ANN VERDOODT

Continued fractions for finite sums

Annales mathématiques Blaise Pascal, tome 1, n° 2 (1994), p. 71-84

<http://www.numdam.org/item?id=AMBP_1994__1_2_71_0>
CONTINUED FRACTIONS FOR FINITE SUMS

Ann Verdooit

Abstract

Our aim in this paper is to construct continued fractions for sums of the type
\[
\sum_{i=0}^{n} b_i z^{c(i)} \text{ or } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}},
\]
where \((b_n)\) is a sequence such that \(b_n\) is different from zero if \(n\) is different from zero, and \(c(n)\) is an element of \(\mathbb{N}\).

Résumé

Le but est de construire des fractions continues pour des sommes du type
\[
\sum_{i=0}^{n} b_i z^{c(i)} \text{ or } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}},
\]
où \((b_n)\) est une suite telle que \(b_n\) est différent de zéro pour \(n\) différent de zéro, et \(c(n)\) est un élément de \(\mathbb{N}\).

1. Introduction

\([a_0, a_1, a_2, \ldots]\) denotes the continued fraction
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}},
\]
and \([a_0, a_1, \ldots, a_n]\) denotes
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots a_{n-1} + \frac{1}{a_n}}},
\]
The \(a_i\)'s are called the partial quotients (or simply the quotients), and \([a_0, a_1, \ldots, a_n]\) is called a finite continued fraction.

Our aim in this paper is to construct continued fractions for sums of the type
\[
\sum_{i=0}^{n} b_i z^{c(i)} \text{ or } \sum_{i=0}^{n} \frac{b_i}{z^{c(i)}},
\]
where \(c(i)\) is an element of \(\mathbb{N}\).
In section 2, we find continued fractions for finite sums of the type \[ \sum_{i=0}^{n} b_i z^i \quad (c(i) = i) \]
or \[ \sum_{i=0}^{n} b_i z^i \quad (c(i) = q^i), \]
where \( (b_n) \) is a sequence such that \( b_n \) is different from zero if \( n \) is different from zero, and where \( q \) is a natural number different from zero and one.

Therefore, we start by giving a continued fraction for the sum \[ \sum_{i=0}^{n} b_i T^{3i} \], where \( b_i \) is different from zero for all \( i \) different from zero \( (b_i \) is a constant in \( T \) \). This can be found in theorem 1.

If we replace \( b_i \) by \( b_i z^i \) in theorem 1, and we put \( T \) equal to one, we find a continued fraction for \[ \sum_{i=0}^{n} b_i z^i \quad (\text{theorem } 2), \]
and if we replace \( b_i \) by \( b_i z^i \) in theorem 1, and we put \( T \) equal to one, we find a continued fraction for \[ \sum_{i=0}^{n} b_i z^i \quad (\text{theorem } 3) \quad (q \) is a natural number different from zero and one \).

In section 3 we find continued fractions for finite sums of the type \[ \sum_{i=0}^{n} b_i z^i \], for some sequences \( (b_n) \) and \( (c(n)) \), where \( c(n) \) is a natural number.

In theorem 4, we find a result for \( c(i) \) equal to \( 2^i \) \( \text{} \) for all \( i \).

Finally, in theorem 5, we give a continued fraction for \[ \sum_{i=0}^{n} \frac{b_i}{z^i} \], where \( c(0) \) equals zero, and \( c(n+1) - 2c(n) \geq 0 \).

The results in this paper are extensions of results that can be found in [2], [3] and [4].

Acknowledgement: I thank professor Van Hamme for the help and the advice he gave me during the preparation of this paper.
2. Continued fractions for sums of the type $\sum_{i=0}^{n} b_i z^i$

All the proofs in sections 2 and 3 can be given with the aid of the following simple lemma:

Lemma

Let

\begin{align*}
    p_0 &= a_0, & q_0 &= 1, & p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\
    p_n &= a_n p_{n-1} + p_{n-2}, & q_n &= a_n q_{n-1} + q_{n-2} \quad (n \geq 2),
\end{align*}

then we have

\begin{enumerate}
    \item $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$ \\
    \item $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \quad (n \geq 1)$ \\
    \item $\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1] \quad (n \geq 1)$
\end{enumerate}

These well-known results can e.g. be found in [1].

First we give a continued fraction for the sum $\sum_{i=0}^{n} b_i z^i$, where $b_i$ is different from zero for all $i$ different from zero ($b_i$ is a constant in $T$):

Theorem 1

Let $(b_n)$ be a sequence such that $b_n \neq 0$ for all $n > 0$.

Define a sequence $(x_n)$ by putting $x_0 = [b_0 T]$ , $x_1 = [b_0 T, b_1 T^3]$ , and if

\[ x_n = [a_0, a_1, \ldots, a_{2n-1}] \]

then setting

\[ x_{n+1} = [a_0, a_1, \ldots, a_{2n-1}, -b_{2n}/b_{n+1} T^{3n}, -a_{2n-1}, \ldots, -a_1] \]

Then

\[ x_n = \sum_{i=0}^{n} b_i T^{3i} \quad \text{for all } n \in \mathbb{N}. \]

Proof

For $n = 0$ the theorem clearly holds.

If $n$ is at least one, we prove that $x_n = \sum_{i=0}^{n} b_i T^{3i}$ and $q_{2n-1} = b_n T^{3n}$.

We prove this by induction. For $n = 1$ the assertion holds.
Suppose it holds for \( 1 \leq n \leq j \). We then prove the assertion for \( n = j + 1 \).

\[ x_{j+1} = [a_0, a_1, ..., a_{2j+1}, a_{2j+2}, a_{2j+3}, ..., a_{3j+1}] \]

( using the definition of a continued fraction )

\[ \frac{-q_{2j-1} p_{2j+1} + q_{2j} \ p_{2j+2}}{-q_{2j-1} q_{2j+2} + q_{2j-2} q_{2j+1}} \quad \text{(by i), ii) and iv) of the lemma )} \]

\[ \frac{-q_{2j-1} (a_{2j} p_{2j+1} + p_{2j+2}) + q_{2j-2} \ p_{2j+1}}{-q_{2j-1} (a_{2j} q_{2j+1} + q_{2j+2}) + q_{2j-2} q_{2j+1}} \quad \text{(by i) of the lemma) } \]

now we have \( p_{2j+1} q_{2j} - p_{2j} q_{2j-1} = (-1)^{2j-2} = 1 \) (by iii) of the lemma

\[ \frac{p_{2j+1}}{q_{2j}} = \frac{1}{a_{2j} (q_{2j})^2} \]

now \( a_{2j} (q_{2j})^2 = -T^{3j} a_{j+1} \) (by the induction hypothesis)

We still have to prove \( q_{2j+1} = b_{j+1} T^{3j+1} \). Let \( k \) be at least one.

Then \( p_k \) and \( q_k \) are polynomials in \( U = T^{-1} \). \( \deg q_k > \deg q_{k-1} \), and the term with the highest degree in \( q_k \) is given by \( a_k \cdot a_{k-1} \cdot ... \cdot a_1 \). This follows from i).

If \( r \) is a polynomial in \( U \) that divides \( p_k \) and \( q_k \), then \( r \) must be a constant in \( U \). This immediately follows from iii). If \( r \) divides \( p_k \) and \( q_k \), then \( r \) divides \( (-1)^{k-1} \). So \( r \) must be a constant.

Since

\[ \sum_{i=0}^{j+1} b_i T^{3i} = [a_0, a_1, ..., a_{2j+1}, a_{2j+2}, a_{2j+3}, ..., a_{3j+1}] = \frac{p_{2j+1}}{q_{2j+1}} \]

we have

\[ \frac{p_{2j+1}}{q_{2j+1}} = \sum_{i=0}^{j+1} b_i T^{3i} \frac{T^{3j+1}}{T^{3i}} = \sum_{i=0}^{j+1} b_i \frac{U^{3i+1} \cdot 3i}{U^{3i+1}} = \frac{b_{j+1} + \sum_{i=0}^{j} b_i U^{3i+1} \cdot 3i}{U^{3j+1}} \]

and we conclude that \( q_{2j+1} = C U^{3j+1} = C T^{3j+1} \) where \( C \) is a constant.

By the previous remark, we have that

\[ q_{2j+1} = C T^{3j+1} = C U^{3j+1} = a_1 \cdot a_2 \cdot ... \cdot a_{2j+1} \]

\[ = (-1)^{2j-1} (a_1 \cdot a_2 \cdot ... \cdot a_{2j+1})^2 \cdot a_{2j} = - (q_{2j-1})^2 \cdot a_{2j} \]

(by the induction hypothesis, since \( q_{2j-1} = b_j T^{3j} = a_1 \cdot a_2 \cdot ... \cdot a_{2j-1} \))
Continued fractions for finite sums

= \left( b_j^{-1} T^{-3j} \right)^2 \cdot \left( - T^{-3j} \frac{b_j^2}{b_{j+1}} \right) = \frac{T^{-3j+1}}{b_{j+1}} \quad \text{which we wanted to prove.}

We immediately have the following

**Proposition**

Let \( x_0 = [a_0] \), \( x_1 = [a_0, a_1] \) and if \( x_n = [a_0, a_1, ..., a_{2n-1}] \), then

\[ x_{n+1} = [a_0, a_1, ..., a_{2n-1}, a_{2n}, -a_{2n-1}, ..., -a_1] \]

If \( n \) is at least two, then the continued fraction of \( x_n \) consists only of the partial quotients

\( a_{2n-1}, a_{2n-2}, -a_{2n-2}, ..., a_1, -a_1 \) and \( a_0 \).

Then the distribution of the partial quotients for \( x_n \) is as follows (\( n \geq 2 \)):

<table>
<thead>
<tr>
<th>partial quotient</th>
<th>number of occurrences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{2n-1} )</td>
<td>1</td>
</tr>
<tr>
<td>( a_{2n-2} )</td>
<td>1</td>
</tr>
<tr>
<td>( -a_{2n-2} )</td>
<td>2</td>
</tr>
</tbody>
</table>
| \( a_{2n-3} \)   | 2\n\n| \( -a_{2n-3} \)  | 2\n\n| \( ... \)       | \( 2^{n-i-2} \) \n\n| \( a_2 \)        | 2\n\n| \( -a_2 \)       | 2\n\n| \( ... \)       | \( 2n-2 \) 
\n**Proof**

We give a proof by induction on \( n \).

\( x_2 = [a_0, a_1, a_2, a_3] = [a_0, a_1, a_2, -a_1] \), so the quotients \( a_0, a_1, -a_1, a_2 \), occur once.

So for \( n \) equal to 2 the assertion holds. Suppose it holds for \( 2 \leq n \leq j \). Then we prove it holds for \( n = j+1 \). Since \( x_{j+1} = [a_0, a_1, ..., a_{2j+1}] = [a_0, a_1, ..., a_{2j-1}, a_{2j}, -a_{2j-1}, ..., -a_1] \), it is clear that the partial quotients \( a_2 \) and \( a_0 \) occur only once.

In the partial quotients \( a_1, ..., a_{2j-1} \) we have

<table>
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</tr>
<tr>
<td>( a_{2j-2} )</td>
<td>1</td>
</tr>
<tr>
<td>( -a_{2j-2} )</td>
<td>2</td>
</tr>
</tbody>
</table>
| \( a_{2j-3} \)   | 2\n\n| \( -a_{2j-3} \)  | 2\n\n| \( ... \)       | \( 2^{j-i-2} \) \n\n| \( a_1 \)        | 2\n\n| \( -a_1 \)       | 2\n
so in the partial quotients \( -a_1, ..., -a_{2j-1} \) we have

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| \( a_{2j-3} \)   | 2\n\n| \( -a_{2j-3} \)  | 2\n\n| \( ... \)       | \( 2^{j-i-2} \) \n\n| \( a_1 \)        | 2\n\n| \( -a_1 \)       | 2\n
This proves the proposition.

Using theorem 1, we immediately have the following:

**Theorem 2**

Let \((b_n)\) be a sequence such that \(b_n\) is different from zero for all \(n\) different from zero.

Define a sequence \((x_n)\) by putting \(x_0 = [b_0]\), \(x_1 = [b_0, b_1, z^{-1}]\) and if \(x_n = [a_0, a_1, ..., a_{2^n-1}]\) then setting \(x_{n+1} = [a_0, a_1, ..., a_{2^n-1}, -b_n^{2} / b_{n+1} z^{n-1}, -a_{2^n-1}, ..., -a_1]\),

then \(x_n = \sum_{i=0}^{n} b_i z^i\) for all \(n \in \mathbb{N}\).

**Proof**

Replace \(b_i\) by \(b_i z^i\) in theorem 1, and put \(T\) equal to one.

**Some examples**

1) Let \(x_n = \sum_{i=0}^{n} x^i\) (i.e. \(b_i = 1\) for all \(i\)). Then \(a_0 = 1\), \(a_1 = x^{-1}\) and \(a_{2^n} = -x^{n-1}\) (\(n \geq 1\)).

2) Let \(x_n = \sum_{i=0}^{n} \frac{x^i}{i!}\) (i.e. \(\lim_{n \to \infty} x_n = e^x\)).

Then \(a_0 = 1\), \(a_1 = x^{-1}\) and \(a_{2^n} = -\frac{n+1}{n!} x^{n-1}\) (\(n \geq 1\)).

3) Let \(x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}\) (i.e. \(\lim_{n \to \infty} x_n = \cos x\)).

Then \(a_0 = 1\), \(a_1 = -2x^{-2}\) and \(a_{2^n} = (-1)^n \frac{(2n+2)(2n+1)}{(2n)!} x^{2n-2}\) (\(n \geq 1\)).

4) Let \(x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}\) (i.e. \(\lim_{n \to \infty} x_n = \sin x\)).

Then \(a_0 = x\), \(a_1 = -6x^{-3}\) and \(a_{2^n} = (-1)^n \frac{(2n+3)(2n+2)}{(2n+1)!} x^{2n-1}\) (\(n \geq 1\)).

In an analogous way as in the previous theorem, we have
Theorem 3

Let \( (b_n) \) be a sequence such that \( b_n \) is different from zero for all \( n \) different from zero, and let \( q \) be a natural number different from zero and one.

Define a sequence \( (x_n) \) by putting \( x_0 = \lfloor b_0 z \rfloor \), \( x_1 = \lfloor b_0 z, b_{-1} z^{-q} \rfloor \) and if \( x_n = \lfloor a_0, a_1, ..., a_{2n-1} \rfloor \) then setting \( x_{n+1} = \lfloor a_0, a_1, ..., -a_{2n-1}, ..., -a_1 \rfloor \).

Then \( x_n = \sum_{i=0}^{n} b_i z^i \) for all \( n \in \mathbb{N} \).

Proof

Replace \( b_i \) by \( b_i z_i \) in theorem 1, and put \( T \) equal to one.

An Example

In [4] we find the following:

Let \( F_q \) be the finite field of cardinality \( q \). Let \( A = F_q[X], K = F_q(X), K_\infty = F_q((1/X)) \) and let \( \Omega \) be the completion of an algebraic closure of \( K_\infty \). Then \( A, K, K_\infty, \Omega \) are well-known analogous of \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) respectively.

Let \( [i] = X^d - X \) (the symbol \( [i] \) does not have the same meaning as in \( x_0 = [a_0] \)). This is just the product of monic irreducible elements of \( A \) of degree dividing \( i \).

Let \( D_0 = 1, D_i = [i] D_{i-1} \) if \( i > 0 \). This is the product of monic elements of \( A \) of degree \( i \).

Let us introduce the following function: \( e(Y) = \sum_{i=0}^{\infty} \frac{Y^i}{D_i} \) \( (Y \in \Omega) \).

Then Thakur gives the following theorem:

Define a sequence \( x_n \) by setting \( x_1 = \lfloor 0, Y^d D_1 \rfloor \) and if \( x_n = \lfloor a_0, a_1, ..., a_{2n-1} \rfloor \) then setting

\[ x_{n+1} = \lfloor a_0, a_1, ..., a_{2n-1}, -Y^{q^n(q-2)} D_{n+1}/D_n^2, -a_{2n-1}, ..., -a_1 \rfloor, \text{ then } x_n = \sum_{i=1}^{n} \frac{Y^i}{D_i} \text{ for all } n \in \mathbb{N}. \]

In particular, \( e(Y) = Y + \lim_{n \to \infty} x_n \).

If we put \( b_i = D_{-1}^i \) if \( i > 0 \), and \( b_0 = 0 \) in theorem 3, then we find the result of Thakur.
3. Continued fractions for sums of the type $\sum_{i=0}^{n} \frac{b_i}{z^{c(i)}}$

In this section, $b_i$ is a constant in $z$, and $c(i)$ is a natural number. Our first theorem in this section gives the continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$ (i.e. $c(i) = 2i$ for all $i$):

**Theorem 4**

Let $(b_n)$ be a sequence such that $b_n$ is different from zero for all $n$. A continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$ can be given as follows:

Put $x_0 = [0, \frac{z}{b_0}]$, $x_1 = [0, \frac{z}{b_0} - \frac{b_1}{b_0}, \frac{b_2z}{b_1} + \frac{b_0}{b_1}]$ and if $x_k = [a_0, a_1, ..., a_{2k}]$ then setting

$x_{k+1} = [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^2 a_{2k} - \gamma_{k+1}^{-1}, a_{2k+2}, ..., a_{2k+1}]$ where $\gamma_{k+1} = \frac{b_{k+1}}{(b_0)^{2k+1}} (b_1)^{-2k+1}$,

$a_{2k+i} = \gamma_{k+1}^2 a_{2k+i-1}$ if $i$ is even, and $a_{2k+i} = \gamma_{k+1}^{-1} a_{2k+i+1}$ if $i$ is odd ($2 \leq i \leq 2k$),

then $x_k = \sum_{i=0}^{k} \frac{b_i}{z^{2i}}$ for all $k \in \mathbb{N}$.

**Proof**

If we have $x_n = [a_0, a_1, ..., a_{2n}] = \frac{p_{2n}}{q_{2n}}$, we show by induction that $x_n$ equals $\sum_{i=0}^{n} \frac{b_i}{z^{2i}}$, and that $q_{2n}$ equals $z^{2n} \frac{b_{2n}}{b_1}$. For $n = 0, 1$ this follows by an easy calculation.

Suppose the assertion holds for $0 \leq n \leq k$. Then we show it holds for $n = k+1$.

The first part of the proof, i.e. showing that $x_{k+1} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2i}}$ is analogous to the first part of the proof of [2], theorem 1.

$x_{k+1} = [a_0, a_1, ..., a_{2k}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^2 a_{2k} - \gamma_{k+1}^{-1}, a_{2k+2}, ..., a_{2k+1}]$

$= [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^2 a_{2k} - \gamma_{k+1}^{-1}, a_{2k+1} + a_{2k+2}, a_{2k+3}, ..., a_{2}, a_{2} ]$

( using the definition of a continued fraction )

Now if $[a_0, a_1, ..., a_{2k}] = \frac{p_{2k}}{q_{2k}}$, then $[a_0, a_1, ..., a_{2k-1}] = \frac{p_{2k-1}}{q_{2k-1}}$ and so
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[\[ a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1} \] = \frac{(a_{2k} + \gamma_{k+1})p_{2k-1} + p_{2k-2}}{(a_{2k} + \gamma_{k+1})q_{2k-1} + q_{2k-2}} = \frac{p_{2k} + \gamma_{k+1}p_{2k-1}}{q_{2k} + \gamma_{k+1}q_{2k-1}}

(by i) and (ii) of the lemma)

Then [\[ a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+2}, a_{2k} - \gamma_{k+1} \] = \frac{(\gamma_{k+2} - \gamma_{k+1})p_{2k} + \gamma_{k+1}p_{2k-1} + p_{2k-1}}{(\gamma_{k+2} - \gamma_{k+1})q_{2k} + \gamma_{k+1}q_{2k-1} + q_{2k-1}}

(by i) and (ii) of the lemma)

And so

[\[ a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, a_{2k} - \gamma_{k+1}, a_{2k} - \gamma_{k+1}, a_{2k} - \gamma_{k+1} \] = \frac{a_{2k}q_{2k-1}p_{2k} + \gamma_{k+1}a_{2k}q_{2k-1}p_{2k-1} - \gamma_{k+1}q_{2k} + \gamma_{k+1}q_{2k-1}p_{2k} + \gamma_{k+1}q_{2k-1}p_{2k-1}}{a_{2k}q_{2k-1}q_{2k} + \gamma_{k+1}a_{2k}q_{2k-1}q_{2k-1} - \gamma_{k+1}q_{2k} + \gamma_{k+1}q_{2k-1}q_{2k} + \gamma_{k+1}q_{2k-1}q_{2k-1}}

(by iv) of the lemma)

If we use the following equalities

(p_n - p_{n-2})q_{n-1} = a_n p_{n-1} q_{n-1}
(q_n - q_{n-2})p_n = a_n p_{n-1} q_{n-1}

then we find that the numerator equals q_{2k}p_{2k} + \gamma_{k+1} (by iii) of the lemma) and the denominator equals (q_{2k})^2.

So we conclude

[\[ x_{k+1} = \frac{p_{2k}}{q_{2k}} + \frac{\gamma_{k+1}}{(q_{2k})^2} = \sum_{i=0}^{k} \frac{b_i}{z^{2i}} + \frac{(b_1)^{2k+1}}{(b_1)^{2k+1}} b_{k+1} + \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2i}}

We still have to show q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}}.

In the same way as in the proof of theorem 1, we find that q_{2k+1} = C z^{2k+1} where C is a constant.

Let \alpha_i be the coefficient of z in a_i.

Then for C, the coefficient of z^{2k+1} in q_{2k+1}, we have

C = \alpha_1 \alpha_2 ... \alpha_{2k-1} \alpha_{2k} (\gamma_{k+1} \alpha_{2k})(\gamma_{k+1} \alpha_{2k-1})(\gamma_{k+1} \alpha_{2k-2})(\gamma_{k+1} \alpha_{2k-3}) ... (\gamma_{k+1} \alpha_1)

= (\alpha_1 \alpha_2 ... \alpha_{2k-1} \alpha_{2k})^2 = \text{(coefficient of } z^k \text{ in } q_k)^2 = \left( \frac{(b_0)^{2k}}{(b_1)^{2k}} \right)^2 = \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}}

and we conclude q_{2k+1} = z^{2k+1} \frac{(b_0)^{2k+1}}{(b_1)^{2k+1}}. This finishes the proof.
Some examples

1) If we put $b_i$ equal to one for all $i$, and $z$ is an integer at least 3, then we find theorem 1 of [2]:

Let $B(u,v) = \sum_{i=0}^{v} \frac{1}{u^i} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^4} + \ldots + \frac{1}{u^{2v}}$  \quad (u \geq 3, u \text{ an integer})

Then $B(u,0) = [0,u]$, $B(u,1) = [0,u-1,u+1]$, and if $B(u,v) = [a_0, a_1, ..., a_n] = \frac{p_n}{q_n}$
then $B(u,v+1) = [a_0, a_1, ..., a_{n-1}, a_n, a_n-1, a_{n-1}, a_{n-2}, ..., a_2, a_1]$.

2) Put $b_i = \lambda^i$. Then we have $x_0 = [0,u]$, $x_1 = [0,u - \frac{u}{\lambda} + \frac{1}{\lambda^2}]$ and if $x_k = [a_0, a_1, ..., a_{2k}]$,
then $x_{k+1} = [a_0, a_1, ..., a_{2k-1}, a_{2k} + \gamma_{k+1}, a_{2k-1}, a_{2k-2}, ..., a_2, a_1]$, where $\gamma_{k+1} = \lambda^k + 2k+1$,
$a_{2k+1} = \gamma_{k+1} a_{2k+i}$ if $i$ is even, and $a_{2k+i} = \gamma_{k+1} a_{2k-i+1}$ if $i$ is odd ($2 \leq i \leq 2k$),
then $x_k = \frac{\lambda^i}{u^{2i}}$ for all $k \in \mathbb{N}$.

For some some sequences $(b_n)$ and $(c(n))$, we can give a continued fraction for the sum

$$\sum_{i=0}^{v} \frac{b_i}{z^{c(i)}}$$

as follows:

**Theorem 5**

Let $(b_n)$ be a sequence such that $b_n \neq 0$ for all $n$, and $b_0 \neq 0$, 1, -1, and 1/2, and let $(c(n))$ be a sequence such that $c(0) = 0$, and $c(n+1) - 2c(n) \geq 0$.

Put $x_0 = [-b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1] = [a_0, a_1, a_2] = \frac{p_2}{q_2} = \frac{p(n)}{q(n)}$,
and if $x_v = [a_0, a_1, ..., a_n]$ = $\frac{p_n}{q_n} = \frac{p(v)}{q(v)}$,
then setting $x_{v+1} = [a_0, a_1, ..., a_v, a_{v+1}, a_{v+1}, 1, a_{v+1} - 1, a_{v+1} - 2, ..., a_2, a_1]$,
where $d(v) = c(v+1) - 2c(v)$, $\alpha_v = \frac{b_v^2}{b_{v+1}}$ if $v \geq 1$ and $\alpha_0 = \frac{b_0^4}{b_1}$,
then $x_v = \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}}$ for all $v$ in $\mathbb{N}$, and $q_v = \frac{z^{c(v)}}{b_v}$ if $v \geq 1$, $q(0) = \frac{1}{(b_0)^2}$.
Remarks

1) The special form of \( b_0, x_0 = b_0 = [-b_0, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1 ] = [ a_0, a_1, a_2 ] \) is needed since in the expression \( [ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n} \) the integer \( n \) must be even.

2) The value of \( n \) is \( n = 2^{v+1} + 2^v + 2 \) (this can be easily seen by induction).

3) The only partial quotients that appear are \(-b_0, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1, \frac{1}{b_0}, \frac{1}{b_0} - 2, \alpha_v z^{d(v)} - 1, 1\), so \( b_0 \) must be different from \( 0, 1, -1, \) and \( 1/2 \).

Proof

For \( v \) equal to \( 0, 1 \) or \( 2 \) we find this result by an easy computation.

We prove the theorem by induction on \( v \).

Suppose we have \( x_v = \sum_{i=0}^{v} \frac{b_i}{z^{d(i)}} = [ a_0, a_1, ..., a_n ] = \frac{p_n}{q_n} = \frac{p(v)}{q(v)} \) with \( q(v) = \frac{z^{c(v)}}{b_v} \).

Then we show that \( x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, ..., a_2, a_1 ] = \sum_{i=0}^{v+1} \frac{b_i}{z^{d(i)}} \) with \( q(v+1) = \frac{z^{c(v+1)}}{b_{v+1}} \).

The first part of the proof, i.e. showing that \( x_{v+1} = \sum_{i=0}^{v+1} \frac{b_i}{z^{d(i)}} \), is analogous to the first part of the proof of the theorem in [3].

Now, by repeated use of i) and ii) of the lemma, we have

\[
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1 ] = \frac{(\alpha_v z^{d(v)} - 1) p_n + p_{n-1}}{(\alpha_v z^{d(v)} - 1) q_n + q_{n-1}}; \\
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1 ] = \frac{\alpha_v z^{d(v)} p_n + p_{n-1}}{\alpha_v z^{d(v)} q_n + q_{n-1}}; \\
[ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1 ] = \frac{a_n \alpha_v z^{d(v)} p_n + a_n p_{n-1} - p_n}{a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n}; \\
x_{v+1} = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, ..., a_1 ] = [ a_0, a_1, ..., a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, [a_{n-1}, ..., a_1] ]
\]

( using the definition of a continued fraction )
\[ q_n = a_n \alpha_n z(d)(v) p_n + q_{n-2} \alpha_n z(d)(v) p_n + a_n q_{n-1} p_{n-1} - q_{n-1} p_n + q_{n-2} p_{n-1} \]

\[ a_n q_{n-1} \alpha_n z(d)(v) q_n + q_{n-2} \alpha_n z(d)(v) q_n + a_n(q_{n-1})^2 - q_{n-1} q_n + q_{n-2} q_{n-1} \]

(by i), ii) and iv) of the lemma)

\[ = \frac{p_n + 1}{q_n} (q_n)^2 \alpha_n z(d)(v) \]

(by i) and iii) of the lemma since \( n \) is even

\[ \text{So } x_{n+1} = \frac{p_n + 1}{q_n} (q_n)^2 \alpha_n z(d)(v) = \sum_{i=0}^{\infty} \frac{b_i}{z^2(i)} + \frac{(b_n^2 b_{n+1})}{z^2(c)(v) (b_n)^2 z(d)(v)} \text{ since } q_n = q(v) = \frac{z(c)(v)}{b_v}, \alpha_v = \frac{(b_n^2)}{b_{n+1}} \]

We still have to prove \( q(v+1) = q_{2n+2} = (q_n)^2 \alpha_n z(d)(v) \).

We cannot use the same trick here as in the proofs of theorems 1 and 4, since we do not necessarily have \( \deg q_{k+1} > \deg q_k \) (\( q_k \) as a polynomial in \( z \)).

We already know that \( q_{n+1} = (\alpha_n z(d)(v) - 1)q_n + q_{n-1}, q_{n+2} = \alpha_n z(d)(v) q_n + q_{n-1} \)

Repeated use of i) of the lemma gives

\[ q_{n+3} = q_{n+2} + a_n \alpha_n z(d)(v) q_n + a_n q_{n-1} - q_n = r_1 \alpha_n z(d)(v) q_n - q_{n-2} \]

(\( \text{where we put } a_n = r_1 \))

\[ q_{n+4} = q_{n+3} + a_n \alpha_n z(d)(v) q_n + a_n q_{n-2} + q_{n-1} = r_2 \alpha_n z(d)(v) q_n + q_{n-3} \]

(\( \text{where we put } a_n = r_2 \))

\[ q_{n+5} = q_{n+4} + a_n \alpha_n z(d)(v) q_n + a_n q_{n-3} - q_n = r_3 \alpha_n z(d)(v) q_n - q_{n-4} \]

(\( \text{where we put } a_n = r_3 \))

etc...

Continuing this way, we find

\[ q_{n+2} = r_k \alpha_n z(d)(v) q_n + (-1)^k q_{n-(k+1)} \]

\[ q_{n+2} = r_k \alpha_n z(d)(v) q_n + (-1)^k q_{n-(k+2)} \]

Then \( q_{n+2} = (a_n-r_1)q_n + (-1)^{k+1} a_n q_{n-(k+1)} + (-1)^k q_{n-k-1} \)

\[ = r_k \alpha_n z(d)(v) q_n + (-1)^{k+2} q_{n-(k+3)} \]

and finally we have \( q_{2n} = q_{n+2} = r_n \alpha_n z(d)(v) q_n + q_{n-(n-1)} \)
Continued fractions for finite sums

$q_{2n+1} = q_{(n+2)+n-1} = r_{n-1} \alpha_v z^{d(v)} q_n - q_{n-1}$ (we remark that $n$ is even)

and so $q_{2n+2} = q_{(n+2)+n} = r_n \alpha_v z^{d(v)} q_n - a_1 q_0 + q_1 = r_n \alpha_v z^{d(v)} q_n$

So if we want to show that $q_{2n+2} = (q_n)^2 \alpha_v z^{d(v)}$, we must show that $r_n$ equals $q_n$.

For the sequence $(r_n)$ we have $r_0 = 1, r_1 = a_n, r_2 = a_n a_{n+1} + 1 = a_n r_1 + r_0,$

$r_3 = a_n (a_n a_{n+1}) + a_n = a_n r_2 + r_1$, and continuing this way we find $r_{k+2} = a_{n-(k+1)} r_{k+1} + r_k$.

From this it follows that $[1, a_n, ..., a_1] = [1, c_1, ..., c_n] = \frac{t_n}{r_n}$ (we put $a_i = c_{n+1-i}$)

with $t_0 = c_0, t_1 = c_1 c_0 + 1, t_1 = c_1, t_n = c_n t_{n-1} + t_{n-2}, r_n = c_n r_{n-1} + r_{n-2}$ ($n \geq 2$).

Now $n$ can be written as $n = 2k+2$ (see remark 2 following theorem 5) and so

$[a_0, a_1, ..., a_n] = [a_0, a_1, ..., a_k, \alpha_{v-1} z^{d(v-1)} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = \frac{p_n}{q_n}$

and then $[1, a_1, ..., a_k, \alpha_{v-1} z^{d(v-1)} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = [1, a_1,..., a_n] = \frac{p_n}{q_n}$

where the $q_i$ ($0 \leq i \leq n$) stay the same since $q_i$ does not depend on $a_0$.

So $[1, a_1,..., a_{k-1}, a_k - 1, 1, \alpha_{v-1} z^{d(v-1)} - 1, a_k, a_{k-1}, ..., a_1] = [1, a_n,..., a_1] = \frac{t_n}{r_n}$

and we conclude $q_i = r_i$ for $0 \leq i \leq k-1$.

We have to show $q_n = r_n$. Now (by repeated use of 1) of the lemma)

$q_k = a_k q_{k-1} + q_{k-2}, r_k = q_k - q_{k-1};$

$q_{k+1} = \alpha_{v-1} z^{d(v-1)} q_k - q_{k-1} = q_k;$

$q_{k+2} = \alpha_{v-1} z^{d(v-1)} q_k + q_{k-1}, r_{k+2} = \alpha_{v-1} z^{d(v-1)} q_k - q_{k-1};$

$q_{k+3} = q_{(k+2)+1} = \alpha_{v-1} z^{d(v-1)} a_k q_{k} + a_k q_{k-1} - q_k = a_k \alpha_{v-1} z^{d(v-1)} q_k - q_{k-2}$

$= R_1 \alpha_{v-1} z^{d(v-1)} q_{k} - q_{k-2}$, where we put $a_k = R_1$,

$r_{k+3} = r_{(k+2)+1} = a_k \alpha_{v-1} z^{d(v-1)} q_k + q_{k-2} = R_1 \alpha_{v-1} z^{d(v-1)} q_k + q_{k-2};$

$q_{k+4} = q_{(k+2)+2} = (a_k a_{k+1}) \alpha_{v-1} z^{d(v-1)} q_k - a_k q_{k-2} + q_{k+1}$

$= (a_k a_{k+1}) \alpha_{v-1} z^{d(v-1)} q_k + q_{k-3}$

$= R_2 \alpha_{v-1} z^{d(v-1)} q_k + q_{k-3}$, where we put $(a_k a_{k+1}) = R_2$, 

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If we continue this way, we find \( q_{k+2} = R_k \alpha_{v-1}z^{d(v-1)}q_k + (-1)^k q_{k-(i+1)} \) and
\[
\begin{align*}
q_{2k} &= q_{(k+2) + k-2} = R_k \alpha_{v-1}z^{d(v-1)}q_k + q_{k-(k-1)} = R_{k-1} \alpha_{v-1}z^{d(v-1)}q_k - q_{k-k} \\
r_{2k} &= r_{(k+2) + k-2} = R_k \alpha_{v-1}z^{d(v-1)}q_k - q_{k-(k-1)} = R_{k-1} \alpha_{v-1}z^{d(v-1)}q_k + q_{k-k}
\end{align*}
\]
and thus \( q_{2k+2} = q_{(k+2) + k} = R_k \alpha_{v-1}z^{d(v-1)}q_k + a_1q_0 + q_1 = R_k \alpha_{v-1}z^{d(v-1)}q_k \),
and \( r_{2k+2} = r_{(k+2) + k} = R_k \alpha_{v-1}z^{d(v-1)}q_k + a_1q_0 - q_1 = R_k \alpha_{v-1}z^{d(v-1)}q_k \),
So we conclude that \( q_{2k+2} = q_n \) equals \( r_{2k+2} = r_n \). This finishes the proof.

The case \( b_i \) equal to one, where \( z \) is an integer at least two, is studied by Shallit ([3]):

Let \( (c(k)) \) be a sequence of positive integers such that \( c(v+1) \geq 2c(v) \) for all \( v \geq v' \), where \( v' \) is a non-negative integer. Let \( d(v) = c(v+1) - 2c(v) \). Define \( S(u,v) \) as follows:

\[
S(u,v) = \sum_{i=0}^{v} u^{-c(i)} , \text{ where } u \text{ is an integer, } u \geq 2 . \text{ Then Shallit proved the following theorem :}
\]

Suppose \( v \geq v' \). If \( S(u,v) = [a_0, a_1, \ldots, a_n] \) and \( n \) is even, then
\[
S(u,v+1) = [a_0, a_1, \ldots, a_n, u^{d(v)-1}, 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].
\]

References


