\{|\varphi_\varepsilon| > 1-\varepsilon\}$, we use that $m_\varepsilon(\varphi_\varepsilon) = \varepsilon(2-\varepsilon)$ to deduce

$$\begin{aligned}(II)_\varepsilon &= \varepsilon^2(2-\varepsilon) \int\limits_{\{|\varphi_\varepsilon|>1-\varepsilon\}} \Psi'_{\ln}(\varphi_\varepsilon)\,\operatorname{div}\boldsymbol{\eta}\,dx\,dt \\ &\leqslant C\varepsilon^2 \|\Psi'_{\ln}(\varphi_\varepsilon)\|_{L^2(Q_T)} \\ &= C\sqrt{\varepsilon}\big(\varepsilon^{\frac{3}{2}}\|\Psi'_{\ln}(\varphi_\varepsilon)\|_{L^2(Q_T)}\big) \to 0,\end{aligned}$$

since the last term in brackets is bounded by the energy estimate from Lemma 3.7.

For the relation of $\widehat{\mathbf{J}}$ and $\mathbf{J}$ we note that due to $\widehat{\mathbf{J}}_\varepsilon \rightharpoonup \widehat{\mathbf{J}}$, $\mathbf{J}_\varepsilon \rightharpoonup \mathbf{J}$ in $L^2(Q_T)$, $\mathbf{J}_\varepsilon = \sqrt{m_\varepsilon(\varphi_\varepsilon)}\widehat{\mathbf{J}}_\varepsilon$ and $\sqrt{m_\varepsilon(\varphi_\varepsilon)} \to \sqrt{m(\varphi)}$ a.e. in Q_T from (3.22) we can conclude

$$\mathbf{J} = \sqrt{m(\varphi)}\,\widehat{\mathbf{J}}.$$

From the weak convergence $\widehat{\mathbf{J}}_\varepsilon \rightharpoonup \widehat{\mathbf{J}}$ in $L^2(Q_T)$ we can conclude that

$$\int\limits_{Q_{(s,t)}} |\widehat{\mathbf{J}}|^2\,dx\,d\tau \leqslant \liminf_{\varepsilon\to 0} \int\limits_{Q_{(s,t)}} m_\varepsilon(\varphi_\varepsilon)|\nabla\mu_\varepsilon|^2\,dx\,d\tau$$

for all $0 \leqslant s \leqslant t \leqslant T$ and this is enough to proceed as in Abels, Depner and Garcke [4] to show the energy estimate.

Finally we just remark that the continuity properties and the initial conditions can be derived with the same arguments as in [4, Secs. 5.2, 5.3], so that altogether we proved Theorem 3.5.

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