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*p-adic Clifford algebras*


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P-ADIC CLIFFORD ALGEBRAS

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I - INTRODUCTION

Let $K$ be a field of characteristic $\neq 2$ and $E$ a vector space over $K$ of finite dimension $n$. A mapping $q : E \to K$ is a quadratic form over $E$ if there exists a bilinear symmetric form $f : E \times E \to K$ such that

$$q(x) = f(x, x) \quad \text{and} \quad f(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)]$$

We assume that $q$ is regular, that is $f$ is non-degenerated.

An element $x \in E$ is isotropic if $q(x) = 0$. Let $V$ be a subspace of $E$; the orthogonal subspace of $V$ is the set $V^\perp = \{y \in E/ f(x, y) = 0 \text{ for all } x \in V\}$. The subspace $V$ is called totally isotropic if $V \subset V^\perp$. It is well known (cf. for example [1]) that any totally isotropic subspace is contained in a maximal totally isotropic subspace. The maximal totally isotropic subspaces have the same dimension $\nu$, called the index of $q$ and $2\nu \leq n$. If $2\nu = n$, then $(E, q)$ is called a hyperbolic space and for the case $n = 2$, one says hyperbolic plane. The index $\nu = 0$ iff $q(x) \neq 0$ for $x \neq 0$ i.e. $(E, q)$ is anisotropic.
Let $E = K^n$ and $B = (e_1, \ldots, e_n)$ be the canonical basis of $E$; the standard quadratic form $q_0$ is the quadratic form associated to the bilinear form

$$< x, y > = \sum_{j=1}^{n} x_j y_j ; \quad \text{where} \quad x = \sum_{j=1}^{n} x_j e_j \quad \text{and} \quad y = \sum_{j=1}^{n} y_j e_j ;$$

hence $q_0(x) = < x, x > = \sum_{j=1}^{n} x_j^2$.

Let $(E, q)$ be a quadratic space, possibly non regular; an algebra $C = C(E, q)$ over $K$, with unit 1, is said to be a Clifford algebra for $(E, q)$ if

(i) There exists a one-to-one linear mapping $\rho : E \to C$ such that $\rho(x)^2 = q(x) \cdot 1$.

(ii) For every algebra $A$ with unit 1 and linear mapping $\phi : E \to A$ satisfying $\phi(x)^2 = q(x) \cdot 1$, there exists an algebra homomorphism $\phi : C \to A$ such that $\phi \circ \rho = \phi$.

Clifford algebra exists and is unique up algebra isomorphism (cf. for instance [1] or [3]). For example, let $K < X_1, \ldots, X_n >$ be the free algebra with free system of generators $X_1, \ldots, X_n$ and $I$ be the two-sided ideal of $K < X_1, \ldots, X_n >$ generated by $X_iX_j + X_jX_i - 2f(e_i, e_j) \cdot 1, 1 \leq i, j \leq n$, where $(e_1, \ldots, e_n)$ is an orthogonal basis of $(E, q)$; then $C(E, q) = K < X_1, \ldots, X_n > / I$.

II - THE P-ADIC STANDARD QUADRATIC FORM $q_0$

II - 1. The index of $q_0$

Let $p$ be a prime number and $Q_p$ be the p-adic field i.e. the completion of the field of rational numbers $Q$ for the p-adic absolute value.

We denote by $[\alpha]$ the integral part of the real number $\alpha$.

**Proposition 1** [2]

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ over $E = Q_p^n$ has index

(i) $\nu = \left\lfloor \frac{n}{2} \right\rfloor$ if $p \equiv 1 \pmod{4}$

(ii) $\nu = \left\lfloor \frac{n}{2} \right\rfloor$ if $p \equiv 3 \pmod{4}$ and $n \not\equiv 2 \pmod{4}$

(iii) $\nu = \left\lfloor \frac{n}{2} \right\rfloor - 1$ if $p \equiv 3 \pmod{4}$ and $n \equiv 2 \pmod{4}$
Proof:

1°) If $p \equiv 1 (\text{mod.} 4)$, it is well known that $i = \sqrt{-1} \in \mathbb{Q}_p$. Let $\nu = \left\lfloor \frac{n}{2} \right\rfloor$ and $e_j = i e_{2j-1} + e_{2j}$, $1 \leq j \leq \nu$, then $V = \bigoplus_{j=1}^{\nu} \mathbb{Q}_p e_j$ is a maximal totally isotropic subspace of $E = \mathbb{Q}_p^n$.

2°) $p \equiv 3 (\text{mod.} 4)$

Therefore $i \not\in \mathbb{Q}_p$ and if $n = 2$ the index of $q_0$ is 0.

If $n = 3$, applying Chevalley's theorem and Newton's method to $q_0(x) = x_1^2 + x_2^2 + x_3^2$ we find $a, b \in \mathbb{Q}_p$, $a \neq 0$, $b \neq 0$, such that $a^2 + b^2 + 1 = 0$. Therefore $e_1 = a e_1 + b e_2 + e_3$ is isotropic in $\mathbb{Q}_p^3$ and $\nu = \left\lfloor \frac{3}{2} \right\rfloor = 1$.

(a) For $n = 4m$, put $e_{2j-1} = a e_{4j-3} + b e_{4j-2} + e_{4j-1}$ and $e_{2j} = -b e_{4j-3} + a e_{4j-2} + e_{4j}$, $1 \leq j \leq m$. It is clear that $q_0(e_{2j-1}) = q_0(e_{2j}) = a^2 + b^2 + 1 = 0$ and $\langle e_{2j-1}, e_{2j} \rangle = -ab + ab = 0$. Therefore $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ is a totally isotropic subspace of $\mathbb{Q}_p^n$ and $\nu = 2m = \left\lfloor \frac{n}{2} \right\rfloor$.

If $n = 4m + 1$, with the same notations as above the subspace $V$ is totally isotropic in $\mathbb{Q}_p^n$ and $\nu = 2m = \left\lfloor \frac{n}{2} \right\rfloor$.

On the other hand if $n = 4m + 3$ the subspaces $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ and $\mathbb{Q}_p e_{2m+1}$ where $e_{2m+1} = a e_{4m+1} + b e_{4m+2} + e_{4m+3}$, are totally isotropic and orthogonal. Therefore $V_0 = V \oplus \mathbb{Q}_p e_{2m+1}$ is totally isotropic and $\nu = 2m + 1 = \left\lfloor \frac{n}{2} \right\rfloor$.

(b) If $n = 4m + 2$, let $V = \bigoplus_{j=1}^{m} (\mathbb{Q}_p e_{2j-1} \oplus \mathbb{Q}_p e_{2j})$ be as above. It is easy to verify that if $x \in \mathbb{Q}_p^n$ is isotropic and $x$ is orthogonal to $V$ then $x \in V$. Therefore $V$ is a maximal totally isotropic subspace of $\mathbb{Q}_p^n$ and $\nu = 2m = \left\lfloor \frac{n}{2} \right\rfloor - 1$. 
Proposition 2: Let $p = 2$.

Let $n = 8m + s$, $0 \leq s \leq 7$.

The standard quadratic form $q_0(x) = \sum_{j=1}^{n} x_j^2$ over $E = \mathbb{Q}_2^n$ has index

(i) $\nu = 4m$ if $0 \leq s \leq 4$

(ii) $\nu = 4m + t$ if $s = 4 + t$, $1 \leq t \leq 3$

Proof:

1°) If $1 \leq n \leq 4$, then the index of $q_0$ is 0.

Indeed, this is clear when $n = 1$.

If $n = 2$, let $x = x_1e_1 + x_2e_2 \in \mathbb{Q}_2^2$ be isotropic and different from 0 i.e. $q_0(x) = x_1^2 + x_2^2 = 0$ and say $x_2 \neq 0$. Therefore $1 + a^2 = 0$ with $a = x_1x_2^{-1}$ and $v_2(a) = 0$ i.e. $a = 1 + 2^\mu a_0$, $\mu \geq 1$, $v_2(a_0) = 0$.

Then $1 + a^2 = 2 + 2^{\mu+1}a_0 + 2^{2\mu}a_0^2 = 0$ or $1 + 2^\mu a_0 + 2^{2\mu-1}a_0 = 0$; in other words $1 \equiv 0 \pmod{2}$; a contradiction.

In the same way, one shows that if $n = 3$ or 4, the index of $q_0$ is 0.

2°) $n = 5$

Let $x_0 = 2e_1 + e_2 + e_3 + e_4 + e_5 \in \mathbb{Q}_2^5$, then $q_0(x_0) = 8$ and $\frac{\partial q_0}{\partial x_j}(x_0) = 2 \not\equiv 0 \pmod{4}$, $2 \leq j \leq 5$.

By Newton's method there exists $x = \sum_{j=1}^{5} x_j e_j \in \mathbb{Q}_2^5$ such that $q_0(x) = 0$ with $x_1 \equiv 2 \pmod{8}, x_j \equiv 1 \pmod{8}$, $2 \leq j \leq 5$.

Put $a = x_1x_5^{-1}$, $b = x_2x_5^{-1}$, $c = x_3x_5^{-1}$, $d = x_4x_5^{-1}$, then $a^2 + b^2 + c_2 + d^2 + 1 = 0$.

The two following elements of $\mathbb{Q}_2^5$

$e_1 = a e_1 + b e_2 + c e_3 + d e_4 + e_5$

$e'_1 = -a e_1 - b e_2 - c e_3 - d e_4 + e_5$

are isotropic with $<e_1,e'_1>=2$. Hence $H = \mathbb{Q}_2e_1 \oplus \mathbb{Q}_2e'_1$ is a hyperbolic plane in $\mathbb{Q}_2^5$. Let $U = H^\perp$ be the orthogonal subspace of $H$ in $\mathbb{Q}_2^5$. The following three elements of $\mathbb{Q}_2^5$:

$u_1 = b e_1 - a e_2 + d e_3 - c e_4$

$u_2 = e_1 - \frac{ac + bd}{c^2 + d^2} e_3 + \frac{bc - ad}{c^2 + d^2} e_4$
\[ u_3 = e_2 + \frac{ad - bc}{c^2 + d^2} e_3 - \frac{ac + bd}{c^2 + d^2} e_4 \]
are elements of \( U \), with
\[ q_0(u_1) = -1, \quad q_0(u_2) = -\frac{1}{c^2 + d^2} = q_0(u_3) \]
Furthermore \( \langle u_i, u_j \rangle = 0 \) if \( 1 \leq i \neq j \leq 3 \), and \( (u_1, u_2, u_3) \) is a basis of \( U \).

For every \( u = y_1 u_1 + y_2 u_2 + y_3 u_3 \in U \) we have \( q_0(u) = y_1^2 q_0(u_1) + y_2^2 q_0(u_2) + y_3^2 q_0(u_3) = -\frac{c^2 y_1^2 + d^2 y_2^2 + y_3^2 + y_3^2}{c^2 + d^2} \) and \( q_0(u) = 0 \) iff \( u = 0 \) because the standard quadratic form of rank 4 is anisotropic. In other words \( (U, q_0) \) is anisotropic and \( \mathbb{Q}_2^5 = H \perp U \) is a Witt decomposition of \( (\mathbb{Q}_2^5, q_0) \). Hence the index of \( q_0 \) is 1.

3°) \( n = 8m + s \), \( 0 \leq s \leq 4 \).
Put, for \( 0 \leq j \leq m - 1 \)

(1)
\[
\begin{align*}
\epsilon_{j,1} &= a \epsilon_{8j+1} + b \epsilon_{8j+2} + c \epsilon_{8j+3} + d \epsilon_{8j+4} + \epsilon_{8j+5} \\
\epsilon_{j,2} &= -b \epsilon_{8j+1} + a \epsilon_{8j+2} + d \epsilon_{8j+3} - c \epsilon_{8j+4} + \epsilon_{8j+6} \\
\epsilon_{j,3} &= -d \epsilon_{8j+1} + c \epsilon_{8j+2} - b \epsilon_{8j+3} + a \epsilon_{8j+4} + \epsilon_{8j+7} \\
\epsilon_{j,4} &= c \epsilon_{8j+1} + d \epsilon_{8j+2} - a \epsilon_{8j+3} - b \epsilon_{8j+4} + \epsilon_{8j+8}
\end{align*}
\]

and

(2)
\[
\begin{align*}
\epsilon'_{j,1} &= -a \epsilon_{8j+1} - b \epsilon_{8j+2} - c \epsilon_{8j+3} - d \epsilon_{8j+4} + \epsilon_{8j+5} \\
\epsilon'_{j,2} &= b \epsilon_{8j+1} - a \epsilon_{8j+2} - d \epsilon_{8j+3} + c \epsilon_{8j+4} + \epsilon_{8j+6} \\
\epsilon'_{j,3} &= d \epsilon_{8j+1} - c \epsilon_{8j+2} + b \epsilon_{8j+3} - a \epsilon_{8j+4} + \epsilon_{8j+7} \\
\epsilon'_{j,4} &= -c \epsilon_{8j+1} - d \epsilon_{8j+2} + a \epsilon_{8j+3} + b \epsilon_{8j+4} + \epsilon_{8j+8}
\end{align*}
\]
A straightforward computation shows that \( \langle \epsilon_{i,k}, \epsilon_{j,l} \rangle = 0 \Rightarrow \langle \epsilon'_{i,k}, \epsilon'_{j,l} \rangle \), \( 0 \leq i, j \leq m - 1; 1 \leq k, l \leq 4 \) and \( \langle \epsilon_{j,l}, \epsilon'_{j,l} \rangle = 2 \); \( 0 \leq j \leq m - 1; 1 \leq l \leq 4 \). Furthermore \( \langle \epsilon_{i,k}, \epsilon'_{j,l} \rangle = 0 \) if \( (i, k) \neq (j, l) \).

Hence the subspaces \( V = \bigoplus_{j=0}^{m-1} \mathbb{Q}_2 \epsilon_{j,l} \) and \( W = \bigoplus_{j=0}^{m-1} \mathbb{Q}_2 \epsilon'_{j,l} \) are isotropic with \( V \cap W = (0) \).

Therefore \( H = V \oplus W \) is a hyperbolic subspace of \( E = \mathbb{Q}_2^{8m+s} \), with \( \text{dim} V = \text{dim} W = 4m \).
But $E = E_m \perp E_s$ (orthogonal sum) where $E_m = \bigoplus_{j=1}^{8m} Q_2e_j$ and $E_s = \bigoplus_{k=1}^{s} Q_2e_{8m+k} \cong Q_2^s$.

If $s = 0$, we have $E = E_m = V \oplus W = H$ and $(E, q_0)$ is a hyperbolic space with index $4m$.

If $1 \leq s \leq 4$; $E = E_m \perp E_s$ with $E_m = V \oplus W = H$. Since $1 \leq \dim E_s = s \leq 4$, the standard quadratic space $(E_s, q_0)$ is anisotropic. Consequently $E = (V \oplus W) \perp E_s$ is a Witt decomposition of $E$ and the index of $q_0$ is $4m$.

$4^o)$ $n = 8m + 4 + t$, $1 \leq t \leq 3$.

a) $n = 8m + 5$

With the same notations as above, we have $E = E_m \perp E_5$ where $E_5 = \bigoplus_{k=1}^{5} Q_2e_{8m+k} \cong Q_2^5$.

Let us write, as for $n = 5$,

\[
\begin{align*}
\begin{cases}
\epsilon_{4m+1} &= a\, \epsilon_{8m+1} + b\, \epsilon_{8m+2} + c\, \epsilon_{8m+3} + d\, \epsilon_{8m+4} + \epsilon_{8m+5} \\
\epsilon'_{4m+1} &= -a\, \epsilon_{8m+1} - b\, \epsilon_{8m+2} - c\, \epsilon_{8m+3} - d\, \epsilon_{8m+4} + \epsilon_{8m+5}
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
u_{m+1} &= b\, \epsilon_{8m+1} - a\, \epsilon_{8m+2} + d\, \epsilon_{8m+3} - c\, \epsilon_{8m+4} \\

\nu_{m+2} &= \epsilon_{8m+1} - \frac{ac + bd}{c^2 + d^2} \epsilon_{8m+3} + \frac{bc - ad}{c^2 + d^2} \epsilon_{8m+4} \\

\nu_{m+3} &= \epsilon_{8m+2} + \frac{ac - bd}{c^2 + d^2} \epsilon_{8m+3} - \frac{bc + ad}{c^2 + d^2} \epsilon_{8m+4}
\end{cases}
\end{align*}
\]

The subspace $U_5 = \bigoplus_{h=1}^{3} Q_2u_{m+h}$ of $E_5$ is anisotropic. On the other hand, $q_0(\epsilon_{4m+1}) = 0 = q_0(\epsilon'_{4m+1})$; $<\epsilon_{4m+1}, \epsilon'_{4m+1}> = 2$ and $\epsilon_{4m+1}, \epsilon'_{4m+1}$ are orthogonal to $U_5$. Therefore $V_0 = V \oplus Q_2\epsilon_{4m+1}$ and $W_0 = W \oplus Q_2\epsilon'_{4m+1}$ are isotropic subspaces of $E$ and $E = (V_0 \oplus W_0) \perp U_5$ is a Witt decomposition of $E$. Hence the index of $q_0$ is $\dim V_0 = \dim W_0 = 4m+1$.

(b) $n = 8m + 6$.

As before, we have $E = E_m \perp E_6$ where $E_6 = \bigoplus_{k=1}^{6} Q_2e_{8m+k} \supset E_5$; hence $\epsilon_{4m+1}$ and $\epsilon'_{4m+1} \in E_6$. 
Let us put
\[
\begin{cases}
\epsilon_{4m+2} = -b e_{8m+1} + a e_{8m+2} + d e_{8m+3} - c e_{8m+4} + e_{8m+6} \\
\epsilon'_{4m+2} = b e_{8m+1} - a e_{8m+2} - d e_{8m+3} + c e_{8m+4} + e_{8m+6}
\end{cases}
\]

and
\[
\begin{cases}
\omega_{m+1} = e_{8m+1} + \frac{bd-ad}{c^2+d^2} e_{8m+3} - \frac{ad+bc}{c^2+d^2} e_{8m+4} \\
\omega_{m+2} = e_{8m+2} - \frac{bc+ad}{c^2+d^2} e_{8m+3} + \frac{ac-bd}{c^2+d^2} e_{8m+4}
\end{cases}
\]

The subspace \( U_6 = \mathbb{Q}_2 \omega_{m+1} \oplus \mathbb{Q}_2 \omega_{m+2} \) of \( E_6 \) is anisotropic. Moreover, \( q_0(\epsilon_{4m+2}) = 0 = q_0(\epsilon'_{4m+2}) \); \(< \epsilon_{4m+2}, \epsilon'_{4m+2} >= 2 \) and \( \epsilon_{4m+2}, \epsilon'_{4m+2} \) are orthogonal to \( U_6 \). Therefore \( V_1 = V_0 \oplus \mathbb{Q}_2 \epsilon_{4m+2} \) and \( W_1 = W_0 \oplus \mathbb{Q}_2 \epsilon'_{4m+2} \) are isotropic subspaces of \( E \) and \( E = (V_1 \oplus W_1) \perp U_6 \) is a Witt decomposition of \( E \). Hence the index of \( q_0 \) is \( \dim V_1 = \dim W_1 = 4m + 2 \).

\( \text{(c)} \)

\[ u = 8m + 7. \]

We have \( E = E_m \perp E_7 \), where \( E_7 = \bigoplus_{k=1}^{7} \mathbb{Q}_2 e_{8m+k} \subseteq E_6 \).

Let us write
\[
\begin{cases}
\epsilon_{4m+3} = -d e_{8m+1} + c e_{8m+2} - b e_{8m+3} + a e_{8m+4} + e_{8m+7} \\
\epsilon'_{4m+3} = d e_{8m+1} - c e_{8m+2} + b e_{8m+3} - a e_{8m+4} + e_{8m+7}
\end{cases}
\]

and
\[
u_m = c e_{8m+1} + d e_{8m+2} - a e_{8m+3} - b e_{8m+4}
\]

The subspace \( U_7 = \mathbb{Q}_2 u_m \) of \( E_7 \) is anisotropic. Furthermore \( q_0(\epsilon_{4m+3}) = 0 = q_0(\epsilon'_{4m+3}) \); \(< \epsilon_{4m+3}, \epsilon'_{4m+3} >= 2 \) and \( \epsilon_{4m+3}, \epsilon'_{4m+3} \) are orthogonal to \( U_7 \). Therefore \( V_2 = V_1 \oplus \mathbb{Q}_2 \epsilon_{4m+2} \) and \( W_2 = W_1 \oplus \mathbb{Q}_2 \epsilon'_{4m+2} \) are isotropic subspaces of \( E \) and \( E = (V_2 \oplus W_2) \perp U_7 \) is a Witt decomposition of \( E \). Hence the index of \( q_0 \) is \( \dim V_2 = \dim W_2 = 4m + 3 \).

**Remark**

Let \( K \) be a non formally real field. The level of \( K \) is the least integer \( s \) such that
\[ -1 = \sum_{j=1}^{s} a_j^2 \text{ where } a_j \in K, a_j \neq 0. \]It is well known that \( s = 2^r, \ r \geq 0 \) (c.f. [3] or [4]).
The level of a p-adic field is 1 if \( p \equiv 1 \pmod{4} \); 2 if \( p \equiv 3 \pmod{4} \) and 4 if \( p = 2 \).

If the level of a field \( K \) is 1 (resp. 2, resp. 4) then the index of the standard quadratic form over \( K^n \) is given by Proposition 1 - (i) [resp. Prop.1 - (ii) - (iii), resp. Prop.2].

More generally let \( K \) be a field of level \( s = 2^r, r \geq 0 \). If we write for any integer \( n, n = m2^{r+1} + a \) where \( 0 \leq a \leq 2^{r+1} - 1 \); then the index of the standard quadratic form over \( K^n \) is

\[
\begin{align*}
(i) & \quad \nu = m2^r \quad \text{if} \quad 0 \leq a \leq 2^r \\
(ii) & \quad \nu = m2^r + t \quad \text{if} \quad a = 2^r + t, \quad 1 \leq t \leq 2^r - 1.
\end{align*}
\]

II - 2 The Clifford algebra \( C(Q_p^n, q_0) \)

The following results can be deduced from a general setting (cf. [3] p. 128-129). Here we establish them by using the computation of the index of \( q_0 \) made in II-1.

Let us recall that if \( E \) is a vector space over a field \( K \) then the exterior algebra \( \wedge(E) \) is the Clifford algebra associated to the null quadratic form over \( E \).

On the other hand, let \((E, q)\) be a regular quadratic space over \( K \). If \( E = V \oplus W \) is a hyperbolic space (\( V \) and \( W \) being maximal totally isotropic subspaces), it is well known that the Clifford algebra \( C(E, q) \) is isomorphic to \( \text{End}(\wedge(V)) \), the space of linear endomorphisms of the vector space \( \wedge(V) \). Furthermore the subalgebra of the even elements of \( C(E, q) \), say \( C_+(E, q) \) is isomorphic to \( \text{End}(\wedge_+(V)) \times \text{End}(\wedge_-(V)) \) where \( \wedge_+(V) \) (resp. \( \wedge_-(V) \)) is the subspace of the even (resp. odd) elements of \( \wedge(V) \).

Generally, if \( E = (V \oplus W) \perp U \) is a Witt decomposition of \( E \), then \( C(E, q) \cong \text{End}(\wedge(V)) \otimes_2 C(U, q) \), the tensor product of \( \mathbb{Z}/2\mathbb{Z} \) - graded algebras (cf. for example [1]).

If \( \dim E = n \), then \( \dim C(E, q) = 2^n = \dim \wedge(E) \).

If \( a, b \in K^* \), we denote by \( \left( \frac{a, b}{K} \right) \) the associated quaternion algebra: i.e. the algebra generated by \( i, j \) with \( i^2 = a \); \( j^2 = b \); \( ij = -ji \). Also \( \left( \frac{a, b}{K} \right) \) is the Clifford algebra of the rank 2 quadratic form \( q(x) = ax_1^2 + bx_2^2 \).

Let us write \( M(n, K) \) the algebra of the \( n \times n \) matrices with coefficients in \( K \).
Theorem 1: $p \equiv 1 \pmod{4}$

(i) If $n = 2m$, then $C(Q_p^n, q_0) \simeq M(2^m, Q_p)$

(ii) If $n = 2m + 1$, then $C(Q_p^n, q_0) \simeq M(2^m, Q_p) \oplus M(2^m, Q_p)$

Proof

Indeed, if $n = 2m$, then $(Q_p^n, q_0)$ is a hyperbolic space. It follows that $C(Q_p^n, q_0) \simeq \text{End}(\wedge(Q_p^n))$.

And, if $n = 2m + 1$, we have a Witt decomposition $Q_p^n = (V \oplus W) \perp U$ where $U = Q_p e_n$.

It follows that $C(U, q_0) \simeq Q_p \oplus Q_p$ which gives (ii).

Theorem 2: $p \equiv 3 \pmod{4}$

(i) If $n = 4m$, then $C(Q_p^n, q_0) \simeq M(2^{2m}, Q_p)$

(ii) If $n = 4m + 1$, then $C(Q_p^n, q_0) \simeq M(2^{2m}, Q_p) \oplus M(2^{2m}, Q_p)$

(iii) If $n = 4m + 2$, then $C(Q_p^n, q_0) \simeq M(2^{2m+1}, Q_p)$

(iv) If $n = 4m + 3$, then $C(Q_p^n, q_0) \simeq M(2^{2m+1}, Q_p[i])$

with $i = \sqrt{-1}$.

Proof:

The case (i) is evident, since $Q_p^{2m}$ is a hyperbolic space.

If $n = 4m + 1$, we have a Witt decomposition $Q_p^n = (V \oplus W) \perp U$ where $U = Q_p u$ with $u = a e_{4m-3} + b e_{4m-2} + e_{4m-1} - e_{4m+1}$ and $q_0(u) = a^2 + b^2 + 1 + 1 = 1$. It follows that $C(U, q_0) \simeq Q_p \oplus Q_p$, which gives (ii).

If $n = 4m + 2$, we have a Witt decomposition $Q_p^n = (V \oplus W) \perp U$ where $U = Q_p u_1 \oplus Q_p u_2$ and $u_1 = a e_{4m-3} + b e_{4m-2} + e_{4m-1} + e_{4m+1} + b e_{4m+2}$

$u_2 = -b e_{4m-3} + e_{4m-2} + e_{4m} - b e_{4m+1} + a e_{4m+2}$

Furthermore $<u_1, u_2> = 0$, $q_0(u_1) = -1 = q_0(u_2)$ and $C(U, q_0) \simeq \left(\frac{-1, -1}{Q_p}\right)$. This quaternion algebra contains an element $z$ with $N(z) = a^2 + b^2 + 1 = 0$. Hence $\left(\frac{-1, -1}{Q_p}\right) \simeq M(2, Q_p)$ and finally we have $C(Q_p^n, q_0) \simeq M(2^{2m}, Q_p) \otimes M(2, Q_p) \simeq M(2^{2m+1}, Q_p)$.

If $n = 4m + 3$, we have a Witt decomposition $Q_p^n = (V \oplus W) \perp U$ where $U = Q_p u$, with $u = -b e_{4m+1} + a e_{4m+2}$ and $q_0(u) = b^2 + a^2 = -1$. Hence $C(U, q_0) \simeq Q_p[i]$, because $u^2 = q_0(u) = -1$. 


We conclude that $C(\mathbb{Q}_p^n, q_0) \simeq M(2^{2m+1}, \mathbb{Q}_p[i])$.

In the proof of the forthcoming theorem, one needs the following lemma

Lemma:

Let $K$ be a field (char. $\neq 2$), $c, d \in K^*$ such that $c^2 + d^2 \neq 0$.

If $\sigma = \frac{1}{c^2 + d^2}$, then $\left( \frac{-\sigma, -\sigma}{K} \right) \simeq \left( \frac{-1, -1}{K} \right)$.

If the two-rank quadratic forms $q_1(x) = -\sigma x_1^2 - \sigma x_2^2$ and $q_2(x) = -x_1^2 - x_2^2$ are equivalent, then their Clifford algebras are isomorphic. But, putting $x_1 = cx_1' + dx_2'$ and $x_2 = dx_1' - cx_2'$, we have $q_1(u(x')) = -\sigma(cx_1' + dx_2')^2 - \sigma(dx_1' - cx_2')^2 = -\sigma(c^2 + d^2)(x_1'^2 + x_2'^2) = q_2(x')$. Hence $q_1$ and $q_2$ are equivalent and the lemma is proved.

Remark

The quaternion algebra $\left( \frac{-1, -1}{\mathbb{Q}_2} \right) = \mathbb{H}_2$ is a skew field.

Indeed, for any $z \in \mathbb{H}_2 = \left( \frac{-1, -1}{\mathbb{Q}_2} \right), z \neq 0$, the norm of $z$ is $N(z) = x_0^2 + x_1^2 + x_2^2 + x_3^2 \neq 0$ (the standard quadratic form of rank 4 over $\mathbb{Q}_2$ is anisotropic).

Theorem 3: $p = 2$

The Clifford algebra $C(\mathbb{Q}_2^n, q_0)$ is isomorphic to:

1. $\text{End}(\wedge(\mathbb{Q}_2^4)) \simeq M(2^{4m}, \mathbb{Q}_2)$, if $n = 8m$
2. $M(2^{4m}, \mathbb{Q}_2) \oplus M(2^{4m}, \mathbb{Q}_2)$, if $n = 8m + 1$
3. $M(2^{4m+1}, \mathbb{Q}_2)$, if $n = 8m + 2$
4. $M(2^{4m+1}, \mathbb{Q}_2[i])$, with $i = \sqrt{-1}$, if $n = 8m + 3$
5. $M(2^{4m+1}, \mathbb{H}_2)$, if $n = 8m + 4$
6. $M(2^{4m+2}, \mathbb{H}_2)$, if $n = 8m + 5$
7. $M(2^{4m+3}, \mathbb{Q}_2[i])$, if $n = 8m + 7$

Proof

According to the proof of Proposition 2, if $n = 8m + s, 0 \leq s \leq 7$, then $\mathbb{Q}_2^n = (V \oplus W) \perp E_s$ where $V$ and $W$ are totally isotropic subspaces of dimension $4m$, and $(E_s, q_0) \simeq$
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(Q_{2^n}, q_0). It follows that \( C(Q_{2^n}, q_0) \simeq \text{End}(\Lambda(Q_{p}^{m})) \otimes_2 C(Q_{2^n}, q_0) \). It is easy to see that
\[ C(Q_{2}, q_0) \simeq Q_2 \oplus Q_2 \; ; \; \; C(Q_{2^2}, q_0) \simeq \left( \frac{1,1}{Q_2} \right) \simeq M(2, Q_2) \] and \( C(Q_{2^n}, q_0) \simeq M(2, Q_2[i]) \).

If \( s = 4 \), the subalgebra, generated by \( e_1 e_2, e_2 e_4 \) and \( e_1 e_4 \), is isomorphic to \( \left( \frac{-1, -1}{Q_2} \right) = H_2 \). Hence \( C(Q_{2^4}, q_0) \simeq M(2, H_2) \).

If \( s = 5 \), then \( Q_{2^5} = F \perp U \), where \( F \) is a hyperbolic plane and \( U \) a three-dimensional anisotropic subspace, with orthogonal basis \((u_1, u_2, u_3)\) satisfying \( q_0(u_1) = -1, q_0(u_2) = -\sigma = q_0(u_3) \left( \sigma = \frac{1}{\sigma+1} \right) \) and \( a, b, c, d \in Q_2 \) such that \( a^2 + b^2 + c^2 + d^2 + 1 = 0 \).

Therefore \( C(U, q_0) \simeq \left( \frac{-\sigma, -\sigma}{Q_2} \right) \simeq H_2 \); \( C_+ \) stands for the even subalgebra. But in \( C(U, q_0), (u_1 u_2 u_3)^2 = \sigma^2 \) is a square in \( Q_2 \); therefore \( C(U, q_0) \simeq H_2 \oplus H_2 \). Furthermore \( C(Q_{2^5}, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2, H_2) \oplus M(2, H_2) \), because \( C(F, q_0) \simeq M(2, Q_2) \).

If \( s = 6 \), then \( Q_{2^6} = F \perp U \), where \( F \) is a hyperbolic space of dimension 4 and \( U \) a two-dimensional anisotropic subspace with an orthogonal basis \((u_1, u_2)\) satisfying \( q(u_1) = -\sigma = q(u_2) \). Therefore \( C(U, q_0) \simeq \left( \frac{-\sigma, -\sigma}{Q_2} \right) \simeq H_2 \). And consequently \( C(Q_{2^6}, q_0) \simeq C(F, q_0) \otimes_2 C(U, q_0) \simeq M(2^2, H_2) \).

If \( s = 7 \), then \( Q_{2^7} = F \perp U \), where \( F \) is a hyperbolic space of dimension 6 and \( U = Q_2 u \), with \( q_0(u) = -1 \). Hence \( C(U, q_0) \simeq Q_2[i] \) and \( C(Q_{2^7}, q_0) \simeq M(2^3, Q_2[i]) \).

One deduces the isomorphisms of the theorem from \( C(Q_{2^n}, q_0) \simeq M(2^{4m}, Q_2) \otimes_2 C(Q_{2^n}, q_0) \).

N.B.: A classical way to prove the above theorems is based on the isomorphisms
\[ C(K^{n+2}, q_0) \simeq C(K^n, -q_0) \otimes C(K^2, q_0) \]
and \( C(K^{n+2}, -q_0) \simeq C(K^n, q_0) \otimes C(K^2, -q_0) \)
which give first 8-periodicity, etc...

\((-q_0 \text{ is the opposite of the standard quadratic form } q_0)\)
III - THE FAMILIES OF P-ADIC CLIFFORD ALGEBRAS

III-1. Equivalent classes of the p-adic quadratic forms

Let \( a, b \in \mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\} \). The Hilbert symbol \((a, b)\) is defined by \((a, b) = 1\) if the quadratic form of rank 3, \( q'(x) = x_0^2 - ax_1^2 - bx_2^2 \) is isotropic \((a, b) = -1\) otherwise.

N.B. \((a, b) = 1\) iff \( \left( \frac{a, b}{\mathbb{Q}_p} \right) = M(2, \mathbb{Q}_p) \).

Let \( E \) be a vector space over \( \mathbb{Q}_p \) of dimension \( n \). Let us consider a regular quadratic form \( q \) over \( E \). If \((e_j)_{1 \leq j \leq n}\) is an orthogonal basis of \( E \) and \( a_j = q(e_j) \); then the discriminant \( d(q) \) of \( q \) is equal to \( a_1 \ldots a_n \) in the group \( M_p = \mathbb{Q}_p^*/\mathbb{Q}_p^2 \). Let \( \epsilon(q) = \prod_{1 \leq i < j \leq n} (a_i, a_j) \).

**Theorem A**

(i) The p-adic regular quadratic forms \( q \) and \( q' \) of rank \( n \) are equivalent iff \( d(q) = d(q') \) and \( \epsilon(q) = \epsilon(q') \).

(ii) Let \( d \in M_p \) and \( \epsilon = \pm 1 \). There exists a p-adic regular quadratic form \( q \) such that \( d(q) = d \) and \( \epsilon(q) = \epsilon \) iff

\[
\begin{align*}
(a) & \quad n = 1 \quad \text{and} \quad \epsilon = 1 \\
(b) & \quad n = 2 \quad \text{and} \quad (d, \epsilon) \neq (-1, -1) \\
(c) & \quad n \geq 3
\end{align*}
\]

**Proof:** cf. [5]

According to that proof of Theorem A, one can give, explicitly, representatives of the equivalence classes of p-adic regular quadratic forms.

Let us recall that \( M_2 = \{ \pm 1, \pm 2, \pm 5, \pm 10 \} \) and \( M_p = \{ 1, p, \omega, \omega p \} \) if \( p \neq 2 \), where \( \omega \) is a unit such that \( \left( \frac{\omega}{p} \right) = -1 \); \( \left( \frac{-1}{p} \right) = \) the Legendre symbol. Furthermore \(-1 = 1\) in \( M_p \) if \( p \equiv 1 \) \((\text{mod.} 4)\) and \( M_p = \{ 1, p, -1, -p \} \) if \( p \equiv 3 \) \((\text{mod.} 4)\).

We are content ourself here, with the primes \( p \) different from \( 2 \). Then a complete set of representatives of the equivalent classes of regular p-adic quadratic forms is obtained as follows.
(a) \( n = 1 \)
Then \( q^n(x) = ax^2, a \in M_p; \) and the Clifford algebras \( C(Q_p, q^n) \) are isomorphic respectively to \( Q_p \oplus Q_p, Q_p[\sqrt{p}], Q_p[\sqrt{\omega}] \) and \( Q_p[\sqrt{\omega^2}] \).

(b) \( n = 2 \)
Then we have over \( Q^2_p \) (with \( \omega = -1 \) if \( p \equiv 3 \) (mod.4))
\[
\begin{align*}
q_0(x) &= x_1^2 + x_2^2 \\
q_1(x) &= x_1^2 + p x_2^2 \\
q_2(x) &= \omega x_1^2 + \omega p x_2^2 \\
q_3(x) &= x_1^2 + \omega x_2^2 \\
q_4(x) &= p x_1^2 + \omega p x_2^2 \quad \text{if } p \equiv 1 \text{ (mod.4)} \quad \text{(resp.)} \quad q_4(x) &= p x_1^2 + p x_2^2 \quad \text{if } p \equiv 3 \text{ (mod.4)} \\
q_5(x) &= x_1^2 + \omega p x_2^2 \\
q_6(x) &= p x_1^2 + \omega x_2^2 \\
\end{align*}
\]
Furthermore \( \epsilon(q_\ell) = 1 \) if \( \ell = 0, 1, 3, 5 \) and \( \epsilon(q_\ell) = -1 \) if \( \ell = 2, 4, 6 \).

N.B : If \( p = 2 \), then for \( n = 2 \), one has
8 regular quadratic forms \( q \) such that \( \epsilon(q) = 1 \)
and 7 regular quadratic forms \( q \) such that \( \epsilon(q) = -1 \).

(c) \( n = 3 \)
If \( (e_1, e_2, e_3) \) is the canonical basis of \( Q^3_p \), then
- \( q_\ell(x) = q_\ell(x_1 e_1 + x_2 e_2) + x_3^2, \ 0 \leq \ell \leq 6 \)
and
- \( q_\ell(x) = p x_1^2 + \omega x_2^2 + \omega p x_3^2 = q_6(x_1 e_1 + x_2 e_2) + \omega p x_3^2 \quad \text{if } p \equiv 1 \text{ (mod.4)} \)
resp.
- \( q_\ell(x) = p x_1^2 - x_2^2 + p x_3^2 = q_6(x_1 e_1 + x_2 e_2) + p x_3^2 \quad \text{if } p \equiv 3 \text{ (mod.4)} \)
Furthermore \( d(q_\ell') = d(q_\ell), \epsilon(q_\ell') = \epsilon(q_\ell), \ 0 \leq \ell \leq 6 \) and \( d(q_\ell') = -1, \epsilon(q_\ell') = -1 \).

(d) \( n \geq 4 \)
Let \( (e_j)_{1 \leq j \leq n} \) be the canonical basis of \( Q^n_p \), then
- \( q_\ell''(x) = q_\ell(x_1 e_1 + x_2 e_2) + \sum_{j=3}^{n} x_j^2, \ 0 \leq \ell \leq 6. \)
In other words \( q_\ell''(x) = q_\ell(x_1 e_1 + x_2 e_2) + q_0 \left( \sum_{j=3}^{n} x_j e_j \right) \)
i.e. \( (Q^n_p, q_\ell'') \cong (Q^2_p, q_\ell) \perp (Q^{n-2}_p, q_0), \ 0 \leq \ell \leq 6 \)
and
If \( n = 3 \), then the classes of regular quadratic forms have 15 representative forms \( q' \) with \( \epsilon(q') = 1 \), resp. \( \epsilon(q') = -1 \) and \( d(q') \neq -1 \), obtained from corresponding representative quadratic forms of ranks 2 by adding the rank 1 form \( x_3^2 \). The other representative form is \( q_1^{15} = -x_1^2 - x_2^2 - x_3^2 \) with \( \epsilon(q_1^{15}) = -1 \) and \( d(q_1^{15}) = -1 \).

And if \( n \geq 4 \), one proceeds as above.

### III - 2 The \( p \)-adic Clifford algebras

With the above notations, we have the following concrete propositions

**Proposition 3:** \( p \neq 2 \)

(i) \( C(\mathbb{Q}_p^2, q_{\ell}) \simeq M(2, \mathbb{Q}_p) \) if \( \ell = 0, 1, 3, 5 \).

(ii) \( C(\mathbb{Q}_p^2, q_{\ell}) \simeq \left( \frac{p, \omega}{\mathbb{Q}_p} \right) = \mathbb{H}_p = \text{the } p\text{-adic quaternion field}, \text{ if } \ell = 2, 4, 6. \)

**Proof**

(i) Indeed, if \( \ell = 0, 1, 3, 5 \); then \( \epsilon(q_{\ell}) = 1 \). Therefore \( C(\mathbb{Q}_p^2, q_{\ell}) \simeq M(2, \mathbb{Q}_p) \).

(ii) If \( \ell = 2, 4, 6 \) then the Clifford algebras \( C(\mathbb{Q}_p^2, q_{\ell}) \) are isomorphic to the quaternion algebras with norm respectively,

\[
N_2(z) = x_0^2 - \omega x_1^2 - \omega x_2^2 + \omega^2 p x_3^2 ;
\]

\[
N_4(z) = x_0^2 - p x_1^2 - \omega p x_2^2 + \omega p^2 x_3^2 \text{ if } p \equiv 1 \pmod{4} ;
\]

(resp. \( N_4(z) = x_0^2 - p x_1^2 - p x_2^2 + p^2 x_3^2 \) if \( p \equiv 3 \pmod{4} \))

and \( N_6(z) = x_0^2 - p x_1^2 - \omega x_2^2 + \omega p x_3^2 \).

It is easily seen that these quadratic forms are anisotropic and equivalent. Therefore

\[
C(\mathbb{Q}_p^2, q_2) \simeq C(\mathbb{Q}_p^2, q_4) \simeq C(\mathbb{Q}_p^2, q_6) \simeq \left( \frac{p, \omega}{\mathbb{Q}_p} \right) = \mathbb{H}_p \text{ is a skew field. Hence } \mathbb{H}_p \text{ is the unique quaternion field over } \mathbb{Q}_p \text{ (according isomorphism). This result obtained directly here is a general result for local fields (cf. [3])}.\]
Proposition 4 : \( p \equiv 1 \pmod{4} \)

The Clifford algebra \( C(Q_p^3, q'_p) \) is isomorphic to

(i) \( M(2, Q_p) \oplus M(2, Q_p) \) if \( \ell = 0 \)

(ii) \( M(2, Q_p[\sqrt{-p}]) \) if \( \ell = 1, 2 \)

(iii) \( M(2, Q_p[\sqrt{-q}]) \) if \( \ell = 3, 4 \)

(iv) \( M(2, Q_p[\sqrt{p}]) \) if \( \ell = 5, 6 \)

(v) \( H_p \oplus H_p \) if \( \ell = 7 \)

Similarly we have

Proposition 4' : \( p \equiv 3 \pmod{4} \)

The Clifford algebra \( C(Q_p^3, q'_p) \) is isomorphic to

(i) \( M(2, Q_p[i]) \) if \( \ell = 0, 4 \)

(ii) \( M(2, Q_p[\sqrt{-p}]) \) if \( \ell = 1, 2 \)

(iii) \( M(2, Q_p) \oplus M(2, Q_p) \) if \( \ell = 3 \)

(iv) \( M(2, Q_p[\sqrt{-q}]) \) if \( \ell = 5, 6 \)

(v) \( H_p \oplus H_p \) if \( \ell = 7 \)

Proof of Propositions 4 and 4'

Let us recall that if \( (E, q) \) is a regular quadratic space over a field \( K \) with \( n = \dim E \) odd, then \( C(E, q) \simeq Z \otimes C_+(E, q) \), where \( Z \) is the centre of \( C(E, q) \) and \( C_+(E, q) \) the subalgebra of even elements. Furthermore, if \( (e_1, \ldots, e_n) \) is an orthogonal basis of \( (E, q) \) then \( u = e_1 \cdots e_n \) is such that \( u^2 = (-1)^{\frac{n(n-1)}{2}} d(q) \) and \( Z = K[u] \).

In particular for \( n = 3 \) and \( q(x) = \alpha \, x_1^2 + \beta \, x_2^2 + \gamma \, x_3^2 \), we have \( e_1^2 = \alpha, e_2^2 = \beta, e_3^2 = \gamma ; u^2 = -\alpha \beta \gamma = \delta \neq 0 \) and \( C_+(E, q) = \langle 1, e_1 e_2, e_1 e_3, e_2 e_3 \rangle \) = subspace generated by \( 1, \ldots, e_2 e_3 \). Put \( E_1 = e_1 e_2, E_2 = e_1 e_3, E_3 = -\alpha e_2 e_3 \), hence \( C_+(E, q) = \langle 1, E_1, E_2, E_3 \rangle \) with \( E_1^2 = -\alpha \beta, E_2^2 = -\alpha \gamma, E_1 E_2 = E_3 = -E_2 E_1 \). Therefore \( C_+(E, q) \simeq \left( \frac{-\alpha \beta, -\alpha \gamma}{K} \right) \).

Consequently (1) if \( \delta \in K^{*2} \), then \( Z \simeq K \oplus K \) and \( C(E, q) \simeq \left( \frac{-\alpha \beta, -\alpha \gamma}{K} \right) \oplus \left( \frac{-\alpha \beta, -\alpha \gamma}{K} \right) \)

(2) if \( \delta \notin K^{*2} \), then \( Z = K[u] \) is a field and \( C(E, q) \simeq \left( \frac{-\alpha \beta, -\alpha \gamma}{K[u]} \right) \).

Applying these remarks to Propositions 4 and 4', one finds the desired isomorphisms. For example if \( p \equiv 1 \pmod{4} \) and \( \ell = 2 \), then \( \delta = -\omega^2 p = (i\omega)^2 p \) and \( Z = Q_p[\sqrt{p}] \), hence
Let $C(Q_p^2, q'_2) \simeq \left( -\frac{\omega^2 p, -\omega}{Q_p[\sqrt{p}]} \right) \simeq M(2, Q_p[\sqrt{p}])$; \( \tilde{q}(v) = p x^2 + \omega y^2 \) represents 1 over \( Q_p[\sqrt{p}] \). Also if \( \ell = 7 \), then \( \delta = -p^2 \omega^2 = (iwp)^2 \), hence \( Z \simeq Q_p \oplus Q_p \) and since \( \left( -\frac{p \omega, -\omega^2}{Q_p} \right) \simeq \left( \frac{p \omega, \omega}{Q_p} \right) \simeq H_p \) we have \( C(Q_p^3, q'_7) \simeq H_p \oplus H_p \).

In the case \( p \equiv 3(\text{mod. } 4) \), for example if \( \ell = 0(\text{resp. } \ell = 3) \) we have \( \delta = -1 \) (resp. \( \delta = 1 \)) \( Z \simeq Q_p[i], \text{(resp. } Z \simeq Q_p \oplus Q_p \). Hence \( C(Q_p^3, q'_0) \simeq \left( \frac{-1}{Q_p[i]} \right) \simeq M(2, Q_p[i]), \text{(resp. } C(Q_p^3, q'_3) \simeq \left( \frac{-1}{Q_p} \right) \oplus \left( \frac{-1}{Q_p} \right) \simeq M(2, Q_p) \oplus M(2, Q_p) \).

The other verifications are left to the reader.

Lemma 2 : \( p \neq 2 \)
\( C(Q_p^4, q''_7) \simeq M(2, H_p). \)

Indeed, since \( q''_7 = p x_1^2 + \omega x_2^2 + (\omega') p x_3^2 + x_4^2 \) where \( \omega' = \omega \) if \( p \equiv 1(\text{mod. } 4) \) and \( \omega = -1, \omega' = 1 \) if \( p \equiv 3(\text{mod. } 4) \); \( \text{we have } C(Q_p^4, q''_7) \simeq \left( \frac{p, \omega}{Q_p} \right) \otimes_2 \left( \frac{\omega', p, 1}{Q_p} \right) \simeq H_p \otimes_2 M(2, Q_p) \simeq M(2, H_p). \)

Theorem 4 : \( p \equiv 1(\text{mod. } 4) \); \( n \geq 4 \)
1°) If \( n = 2m \), then the Clifford algebra \( C(Q_p^n, q''_t) \) is isomorphic to
   (i) \( M(2^m, Q_p) \) if \( \ell = 0, 1, 3, 5 \)
   (ii) \( M(2^{m-1}, H_p) \) if \( \ell = 2, 4, 6, 7 \)
2°) If \( n = 2m + 1 \), then the Clifford algebra \( C(Q_p^n, q''_t) \) is isomorphic to
   (i) \( M(2^m, Q_p) \oplus M(2^m, Q_p) \) if \( \ell = 0 \)
   (ii) \( M(2^m, Q_p[\sqrt{\tau}]) \) if \( \ell = 1, 2, 3, 4, 5, 6 \)
   with \( \tau = p \) (resp. \( \omega \), resp. \( \omega p \)) for \( \ell = 1, 2 \) (resp. \( \ell = 3, 4 \); resp. \( \ell = 5, 6 \)).
   (iii) \( M(2^{m-1}, H_p) \oplus M(2^{m-1}, H_p) \) if \( \ell = 7 \)

Proof :
1°) \( n = 2m \)
Notice that \( C(Q_p^n, q''_t) \simeq C(Q_p^{n-2}, q_0) \otimes_2 C(Q_p^2, q_t), 0 \leq \ell \leq 6 \). But by Proposition 3, we have \( C(Q_p^2, q_t) \simeq M(2, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^2, q_t) \simeq H_p \) if \( \ell = 7 \).
Since \( C(\mathbb{Q}_p^{n-2}, q_0) \simeq M(2^{m-1}, \mathbb{Q}_p) \) by Theorem 1- (i) - , we have \( C(\mathbb{Q}_p^n, q''_\ell) \simeq M(2^{m-1}, \mathbb{Q}_p) \otimes_2 M(2, \mathbb{Q}_p) \simeq M(2^m, \mathbb{Q}_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(\mathbb{Q}_p^n, q''_\ell) \simeq M(2^{m-1}, \mathbb{Q}_p) \otimes_2 H_p \simeq M(2^{m-1}, H_p) \) if \( \ell = 2, 4, 6 \).

For \( \ell = 7 \) , applying Lemma 2 and Theorem 1 - (i) - we obtain \( C(\mathbb{Q}_p^n, q''_\ell) \simeq C(\mathbb{Q}_p^{n-4}, q_0) \otimes_2 C(\mathbb{Q}_p^4, q''_\ell) \simeq M(2^{m-2}, \mathbb{Q}_p) \otimes_2 M(2, H_p) \simeq M(2^{m-1}, H_p) \).

2°) \( n = 2m + 1 \)

If \( 1 \leq \ell \leq 6 \) , then we have \( C(\mathbb{Q}_p^n, q''_\ell) \simeq C(\mathbb{Q}_p^{n-4}, q_0) \otimes_2 C(\mathbb{Q}_p^3, q''_\ell) \simeq M(2^{m-1}, \mathbb{Q}_p) \otimes_2 C(\mathbb{Q}_p^3, q''_\ell) \).

Applying Proposition 4, we obtain the isomorphism \( C(\mathbb{Q}_p^n, q''_\ell) \simeq M(2^m, \mathbb{Q}_p[\sqrt{\tau}]) \) as claimed.

The case \( \ell = 0 \) is Theorem 1 - (ii) -

If \( \ell = 7 \) , then \( C(\mathbb{Q}_p^3, q''_\ell) \simeq H_p \oplus H_p \) and \( C(\mathbb{Q}_p^n, q''_\ell) \simeq M(2^{m-1}, \mathbb{Q}_p) \otimes_2 (H_p \oplus H_p) \simeq M(2^{m-1}, H_p) \oplus M(2^{m-1}, H_p) \).

Theorem 5 : \( p \equiv 3 \pmod{4} ; n \geq 4 \)

The Clifford algebra \( C(\mathbb{Q}_p^n, q''_\ell) \) is isomorphic to the following matrix algebra or direct sum of two matrix algebras.

1°) \( n = 4m \)
(i) \( M(2^{2m}, \mathbb{Q}_p) \) if \( \ell = 0, 1, 3, 5 \)
(ii) \( M(2^{2m-1}, H_p) \) if \( \ell = 2, 4, 6, 7 \)

2°) \( n = 4m + 1 \)
(i) \( M(2^{2m}, \mathbb{Q}_p) \oplus M(2^{2m}, \mathbb{Q}_p) \) if \( \ell = 0, 4 \)
(ii) \( M(2^{2m}, \mathbb{Q}_p[\sqrt{\tau}]) \) if \( \ell = 1, 2, 3, 5, 6, 7 \)

with \( \tau = p \) (resp. \(-1\), resp. \(-p\)) for \( \ell = 1, 2 \) (resp. \(\ell = 3, 7\), resp. \(\ell = 5, 6\)).

3°) \( n = 4m + 2 \)
(i) \( M(2^{2m+1}, \mathbb{Q}_p) \) if \( \ell = 0, 1, 3, 5 \)
(ii) \( M(2^{2m}, H_p) \) if \( \ell = 2, 4, 6, 7 \)
4°) \( n = 4m + 3 \)

(i) \( M(2^{2m+1}, Q_p) \oplus M(2^{2m+1}, Q_p) \) if \( \ell = 3 \)

(ii) \( M(2^{2m+1}, Q_p[\sqrt{\tau}]) \) if \( \ell = 0, 1, 2, 4, 5, 6, \)

with \( \tau = -1 \) (resp. \( -p \), resp. \( p \)) for \( \ell = 0, 4 \) (resp. \( \ell = 1, 2, \) resp. \( \ell = 5, 6 \)).

(iii) \( M(2^{2m}, H_p) \oplus M(2^{2m}, H_p) \) if \( \ell = 7 \).

Proof :

1°) \( n = 4m \)

As in Lemma 2, it is readily seen that \( C(Q_p^4, q_\ell^\prime) \simeq M(2^2, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^4, q_\ell^\prime) \simeq M(2, H_p) \) if \( \ell = 2, 4, 6, 7 \).

If \( n = 4m, m \geq 2 \), we have \( C(Q_p^n, q_\ell^\prime) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^4, q_\ell^\prime) \). But Theorem 3 - (i) - gives \( C(Q_p^{n-4}, q_0) \simeq M(2^{2m-2}, Q_p) \). Therefore \( C(Q_p^n, q_\ell^\prime) \simeq M(2^{2m-2}, Q_p) \otimes_2 M(2^2, Q_p) \simeq M(2^{2m-1}, H_p) \) if \( \ell = 2, 4, 6, 7 \). On the other hand \( C(Q_p^n, q_\ell^\prime) \simeq Z \otimes M(2, H_p) \), which proves the isomorphisms.

2°) \( n = 4m + 1 \)

With notations used in the proof of Propositions 4 and 4' we have \( C(Q_p^n, q_\ell^\prime) \simeq Z \otimes C_+(Q_p^n, q_\ell^\prime) \) and \( Z = Q_p[u] \) where \( u^2 = d(q_\ell^\prime) \). Hence \( Z \) is isomorphic to \( Q_p \oplus Q_p \) if \( \ell = 0, 4 \); resp. \( Q_p[\sqrt{\tau}] \) if \( \ell = 1, 2 \); resp. \( Q_p[\sqrt{-1}] \) if \( \ell = 3, 7 \); resp. \( Q_p[\sqrt{-p}] \) if \( \ell = 5, 6 \). On the other hand \( C_+(Q_p^n, q_\ell^\prime) \simeq C_+(Q_p \cdot Z, x_n, x_n^2) \otimes C(Q_p^{n-1}, -q_\ell^\prime) \simeq C(Q_p^{n-1}, -q_\ell^\prime) \simeq M(2^{2m}, Q_p) \). Hence \( C(Q_p^n, q_\ell^\prime) \simeq Z \otimes M(2^{2m}, Q_p) \) which proves the isomorphisms.

3°) \( n = 4m + 2 \)

Since \( n - 2 = 4m \), we obtain \( C(Q_p^n, q_\ell^\prime) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^2, q_\ell) \).

By Theorem 2 - (i) - one has \( C(Q_p^{4m}, q_0) \simeq M(2^{2m}, Q_p) \) and by Proposition 3, \( C(Q_p^2, q_\ell) \simeq M(2, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^2, q_\ell) \simeq H_p \) if \( \ell = 2, 4, 6 \). It follows that \( C(Q_p^n, q_\ell^\prime) \simeq M(2^{2m+1}, Q_p) \) if \( \ell = 0, 1, 3, 5 \) and \( C(Q_p^n, q_\ell^\prime) \simeq M(2^{2m+1}, H_p) \) if \( \ell = 2, 4, 6 \).

For the case \( \ell = 7 \), since \( n - 4 = 4(m - 1) + 2 \) we have \( C(Q_p^n, q_\ell^\prime) \simeq C(Q_p^{n-4}, q_0) \otimes_2 C(Q_p^2, q_\ell) \). By theorem 2 - (iii) -, \( C(Q_p^{n-4}, q_0) \simeq M(2^{2m+1}, Q_p) \) and by Lemma 2, \( C(Q_p^2, q_\ell) \simeq M(2, H_p) \). Hence \( C(Q_p^n, q_\ell^\prime) \simeq M(2^{2m}, H_p) \).

Notice that in 1°) and 3°) the exponent of 2 is \( \frac{3}{2} \).
4°) \( n = 4m + 3 \)

Here, \( n-3 = 4m \) and \( C\left( Q_p^n, q''_\ell \right) \simeq C\left( Q_p^{4m}, q_0 \right) \otimes_2 C\left( Q_p^3, q'_\ell \right) \). But \( C\left( Q_p^{4m}, q_0 \right) \simeq M(2^{2m}, Q_p) \) and by Proposition 4', \( C\left( Q_p^3, q'_\ell \right) \) is isomorphic to \( M(2, Q_p) \oplus M(2, Q_p) \) if \( \ell = 3 \), resp. \( H_p \oplus H_p \) if \( \ell = 7 \), resp. \( M(2, Q_p[\sqrt{\tau}]) \) if \( \ell = 0, 1, 2, 4, 5, 6 \) with \( \tau = -1 \) for \( \ell = 0, 4 \); \( \tau = -p \) for 1, 2 and \( \tau = p \) for \( \ell = 5, 6 \).

Taking tensor product we obtain the desired isomorphisms.

Remark :

As for \( C\left( Q_p^n, q_0 \right) \), for the other Clifford algebras \( C\left( Q_p^n, q''_\ell \right) \) we have 2-periodicity when \( p \equiv 1 \) (mod. 4) and 4-periodicity when \( p \equiv 3 \) (mod. 4).

N.B. When \( p = 2 \), in the same way one can give as above the table of the 2-adic Clifford algebras.

REFERENCES


