STANY DE SMEDT

Orthonormal bases for $p$-adic continuous and continuously differentiable functions


<http://www.numdam.org/item?id=AMBP_1995__2_1_275_0>
ORTHONORMAL BASES FOR P-ADIC CONTINUOUS AND
CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

Stany De Smedt

Abstract. In this paper we adapt the well-known Mahler and van der Put base of the Banach space of continuous functions to the case of the n-times continuously differentiable functions in one and several variables.

1991 Mathematics subject classification : 46S10

1. Introduction

Let \( K \) be an algebraic extension of \( \mathbb{Q}_p \), the field of p-adic numbers. As usual, we write \( \mathbb{Z}_p \) for the ring of p-adic integers and \( C(\mathbb{Z}_p \to K) \) for the Banach space of continuous functions from \( \mathbb{Z}_p \) to \( K \). We have the following well-known bases for \( C(\mathbb{Z}_p \to K) \) : on one hand, we have the Mahler base \( \binom{x}{n} \) \((n \in \mathbb{N})\), consisting of polynomials of degree \( n \) and on the other hand we have the van der Put base \( \{e_n \mid n \in \mathbb{N}\} \) consisting of locally constant functions \( e_n \) defined as follows : \( e_0(x) = 1 \) and for \( n > 0 \), \( e_n \) is the characteristic function of the ball \( \{ \alpha \in \mathbb{Z}_p \mid |\alpha - n| < 1/n \} \). For every \( f \in C(\mathbb{Z}_p \to K) \) we have the following uniformly convergent series

\[
f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \quad \text{where} \quad a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j)
\]

\[
f(x) = \sum_{n=0}^{\infty} b_n e_n(x) \quad \text{where} \quad b_0 = f(0) \text{ and } b_n = f(n) - f(n_-).
\]

Here \( n_- \) is defined as follows. For every \( n \in \mathbb{N}_0 \), we have a Hensel expansion \( n = n_0 + n_1 p + \ldots + n_s p^s \) with \( n_s \neq 0 \). Then \( n_- = n_0 + n_1 p + \ldots + n_s p^s - 1 \). We further put \( \gamma_0 = 1, \gamma_n = n - n_- = n_s p^s, \delta_0 = 1, \delta_n = p^s \) and \( n_- = n - \delta_n \). Remark that \( |\delta_n| = |\gamma_n| \).
In the sequel, we will also use the following notation, for \( m, x \in \mathbb{Q}_p \), \( x = \sum_{j=-\infty}^{\infty} a_j p^j : m \prec x \) if \( m = \sum_{j=-\infty}^{i} a_j p^j \) for some \( i \in \mathbb{Z} \). We sometimes refer to the relation \( \prec \) between \( m \) and \( x \) as "\( m \) is an initial part of \( x \)" or "\( x \) starts with \( m \)."

Let \( f : \mathbb{Z}_p \rightarrow K \). The (first) difference quotient \( \phi_1 f : \nabla^2 \mathbb{Z}_p \rightarrow K \) is defined by \( \phi_1 f(x, y) = \frac{f(y) - f(x)}{y - x} \), where \( \nabla^2 \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{ (x, x) \mid x \in \mathbb{Z}_p \} \). \( f \) is called continuously differentiable (or strictly differentiable, or uniformly differentiable) at \( a \in \mathbb{Z}_p \) if \( \lim_{(x, y) \to (a, a)} \phi_1 f(x, y) \) exists. We will also say that \( f \) is \( C^1 \) at \( a \). In a similar way, we may define \( C^n \)-functions as follows: for \( n \in \mathbb{N} \), we define \( \nabla^{n+1} \mathbb{Z}_p = \{ (x_1, ..., x_{n+1}) \in \mathbb{Z}_p^{n+1} \mid x_i \neq x_j \text{ if } i \neq j \} \) and the \( n \)-th difference quotient \( \phi_n f : \nabla^{n+1} \mathbb{Z}_p \rightarrow K \) by \( \phi_0 f = f \) and

\[
\phi_n f(x_1, x_2, ..., x_{n+1}) = \frac{\phi_{n-1} f(x_2, x_3, ..., x_{n+1}) - \phi_{n-1} f(x_1, x_3, ..., x_{n+1})}{x_2 - x_1}.
\]

A function \( f \) is called a \( C^n \)-function if \( \phi_n f \) can be extended to a continuous function \( \bar{\phi}_n f \) on \( \mathbb{Z}_p^{n+1} \). Recall from \([4],[5]\) that \( \bar{\phi}_n f(x, x, ..., x) = \frac{f^{(n)}(x)}{n!} \), for all \( x \in \mathbb{Z}_p \). The set of all \( C^n \)-functions from \( \mathbb{Z}_p \) to \( K \) will be denoted by \( C^n(\mathbb{Z}_p \rightarrow K) \). For any \( C^n \)-function \( f \), we define \( \|f\|_n = max\{ \|\phi_j f\|_s \mid 0 \leq j \leq n \} \) where \( \| \cdot \|_s \) is the sup norm. (For \( f : X \rightarrow K, \|f\|_s = max_{x \in X} |f(x)| \) \( \| \cdot \|_n \) is a norm on \( C^n \), making \( C^n \) into a Banach space.

2. Generalization of the Mahler base for \( C(\mathbb{Z}_p \rightarrow \mathbb{Q}_p) \)

One can construct other orthonormal bases of \( C(\mathbb{Z}_p \rightarrow K) \) by generalizing the procedure used to define the Mahler base as did Y. Amice. In general, we have the following characterization of the polynomial sequences \( \varepsilon_n \in K[x], n \geq 0 \) such that \( deg(\varepsilon_n) = n \) and which are orthonormal bases of the space \( C(\mathbb{B} \rightarrow K) \), where \( B = \{ x \in K \mid |x| \leq 1 \} \).

**Theorem:** Let \( (\varepsilon_n)_{n \geq 0} \) be a sequence of polynomials in \( K[x] \) of degree \( n \). They form an orthonormal base of \( C(\mathbb{B} \rightarrow K) \) if and only if \( ||\varepsilon_n||_s = 1 \) and \( ||\varepsilon_n||_G = |\text{coeff } x^n| = |\pi^{-(n-s(n))}/(q-1)| \) where \( \pi \) is a uniformizing parameter of \( K \), \( q \) the cardinality of the residue class field of \( K \) and \( s(n) \) the sum of the digits of \( n \) in base \( q \). By the way, for a polynomial \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( ||f||_G = max_{i \leq n} |a_i| \).

Given an orthonormal base, we can construct other orthonormal bases by taking a certain linear combination of the given base as will be stated in the following theorem.

**Theorem:** Let \( \varepsilon_n (n \in \mathbb{N}) \) be an orthonormal base of \( C(\mathbb{Z}_p \rightarrow K) \) and put \( p_n = \sum_{j=0}^{n} a_{n,j} \varepsilon_j \) where \( a_{n,j} \in K \) and \( a_{n,n} \neq 0 \). The \( p_n (n \in \mathbb{N}) \) form an orthonormal base for
C(Z_p \to K) if and only if |a_{n,j}| \leq 1 for all j \leq n and |a_{n,n}| = 1.
We can generalize the Mahler base also by changing the degree of the polynomials as follows.

**Theorem:** The polynomials $q_n(x) = \binom{px}{pn}$ (n \in \mathbb{N}) form an orthonormal base for $C(Z_p \to \mathbb{Q}_p)$ and every continuous function $f : Z_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum a_n \binom{px}{pn}$

with $a_n = \sum (-1)^{n-k} \binom{pn}{pk} \alpha_{n-k} f(k)$
and $\alpha_0^{(p)} = 1, \alpha_m^{(p)} = \sum \frac{(-1)^{r+m} \binom{pm}{pl_1...pl_r}}{1 \leq l_1 \leq m \leq p}$

If we mix the Mahler and van der Put base together, we obtain a new orthonormal base.

**Theorem:** The sequence $q_n(x) = \binom{x}{n} e_n(x)$ (n \in \mathbb{N}) forms an orthonormal base for $C(Z_p \to \mathbb{Q}_p)$. Moreover, every continuous function $f : Z_p \to \mathbb{Q}_p$ can be written as a uniformly convergent series $f(x) = \sum a_i \binom{x}{i} e_i(x)$

with $a_i = \sum \alpha_{i,j} f(j)$
and $\alpha_{i,i} = 1, \alpha_{i,j} = \sum_{j=k_0 \leq k_1 \leq \ldots \leq k_n = i} (-1)^n \binom{i}{k_{n-1}} \binom{k_{n-2}}{k_{n-2}} \ldots \binom{j}{k}$

3. Differentiable functions

For $C^n$-functions the polynomials $\binom{x}{i}$ (i \in \mathbb{N}) still remain a base, we only have to add the factor $\gamma_i \gamma_{i/2} \ldots \gamma_{i/n}$ where $\gamma_i = i - \lfloor \alpha \rfloor $ and \lfloor \alpha \rfloor denotes the integer part of $\alpha$, to obtain the orthonormal base $\gamma_i \gamma_{i/2} \ldots \gamma_{i/n} \binom{x}{i}$. The proof is based on the following lemma in case $n = 2$.

**Lemma** Let $f$ be a continuous function with interpolation coefficients $a_n$. Then $f$ is a $C^2$-function if and only if $\left| \frac{a_{i+j+k+2}}{(k+1)(j+k+2)} \right| \to 0$ as $i + j + k$ approach infinity.

**Corollary** If $f$ is a $C^2$-function, then $||\phi_2 f||_s = \sup_n \left| \frac{a_n}{\gamma_n \gamma_{[n/2]}} \right|$}

A similar property does not hold for the van der Put base.
In case $n = 1$, we know that $\{ \gamma_i e_i(x) \mid i \in \mathbb{N} \} \cup \{(x-i).e_i(x) \mid i \in \mathbb{N} \}$ is an orthonormal base for $C^1(Z_p \to K)$. Therefore every continuously differentiable function $f$ can be written
under the form \( f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n (x-n) x^n \) where \( a_0 = f(0) \), \( a_n = f(n) - f(n-) - (n - n-) f'(n-) \), \( b_0 = f'(0) \) and \( b_n = f'(n) - f'(n-) \). For details we refer to [6].

The case \( n = 2 \), can be treated as follows.

**Theorem** : Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n (x-n) x^n \in C^1(\mathbb{Z}_p \to K) \).

\( f \in C^2(\mathbb{Z}_p \to K) \) if and only if \( \lim_{n \to a} \frac{a_n}{\gamma_n^2} \) and \( \lim_{n \to a} \frac{b_n}{\gamma_n} \) exist for all \( a \in \mathbb{Z}_p \), and \( \lim_{n \to a} \frac{b_n}{\gamma_n} = 2 \lim_{n \to a} \frac{a_n}{\gamma_n^2} \).

**Theorem** : \( \{ \gamma_n^2 x^n, \gamma_n (x-n) x^n, (x-n)^2 x^n \mid n \in \mathbb{N} \} \) is an orthonormal base for \( C^2(\mathbb{Z}_p \to K) \) and for every \( f \in C^2(\mathbb{Z}_p \to K) \) we have

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n (x-n) x^n + \sum_{n=0}^{\infty} c_n \frac{(x-n)^2}{2} x^n \quad \text{with}
\]

\[
a_0 = f(0)
\]

\[
a_n = f(n) - f(n-) - (n - n-) f'(n-) - \frac{(n - n-)^2}{2} f''(n-) \quad \text{for } n \neq 0
\]

\[
b_0 = f'(0)
\]

\[
b_n = f'(n) - f'(n-) - (n - n-) f''(n-) \quad \text{for } n \neq 0
\]

\[
c_0 = f''(0)
\]

\[
c_n = f''(n) - f''(n-) \quad \text{for } n \neq 0
\]

The construction of this orthonormal base, which is very technical, is based on the use of an antiderivation map \( P_n : C^{n-1}(\mathbb{Z}_p \to K) \to C^n(\mathbb{Z}_p \to K) \) defined by \( P_n f(x) = \sum_{m=0}^{\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(x_m)}{(j+1)!} (x_{m+1} - x_m)^{j+1} \) with \( x_m = \sum_{j=-\infty}^{\infty} a_j p^j \) and on the following lemmas.

**Lemma** : For \((t_1, \ldots, t_k) \in \nabla^k X = \{(x_1, x_2, \ldots, x_k) \mid x_i \neq x_j \text{ if } i \neq j \} \) with \( t_1 = x, t_i = y \) and \( t_k = z \), we have

\[
\phi_2 f(x, y, z) = \sum_{j=2}^{k-1} \mu_j \phi_2 f(t_j-1, t_j, t_j+1) \quad \text{with } \mu_j = \begin{cases} \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(z-x)(y-x)} & \text{for } j \geq i \\ \frac{(t_{j+1}-t_{j-1})(t_j-t_k)}{(z-x)(y-z)} & \text{for } j \leq i \end{cases}
\]

Moreover, \( \sum_{j=2}^{k-1} \mu_j = 1 \).

**Lemma** : Let \( S \) be a ball in \( K \) and \( f \in C(\mathbb{Z}_p \to K) \).

Suppose that \( \phi_2 f(n, n - \delta_n, n + p^k \delta_n) \in S \) for all \( n \in \mathbb{N}_0, k \in \mathbb{N} \), then \( \phi_2 f(x, y, z) \in S \) for all \( x, y, z \in \mathbb{Z}_p, x \neq y, x \neq z, y \neq z \).

4. Several variables

We can also construct the Mahler and van der Put base for functions of several vari-
ables. This brings us to the following results.
Theorem: The family \( \max\{\gamma_n, \gamma_m\} \cdot \binom{x}{n} \cdot \binom{y}{m} \) \((n, m \in \mathbb{N})\) forms an orthonormal base for \( C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \).

The proof is based on

Theorem: \( f(x, y) = \sum_{n,m} a_{n,m} \binom{x}{n} \binom{y}{m} \) is a \( C^1 \)-function if and only if \( \left| \frac{a_{i+j+1,k}}{j+1} \right| \to 0 \)

and \( \left| \frac{a_{i,j+k+1}}{k+1} \right| \to 0 \) as \( i+j+k \) approach infinity or equivalently \( \frac{a_{n,m}}{\gamma_n} \to 0 \) and \( \frac{a_{n,m}}{\gamma_m} \to 0 \) as \( n+m \) approach infinity.

Starting with the van der Put base \( e_n(n \in \mathbb{N}) \) of \( C(\mathbb{Z}_p \to K) \), we get

Theorem: The family \( e_n(x)e_m(y), (x-n)e_n(x)e_m(y), (y-m)e_n(x)e_m(y) \)

\((n, m \in \mathbb{N})\) forms an orthogonal base for \( C^1(\mathbb{Z}_p \times \mathbb{Z}_p \to K) \) and every \( C^1 \)-function \( f \) can be written as

\[
f(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}e_i(x)e_j(y) + b_{i,j}(x-i)e_i(x)e_j(y) + c_{i,j}(y-j)e_i(x)e_j(y)
\]

with

\[
a_{0,0} = f(0,0)
\]

\[
a_{n,0} = f(n,0) - f(n-,0) - \gamma_n \frac{\partial f}{\partial x}(n-,0) \quad \text{for } n \neq 0
\]

\[
a_{0,m} = f(0,m) - f(0,m-) - \gamma_m \frac{\partial f}{\partial y}(0,m-) \quad \text{for } m \neq 0
\]

\[
a_{n,m} = f(n,m) - f(n-,m) - f(n,m-) + f(n-,m-) - \gamma_n \left( \frac{\partial f}{\partial x}(n-,m) - \frac{\partial f}{\partial x}(n-,m-) \right)
\]

\[
- \gamma_m \left( \frac{\partial f}{\partial y}(n,m-) - \frac{\partial f}{\partial y}(n-,m-) \right) \quad \text{for } n \neq 0 \text{ and } m \neq 0
\]

\[
b_{0,0} = \frac{\partial f}{\partial x}(0,0)
\]

\[
b_{n,0} = \frac{\partial f}{\partial x}(n,0) - \frac{\partial f}{\partial x}(n-,0) \quad \text{for } n \neq 0
\]

\[
b_{0,m} = \frac{\partial f}{\partial x}(0,m) - \frac{\partial f}{\partial x}(0,m-) \quad \text{for } m \neq 0
\]

\[
b_{n,m} = \frac{\partial f}{\partial x}(n,m) - \frac{\partial f}{\partial x}(n-,m) - \frac{\partial f}{\partial x}(n,m-) + \frac{\partial f}{\partial x}(n-,m-) \quad \text{for } n \neq 0 \text{ and } m \neq 0
\]

\[
c_{0,0} = \frac{\partial f}{\partial y}(0,0)
\]

\[
c_{n,0} = \frac{\partial f}{\partial y}(n,0) - \frac{\partial f}{\partial y}(n-,0) \quad \text{for } n \neq 0
\]

\[
c_{0,m} = \frac{\partial f}{\partial y}(0,m) - \frac{\partial f}{\partial y}(0,m-) \quad \text{for } m \neq 0
\]

\[
c_{n,m} = \frac{\partial f}{\partial y}(n,m) - \frac{\partial f}{\partial y}(n-,m) - \frac{\partial f}{\partial y}(n,m-) + \frac{\partial f}{\partial y}(n-,m-) \quad \text{for } n \neq 0 \text{ and } m \neq 0
\]

Remark: To obtain an orthonormal base, the \( e_i(x)e_j(y) \) should be multiplied by
max\{\gamma_i, \gamma_j\}; the \((x-i)e_i(x)e_j(y)\) by max \(\left\{\frac{1}{p^{\gamma_i}}, 1, \frac{\gamma_j}{p^{\gamma_i}}\right\}\) in case \(i \neq 0\) and by \(\gamma_j\) in case \(i = 0\) and analogous for \((y-j)e_i(x)e_j(y)\).

**Generalization:** The sequence \((x-i)^k(y-j)^l e_i(x)e_j(y)\) with \(0 \leq k + l \leq n, i \in \mathbb{N}\) and \(j \in \mathbb{N}\) forms an orthogonal base for \(C^n(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{K})\) whereby every \(C^n\)-function \(f\) can be written as \(f(x, y) = \sum_{i,j=0}^{\infty} \sum_{k+l=0}^{n} a_{i,j}^k \frac{(x-i)^k (y-j)^l}{k!} e_i(x)e_j(y)\) with

\[
a_{i,j}^k = \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(i, j) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha} f}{\partial x^k \partial y^{l+\alpha}}(i, j) \frac{\gamma_i^\alpha}{\alpha!} - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^{l+\beta}}(i, j) \frac{\gamma_j^\beta}{\beta!} + \sum_{\alpha+\beta=0}^{n-k-l} \frac{\partial^{k+l+\alpha+\beta} f}{\partial x^k \partial y^{l+\beta}}(i, j) \frac{\gamma_i^\alpha \gamma_j^\beta}{\alpha! \beta!} \quad \text{for } i \neq 0 \text{ and } j \neq 0
\]

\[
a_{i,0}^k = \frac{\partial^{k+l} f}{\partial x^k \partial y^{l+1}}(0, 0) - \sum_{\alpha=0}^{n-k-l} \frac{\partial^{k+l+\alpha} f}{\partial x^k \partial y^{l+\alpha}}(0, 0) \frac{\gamma_i^\alpha}{\alpha!} \quad \text{for } i \neq 0
\]

\[
a_{0,j}^k = \frac{\partial^{k+l} f}{\partial x^k \partial y^{j+1}}(0, 0) - \sum_{\beta=0}^{n-k-l} \frac{\partial^{k+l+\beta} f}{\partial x^k \partial y^{j+\beta}}(0, 0) \frac{\gamma_j^\beta}{\beta!} \quad \text{for } j \neq 0
\]

and \(a_{0,0}^k = \frac{\partial^{k+l} f}{\partial x^k \partial y^{j+1}}(0, 0)\).

The previous theorems show that \(C^n(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{K})\) is not the complete tensor product of \(C^n(\mathbb{Z}_p \rightarrow \mathbb{K})\) with \(C^n(\mathbb{Z}_p \rightarrow \mathbb{K})\) as one may expect, considering the case \(C(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{K})\). Therefore we define a finer structure for functions of two variables.

**Definition:**
\(\phi_{0,0} f(x_0, y_0) = f(x_0, y_0)\)
\(\phi_{1,0} f(x_0, x_1, y_0) = \frac{f(x_0, y_0) - f(x_1, y_0)}{x_0 - x_1} \quad \text{for } x_0 \neq x_1\)
\(\phi_{0,1} f(x_0, y_0, y_1) = \frac{f(x_0, y_0) - f(x_0, y_1)}{y_0 - y_1} \quad \text{for } y_0 \neq y_1\)
\[:\]
\(\phi_{i,j} f(x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_j) \quad = \frac{\phi_{i-1,j} f(x_0, \ldots, x_{i-1}, y_0, \ldots, y_j) - \phi_{i-1,j} f(x_0, \ldots, x_{i-2}, x_i, y_0, \ldots, y_j)}{x_{i-1} - x_i} \quad \text{for } (x_0, x_1, \ldots, x_i, y_0, y_1, \ldots, y_j) \in \mathbb{V}^{i+1} \mathbb{Z}_p \times \mathbb{V}^{j+1} \mathbb{Z}_p \) is the differencequotient of order \(i\) in the first variable and order \(j\) in the second variable of the function \(f\) from \(\mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{K}\).

**Definition:** \(f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{K}\) is \(m\) times strictly differentiable in his first variable and \(n\) times strictly differentiable in his second variable (for short: a \(C^{m,n}\)-function) if and
only if \(\phi_{m,n}f\) can be extended to a continuous function \(\overline{\phi_{m,n}f}\) on \(\mathbb{Z}_p^{m+n+2}\). The set of all \(C^{m,n}\)-functions \(f : \mathbb{Z}_p \times \mathbb{Z}_p \to K\) is denoted \(C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)\). For \(f : \mathbb{Z}_p \times \mathbb{Z}_p \to K\), set \(\|f\|_{m,n} = \max_{0 \leq i,j \leq m} ||\phi_{i,j}f||_s\).

For these functions, we get the following equivalent of the Mahler base.

**Theorem:** The family \(\gamma_i \gamma_{i/2} \ldots \gamma_{i/m} \gamma_j \gamma_{j/2} \ldots \gamma_{j/n}\) \((x_i, y_j) (i, j \in \mathbb{N})\) forms an orthonormal base for \(C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)\).

Since it can be easily seen that there is an isometry between the complete tensor product \(C^{m}(\mathbb{Z}_p \to K) \otimes C^{n}(\mathbb{Z}_p \to K)\) and \(C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)\), the van der Put base for \(C^{m,n}\)-functions is given as follows.

**Theorem:** The family \(\gamma_i^{m-k}(x-i)^k \gamma_j^{n-l}(y-j)^l e_i(x)e_j(y)\) with \(0 \leq k \leq m, 0 \leq l \leq n, i \in \mathbb{N}\) and \(j \in \mathbb{N}\) forms an orthonormal base for \(C^{m,n}(\mathbb{Z}_p \times \mathbb{Z}_p \to K)\) whereby every \(C^{m,n}\)-function \(f\) can be written as \(f(x, y) = \sum_{i,j=0}^{\infty} \sum_{k=0}^{m} \sum_{l=0}^{n} a_{i,j}^{k,l}(x-i)^k(y-j)^l e_i(x)e_j(y)\) with

\[
a_{i,j}^{k,l} = \frac{\frac{\partial^{k+l}f}{\partial x^k \partial y_l}(i,j)}{k!} - \sum_{\alpha=0}^{m-k} \frac{\frac{\partial^{k+l+\alpha}f}{\partial x^{k+\alpha} \partial y_l}(i,-j)}{\alpha!} \frac{\gamma_i^\alpha}{\alpha!} \frac{\gamma_j^\beta}{\beta!} - \sum_{\beta=0}^{n-l} \frac{\frac{\partial^{k+l+\beta}f}{\partial x^k \partial y_{l+\beta}}(i,j)}{\beta!} \frac{\gamma_i^\alpha}{\alpha!} \frac{\gamma_j^\beta}{\beta!}
\]

\[
+ \sum_{\alpha=0}^{m-k} \sum_{\beta=0}^{n-l} \frac{\frac{\partial^{k+l+\alpha+\beta}f}{\partial x^{k+\alpha} \partial y_{l+\beta}}(i,-j)}{\alpha!} \frac{\gamma_i^\alpha}{\alpha!} \frac{\gamma_j^\beta}{\beta!} \quad \text{for } i \neq 0 \text{ and } j \neq 0.
\]

\[
a_{i,0}^{k,l} = \frac{\frac{\partial^{k+l}f}{\partial x^k \partial y_l}(i,0)}{k!} - \sum_{\alpha=0}^{m-k} \frac{\frac{\partial^{k+l+\alpha}f}{\partial x^{k+\alpha} \partial y_l}(i,-0)}{\alpha!} \frac{\gamma_i^\alpha}{\alpha!} \quad \text{for } i \neq 0
\]

\[
a_{0,j}^{k,l} = \frac{\frac{\partial^{k+l}f}{\partial x^k \partial y_l}(0,j)}{k!} - \sum_{\beta=0}^{n-l} \frac{\frac{\partial^{k+l+\beta}f}{\partial x^k \partial y_{l+\beta}}(0,-j)}{\beta!} \frac{\gamma_j^\beta}{\beta!} \quad \text{for } j \neq 0
\]

\[
\text{and } a_{0,0}^{k,l} = \frac{\frac{\partial^{k+l}f}{\partial x^k \partial y_l}(0,0)}{k!}
\]

**REFERENCES**


Vrije Universiteit Brussel,
Faculteit Toegepaste Wetenschappen,
Pleinlaan 2
B 1050 BRUSSEL,
Belgium