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Annales mathématiques Blaise Pascal, tome 2, n° 1 (1995), p. 299-305

http://www.numdam.org/item?id=AMBP_1995__2_1_299_0

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THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS ON V_q , WITH THE AID OF OPERATORS

Ann Verdoodt

Abstract. Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity, and let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \dots\}$. K is a non-archimedean valued field, K contains $\mathbb{Q}_p$, and K is complete for the valuation $|\cdot|$, which extends the p -adic valuation. $C(V_q \rightarrow K)$ is the Banach space of continuous functions from V_q to K , equipped with the supremum norm. Let $\mathcal{E}$ and D_q be the operators on $C(V_q \rightarrow K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_q f)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with D_q), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \rightarrow K)$. If f is an element of $C(V_q \rightarrow K)$, then there exist elements α_n of K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n .

1991 Mathematics subject classification : 46S10

1. Introduction

Let p be a prime, $\mathbb{Z}_p$ the ring of the p -adic integers, $\mathbb{Q}_p$ the field of the p -adic numbers. K is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that K is complete for the valuation $|\cdot|$, which extends the p -adic valuation. Let a and q be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), q not a root of unity. Let V_q be the closure of the set $\{aq^n \mid n = 0, 1, 2, \dots\}$. We denote by $C(V_q \rightarrow K)$ (resp. $C(\mathbb{Z}_p \rightarrow K)$) the set of all continuous functions $f : V_q \rightarrow K$ (resp. $f : \mathbb{Z}_p \rightarrow K$) equipped with the supremum norm. If f is an element of $C(V_q \rightarrow K)$ then we define the operators $\mathcal{E}$ and D_q as follows :

$$(\mathcal{E}f)(x) = f(qx)$$

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

We remark that the operator $\mathcal{E}$ does not commute with D_q . Furthermore, the operator D_q lowers the degree of a polynomial with one, whereas the operator $\mathcal{E}$ does not.

If $\mathcal{L}$ is a non-archimedean Banach space over a non-archimedean valued field L , and $e_1, e_2, \dots$ is a finite or infinite sequence of elements of $\mathcal{L}$, then we say that this sequence is orthogonal if $\|e_1 e_1 + \dots + e_k e_k\| = \max\{\|e_i e_i\| : i = 1, \dots, k\}$ for all k in $\mathbb{N}$ (or for all k that do not exceed the length of the sequence) and for all $e_1, \dots, e_k$ in L . An orthogonal sequence $e_1, e_2, \dots$ is called orthonormal if $\|e_i\| = 1$ for all i . A family (e_i) of elements of $\mathcal{L}$ forms a(n) (ortho)normal basis of $\mathcal{L}$ if the family (e_i) is orthonormal and also a basis. We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if p_n is exactly of degree n for all natural numbers n .

The aim here is to find normal bases for $C(V_q \rightarrow K)$, which consist of polynomial sequences. Therefore we will use linear, continuous operators which commute with D_q or with $\mathcal{E}$. If $(r_n(x))$ is such a polynomial sequence, and if f is an element of $C(V_q \rightarrow K)$,

there exist coefficients α_n in K such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-

hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients α_n .

We remark that all the results (with proofs) in this paper can be found in [5], except for theorem 5.

2. Notations.

Let V_q , K and $C(V_q \rightarrow K)$ be as in the introduction. The supremum norm on $C(V_q \rightarrow K)$ will be denoted by $\|\cdot\|$. We introduce the following:

$$A_0(x) = 1, A_n(x) = (x - aq^{n-1})A_{n-1}(x) \quad (n \geq 1),$$

$$B_n(x) = A_n(x)/A_n(aq^n), C_n(x) = a^n q^{n(n-1)/2} (q-1)^n B_n(x)$$

It is clear that $(A_n(x))$, $(B_n(x))$ and $(C_n(x))$ are polynomial sequences. The sequence $(C_n(x))$ forms a basis for $C(V_q \rightarrow K)$ and the sequence $(B_n(x))$ forms a normal basis for $C(V_q \rightarrow K)$. From this it follows that $\|B_n\| = 1$ and $\|C_n\| = |(q-1)^n|$. Let $\mathcal{E}$ and D_q be as in the introduction. Then we introduce the following:

Definition. Let f be a function from V_q to K . We define the following operators:

$$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$$

$$(\mathcal{E}^n f)(x) = f(q^n x)$$

$$\mathcal{D}f(x) = \mathcal{D}^{(1)}f(x) = f(qx) - f(x) = ((\mathcal{E} - 1)f)(x)$$

$$\mathcal{D}^{(n)}f(x) = ((\mathcal{E} - 1) \dots (\mathcal{E} - q^{n-1})f)(x), \mathcal{D}^{(0)}f(x) = f(x)$$

The operator D_q does not commute with $\mathcal{D}$. The following properties are easily verified:

$D_q^j C_k(x) = C_{k-j}(x)$ if $k \geq j$, $D_q^j C_k(x) = 0$ if $j > k$. So D_q^j lowers the degree of a polynomial with j

$\mathcal{D}^{(j)}B_k(x) = (x/a)^j q^{j(j-k)} B_{k-j}(x)$ if $j \leq k$, $\mathcal{D}^{(j)}B_k(x) = 0$ if $j > k$

If $p(x)$ is a polynomial of degree n , then $(\mathcal{D}^{(j)}p)(x)$ is a polynomial of degree n if n is at least j , and $(\mathcal{D}^{(j)}p)(x)$ is the zero-polynomial if n is strictly smaller than j .

If f is an element of $C(V_q \rightarrow K)$, then we also have

$$\text{i) } (\mathcal{D}^{(n)}f)(x) = x^n q^{n(n-1)/2} (q-1)^n (D_q^n f)(x)$$

$$\text{ii) } (q-1)^n D_q^n f(x) \rightarrow 0 \text{ uniformly}$$

$$\text{iii) } \mathcal{D}^{(n)}f(x) \rightarrow 0 \text{ uniformly}$$

(i) can be found in [1], p. 60, ii) can be found in [3], p. 124-125, iii) follows from i) and ii).

3. Linear Continuous Operators which Commute with $\mathcal{E}$ or with D_q

Let us start this section with the following known result :

If f is an element of $C(\mathbb{Z}_p \rightarrow K)$, then the translation operator E on $C(\mathbb{Z}_p \rightarrow K)$ is the operator defined by $Ef(x) = f(x+1)$.

If we put $G_n(x) = \binom{x}{n}$ (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem :

A linear, continuous operator Q on $C(\mathbb{Z}_p \rightarrow K)$ commutes with the translation operator E if and only if the sequence (g_n) is bounded, where $g_n = QG_n(0)$.

Such an operator Q can be written in the following way : $Q = \sum_{i=0}^{\infty} g_i \Delta^i$, where Δ is the operator defined as follows : $(\Delta f)(x) = f(x+1) - f(x)$

We can prove analogous theorems for the operators $\mathcal{E}$ and D_q on $C(V_q \rightarrow K)$:

Theorem 1 *An operator Q on $C(V_q \rightarrow K)$ is continuous, linear and commutes with $\mathcal{E}$ if and only if the sequence (b_n) is bounded, where $b_n = (QB_n)(a)$.*

From the proof of the theorem it follows that Q can be written in the form $Q = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$.

If f is an element of $C(V_q \rightarrow K)$, then $(Qf)(x) = \sum_{i=0}^{\infty} b_i (\mathcal{D}^{(i)}f)(x)$ and the series on the right-hand-side is uniformly convergent (since $\mathcal{D}^{(n)}f(x) \rightarrow 0$ uniformly). Clearly we have

$$b_n = (QB_n)(a), \text{ since } (QB_n)(a) = \left(\sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)} B_n \right)(a) = \left(\sum_{i=0}^n b_i (x/a)^i q^{i(i-n)} B_{n-i} \right)(a) = b_n.$$

Furthermore, Qx^n is a K -multiple of x^n .

If $b_0 = \dots = b_{N-1} = 0, b_N \neq 0$, and if $p(x)$ is a polynomial, then x^N divides $(Qp)(x)$.

Some examples

1) For the operator $\mathcal{E}$ we have : $(\mathcal{E}B_n)(x) = B_n(qx)$, so $(\mathcal{E}B_0)(a) = 1$, $(\mathcal{E}B_1)(a) = 1$, and $(\mathcal{E}B_n)(a) = 0$ if $n \geq 2$. This gives us $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$.

2) The operator $\mathcal{E} \circ \mathcal{D} = \mathcal{E}\mathcal{D}$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}\mathcal{D})B_0)(a) = 0$, and since $(n \geq 1)$ $((\mathcal{E}\mathcal{D})B_n)(x) = (\mathcal{E}(\frac{x}{a}q^{1-n}B_{n-1}))(x) = \frac{qx}{a}q^{1-n}B_{n-1}(qx)$, we find $((\mathcal{E}\mathcal{D})B_1)(a) = q$, $((\mathcal{E}\mathcal{D})B_2)(a) = 1$ and $((\mathcal{E}\mathcal{D})B_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}\mathcal{D} = q\mathcal{D}^{(1)} + \mathcal{D}^{(2)}$.

Analogous to theorem 1 we have :

Theorem 2 An operator Q on $C(V_q \rightarrow K)$ is continuous, linear and commutes with D_q if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.

Such an operator Q can be written in the form $Q = \sum_{i=0}^{\infty} c_i D_q^i$, and if f is an element of

$C(V_q \rightarrow K)$ it follows that $(Qf)(x) = \sum_{i=0}^{\infty} c_i (D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \rightarrow 0$ uniformly). Furthermore, we have $c_n = (QC_n)(a)$ since

$$(QC_n)(a) = \left(\sum_{i=0}^{\infty} c_i D_q^i C_n \right)(a) = \sum_{i=0}^n c_i C_{n-i}(a) = c_n.$$

Remarks

1) Let R and Q be linear, continuous operators on $C(V_q \rightarrow K)$, with R of the form $R = \sum_{i=1}^{\infty} b_i D^{(i)}$ (i.e. R commutes with $\mathcal{E}$, $b_0 = 0$), and Q of the form $Q = \sum_{i=1}^{\infty} c_i D_q^i$ (i.e. Q commutes with D_q , $c_0 = 0$). The main difference between the operators Q and R is that Q lowers the degree of each polynomial with at least one, where R does not necessarily lower the degree of a polynomial.

2) If Q_1 and Q_2 both commute with D_q and if $Q_1 = \sum_{i=0}^{\infty} c_{1;i} D_q^i$,

$$Q_2 = \sum_{i=0}^{\infty} c_{2;i} D_q^i, \text{ then } (Q_1 \circ Q_2)(f) = (Q_2 \circ Q_1)(f) = \sum_{k=0}^{\infty} D_q^k f \left(\sum_{j=0}^k c_{1;j} c_{2;k-j} \right).$$

If we take two formal power series $q_1(t) = \sum_{i=0}^{\infty} c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^{\infty} c_{2;i} t^i$, then

$q_1(t) \cdot q_2(t) = \sum_{k=0}^{\infty} t^k \left(\sum_{j=0}^k c_{1;j} c_{2;k-j} \right)$, so the composition of two operators which commute with D_q , corresponds with multiplication of power series.

This is not the case if we take two operators which commute with $\mathcal{E}$: Take e.g. $\mathcal{E} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)}$ and $\mathcal{D}^{(1)}$, then $\mathcal{E} \circ \mathcal{D}^{(1)} = \mathcal{E} \mathcal{D}^{(1)} = q \mathcal{D}^{(1)} + \mathcal{D}^{(2)}$, whereas for power series this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \rightarrow K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If Q is an operator as found in theorem 1, with b_0 equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with Q . We remark that the operator $R = \sum_{i=0}^{\infty} b_i \mathcal{D}^{(i)}$ does not necessarily lowers the degree of a polynomial.

Proposition 1 Let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 1$) with $|b_N| > |b_n|$ if $n > N$. There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

In the same way as in proposition 1 we have.

Proposition 2 Let $Q = \sum_{i=N}^{\infty} c_i \mathcal{D}_q^{(i)}$ ($N \geq 1$), $c_N \neq 0$, $(c_n/(q-1)^n)$ bounded.

Then there exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which form normal bases for $C(V_q \rightarrow K)$. If f is an element of $C(V_q \rightarrow K)$, there exist coefficients α_n such that $f(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases, it is also possible to give an expression for the coefficients α_n .

Theorem 3 Let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 1$) with $|b_n| < |b_N| = 1$ if $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of Q equals one.

2) If f is an element of $C(V_q \rightarrow K)$, then f can be written as a uniformly convergent series $f(x) = \sum_{n=0}^{\infty} \beta_n p_n(x)$, $\beta_n = ((D^{(i)}(x^{-N}Q)^k)f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with

$\|f\| = \max_{0 \leq k; 0 \leq i < N} |((D^{(i)}(x^{-N}Q)^k)f)(a)|$, where $x^{-N}Q$ is a linear continuous operator with norm equal to one.

And analogous to theorem 3 we have

Theorem 4 Let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ ($N \geq 1$) with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$.

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of Q equals one.

2) If f is an element of $C(V_q \rightarrow K)$, there exists a unique, uniformly convergent expansion of the form $f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x)$, where $\gamma_n = a^i (q-1)^i q^{i(i-1)/2} (D_q^i Q^k f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with $\|f\| = \max_{0 \leq k; 0 \leq i < N} \{|(q-1)^i (D_q^i Q^k f)(a)|\}$.

Remark. Here we have $|c_n| \leq |c_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|$ ($n > N$).

An example

Let us consider the following operator $Q = (q-1)D_q$. Then $c_1 = (q-1)$ and $c_k = 0$ if $k \neq 1$. The polynomials $p_k(x)$ are given by $p_k(x) = C_k(x)/(q-1)^k$, and they form a normal basis for $C(V_q \rightarrow K)$. The expansion $f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_q^k f)(a) p_k(x) = \sum_{k=0}^{\infty} (D_q^k f)(a) C_k(x)$ is known as Jackson's interpolation formula ([2],[3]).

If Q is an operator as found in theorem 4, with N equal to one, then we can prove a theorem analogous to theorem 2:

Theorem 5 Let Q be an operator such that $Q = \sum_{i=1}^{\infty} c_i D_q^i$, with $|c_1| = |(q-1)|$,

$|c_n| \leq |(q-1)^n|$ if $n > 1$, and let $p_n(x)$ be the polynomial sequence as found in theorem 4.

An operator T on $C(V_q \rightarrow K)$ is continuous, linear and commutes with D_q if and only if T is of the form $T = \sum_{i=0}^{\infty} d_i Q^i$, where the sequence (d_n) is bounded, where $d_n = (Tp_n)(a)$.

Remark. In theorem 2 the sequence $(c_n/(q-1)^n)$ must be bounded, whereas here the sequence (d_n) must be bounded. This follows from the fact that the norm of the operator D_q equals $|q-1|^{-1}$, whereas the norm of the operator Q equals 1.

5. More Normal Bases

We want to make more normal bases, using the ones we found in theorems 3 and 4. For operators which commute with $\mathcal{E}$ we can prove the following theorem:

Theorem 6 Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \rightarrow K)$, and let $Q = \sum_{i=N}^{\infty} b_i \mathcal{D}^{(i)}$ ($N \geq 0$) with $1 = |b_N| > |b_k|$ if $k > N$. If $Qp_n(x) = x^N r_{n-N}(x)$ ($n \geq N$), then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \rightarrow K)$.

And analogous for operators which commute with the operator D_q we have :

Theorem 7 Let $(p_n(x))$ be a polynomial sequence which forms a normal basis for $C(V_q \rightarrow K)$, and let $Q = \sum_{i=N}^{\infty} c_i D_q^i$ ($N \geq 0$) with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$. If $(Qp_n)(x) = r_{n-N}(x)$ ($n \geq N$), then the polynomial sequence $(r_k(x))$ forms a normal basis for $C(V_q \rightarrow K)$.

We remark that analogous results can be found on the space $C(\mathbb{Z}_p \rightarrow K)$ for linear continuous operators which commute with the translation operator E . The result analogous to theorems 3 and 4 for the case N equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper.

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