ANN VERDOODT

Continued fractions for finite sums

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CONTINUED FRACTIONS FOR FINITE SUMS

Ann Verdooodt

Abstract

Our aim in this paper is to construct continued fractions for sums of the type
\[ \sum_{i=0}^{n} b_i z^{c(i)} \text{ or } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \], where \( (b_n) \) is a sequence such that \( b_n \) is different from zero if \( n \) is different from zero, and \( c(n) \) is an element of \( \mathbb{N} \).

Résumé

Le but est de construire des fractions continues pour des sommes du type
\[ \sum_{i=0}^{n} b_i z^{c(i)} \text{ ou } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \], où \( (b_n) \) est une suite telle que \( b_n \) est différent de zéro pour \( n \) différent de zéro, et \( c(n) \) est un élément de \( \mathbb{N} \).

1. Introduction

\[ [a_0, a_1, a_2, \ldots] \] denotes the continued fraction \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \),

and \([a_0, a_1, \ldots, a_n]\) denotes \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + a_{n-1} + \frac{1}{a_n}}} \).

The \( a_i \)'s are called the partial quotients (or simply the quotients), and \([a_0, a_1, \ldots, a_n]\) is called a finite continued fraction.

Our aim in this paper is to construct continued fractions for sums of the type \( \sum_{i=0}^{n} b_i z^{c(i)} \text{ or } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \), where \( c(i) \) is an element of \( \mathbb{N} \).
In section 2, we find continued fractions for finite sums of the type \( \sum_{i=0}^{n} b_i z^i \) (c(i) = i) or \( \sum_{i=0}^{n} b_i z^i \) (c(i) = q^i), where \((b_n)\) is a sequence such that \(b_n\) is different from zero if \(n\) is different from zero, and where \(q\) is a natural number different from zero and one.

Therefore, we start by giving a continued fraction for the sum \( \sum_{i=0}^{n} b_i T^{3i} \), where \(b_i\) is different from zero for all \(i\) different from zero (\(b_i\) is a constant in \(T\)). This can be found in theorem 1.

If we replace \(b_i\) by \(b_i z^i\) in theorem 1, and we put \(T\) equal to one, we find a continued fraction for \( \sum_{i=0}^{n} b_i z^i \) (theorem 2), and if we replace \(b_i\) by \(b_i z^i\) in theorem 1, and we put \(T\) equal to one, we find a continued fraction for \( \sum_{i=0}^{n} b_i z^i \) (theorem 3) (\(q\) is a natural number different from zero and one).

In section 3 we find continued fractions for finite sums of the type \( \sum_{i=0}^{n} \frac{b_i}{z^i} \), for some sequences \((b_n)\) and \((c(n))\), where \(c(n)\) is a natural number.

In theorem 4, we find a result for \(c(i)\) equal to \(2^i\) (for all \(i\)).

Finally, in theorem 5, we give a continued fraction for \( \sum_{i=0}^{n} \frac{b_i}{z^i} \), where \(c(0)\) equals zero, and \(c(n+1) - 2c(n) \geq 0\).

The results in this paper are extensions of results that can be found in [2], [3] and [4].

Acknowledgement: I thank professor Van Hamme for the help and the advice he gave me during the preparation of this paper.
2. Continued fractions for sums of the type $\sum_{i=0}^{n} b_i z^i$

All the proofs in sections 2 and 3 can be given with the aid of the following simple lemma:

Lemma

Let i) $p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1$,

\[ p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 2), \]

then we have

ii) $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$

iii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad (n \geq 1)$

iv) $\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1] \quad (n \geq 1)$

These well-known results can e.g. be found in [1].

First we give a continued fraction for the sum $\sum_{i=0}^{n} b_i z^i$, where $b_i$ is different from zero for all $i$ different from zero ($b_i$ is a constant in $T$):

Theorem 1

Let $(b_n)$ be a sequence such that $b_n \neq 0$ for all $n > 0$.

Define a sequence $(x_n)$ by putting $x_0 = [b_0 T]$, $x_1 = [b_0 T, b_1 T^{-3}]$, and if

\[ x_n = [a_0, a_1, \ldots, a_{2n-1}] \]

then setting $x_{n+1} = [a_0, a_1, \ldots, a_{2n-1}, -b_{2n}/b_{n+1} T^{-3n}, -a_{2n-1}, \ldots, -a_1]$.

Then $x_n = \sum_{i=0}^{n} b_i T^{3i}$ for all $n \in \mathbb{N}$.

Proof

For $n = 0$ the theorem clearly holds.

If $n$ is at least one, we prove that $x_n = \sum_{i=0}^{n} b_i T^{3i}$ and $q_{2n-1} = b_n^{-1} T^{-3n}$.

We prove this by induction. For $n = 1$ the assertion holds.
Suppose it holds for $1 \leq n \leq j$. We then prove the assertion for $n = j+1$.

$x_{j+1} = [a_0, a_1, ... , a_{2j+1-1}]$.

$= \frac{-q_{2j+1} p_{2j+1} + q_{2j+2} p_{2j+1}}{-q_{2j+1} q_{2j+2} q_{2j+1}}$ (using the definition of a continued fraction)

(by i), ii) and iv) of the lemma)

$= \frac{-q_{2j-1} (a_{2j} p_{2j-1} + p_{2j}) + q_{2j} p_{2j-1}}{-q_{2j-1} (a_{2j} q_{2j-1} + q_{2j-2}) + q_{2j-2} q_{2j-1}}$ (by i) of the lemma)

Now we have $p_{2j-1} q_{2j-2} - p_{2j-2} q_{2j-1} = (-1)^{2j-2} = 1$ (by iii) of the lemma)

We still have to prove $q_{2j+1-1} = b_{j+1} T^{3j+1}$. Let $k$ be at least one.

Then $p_k$ and $q_k$ are polynomials in $U = T^{-1}$. deg $q_k >$ deg $q_{k-1}$, and the term with the highest degree in $q_k$ is given by $a_k \cdot a_{k-1} \cdot ... \cdot a_1$. This follows from i).

If $r$ is a polynomial in $U$ that divides $p_k$ and $q_k$, then $r$ must be a constant in $U$. This immediately follows from iii). If $r$ divides $p_k$ and $q_k$, then $r$ divides $(-1)^{k-1}$. So $r$ must be a constant.

Since $\sum_{i=0}^{i+1} b_i T^{3i} = [a_0, a_1, ..., a_{2j+1-1}] = \frac{p_{2j+1-1}}{q_{2j+1-1}}$, we have

$\frac{p_{2j+1-1}}{q_{2j+1-1}} = \sum_{i=0}^{i+1} b_i T^{3i} T^{-3i+1} = \sum_{i=0}^{i+1} b_i U^{3i+1-3i} = b_{j+1} + \sum_{i=0}^{j} b_i U^{3i+1-3i}$

and we conclude that $q_{2j+1-1} = C U^{3j+1} = C T^{3j+1}$ where $C$ is a constant.

By the previous remark, we have that

$q_{2j+1-1} = C T^{-3j+1} = C U^{3j+1} = a_1 \cdot a_2 \cdot ... \cdot a_{2j+1-1}$

$= (-1)^{2j-1}(a_1 \cdot a_2 \cdot ... \cdot a_{2j-1})^2 \cdot a_{2j} = -(q_{2j-1})^2 \cdot a_{2j}$

(by the induction hypothesis, since $q_{2j-1} = b_j T^{-3j} = a_1 \cdot a_2 \cdot ... \cdot a_{2j-1}$)
Continued fractions for finite sums

\[ \frac{T^{-3j+1}}{b_{j+1}} \] \( \cdot \) \( ( -T^{-3j} \frac{b_j^2}{b_{j+1}} ) \] which we wanted to prove.

We immediately have the following

**Proposition**

Let \( x_0 = [ a_0 ] \), \( x_1 = [ a_0, a_1 ] \) and if \( x_n = [ a_0, a_1, ..., a_{2n-1} ] \), then

\[ x_{n+1} = [ a_0, a_1, ..., a_{2n-1} , a_{2n}, -a_{2n-1} , ..., -a_1 ] . \]

If \( n \) is at least two, then the continued fraction of \( x_n \) consists only of the partial quotients \( a_{2n-1} , a_{2n-2} , -a_{2n-2} , ..., a_1 , -a_1 \) and \( a_0 \).

Then the distribution of the partial quotients for \( x_n \) is as follows ( \( n \geq 2 \)):

<table>
<thead>
<tr>
<th>partial quotient</th>
<th>number of occurrences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{2n-1} )</td>
<td>1</td>
</tr>
<tr>
<td>( a_{2n-2} )</td>
<td>1</td>
</tr>
<tr>
<td>( -a_{2n-2} )</td>
<td>2</td>
</tr>
<tr>
<td>( a_{2n-3} )</td>
<td>2</td>
</tr>
<tr>
<td>( -a_{2n-3} )</td>
<td>2</td>
</tr>
<tr>
<td>( a_{2i} )</td>
<td>( 2n-i-2 )</td>
</tr>
<tr>
<td>( -a_{2i} )</td>
<td>( 2n-i-2 )</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>( 2n-2 )</td>
</tr>
<tr>
<td>( -a_1 )</td>
<td>( 2n-2 )</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>1</td>
</tr>
</tbody>
</table>

**Proof**

We give a proof by induction on \( n \).

\( x_2 = [ a_0, a_1, a_2, a_3 ] = [ a_0, a_1, -a_1, a_2 ] \), so the quotients \( a_0, a_1, -a_1, a_2 \), occur once.

So for \( n \) equal to 2 the assertion holds. Suppose it holds for \( 2 \leq n \leq j \). Then we prove it holds for \( n = j+1 \). Since \( x_{j+1} = [ a_0, a_1, ..., a_{2j+1} ] = [ a_0, a_1, ..., a_{2j+1} , a_{2j}, -a_{2j+1}, ..., -a_1 ] \), it is clear that the partial quotients \( a_{2j} \) and \( a_0 \) occur only once.

In the partial quotients \( a_1, ..., a_{2j-1} \) we have

<table>
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<td>( a_{2j-1} )</td>
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</tr>
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<td>( -a_{2j-2} )</td>
<td>2</td>
</tr>
<tr>
<td>( a_{2j-3} )</td>
<td>2</td>
</tr>
<tr>
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<tr>
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<td>( 2j-i-2 )</td>
</tr>
<tr>
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</tr>
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<td>( a_1 )</td>
<td>( 2j-2 )</td>
</tr>
<tr>
<td>( -a_1 )</td>
<td>( 2j-2 )</td>
</tr>
</tbody>
</table>

so in the partial quotients \( -a_1, ..., -a_{2j-1} \) we have

<table>
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</tr>
<tr>
<td>( -a_1 )</td>
<td>( 2j-2 )</td>
</tr>
</tbody>
</table>
This proves the proposition.

Using theorem 1, we immediately have the following:

**Theorem 2**

Let \((b_n)\) be a sequence such that \(b_n\) is different from zero for all \(n\) different from zero.

Define a sequence \((x_n)\) by putting \(x_0 = [b_0]\), \(x_1 = [b_0, b_1 z^{-1}]\) and if \(x_n = [a_0, a_1, ..., a_{2n-1}]\) then setting \(x_{n+1} = [a_0, a_1, ..., a_{2n-1}, -b^{2n}/b_{n+1} z^{n-1}, -a_{2n-1}, ..., -a_1]\),

then \(x_n = \sum_{i=0}^{n} b_i z^i\) for all \(n \in \mathbb{N}\).

**Proof**

Replace \(b_i\) by \(b_i z^i\) in theorem 1, and put \(T\) equal to one.

**Some examples**

1) Let \(x_n = \sum_{i=0}^{n} x^i\) (i.e. \(b_i = 1\) for all \(i\)). Then \(a_0 = 1\), \(a_1 = x^1\) and \(a_{2n} = -x^{n-1} (n \geq 1)\).

2) Let \(x_n = \sum_{i=0}^{n} \frac{x^i}{i!}\) (i.e. \(\lim_{n \to \infty} x_n = e^x\)).

Then \(a_0 = 1\), \(a_1 = x^1\) and \(a_{2n} = -\frac{n+1}{n!} x^{n-1} (n \geq 1)\).

3) Let \(x_n = \sum_{i=0}^{n} (-1)^i \frac{x^{2i}}{(2i)!}\) (i.e. \(\lim_{n \to \infty} x_n = \cos x\)).

Then \(a_0 = 1\), \(a_1 = -2x^2\) and \(a_{2n} = (-1)^n \frac{(2n+2)(2n+1)}{(2n)!} x^{2n-2} (n \geq 1)\).

4) Let \(x_n = \sum_{i=0}^{n} (-1)^i \frac{x^{2i+1}}{(2i+1)!}\) (i.e. \(\lim_{n \to \infty} x_n = \sin x\)).

Then \(a_0 = x\), \(a_1 = -6x^3\) and \(a_{2n} = (-1)^n \frac{(2n+3)(2n+2)}{(2n+1)!} x^{2n-1} (n \geq 1)\).

In an analogous way as in the previous theorem, we have...
Theorem 3

Let \((b_n)\) be a sequence such that \(b_n\) is different from zero for all \(n\) different from zero, and let \(q\) be a natural number different from zero and one.

Define a sequence \((x_n)\) by putting \(x_0 = [b_0 z]\), \(x_1 = [b_0 z, b_1^{-1} z^{-q}]\) and if \(x_n = [a_0, a_1, ..., a_{2n-1}]\) then setting \(x_{n+1} = [a_0, a_1, ..., a_{2n-1}, -b_0^2/b_{n+1} z^{-q(q-2)} -a_{2n-1}, ..., -a_1]\).

Then \(x_n = \sum_{i=0}^{n} b_i z^i\) for all \(n \in \mathbb{N}\).

Proof

Replace \(b_i\) by \(b_i z^i\) in theorem 1, and put \(T\) equal to one.

An Example

In [4] we find the following:

Let \(\mathbb{F}_q\) be the finite field of cardinality \(q\). Let \(A = \mathbb{F}_q[X], K = \mathbb{F}_q((1/X)), K_\infty = \mathbb{F}_q((1/X))\) and let \(\Omega\) be the completion of an algebraic closure of \(K_\infty\). Then \(A, K, K_\infty, \Omega\) are well-known analogous of \(\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\) respectively.

Let \([i] = X^i - X\) (the symbol \([i]\) does not have the same meaning as in \(x_0 = [a_0]\)). This is just the product of monic irreducible elements of \(A\) of degree dividing \(i\).

Let \(D_0 = 1, D_i = [i] D_{i-1}\) if \(i > 0\). This is the product of monic elements of \(A\) of degree \(i\).

Let us introduce the following function: \(e(Y) = \sum_{i=0}^{\infty} \frac{Y^q^i}{D_i}\) (\(Y \in \Omega\)).

Then Thakur gives the following theorem:

Define a sequence \(x_n\) by setting \(x_1 = [0, Y^q D_1]\) and if \(x_n = [a_0, a_1, ..., a_{2n-1}]\) then setting

\[x_{n+1} = [a_0, a_1, ..., a_{2n-1}, -Y^q(q-2)D_{n+1}/D_n^2, -a_{2n-1}, ..., -a_1]\]

then \(x_n = \sum_{i=1}^{n} \frac{Y^q^i}{D_i}\) for all \(n \in \mathbb{N}\).

In particular, \(e(Y) = Y + \lim_{n \to \infty} x_n\).

If we put \(b_i = D^{-1}_i\) if \(i > 0\), and \(b_0 = 0\) in theorem 3, then we find the result of Thakur.
3. Continued fractions for sums of the type \( \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \)

In this section, \( b_i \) is a constant in \( z \), and \( c(i) \) is a natural number. Our first theorem in this section gives the continued fraction for the sum \( \sum_{i=0}^{n} \frac{b_i}{z^{2i}} \) (i.e. \( c(i) = 2i \) for all \( i \)):

**Theorem 4**

Let \( \{b_n\} \) be a sequence such that \( b_n \) is different from zero for all \( n \). A continued fraction for the sum \( \sum_{i=0}^{n} \frac{b_i}{z^{2i}} \) can be given as follows:

\[
\frac{1}{x_0} = [0, z/b_0], \ x_1 = [0, z \cdot \frac{b_1}{b_0}, \frac{b_3}{b_1^2} + \frac{b_2}{b_1}, \ldots]
\]

and if \( x_k = [a_0, a_1, \ldots, a_{2k}] \) then setting

\[
x_{k+1} = [a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-1} a_{2k} - \gamma_{k+1}^{-1}, a_{2k+1}, \ldots, a_{2k+1}]
\]

where \( \gamma_{k+1} = b_{k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} \),

\[
a_{2k+i} = \gamma_{k+1}^{-1} a_{2k+i+1} \quad \text{if } i \text{ is even}, \quad \text{and} \quad a_{2k+i} = \gamma_{k+1}^{-1} a_{2k+i+1} \quad \text{if } i \text{ is odd (} 2 \leq i \leq 2k \),
\]

then \( x_k = \sum_{i=0}^{k} \frac{b_i}{z^{2i}} \) for all \( k \in \mathbb{N} \).

**Proof**

If we have \( x_n = [a_0, a_1, \ldots, a_{2n}] = \frac{p_{2n}}{q_{2n}} \), we show by induction that \( x_n \) equals \( \sum_{i=0}^{n} \frac{b_i}{z^{2i}} \), and

that \( q_{2n} \) equals \( z^{2n} \frac{b_{2n}}{b_1^{2n}} \). For \( n = 0, 1 \) this follows by an easy calculation.

Suppose the assertion holds for \( 0 \leq n \leq k \). Then we show it holds for \( n = k+1 \).

The first part of the proof, i.e. showing that \( x_{k+1} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2i}} \) is analogous to the first part of the proof of [2], theorem 1.

\[
x_{k+1} = [a_0, a_1, \ldots, a_{2k+1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-1} a_{2k} - \gamma_{k+1}^{-1}, a_{2k+2}, \ldots, a_{2k+1}]
\]

\[
= [a_0, a_1, \ldots, a_{2k+1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-2}, a_{2k+2}, \ldots, a_{2k+1}]
\]

( using the definition of a continued fraction )

Now if \( [a_0, a_1, \ldots, a_{2k}] = \frac{p_{2k}}{q_{2k}} \), then \( [a_0, a_1, \ldots, a_{2k+1}] = \frac{p_{2k+1}}{q_{2k+1}} \) and so
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\[[ a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1} ] = \frac{(a_{2k} + \gamma_{k+1})p_{2k-1} + p_{2k-2}}{(a_{2k} + \gamma_{k+1})q_{2k-1} + q_{2k-2}} = \frac{p_{2k} + \gamma_{k+1}p_{2k-1}}{q_{2k} + \gamma_{k+1}q_{2k-1}} \]

(by i) and ii) of the lemma

Then \[[ a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+2} a_{2k} - \gamma_{k+1} \] = \frac{(\gamma_{k+2}^{-1} a_{2k} - \gamma_{k+1}^{-1})(p_{2k} + \gamma_{k+1}p_{2k-1}) + p_{2k-1}}{(\gamma_{k+1}^{-1} a_{2k} - \gamma_{k+1}^{-1})(q_{2k} + \gamma_{k+1}q_{2k-1}) + q_{2k-1}} \]

(by i) and ii) of the lemma

And so

\[[ a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+2} a_{2k} - \gamma_{k+1} a_{2k-1}, a_{2k-2}, a_{2k-3}, \ldots, a_2, a_1 ] \]

\[= \frac{a_{2k} q_{2k-1} p_{2k} + \gamma_{k+1} a_{2k} q_{2k-1} p_{2k-1} + \gamma_{k+1} a_{2k-1} q_{2k-1} p_{2k-2} + \gamma_{k+1} a_{2k-2} q_{2k-2} p_{2k-1}}{a_{2k} q_{2k-1} q_{2k} + \gamma_{k+1} a_{2k-1} q_{2k} q_{2k-1} + \gamma_{k+1} a_{2k-2} q_{2k-2} q_{2k-1}} \]

(by iv) of the lemma

If we use the following equalities

\[(p_n - p_{n-2})q_{n-1} = a_n p_{n-2} q_{n-1} \]
\[(q_n - q_{n-2})p_n = a_n p_n q_{n-1} \]
\[(q_n - q_{n-2})q_n = a_n q_n q_{n-1} \]
\[(q_n - q_{n-2})q_{n-1} = a_n^2 q_{n-1} \]

(by i) of the lemma

then we find that the numerator equals \(q_{2k} p_{2k} + \gamma_{k+1}\) (by iii) of the lemma) and the denominator equals \((q_{2k})^2\).

So we conclude

\[x_{k+1} = \frac{p_{2k}}{q_{2k}} + \frac{\gamma_{k+1}}{(q_{2k})^2} = \sum_{i=0}^{k} \frac{b_i}{z^{2i} + \frac{(b_i)^{2k+1}}{(b_{2k})^{2k+1}} b_{k+1} = \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2i}} \]

We still have to show \(q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} \).

In the same way as in the proof of theorem 1, we find that \(q_{2k+1} = C z^{2k+1}\) where \(C\) is a constant.

Let \(\alpha_i\) be the coefficient of \(z\) in \(a_i\).

Then for \(C\), the coefficient of \(z^{2k+1}\) in \(q_{2k+1}\), we have

\[C = \alpha_1 \alpha_2 \ldots \alpha_{2k} (\gamma_{k+1}^{-1} \alpha_{2k}) (\gamma_{k+1}^{-2} \alpha_{2k-1}) (\gamma_{k+1}^{-2} \alpha_{2k-2}) (\gamma_{k+1}^{-2} \alpha_{2k-3}) \ldots (\gamma_{k+1}^{-2} \alpha_1) \]

\[= (\alpha_1 \alpha_2 \ldots \alpha_{2k} \alpha_{2k})^2 = (\text{coefficient of } z^{2k} \text{ in } q_{2k})^2 = \left( \frac{(b_0)^2}{(b_1)^{2k}} \right)^2 = \frac{(b_0)^{2k}}{(b_1)^{2k}} \]

and we conclude \(q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}}\). This finishes the proof.
Some examples

1) If we put $b_i$ equal to one for all $i$, and $z$ is an integer at least 3, then we find theorem 1 of [2]:

Let $B(u,v) = \sum_{i=0}^{v} \frac{1}{u^{2^i}} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^4} + \ldots + \frac{1}{u^{2^v}} \quad (u \geq 3, u \text{ an integer})$

Then $B(u,0) = [0,u]$, $B(u,1) = [0,u-1,u+1]$, and if $B(u,v) = [a_0, a_1, \ldots, a_n] = \frac{P_n}{Q_n}$

then $B(u,v+1) = [a_0, a_1, \ldots, a_n-1, a_n, a_n+1, a_n, a_n-1, a_n-2, \ldots, a_2, a_1]$.

2) Put $b_i = \lambda^i$. Then we have $x_0 = [0, u]$, $x_1 = [0, u - \lambda \frac{u}{\lambda} + \frac{1}{\lambda}]$ and if $x_k = [a_0, a_1, \ldots, a_{2k}]$, then $x_{k+1} = [a_0, a_1, \ldots, a_{2k-1}, \gamma_{k+1}, a_{2k}, \gamma_{k+1}, a_{2k+1}, \ldots, a_{2k+1}]$, where $\gamma_{k+1} = \lambda^{k+1-2k+1}$, $a_{2k+i} = \gamma_{k+1} a_{2k+i-1} + \lambda^{k+1} a_{2k+i+1}$ if $i$ is even, and $a_{2k+i} = \gamma_{k+1} a_{2k+i+1}$ if $i$ is odd ($2 \leq i \leq 2k$),

then $x_k = \sum_{i=0}^{k} \frac{\lambda^i}{u^{2^i}}$ for all $k \in \mathbb{N}$.

For some some sequences $(b_n)$ and $(c(n))$, we can give a continued fraction for the sum

$$\sum_{i=0}^{v} \frac{b_i}{2c(i)}$$

as follows:

Theorem 5

Let $(b_n)$ be a sequence such that $b_n \neq 0$ for all $n$, and $b_0 \neq 0$, 1, -1, and 1/2, and let $(c(n))$ be a sequence such that $c(0) = 0$, and $c(n+1) - 2c(n) \geq 0$.

Put $x_0 = [-b_0, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1] = [a_0, a_1, a_2] = \frac{P_2}{Q_2} = \frac{P(0)}{Q(0)}$, and if $x_v = [a_0, a_1, \ldots, a_n] = \frac{P_n}{Q_n} = \frac{P(v)}{Q(v)}$,

then setting $x_{v+1} = [a_0, a_1, \ldots, a_n, \alpha_v 2c(v) - 1, 1, a_{n-1}, a_{n-1}, \ldots, a_2, a_1]$, where $d(v) = c(v+1) - 2c(v)$, $\alpha_v = \frac{b_{2v+2}}{b_{v+1}}$ if $v \geq 1$ and $\alpha_0 = \frac{b_0}{b_1}$,

then $x_v = \sum_{i=0}^{v} \frac{b_i}{2c(i)}$ for all $v$ in $\mathbb{N}$, and $q(v) = \frac{2c(v)}{b_v}$ if $v \geq 1$, $q(0) = \frac{1}{(b_0)^2}$.
Remarks

1) The special form of $b_0, x_0 = b_0 = [ -b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1 ] = [ a_0, a_1, a_2 ]$ is needed since in the expression $[ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n}$ the integer $n$ must be even.

2) The value of $n$ is $n = 2^{v+1} + 2^v + 2$ (this can be easily seen by induction).

3) The only partial quotients that appear are $-b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1, \frac{1}{b_0}, -2, \alpha_v z^{d(v)} - 1, 1$, so $b_0$ must be different from $0, 1, -1,$ and $1/2$.

Proof

For $v$ equal to $0$, $1$ or $2$ we find this result by an easy computation.

We prove the theorem by induction on $v$.

Suppose we have $x_v = \sum_{i=0}^{v} b_i z^{c(i)} = [ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n}$ with $q_v = \frac{z_v}{b_v}$.

Then we show that $x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_{n-1}, a_{n-1}, ..., a_2, a_1 ] = \sum_{i=0}^{v+1} b_i z^{c(i)}$

with $q_{v+1} = \frac{z_{v+1}}{b_{v+1}}$.

The first part of the proof, i.e. showing that $x_{v+1} = \sum_{i=0}^{v+1} b_i z^{c(i)}$, is analogous to the first part of the proof of the theorem in [3].

Now, by repeated use of i) and ii) of the lemma, we have

$$[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1 ] = \frac{(\alpha_v z^{d(v)} - 1)p_n + p_{n-1}}{(\alpha_v z^{d(v)} - 1)q_n + q_{n-1}}$$

$$[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1 ] = \frac{\alpha_v z^{d(v)} p_n + p_{n-1}}{\alpha_v z^{d(v)} q_n + q_{n-1}}$$

$$[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_{n-1} ] = \frac{a_n \alpha_v z^{d(v)} p_n + a_n p_{n-1} - p_n}{a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n}$$

$$x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_{n-1}, a_{n-1}, ..., a_1 ]$$

$$= [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_{n-1}, [a_{n-1}, ..., a_1]]$$

( using the definition of a continued fraction )
\[ \begin{align*}
&= a_n q_{n-1} \alpha_v z^{d(v)} p_n + q_{n-2} \alpha_v z^{d(v)} p_n + a_n q_{n-1} p_{n-1} - q_{n-1} p_n + q_{n-2} p_{n-1} \\
&= a_n q_{n-1} \alpha_v z^{d(v)} q_n + q_{n-2} \alpha_v z^{d(v)} q_n + a_n (q_{n-1})^2 - q_{n-1} q_n + q_{n-2} q_{n-1}
\end{align*} \]

(by i), ii) and iv) of the lemma)

\[ \begin{align*}
&= \frac{p_n + 1}{q_n (q_n)^2 \alpha_v z^{d(v)}} \\
&= \frac{1}{(q_n)^2 \alpha_v z^{d(v)}} \\
&= \sum_{i=0}^{v} b_i z^{c(i)} + \frac{(b_v)^2}{z^{c(v)} (b_v)^2 z^{d(v)}}
\end{align*} \]

(by i) and iii) of the lemma since \( n \) is even

So \( x_{v+1} = \frac{p_n + 1}{q_n (q_n)^2 \alpha_v z^{d(v)}} = \sum_{i=0}^{v+1} b_i z^{c(i)} \)

We still have to prove \( q_{v+1} = q_{2n+2} = (q_n)^2 \alpha_v z^{d(v)} \).

We cannot use the same trick here as in the proofs of theorems 1 and 4, since we do not necessarily have \( \deg q_{k+1} > \deg q_k \) (\( q_k \) as a polynomial in \( z \)).

We already know that \( q_{n+1} = (\alpha_v z^{d(v)} - 1) q_n + q_{n-1}, \)

\( q_{n+2} = \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_{n-2} \) (where we put \( a_n = r_1 \))

\( q_{n+3} = q_{n+2} + q_{n+1} = (q_{n+1})^2 \alpha_v z^{d(v)} q_n + (q_n)^2 \alpha_v z^{d(v)} q_n + q_{n-2} \) (where we put \( a_{n-1}a_n+1 = r_2 \))

\( q_{n+4} = (q_{n+3})^2 + q_{n+2} = (a_{n-1}a_n+1) \alpha_v z^{d(v)} q_n - a_{n-1} q_{n-2} + q_n = r_2 \alpha_v z^{d(v)} q_n + q_{n-3} \)

(\( \) where we put \( a_{n-1}a_n+1 = r_2 \))

\( q_{n+5} = q_{n+4} + q_{n+1} = (a_{n-2}(a_{n-1}a_n+1) + a_n) \alpha_v z^{d(v)} q_n + a_{n-2} q_{n-3} - q_{n-2} \)

\( = r_3 \alpha_v z^{d(v)} q_n - q_{n-4} \) (where we put \( a_{n-2}(a_{n-1}a_n+1) + a_n = r_3 \))

etc...

Continuing this way, we find

\[ q_{n+2+k} = r_k \alpha_v z^{d(v)} q_n + (-1)^k q_{n-(k+1)} \]

(\( q_{n+2+k+1} = r_{k+1} \alpha_v z^{d(v)} q_n + (-1)^{k+1} q_{n-(k+2)} \))

Then \( q_{n+2+k+2} = (a_{n-(k+1)}r_{k+1}+r_k) \alpha_v z^{d(v)} q_n + (-1)^{k+1} a_{n-(k+1)} q_{n-k} + (-1)^k q_{n-k-1} \)

\( = r_{k+2} \alpha_v z^{d(v)} q_n + (-1)^{k+2} q_{n-(k+3)} \)

and finally we have \( q_{2n} = q_{n+2+n-2} = r_{n+2} \alpha_v z^{d(v)} q_n + q_{n-(n-1)} \)
Continued fractions for finite sums

\[ q_{2n+1} = q_{(n+2)+n-1} = r_{n-1} \alpha_v z^{d(v)} q_n - q_{n-1} \] (we remark that \( n \) is even)

and so \( q_{2n+2} = q_{(n+2)+n} = r_n \alpha_v z^{d(v)} q_n - a_1 q_0 + q_1 = r_n \alpha_v z^{d(v)} q_n \)

So if we want to show that \( q_{2n+2} = (q_n)^2 \alpha_v z^{d(v)} \), we must show that \( r_n \) equals \( q_n \).

For the sequence \( (r_n) \) we have \( r_0 = 1 \), \( r_1 = a_n \), \( r_2 = a_n a_{n+1} + 1 = a_{n-1} r_1 + r_0 \),
\( r_3 = a_{n-2}(a_n a_{n+1}) + a_n = a_{n-2} r_2 + r_1 \), and continuing this way we find \( r_{k+2} = a_{n-(k+1)} r_{k+1} + r_k \).

From this it follows that \( [1, a_n, \ldots, a_1] = [1, c_1, \ldots, c_n] = \frac{t_n}{r_n} \) (we put \( a_i = c_{n+1-i} \))

with \( t_0 = c_0 \), \( r_0 = 1 \), \( t_1 = c_1 c_0 + 1 \), \( r_1 = c_1 \),
\( t_n = c_n t_{n-1} + t_{n-2} \), \( r_n = c_n r_{n-1} + r_{n-2} \) \((n \geq 2)\),

Now \( n \) can be written as \( n = 2k+2 \) (see remark 2 following theorem 5) and so
\( [a_0, a_1, \ldots, a_n] = [a_0, a_1, \ldots, a_k, \alpha_{v-1} z^{d(v-1)} - l, 1, a_k - 1, a_k, \ldots, a_1] = \frac{p_n}{q_n} \)

and then \( [1, a_1, \ldots, a_k, \alpha_{v-1} z^{d(v-1)} - l, 1, a_k - 1, a_k, \ldots, a_1] = [1, a_1, \ldots, a_n] = \frac{p_n}{q_n} \)

where the \( q_i \) \((0 \leq i \leq n)\) stay the same since \( q_i \) does not depend on \( a_0 \).

So \( [1, a_1, \ldots, a_{k-1}, a_k - 1, l, \alpha_{v-1} z^{d(v-1)} - l, a_k, a_k, \ldots, a_1] = [1, a_n, \ldots, a_1] = \frac{t_n}{r_n} \)

and we conclude \( q_i = r_i \) for \( 0 \leq i \leq k-1 \).

We have to show \( q_n = r_n \). Now (by repeated use of i) of the lemma)
\( q_k = a_k q_{k-1} + q_{k-2} \), \( r_k = q_k - q_{k-1} \);
\( q_{k+1} = \alpha_{v-1} z^{d(v-1)} q_k - q_{k+1}, \) \( r_{k+1} = q_k \);
\( q_{k+2} = \alpha_{v-1} z^{d(v-1)} q_k + q_{k+1}, \) \( r_{k+2} = \alpha_{v-1} z^{d(v-1)} q_k - q_{k+1} \);
\( q_{k+3} = q_{(k+2)+1} = \alpha_{v-1} z^{d(v-1)} a_k q_k + a_k q_{k-1} - q_k = a_k \alpha_{v-1} z^{d(v-1)} q_k - q_{k-2} \)
\( = R_i \alpha_{v-1} z^{d(v-1)} q_k - q_{k-2} \), where we put \( a_k = R_i \),
\( r_{k+3} = r_{(k+2)+1} = a_k \alpha_{v-1} z^{d(v-1)} q_k + q_{k-2} = R_i \alpha_{v-1} z^{d(v-1)} q_k + q_{k-2} \);
\( q_{k+4} = q_{(k+2)+2} = (a_{k-1} a_{k+1}) \alpha_{v-1} z^{d(v-1)} q_k - a_k q_{k-2} + q_{k+1} \)
\( = (a_{k-1} a_{k+1}) \alpha_{v-1} z^{d(v-1)} q_k + q_{k-3} \)
\( = R_2 \alpha_{v-1} z^{d(v-1)} q_k + q_{k-3} \) where we put \( (a_{k-1} a_{k+1}) = R_2 \),
\[ r_{k+4} = r_{(k+2)+2} = (a_{k-1}a_{k+1}+1)\alpha v \cdot z^{d(v-1)}q_k + a_{k-1}q_{k-2} - q_{k-1} \]
\[ = (a_{k-1}a_{k+1})\alpha v \cdot z^{d(v-1)}q_k - q_{k-3} = R_2\alpha v \cdot z^{d(v-1)}q_k - q_{k-3} \]

.......

If we continue this way, we find \( q_{i+1} = R_i\alpha v \cdot z^{d(v-1)}q_k + (-1)^i q_{k-(i+1)} \), and
\[ r_{(k+2)+1} = R_k\alpha v \cdot z^{d(v-1)}q_k - (-1)^i q_{k-(i+1)} \quad (0 \leq i \leq k, R_0 = 1) \), and so we have
\[ q_{2k} = q_{(k+2)+k-2} = R_k\alpha v \cdot z^{d(v-1)}q_k + q_{k-(k-1)} \], \( q_{2k+1} = q_{(k+2)+k-1} = R_k\alpha v \cdot z^{d(v-1)}q_k - q_{k-k} \) (we remark that \( k \) is even) and thus \( q_{2k+2} = q_{(k+2)+k} = R_k\alpha v \cdot z^{d(v-1)}q_k - a_1q_{k-0} + q_{k-1} = R_k\alpha v \cdot z^{d(v-1)}q_k \),
and \( r_{2k} = r_{(k+2)+k-2} = R_k\alpha v \cdot z^{d(v-1)}q_k - q_{k-(k-1)} \), \( r_{2k+1} = r_{(k+2)+k-1} = R_k\alpha v \cdot z^{d(v-1)}q_k + q_{k-k} \) and thus \( r_{2k+2} = r_{(k+2)+k} = R_k\alpha v \cdot z^{d(v-1)}q_k + a_1q_{k-0} - q_{k-1} = R_k\alpha v \cdot z^{d(v-1)}q_k \),
so we conclude that \( q_{2k+2} = q_n \) equals \( r_{2k+2} = r_n \). This finishes the proof.

The case \( b_i \) equal to one, where \( z \) is an integer at least two, is studied by Shallit ([3]):

Let \( (c(k)) \) be a sequence of positive integers such that \( c(v+1) \geq 2c(v) \) for all \( v \geq v' \), where \( v' \) is a non-negative integer. Let \( d(v) = c(v+1) - 2c(v) \). Define \( S(u,v) \) as follows:

\[ S(u,v) = \sum_{i=0}^{v} u^{-c(i)} \], where \( u \) is an integer, \( u \geq 2 \). Then Shallit proved the following theorem:

Suppose \( v \geq v' \). If \( S(u,v) = [a_0, a_1, ..., a_n] \) and \( n \) is even, then
\[ S(u,v+1) = [a_0, a_1, ..., a_n, u^{d(v)-1}l, a_{n-1}, a_{n-2}, ..., a_2, a_1] \].

References


