ANN VERDOODT

Continued fractions for finite sums

Annales mathématiques Blaise Pascal, tome 1, n° 2 (1994), p. 71-84

<http://www.numdam.org/item?id=AMBP_1994__1_2_71_0>
CONTINUED FRACTIONS FOR FINITE SUMS

Ann Verdoott

Abstract

Our aim in this paper is to construct continued fractions for sums of the type
\[ \sum_{i=0}^{n} b_i z^{c(i)} \] or \[ \sum_{i=0}^{n} b_i / z^{c(i)} \], where \( b_n \) is a sequence such that \( b_n \) is different from zero if \( n \) is different from zero, and \( c(n) \) is an element of \( \mathbb{N} \).

1. Introduction

\([ a_0, a_1, a_2, \ldots ]\) denotes the continued fraction \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \),

and \([ a_0, a_1, \ldots, a_n ]\) denotes \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + a_{n-1} + \frac{1}{a_n}}} \).

The \( a_i \)'s are called the partial quotients (or simply the quotients), and \([ a_0, a_1, \ldots, a_n ]\) is called a finite continued fraction.

Our aim in this paper is to construct continued fractions for sums of the type \( \sum_{i=0}^{n} b_i z^{c(i)} \) or \( \sum_{i=0}^{n} b_i / z^{c(i)} \), where \( c(i) \) is an element of \( \mathbb{N} \).
In section 2, we find continued fractions for finite sums of the type \( \sum_{i=0}^{n} b_i z^i \) (\( c(i) = i \))
or \( \sum_{i=0}^{n} b_i z^i \) (\( c(i) = q_i \)), where \( (b_n) \) is a sequence such that \( b_n \) is different from zero if \( n \) is different from zero, and where \( q \) is a natural number different from zero and one.

Therefore, we start by giving a continued fraction for the sum \( \sum_{i=0}^{n} b_i T^i \), where \( b_i \) is different from zero for all \( i \) different from zero (\( b_i \) is a constant in \( T \)). This can be found in theorem 1.

If we replace \( b_i \) by \( b_i z^i \) in theorem 1, and we put \( T \) equal to one, we find a continued fraction for \( \sum_{i=0}^{n} b_i z^i \) (theorem 2), and if we replace \( b_i \) by \( b_i z^i \) in theorem 1, and we put \( T \) equal to one, we find a continued fraction for \( \sum_{i=0}^{n} b_i z^i \) (theorem 3) (\( q \) is a natural number different from zero and one).

In section 3 we find continued fractions for finite sums of the type \( \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \), for some sequences \( (b_n) \) and \( (c(n)) \), where \( c(n) \) is a natural number.

In theorem 4, we find a result for \( c(i) \) equal to \( 2^i \) (for all \( i \)).

Finally, in theorem 5, we give a continued fraction for \( \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}} \), where \( c(0) \) equals zero, and \( c(n+1) - 2c(n) \geq 0 \).

The results in this paper are extensions of results that can be found in [2], [3] and [4].

Acknowledgement: I thank professor Van Hamme for the help and the advice he gave me during the preparation of this paper.
2. Continued fractions for sums of the type \( \sum_{i=0}^{n} b_i z^i \)

All the proofs in sections 2 and 3 can be given with the aid of the following simple lemma:

**Lemma**

Let

\[
\begin{align*}
p_0 &= a_0, & q_0 &= 1, & p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\
p_n &= a_n p_{n-1} + p_{n-2}, & q_n &= a_n q_{n-1} + q_{n-2} \quad (n \geq 2),
\end{align*}
\]

then we have

ii) \( \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] \)

iii) \( p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad (n \geq 1) \)

iv) \( \frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1] \quad (n \geq 1) \)

These well-known results can e.g. be found in [1].

First we give a continued fraction for the sum \( \sum_{i=0}^{n} b_i T^i \), where \( b_i \) is different from zero for all \( i \) different from zero (\( b_i \) is a constant in \( T \)):

**Theorem 1**

Let \( (b_n) \) be a sequence such that \( b_n \neq 0 \) for all \( n > 0 \).

Define a sequence \( (x_n) \) by putting \( x_0 = [b_0 T] \), \( x_1 = [b_0 T, b_1 T^3] \), and if 

\[
x_n = [a_0, a_1, \ldots, a_{2n-1}] \text{ then setting } \\
x_{n+1} = [a_0, a_1, \ldots, a_{2n-1}, -b_{2n}/b_{n+1} T^{-3n}, -a_{2n-1}, \ldots, -a_1].
\]

Then \( x_n = \sum_{i=0}^{n} b_i T^i \) for all \( n \in \mathbb{N} \).

**Proof**

For \( n = 0 \) the theorem clearly holds.

If \( n \) is at least one, we prove that \( x_n = \sum_{i=0}^{n} b_i T^i \) and \( q_{2n+1} = b_1 T^{-3n} \).

We prove this by induction. For \( n = 1 \) the assertion holds.
Suppose it holds for $1 \leq n \leq j$. We then prove the assertion for $n = j+1$.

$x_{j+1} = \lfloor a_0, a_1, ..., a_{2j+1-1} \rfloor$.

$$x_{j+1} = \frac{-q_{2j-1} p_{2j} + q_{2j-2} p_{2j-1}}{-q_{2j-1} q_{2j} + q_{2j-2} q_{2j-1}}$$  \hspace{1cm} (by i), ii) and iv) of the lemma)

$$= \frac{-q_{2j-1}(a_{2j}p_{2j-1} + p_{2j-2}) + q_{2j-2}p_{2j-1}}{-q_{2j-1}(a_{2j}q_{2j-1} + q_{2j-2}) + q_{2j-2}q_{2j-1}}$$  \hspace{1cm} (by i) of the lemma)

now we have $p_{2j-1} q_{2j-2} - p_{2j-2} q_{2j-1} = (-1)^{2j-2} = 1$  \hspace{1cm} (by iii) of the lemma)

$$= \frac{p_{2j-1}}{q_{2j-1}} - \frac{1}{a_{2j}(q_{2j-1})^2}$$

now $a_{2j}(q_{2j-1})^2 = -T^{3j} b_j^2 (b_j^{-1} T^{3j})^2 = -T^{3j+1} b_j^{-1}$

$$x_{j+1} = \lfloor a_0, a_1, ..., a_{2j+1-1} \rfloor = \sum_{i=0}^{j+1} b_i T^{3i}$$  \hspace{1cm} (by the induction hypothesis)

We still have to prove $q_{2j+1-1} = b_j^{-1} T^{3j+1}$. Let $k$ be at least one.

Then $p_k$ and $q_k$ are polynomials in $U = T^{-1}$. deg $q_k > deg q_{k-1}$, and the term with the highest degree in $q_k$ is given by $a_k \cdot a_{k-1} \cdot ... \cdot a_1$. This follows from i).

If $r$ is a polynomial in $U$ that divides $p_k$ and $q_k$, then $r$ must be a constant in $U$. This immediately follows from iii). If $r$ divides $p_k$ and $q_k$, then $r$ divides $(-1)^{k-1}$. So $r$ must be a constant.

Since $\sum_{i=0}^{j+1} b_i T^{3i} = \lfloor a_0, a_1, ..., a_{2j+1-1} \rfloor = \frac{p_{2j+1-1}}{q_{2j+1-1}}$, we have

$$\frac{p_{2j+1-1}}{q_{2j+1-1}} = \sum_{i=0}^{j+1} b_i T^{3i} = \sum_{i=0}^{j+1} b_i \frac{T^{3i} U^{3i+1-3i}}{U^{3i+1}} = \frac{b_{j+1} + \sum_{i=0}^{j} b_i U^{3j+1-3i}}{U^{3j+1}}$$

and we conclude that $q_{2j+1-1} = C U^{3j+1} = C T^{3j+1}$ where $C$ is a constant.

By the previous remark, we have that

$q_{2j+1-1} = C T^{3j+1} = C U^{3j+1} = a_1 \cdot a_2 \cdot ... \cdot a_{2j+1-1}$

$$= (-1)^{2j-1} (a_1 \cdot a_2 \cdot ... \cdot a_{2j-1})^2 \cdot a_{2j} = - (q_{2j-1})^2 \cdot a_{2j}$$  \hspace{1cm} (by the induction hypothesis, since $q_{2j-1} = b_j^{-1} T^{3j} = a_1 \cdot a_2 \cdot ... \cdot a_{2j-1}$)
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\[ \frac{\frac{b_1^2}{b_{j+1}}}{-\left(b^{-1} T^{-3} + \frac{b_1^2}{b_{j+1}}\right)} = \frac{T^{-3j+1}}{b_{j+1}} \]

which we wanted to prove.

We immediately have the following

Proposition

Let \( x_0 = [a_0] \), \( x_1 = [a_0, a_1] \) and if \( x_n = [a_0, a_1, \ldots, a_{2n-1}] \), then
\[
\begin{align*}
x_{n+1} &= \left[ a_0, a_1, \ldots, a_{2n-1}, a_{2n}, -a_{2n-1}, \ldots, -a_1 \right].
\end{align*}
\]

If \( n \) is at least two, then the continued fraction of \( x_n \) consists only of the partial quotients
\( a_{2n-1}, a_{2n-2}, -a_{2n-2}, \ldots, a_1, -a_1 \) and \( a_0 \).

Then the distribution of the partial quotients for \( x_n \) is as follows (\( n \geq 2 \)):

<table>
<thead>
<tr>
<th>Partial Quotient</th>
<th>Number of Occurrences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{2n-1} )</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
</tr>
<tr>
<td>(-a_{2n-2} )</td>
<td>1</td>
</tr>
<tr>
<td>( a_{2n-3} )</td>
<td>2</td>
</tr>
<tr>
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<td>2</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( 2^{n-i-2} )</td>
</tr>
<tr>
<td>( a_i )</td>
<td>( 2^{n-2} )</td>
</tr>
<tr>
<td>(-a_i )</td>
<td>( 2^{n-2} )</td>
</tr>
<tr>
<td>( a_0 )</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof

We give a proof by induction on \( n \).

\( x_2 = [a_0, a_1, a_2, a_3] = [a_0, a_1, a_2, -a_1] \), so the quotients \( a_0, a_1, -a_1, a_2 \), occur once.

So for \( n \) equal to 2 the assertion holds. Suppose it holds for \( 2 \leq n \leq j \). Then we prove it holds for \( n = j+1 \). Since \( x_{j+1} = [a_0, a_1, \ldots, a_{2j+1}] = [a_0, a_1, \ldots, a_{2j-1}, a_{2j}, -a_{2j-1}, \ldots, -a_1] \), it is clear that the partial quotients \( a_{2j} \) and \( a_0 \) occur only once.

In the partial quotients \( a_1, \ldots, a_{2j-1} \) we have

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so in the partial quotients \(-a_1, \ldots, -a_{2j-1}\) we have

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This proves the proposition.

Using theorem 1, we immediately have the following:

**Theorem 2**

Let \((b_n)\) be a sequence such that \(b_n\) is different from zero for all \(n\) different from zero.

Define a sequence \((x_n)\) by putting \(x_0 = [b_0]\), \(x_1 = [b_0, b_{-1}]\) and if \(x_n = [a_0, a_1, ..., a_{2n-1}]\)

then setting \(x_{n+1} = [a_0, a_1, ..., a_{2n-1}, -b_{2n}/b_{n+1}z^{n-1}, -a_{2n-1}, ..., -a_1]\),

then \(x_n = \sum_{i=0}^{n} b_i z^i\) for all \(n \in \mathbb{N}\).

**Proof**

Replace \(b_i\) by \(b_i z^i\) in theorem 1, and put \(T\) equal to one.

**Some examples**

1) Let \(x_n = \sum_{i=0}^{n} x^i\) (i.e. \(b_i = 1\) for all \(i\)). Then \(a_0 = 1\), \(a_1 = x\) and \(a_{2n} = -x^{n-1}\) (\(n \geq 1\))

2) Let \(x_n = \sum_{i=0}^{n} \frac{x^i}{i!}\) (i.e. \(\lim_{n \to \infty} x_n = e^x\)).

Then \(a_0 = 1\), \(a_1 = x\) and \(a_{2n} = -\frac{n+1}{n!} x^{n-1}\) (\(n \geq 1\))

3) Let \(x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}\) (i.e. \(\lim_{n \to \infty} x_n = \cos x\)).

Then \(a_0 = 1\), \(a_1 = -2x^2\) and \(a_{2n} = (-1)^n \frac{(2n+2)(2n+1)}{(2n)!} x^{2n-2}\) (\(n \geq 1\))

4) Let \(x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}\) (i.e. \(\lim_{n \to \infty} x_n = \sin x\)).

Then \(a_0 = x\), \(a_1 = -6x^3\) and \(a_{2n} = (-1)^n \frac{(2n+3)(2n+2)}{(2n+1)!} x^{2n-1}\) (\(n \geq 1\))

In an analogous way as in the previous theorem, we have
Theorem 3

Let \((b_n)\) be a sequence such that \(b_n\) is different from zero for all \(n\) different from zero, and let \(q\) be a natural number different from zero and one.

Define a sequence \((x_n)\) by putting \(x_0 = [b_0 z]\), \(x_1 = [b_0 z, b_1^{-1} z^q]\) and if \(x_n = [a_0, a_1, \ldots, a_{2n-1}]\) then setting \(x_{n+1} = [a_0, a_1, \ldots, -a_{2n-1}, \ldots, -a_1]\).

Then \(x_n = \sum_{i=0}^{n} b_i z^i\) for all \(n \in \mathbb{N}\).

Proof

Replace \(b_i\) by \(b_i z^i\) in theorem 1, and put \(T\) equal to one.

An Example

In [4] we find the following:

Let \(F_q\) be the finite field of cardinality \(q\). Let \(A = F_q[X]\), \(K = F_q((X))\), \(K_\infty = F_q((1/X))\) and let \(\Omega\) be the completion of an algebraic closure of \(K_\infty\). Then \(A, K, K_\infty, \Omega\) are well-known analogous of \(Z, Q, R, C\) respectively.

Let \([i] = X^i - X\) (the symbol \([i]\) does not have the same meaning as in \(x_0 = [a_0]\)). This is just the product of monic irreducible elements of \(A\) of degree dividing \(i\).

Let \(D_0 = 1, D_i = [i] D_{i-1}\) if \(i > 0\). This is the product of monic elements of \(A\) of degree \(i\).

Let us introduce the following function: \(e(Y) = \sum_{i=0}^{\infty} \frac{Y^q^i}{D_i}\) \((Y \in \Omega)\).

Then Thakur gives the following theorem:

Define a sequence \(x_n\) by setting \(x_1 = [0, Y^q D_1]\) and if \(x_n = [a_0, a_1, \ldots, a_{2n-1}]\) then setting

\[x_{n+1} = [a_0, a_1, \ldots, a_{2n-1}, -Y^q^{n(q-2)}D_{n+1}/D_n^2, -a_{2n-1}, \ldots, -a_1],\]

then \(x_n = \sum_{i=1}^{n} \frac{Y^q^i}{D_i}\) for all \(n \in \mathbb{N}\).

In particular, \(e(Y) = Y + \lim_{n \to \infty} x_n\).

If we put \(b_i = D_i^{-1}\) if \(i > 0\), and \(b_0 = 0\) in theorem 3, then we find the result of Thakur.
3. Continued fractions for sums of the type $\sum_{i=0}^{n} \frac{b_i}{z^{c(i)}}$

In this section, $b_i$ is a constant in $z$, and $c(i)$ is a natural number. Our first theorem in this section gives the continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$ (i.e. $c(i) = 2i$ for all $i$):

**Theorem 4**

Let $(b_n)$ be a sequence such that $b_n$ is different from zero for all $n$. A continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$ can be given as follows:

Put $x_0 = [0, z/b_0]$ , $x_1 = [0, z b_0 - b_1 b_2 b_0 , b_2 b_0 + b_1 b_0]$ and if $x_k = [a_0, a_1, ..., a_{2k}]$ then setting

$x_{k+1} = [a_0, a_1, ..., a_{2k} + \gamma_{k+1} , \gamma_{k+1}^{-2} a_{2k} + \gamma_{k+1}^{-1} , a_{2k+1}, a_{2k+2}, ..., a_{2k+1}]$ where $\gamma_{k+1} = b_{k+1} (b_0)^{2k+1} (b_1)^{2k+1}$,

$a_{2k+i} = \gamma_{k+1}^{-2} a_{2k+i+1}$ if $i$ is even, and $a_{2k+i} = \gamma_{k+1}^{-1} a_{2k+i+1}$ if $i$ is odd $(2 \leq i \leq 2k)$,

then $x_k = \sum_{i=0}^{k} \frac{b_i}{z^{2i}}$ for all $k \in \mathbb{N}$.

**Proof**

If we have $x_n = [a_0, a_1, ..., a_{2n}] = \frac{p_{2n}}{q_{2n}}$, we show by induction that $x_n$ equals $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$, and that $q_{2n}$ equals $z^{2n} \frac{b_{2n}}{b_1}$. For $n = 0, 1$ this follows by an easy calculation.

Suppose the assertion holds for $0 \leq n \leq k$. Then we show it holds for $n = k+1$.

The first part of the proof, i.e. showing that $x_{k+1} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2i}}$ is analogous to the first part of the proof of [2], theorem 1.

$x_{k+1} = [a_0, a_1, ..., a_{2k+1} , a_{2k} + \gamma_{k+1} , \gamma_{k+1}^{-2} a_{2k} + \gamma_{k+1}^{-1} , a_{2k+2}, ..., a_{2k+1}]$

$= [a_0, a_1, ..., a_{2k+1} , a_{2k} + \gamma_{k+1} , \gamma_{k+1}^{-2} a_{2k+1} + \gamma_{k+1}^{-1} [a_{2k+1}, a_{2k+2}, a_{2k+3}, ..., a_{2k+1}] ]$

( using the definition of a continued fraction )

Now if $[a_0, a_1, ..., a_{2k}] = \frac{p_{2k}}{q_{2k}}$, then $[a_0, a_1, ..., a_{2k-1}] = \frac{p_{2k-1}}{q_{2k-1}}$ and so
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\[ [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}] = \frac{(a_{2k} + \gamma_{k+1})p_{2k-1} + p_{2k-2}}{(a_{2k} + \gamma_{k+1})q_{2k-1} + q_{2k-2}} = \frac{p_{2k} + \gamma_{k+1}p_{2k-1}}{q_{2k} + \gamma_{k+1}q_{2k-1}} \]

(by i) and (ii) of the lemma

Then \[ [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1}] = \frac{(\gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1})(p_{2k} + \gamma_{k+1}p_{2k-1}) + p_{2k-1}}{(\gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1})(q_{2k} + \gamma_{k+1}q_{2k-1}) + q_{2k-1}} \]

(by i) and (ii) of the lemma

And so

\[ [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1}] [a_2, a_3, ..., a_{2k-1}, a_{2k-2}, a_{2k-3}, ..., a_2, a_1] \]

\[ = a_{2k} q_{2k-1} p_{2k} + \gamma_{k+1} a_{2k} q_{2k-1} p_{2k-1} - \gamma_{k+1} a_{2k-1} q_{2k} - \gamma_{k+1} q_{2k-1} p_{2k} + \gamma_{k+1} q_{2k-2} p_{2k-1} \]

(by iv) of the lemma

If we use the following equalities

\[ (p_n - p_{n-2})q_{n-1} = a_n p_{n-1} q_{n-1} \]
\[ (q_n - q_{n-2})p_n = a_n q_{n-1} \]
\[ (q_n - q_{n-2})q_n = a_n q_{n-1} \] (by i) of the lemma

then we find that the numerator equals \( q_{2k} p_{2k} + \gamma_{k+1} \) (by iii) of the lemma and the denominator equals \( (q_{2k})^2 \).

So we conclude

\[ x_{k+1} = \frac{p_{2k}}{q_{2k}} + \frac{\gamma_{k+1}}{(q_{2k})^2} = \sum_{i=0}^{k} \frac{b_i}{z^i} + \frac{(b_1)^{2k+1}}{Z^{2k+1}(b_0)^{2k+1}} b_{k+1} (b_0)^{2k+1} (b_1)^{2k+1} = \sum_{i=0}^{k+1} \frac{b_i}{Z^{2i}} \]

We still have to show \( q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} \).

In the same way as in the proof of theorem 1, we find that \( q_{2k+1} = C z^{2k+1} \) where \( C \) is a constant.

Let \( \alpha \) be the coefficient of \( z \) in \( a_i \).

Then for \( C \), the coefficient of \( z^{2k+1} \) in \( q_{2k+1} \), we have

\[ C = \alpha_1 \alpha_2 ... \alpha_{2k-1} \alpha_{2k} (\gamma_{k+1}^{-2} \alpha_{2k})(\gamma_{k+1}^{-2} \alpha_{2k-1})(\gamma_{k+1}^{-2} \alpha_{2k-2})(\gamma_{k+1}^{-2} \alpha_{2k-3}) ... (\gamma_{k+1}^{-2} \alpha_{1}) \]

\[ = (\alpha_1 \alpha_2 ... \alpha_{2k-1} \alpha_{2k})^2 = (\text{coefficient of } z^k \text{ in } q_k)^2 = \left( \frac{(b_0)^{2k}}{(b_1)^{2k}} \right)^2 = \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} \]

and we conclude \( q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} \). This finishes the proof.
Some examples

1) If we put \( b_i \) equal to one for all \( i \), and \( z \) is an integer at least 3, then we find theorem 1 of [2]:

Let \( B(u,v) = \sum_{i=0}^{v} \frac{1}{u^i} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \ldots + \frac{1}{u^v} \quad (u \geq 3, u \text{ an integer}) \).

Then \( B(u,0) = [0, u] \), \( B(u,1) = [0, u-1, u+1] \), and if \( B(u,v) = [a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n} \),

then \( B(u,v+1) = [a_0, a_1, \ldots, a_n-1, a_n+1, a_{n-1}, a_{n-1}, a_{n-2}, \ldots, a_2, a_1] \).

2) Put \( b_i = \lambda^i \). Then we have \( x_0 = [0, u] \), \( x_1 = [0, u - \frac{u}{\lambda^2} + \frac{1}{\lambda}] \) and if \( x_k = [a_0, a_1, \ldots, a_{2k}] \),

then \( x_{k+1} = [a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-1} a_{2k-1}, a_{2k-2}, \ldots, a_{2k+1}] \), where \( \gamma_{k+1} = \lambda^{k+1-2k+1} \),

\( a_{2k+i} = \gamma_{k+1}^2 a_{2k-i+1} \) if \( i \) is even, and \( a_{2k+i} = \gamma_{k+1}^2 a_{2k-i+1} \) if \( i \) is odd (\( 2 \leq i \leq 2k \)),

then \( x_k = \sum_{i=0}^{k} \frac{\lambda^i}{u^{2i}} \) for all \( k \in \mathbb{N} \).

For some some sequences \((b_n)\) and \((c(n))\), we can give a continued fraction for the sum

\[ \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}} \] as follows:

Theorem 5

Let \((b_n)\) be a sequence such that \( b_n \neq 0 \) for all \( n \), and \( b_0 \neq 0, 1, -1, \) and \( 1/2 \), and let \((c(n))\) be a sequence such that \( c(0) = 0 \), and \( c(n+1) - 2c(n) \geq 0 \).

Put \( x_0 = [-b_0, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1] = [a_0, a_1, a_2] = \frac{p_2}{q_2} = \frac{p(n)}{q(n)} \),

and if \( x_v = [a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n} = \frac{p(v)}{q(v)} \),

then setting \( x_{v+1} = [a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1, a_{n-1}, a_{n-1}, \ldots, a_2, a_1] \),

where \( d(v) = c(v+1) - 2c(v) \), \( \alpha_v = \frac{b_v^2}{b_{v+1}} \) if \( v \geq 1 \) and \( \alpha_0 = \frac{b_0^4}{b_1} \),

then \( x_v = \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}} \) for all \( v \) in \( \mathbb{N} \), and \( q_v = \frac{z^{c(v)}}{b_v} \) if \( v \geq 1 \), \( q(0) = \frac{1}{(b_0)^2} \).
Remarks

1) The special form of \( b_0, x_0 = b_0 = [ -b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1 ] = [ a_0, a_1, a_2 ] \) is needed since in the expression \( [ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n} \) the integer \( n \) must be even.

2) The value of \( n \) is \( n = 2^{v+1} + 2^v + 2 \) (this can be easily seen by induction).

3) The only partial quotients that appear are \( -b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1, \frac{1}{b_0} - 2, \alpha_v z^{d(v)} - 1, 1 \), so \( b_0 \) must be different from \( 0, 1, -1, \) and \( 1/2 \).

Proof

For \( v \) equal to \( 0, 1 \) or \( 2 \) we find this result by an easy computation.

We prove the theorem by induction on \( v \).

Suppose we have \( x_v = \sum_{i=0}^{v} b_i z^{c(i)} = [ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n} = \frac{p(v)}{q(v)} \) with \( q(v) = \frac{z^{c(v)}}{b_v} \).

Then we show that \( x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, ..., a_2, a_1 ] = \sum_{i=0}^{v+1} \frac{b_i}{z^{c(i)}} \)

with \( q(v+1) = \frac{z^{c(v+1)}}{b_{v+1}} \).

The first part of the proof, i.e. showing that \( x_{v+1} = \sum_{i=0}^{v+1} \frac{b_i}{z^{c(i)}} \), is analogous to the first part of the proof of the theorem in [3].

Now, by repeated use of i) and ii) of the lemma, we have

\[
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1 ] = \frac{(\alpha_v z^{d(v)} - 1)p_n + p_{n-1}}{(\alpha_v z^{d(v)} - 1)q_n + q_{n-1}};
\]

\[
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1 ] = \frac{\alpha_v z^{d(v)} p_n + p_{n-1}}{\alpha_v z^{d(v)} q_n + q_{n-1}};
\]

\[
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1 ] = \frac{a_n \alpha_v z^{d(v)} p_n + a_n p_{n-1} - p_n}{a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n}
\]

\( x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, ..., a_1 ] \)

\( = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, [a_{n-1}, ..., a_1] ] \)

( using the definition of a continued fraction )
\[
\begin{align*}
\frac{a_n q_{n-1} \alpha_v z^{d(v)} p_n + q_{n-2} \alpha_v z^{d(v)} p_n + a_n q_{n-1} p_n - q_{n-1} p_n + q_{n-2} p_{n-1}}{a_n q_{n-1} \alpha_v z^{d(v)} q_n + q_{n-2} \alpha_v z^{d(v)} q_n + a_n(q_{n-1})^2 - q_{n-1} q_n + q_{n-2} q_{n-1}} = & \ \\
\text{(by i), ii) and iv) of the lemma)}
\end{align*}
\]
\[
\frac{p_n}{q_n} + \frac{1}{(q_n)^2 \alpha_v z^{d(v)}} = & \ \\
\text{(by i) and iii) of the lemma since } n \text{ is even)}
\]
\[
\text{So } x_{v+1} = \frac{p_n}{q_n} + \frac{1}{(q_n)^2 \alpha_v z^{d(v)}} = \sum_{i=0}^{v} b_i z^{c(i)} + \frac{(b_v)^2 b_{v+1}}{z^{c(v)}(b_v)^2 z^{d(v)}} \text{ since } q_n = q(v) = z^{c(v)} b_v, \alpha_v = (by)^2 b_{v+1}i=0
\]
\[
\sum_{i=0}^{v+1} b_i z^{c(i)}
\]

We still have to prove \( q_{(v+1)} = q_{2n+2} = \left(\alpha_v z^{d(v)} - 1\right) q_n + q_{n-1} = \alpha_v z^{d(v)} q_n + q_{n-1} \)

Repeated use of i) of the lemma gives
\[
\begin{align*}
q_{n+3} & = q_{(n+2)+1} = a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n = r_1 \alpha_v z^{d(v)} q_n - q_{n-2} & \text{(where we put } a_n = r_1) \\
q_{n+4} & = q_{(n+2)+2} = (a_n, a_{n+1})\alpha_v z^{d(v)} q_n - a_n q_{n-2} + q_{n-1} = r_2 \alpha_v z^{d(v)} q_n + q_{n-3} \\
q_{n+5} & = q_{(n+2)+3} = (a_n, a_{n+1})\alpha_v z^{d(v)} q_n + a_n q_{n-3} - q_{n-2} \\
& = r_3 \alpha_v z^{d(v)} q_n - q_{n-4} & \text{(where we put } a_n, a_{n+1} = r_2) \\
& \text{etc...}
\end{align*}
\]

Continuing this way, we find
\[
q_{(n+2)+k} = r_k \alpha_v z^{d(v)} q_n + (-1)^k q_{n-(k+1)} \text{, } q_{(n+2)+k+1} = r_{k+1} \alpha_v z^{d(v)} q_n + (-1)^{k+1} q_{n-(k+2)}
\]

Then \( q_{(n+2)+k+2} = (a_n, a_{n+1}) r_{k+1} \alpha_v z^{d(v)} q_n + (-1)^{k+1} a_n(q_{n+1})q_{n-k-2} + (-1)^k q_{n-k-1} \)
\[
= r_{k+2} \alpha_v z^{d(v)} q_n + (-1)^{k+2} q_{n-(k+3)}
\]

and finally we have \( q_{2n} = q_{(n+2)+n-2} = r_{n+2} \alpha_v z^{d(v)} q_n + q_{n-(n-1)} \)
Continued fractions for finite sums

\[ q_{2n+1} = q_{(n+2)+n-1} = r_{n-1} \alpha v z^{d(v)} q_n - q_{n-n} \] (we remark that \( n \) is even)

and so

\[ q_{2n+2} = q_{(n+2)+n} = r_n \alpha v z^{d(v)} q_n - a_1 q_0 + q_1 = r_n \alpha v z^{d(v)} q_n \]

So if we want to show that \( q_{2n+2} = (q_n)^2  \alpha v z^{d(v)} \), we must show that \( r_n \) equals \( q_n \).

For the sequence \( (r_n) \) we have \( r_0 = 1, r_1 = a_n, r_2 = a_n a_n + 1 = a_n r_1 + r_0 \),

\[ r_3 = a_n a_n a_n + 1 + a_n = a_n r_2 + r_1 \], and continuing this way we find \( r_{k+2} = a_n r_k + r_{k+1} \).

From this it follows that \( [1, a_n, ..., a_1] = [1, c_1, ..., c_n] = \frac{t_n}{r_n} \) (we put \( a_i = c_{n-i} \))

with \( t_0 = c_0, r_0 = 1, t_1 = c_1 c_0 + 1, r_1 = c_1, t_n = c_n t_{n-1} + t_{n-2}, r_n = c_n r_{n-1} + r_{n-2} \) \( (n \geq 2) \).

Now \( n \) can be written as \( n = 2k+2 \) (see remark 2 following theorem 5) and so

\[ [a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_k, \alpha v z^{d(v)-1} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = \frac{p_n}{q_n} \]

and then \( [1, a_1, ..., a_k, \alpha v z^{d(v)-1} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = [1, a_1, ..., a_n] = \frac{p_n}{q_n} \)

where the \( q_i \) \( (0 \leq i \leq n) \) stay the same since \( q_i \) does not depend on \( a_0 \).

So \( [1, a_1, ..., a_k-1, a_{k-1}, a_{k-2}, a_{k-1}, ..., a_1] = [1, a_n, ..., a_1] = \frac{t_n}{r_n} \)

and we conclude \( q_i = r_i \) for \( 0 \leq i \leq k-1 \).

We have to show \( q_n = r_n \). Now (by repeated use of i) of the lemma)

\[ q_k = a_k q_{k-1} + q_{k-2}, r_k = q_k - q_{k-1} \]

\[ q_{k+1} = \alpha v z^{d(v-1)} q_k - q_k + q_{k-1}, r_{k+1} = q_k \]

\[ q_{k+2} = \alpha v z^{d(v-1)} q_k + q_{k-1}, r_{k+2} = \alpha v z^{d(v-1)} q_k - q_{k-1} \]

\[ q_{k+3} = q_{(k+2)+1} = \alpha v z^{d(v-1)} a_k q_k + a_k q_{k-1} - q_k = a_k \alpha v z^{d(v-1)} q_k - q_{k-2} \]

\[ = R_1 \alpha v z^{d(v-1)} q_k - q_{k-2}, \text{where we put } a_k = R_1, \]

\[ r_{k+3} = r_{(k+2)+1} = a_k \alpha v z^{d(v-1)} q_k + q_{k-2} = R_1 \alpha v z^{d(v-1)} q_k + q_{k-2} \]

\[ q_{k+4} = q_{(k+2)+2} = (a_{k-1} a_{k+1}) \alpha v z^{d(v-1)} q_k - a_{k-1} q_{k-2} + q_{k-1} \]

\[ = (a_{k-1} a_{k+1}) \alpha v z^{d(v-1)} q_k + q_{k-3} \]

where we put \( (a_{k-1} a_{k+1}) = R_2 \).
\[ r_{k+4} = r_{(k+2)+2} = (a_k 1 a_k + 1) \alpha_{v-1} Z^{d(v-1)} q_k + a_k 1 q_{k-2} - q_{k-1} \]

\[ = (a_k 1 a_k + 1) \alpha_{v-1} Z^{d(v-1)} q_k - q_{k-3} = R_2 \alpha_{v-1} Z^{d(v-1)} q_k - q_{k-3} \]

.....

If we continue this way, we find \( q_i = R_i \alpha_{v-1} Z^{d(v-1)} q_k + (-1)^i q_{k-i} \), and

\[ r_{(k+2)+i} = R_i \alpha_{v-1} Z^{d(v-1)} q_k - (-1)^i q_{k-i+1} \] \( (0 \leq i \leq k, R_0 = 1) \), and so we have

\[ q_{2k} = q_{(k+2)+2} = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k + q_{k-k} \]

\[ q_{2k+1} = q_{(k+2)+k} = R_k 1 \alpha_{v-1} Z^{d(v-1)} q_k - q_{k-k} \] ( we remark that \( k \) is even ) and thus \( q_{2k+2} = q_{(k+2)+k} = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k - a_1 q_{k-0} + q_k = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k \),

and \[ r_{2k} = r_{(k+2)+k} = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k - q_{k-k} \] , \( r_{2k+1} = r_{(k+2)+k} = R_k 1 \alpha_{v-1} Z^{d(v-1)} q_k + q_{k-k} \) and thus \( r_{2k+2} = r_{(k+2)+k} = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k + a_1 q_{k-0} - q_k = R_k 2 \alpha_{v-1} Z^{d(v-1)} q_k \),

So we conclude that \( q_{2k+2} = q_n \) equals \( r_{2k+2} = r_n \). This finishes the proof.

The case \( b_i \) equal to one, where \( z \) is an integer at least two, is studied by Shallit ([3]):

Let \( (c(k)) \) be a sequence of positive integers such that \( c(v+1) \geq 2c(v) \) for all \( v \geq v' \), where \( v' \)

is a non-negative integer. Let \( d(v) = c(v+1) - 2c(v) \). Define \( S(u,v) \) as follows:

\[ S(u,v) = \sum_{i=0}^{v} u^{-c(i)} \], where \( u \) is an integer , \( u \geq 2 \). Then Shallit proved the following theorem:

Suppose \( v \geq v' \). If \( S(u,v) = \{ a_0, a_1, ..., a_n \} \) and \( n \) is even , then

\[ S(u,v+1) = \{ a_0, a_1, ..., a_n, u^{d(v)-1}, 1, a_{n-1}, a_{n-1}, a_{n-2}, ..., a_2, a_1 \} \].

References


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