The construction of normal bases for the space of continuous functions on $V_q$, with the aid of operators


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THE CONSTRUCTION OF NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS ON $V_q$, WITH THE AID OF OPERATORS

Ann Verdooit

Abstract. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity, and let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. $K$ is a non-archimedean valued field, $K$ contains $\mathbb{Q}_p$, and $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. $C(V_q \to K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm. Let $\mathcal{E}$ and $D_q$ be the operators on $C(V_q \to K)$ defined by $(\mathcal{E}f)(x) = f(qx)$ and $(D_qf)(x) = (f(qx) - f(x))/(x(q-1))$. We will find all linear and continuous operators that commute with $\mathcal{E}$ (resp. with $D_q$), and we use these operators to find normal bases $(r_n(x))$ for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $\alpha_n$ of $K$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients $\alpha_n$.

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1. Introduction

Let $p$ be a prime, $\mathbb{Z}_p$ the ring of the $p$-adic integers, $\mathbb{Q}_p$ the field of the $p$-adic numbers. $K$ is a non-archimedean valued field, $K \supset \mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$ (i.e. $|a| = |q| = 1$), $q$ not a root of unity. Let $V_q$ be the closure of the set $\{aq^n \mid n = 0, 1, 2, \ldots\}$. We denote by $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) the set of all continuous functions $f : V_q \to K$ (resp. $f : \mathbb{Z}_p \to K$) equipped with the supremum norm. If $f$ is an element of $C(V_q \to K)$ then we define the operators $\mathcal{E}$ and $D_q$ as follows:

$(\mathcal{E}f)(x) = f(qx)$
We remark that the operator £ does not commute with $D_q$. Furthermore, the operator $D_q$ lowers the degree of a polynomial with one, whereas the operator $E$ does not.

If $\mathcal{L}$ is a non-archimedean Banach space over a non-archimedean valued field $L$, and $e_1, e_2, \ldots$ is a finite or infinite sequence of elements of $\mathcal{L}$, then we say that this sequence is orthogonal if $\sum ||e_i|| = \max\{||e_i|| : i = 1, \ldots, k\}$ for all $k$ in $\mathbb{N}$ (or for all $k$ that do not exceed the length of the sequence) and for all $e_1, \ldots, e_k$ in $L$. An orthogonal sequence $e_1, e_2, \ldots$ is called orthonormal if $||e_i|| = 1$ for all $i$. A family $(e_i)$ of elements of $\mathcal{L}$ forms a(n) (orthono)normal basis of $\mathcal{L}$ if the family $(e_i)$ is orthonormal and also a basis.

We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if $p_n$ is exactly of degree $n$ for all natural numbers $n$.

The aim here is to find normal bases for $C(V_q \rightarrow K)$, which consist of polynomial sequences. Therefore we will use linear, continuous operators which commute with $D_q$ or with $E$. If $(r_n(x))$ is such a polynomial sequence, and if $f$ is an element of $C(V_q \rightarrow K)$, there exist coefficients $\alpha_n$ in $K$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n r_n(x)$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients $\alpha_n$.

We remark that all the results (with proofs) in this paper can be found in [5], except for theorem 5.

2. Notations.

Let $V_q$, $K$ and $C(V_q \rightarrow K)$ be as in the introduction. The supremum norm on $C(V_q \rightarrow K)$ will be denoted by $\| \cdot \|$. We introduce the following:

$A_0(x) = 1, A_n(x) = (x - aq^{n-1})A_{n-1}(x)$ ($n \geq 1$),

$B_n(x) = A_n(x)/A_n(aq^n), \quad C_n(x) = a^nq^{n(n-1)/2}(q - 1)^nB_n(x)$

It is clear that $(A_n(x)), (B_n(x))$ and $(C_n(x))$ are polynomial sequences. The sequence $(C_n(x))$ forms a basis for $C(V_q \rightarrow K)$ and the sequence $(B_n(x))$ forms a normal basis for $C(V_q \rightarrow K)$. From this it follows that $\|B_n\| = 1$ and $\|C_n\| = |(q - 1)^n|$. Let $E$ and $D_q$ be as in the introduction. Then we introduce the following:

**Definition.** Let $f$ be a function from $V_q$ to $K$. We define the following operators:

$(D_q^n f)(x) = (D_q(D_q^{n-1} f))(x)$

$(E^n f)(x) = f(q^n x)$

$D f(x) = D^{(1)} f(x) = f(qx) - f(x) = ((E - 1)f)(x)$

$D^{(n)} f(x) = ((E - 1) \ldots (E - q^{n-1} f))(x)$, $D^{(0)} f(x) = f(x)$

The operator $D_q$ does not commute with $D$. The following properties are easily verified:

$D_q^j C_k(x) = C_{k-j}(x)$ if $k \geq j$, $D_q^j C_k(x) = 0$ if $j < k$. So $D_q^j$ lowers the degree of a polynomial with $j$. 
\[ D^{(j)}B_k(x) = \frac{x}{a}^j q^{(j-k)} B_{k-j}(x) \text{ if } j \leq k , \quad D^{(j)}B_k(x) = 0 \text{ if } j > k \]

If \( p(x) \) is a polynomial of degree \( n \), then \( (D^{(j)}p)(x) \) is a polynomial of degree \( n \) if \( n \) is at least \( j \), and \( (D^{(j)}p)(x) \) is the zero-polynomial if \( n \) is strictly smaller than \( j \).

If \( f \) is an element of \( C(\mathbb{Z}_q \rightarrow K) \), then we also have

\begin{enumerate}[i)]
  \item \( (D^{(n)}f)(x) = x^n q^n(n-1)/2(q-1)^n(D_q^n f)(x) \)
  \item \( (q-1)^n D_q^n f(x) \rightarrow 0 \) uniformly
  \item \( D^{(n)}f(x) \rightarrow 0 \) uniformly
\end{enumerate}

(i) can be found in [1], p. 60, ii) can be found in [3], p. 124-125, iii) follows from i) and ii).

3. Linear Continuous Operators which Commute with \( \mathcal{E} \) or with \( D_q \)

Let us start this section with the following known result:

If \( f \) is an element of \( C(\mathbb{Z}_p \rightarrow K) \), then the translation operator \( \mathcal{E} \) on \( C(\mathbb{Z}_p \rightarrow K) \) is the operator defined by \( Ef(x) = f(x+1) \).

If we put \( G_n(x) = \binom{x}{n} \) (the binomial polynomials), then L. Van Hamme ([4]) proved the following theorem:

A linear, continuous operator \( Q \) on \( C(\mathbb{Z}_p \rightarrow K) \) commutes with the translation operator \( \mathcal{E} \) if and only if the sequence \( (g_n) \) is bounded, where \( g_n = QG_n(0) \).

Such an operator \( Q \) can be written in the following way: \( Q = \sum_{i=0}^{\infty} g_i \Delta^i \), where \( \Delta \) is the operator defined as follows: \( (\Delta f)(x) = f(x+1) - f(x) \).

We can prove analogous theorems for the operators \( \mathcal{E} \) and \( D_q \) on \( C(V_q \rightarrow K) \):

**Theorem 1** An operator \( Q \) on \( C(V_q \rightarrow K) \) is continuous, linear and commutes with \( \mathcal{E} \) if and only if the sequence \( (b_n) \) is bounded, where \( b_n = (QB_n)(a) \).

From the proof of the theorem it follows that \( Q \) can be written in the form \( Q = \sum_{i=0}^{\infty} b_i D^{(i)} \).

If \( f \) is an element of \( C(V_q \rightarrow K) \), then \( (Qf)(x) = \sum_{i=0}^{\infty} b_i(D^{(i)}f)(x) \) and the series on the right-hand-side is uniformly convergent (since \( D^{(n)}f(x) \rightarrow 0 \) uniformly). Clearly we have

\[ b_n = (QB_n)(a) \text{, since } (QB_n)(a) = \sum_{i=0}^{\infty} b_i D^{(i)}B_n(a) = \sum_{i=0}^{n} b_i(x/a)^i q^{(i-n)}B_{n-i}(a) = b_n. \]

Furthermore, \( Qx^n \) is a \( K \)-multiple of \( x^n \).

If \( b_0 = \ldots = b_{N-1} = 0, b_N \neq 0 \), and if \( p(x) \) is a polynomial, then \( x^N \) divides \( (Qp)(x) \).

**Some examples**

1) For the operator \( \mathcal{E} \) we have: \( (\mathcal{E}B_n)(x) = B_n(qx) \), so \( (\mathcal{E}B_0)(a) = 1 \), \( (\mathcal{E}B_1)(a) = 1 \), and \( (\mathcal{E}B_n)(a) = 0 \) if \( n \geq 2 \). This gives us \( \mathcal{E} = D^{(0)} + D^{(1)} \).
2) The operator $\mathcal{E} \circ \mathcal{D} = \mathcal{E}\mathcal{D}$ clearly commutes with $\mathcal{E}$. We have $((\mathcal{E}\mathcal{D})B_0)(a) = 0$, and since $(n \geq 1) ((\mathcal{E}\mathcal{D})B_n)(x) = (\mathcal{E} (q^1 - nb_{n-1}))(x) = \frac{q^x}{a}q^{1-n}b_{n-1}(q x)$, we find $((\mathcal{E}\mathcal{D})B_1)(a) = q$, $((\mathcal{E}\mathcal{D})B_2)(a) = 1$ and $((\mathcal{E}\mathcal{D})B_n)(a) = 0$ if $n \geq 3$. We conclude that $\mathcal{E}\mathcal{D} = q\mathcal{D}(1) + \mathcal{D}(2)$.

Analogous to theorem 1 we have:

**Theorem 2** An operator $Q$ on $C(V_q \to K)$ is continuous, linear and commutes with $D_q$ if and only if the sequence $(c_n/(q-1)^n)$ is bounded, where $c_n = (QC_n)(a)$.

Such an operator $Q$ can be written in the form $Q = \sum_{i=0}^\infty c_i D_q^i$, and if $f$ is an element of $C(V_q \to K)$ it follows that $(Qf)(x) = \sum_{i=0}^\infty c_i(D_q^i f)(x)$, where the series on the right-hand-side converges uniformly (since $(q-1)^n D_q^n f(x) \to 0$ uniformly). Furthermore, we have $c_n = (QC_n)(a)$ since

$$(QC_n)(a) = \left(\sum_{i=0}^\infty c_i D_q^i C_n\right)(a) = \sum_{i=0}^n c_i C_{n-i}(a) = c_n.$$

**Remarks**

1) Let $R$ and $Q$ be linear, continuous operators on $C(V_q \to K)$, with $R$ of the form $R = \sum_{i=1}^\infty b_i D_q^i$ (i.e. $R$ commutes with $\mathcal{E}$, $b_0 = 0$), and $Q$ of the form $Q = \sum_{i=1}^\infty c_i D_q^i$ (i.e. $Q$ commutes with $D_q$, $c_0 = 0$). The main difference between the operators $Q$ and $R$ is that $Q$ lowers the degree of each polynomial with at least one, while $R$ does not necessarily lower the degree of a polynomial.

2) If $Q_1$ and $Q_2$ both commute with $D_q$ and if $Q_1 = \sum_{i=0}^\infty c_{1;i} D_q^i$, $Q_2 = \sum_{i=0}^\infty c_{2;i} D_q^i$, then $(Q_1 \circ Q_2)(f) = (Q_2 \circ Q_1)(f) = \sum_{k=0}^\infty D_q^k f \left(\sum_{j=0}^k c_{1;j}c_{2;k-j}\right)$.

If we take two formal power series $q_1(t) = \sum_{i=0}^\infty c_{1;i} t^i$, $q_2(t) = \sum_{i=0}^\infty c_{2;i} t^i$, then

$q_1(t) \cdot q_2(t) = \sum_{k=0}^\infty t^k \left(\sum_{j=0}^k c_{1;j}c_{2;k-j}\right)$, so the composition of two operators which commute with $D_q$, corresponds with multiplication of power series.
This is not the case if we take two operators which commute with $E$: Take e.g.
$E = D^{(0)} + D^{(1)}$ and $D^{(1)}$, then $E \circ D^{(1)} = E D^{(1)} = q D^{(1)} + D^{(2)}$, whereas for power series
this gives $q_1(t) = 1 + t$, $q_2(t) = t$ and $q_1(t) \cdot q_2(t) = t + t^2$.

4. Normal bases for $C(V_q \rightarrow K)$

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$
which form normal bases for $C(V_q \rightarrow K)$. If $Q$ is an operator as found in theorem 1, with $b_0$ equal to zero, we associate a (unique) polynomial sequence $(p_n(x))$ with $Q$. We remark
that the operator $R = \sum_{i=0}^{\infty} b_i D^{(i)}$ does not necessarily lowers the degree of a polynomial.

Proposition 1 Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)} (N \geq 1)$ with $|b_N| > |b_n|$ if $n > N$. There exists a
unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = x^N p_{n-N}(x)$ if $n \geq N$, $p_n(a^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

In the same way as in proposition 1 we have.

Proposition 2 Let $Q = \sum_{i=N}^{\infty} c_i D^{(i)}_q (N \geq 1)$, $c_N \neq 0$, $(c_n/(q - 1)^n)$ bounded.

Then there exists a unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(a^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$.

We use the operators of theorems 1 and 2 to make polynomials sequences $(p_n(x))$ which
form normal bases for $C(V_q \rightarrow K)$. If $f$ is an element of $C(V_q \rightarrow K)$, there exist coefficients
$\alpha_n$ such that $f(x) = \sum_{n=0}^{\infty} \alpha_n p_n(x)$ where the series on the right-hand-side is uniformly
convergent. In some cases, it is also possible to give an expression for the coefficients $\alpha_n$.

Theorem 3 Let $Q = \sum_{i=N}^{\infty} b_i D^{(i)} (N \geq 1)$ with $|b_n| < |b_N| = 1$ if $n > N$

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Q p_n)(x) = x^N p_{n-N}(x) if
n \geq N$, $p_n(a^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence
forms a normal basis for $C(V_q \rightarrow K)$ and the norm of $Q$ equals one.

2) If $f$ is an element of $C(V_q \rightarrow K)$, then $f$ can be written as a uniformly convergent
series $f(x) = \sum_{n=0}^{\infty} \beta_n p_n(x)$, $\beta_n = ((D^{(i)}(x^{-N}Q)^k)f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with
$||f|| = \max_{0 \leq k, 0 \leq i < N} |((D^{(i)}(x^{-N}Q)^k)f)(a)|$, where $x^{-N}Q$ is a linear continuous operator
with norm equal to one.
And analogous to theorem 3 we have

**Theorem 4**  Let $Q = \sum_{i=N}^{\infty} c_i D_i^q (N \geq 1)$ with $|c_N| = |(q-1)^N|$, $|c_n| \leq |(q-1)^n|$ if $n > N$.

1) There exists a unique polynomial sequence $(p_n(x))$ such that $(Qp_n)(x) = p_{n-N}(x)$ if $n \geq N$, $p_n(aq^i) = 0$ if $n \geq N$, $0 \leq i < N$ and $p_n(x) = B_n(x)$ if $n < N$. This sequence forms a normal basis for $C(V_q \rightarrow K)$ and the norm of $Q$ equals one.

2) If $f$ is an element of $C(V_q \rightarrow K)$, there exists a unique, uniformly convergent expansion of the form $f(x) = \sum_{n=0}^{\infty} \gamma_n p_n(x)$, where $\gamma_n = a^i(q-1)^i q^{(i-1)/2}(D_i^q Q^k f)(a)$ if $n = i + kN$ ($0 \leq i < N$), with $\|f\| = \max_{0 \leq k; 0 \leq i < N \{|(q-1)^i(D_i^q Q^k f)(a)|\}$.

**Remark.** Here we have $|c_n| \leq |c_N|$, in contrast with theorem 3, where we need $|b_n| < |b_N|$ ($n > N$).

An example
Let us consider the following operator $Q = (q-1)D_q$. Then $c_1 = (q-1)$ and $c_k = 0$ if $k \neq 1$. The polynomials $p_k(x)$ are given by $p_k(x) = C_k(x)/(q-1)^k$, and they form a normal basis for $C(V_q \rightarrow K)$. The expansion $f(x) = \sum_{k=0}^{\infty} ((q-1)^k D_q^k f)(a)p_k(x) = \sum_{k=0}^{\infty} (D_q^k f)(a)C_k(x)$ is known as Jackson’s interpolation formula ([2],[3])

If $Q$ is an operator as found in theorem 4, with $N$ equal to one, then we can prove a theorem analogous to theorem 2:

**Theorem 5**  Let $Q$ be an operator such that $Q = \sum_{i=1}^{\infty} c_i D_q^i$, with $|c_1| = |(q-1)|$, $|c_n| \leq |(q-1)^n|$ if $n > 1$, and let $p_n(x)$ be the polynomial sequence as found in theorem 4.

An operator $T$ on $C(V_q \rightarrow K)$ is continuous, linear and commutes with $D_q$ if and only if $T$ is of the form $T = \sum_{i=0}^{\infty} d_i Q^i$, where the sequence $(d_n)$ is bounded, where $d_n = (T p_n)(a)$.

**Remark.** In theorem 2 the sequence $(c_n/(q-1)^n)$ must be bounded, whereas here the sequence $(d_n)$ must be bounded. This follows from the fact that the norm of the operator $D_q$ equals $|q-1|^{-1}$, whereas the norm of the operator $Q$ equals 1.

5. More Normal Bases
We want to make more normal bases, using the ones we found in theorems 3 and 4. For operators which commute with $\mathcal{E}$ we can prove the following theorem:
Theorem 6 Let \( (p_n(x)) \) be a polynomial sequence which forms a normal basis for \( C(V_q \to K) \), and let \( Q = \sum_{i=N}^{\infty} b_i D^{(i)} (N \geq 0) \) with \( 1 = |b_N| > |b_k| \) if \( k > N \). If \( Q p_n(x) = x^N r_{n-N}(x) \) \( (n \geq N) \), then the polynomial sequence \( (r_k(x)) \) forms a normal basis for \( C(V_q \to K) \).

And analogous for operators which commute with the operator \( D_q \) we have:

Theorem 7 Let \( (p_n(x)) \) be a polynomial sequence which forms a normal basis for \( C(V_q \to K) \), and let \( Q = \sum_{i=N}^{\infty} c_i D_q^{(i)} (N \geq 0) \) with \( |c_N| = |(q-1)^N| \), \( |c_n| \leq |(q-1)^n| \) if \( n > N \).

If \( (Q p_n)(x) = r_{n-N}(x) \) \( (n \geq N) \), then the polynomial sequence \( (r_k(x)) \) forms a normal basis for \( C(V_q \to K) \).

We remark that analogous results can be found on the space \( C(\mathbb{Z}_p \to K) \) for linear continuous operators which commute with the translation operator \( E \). The result analogous to theorems 3 and 4 for the case \( N \) equal to one, was found by L. Van Hamme (see [4]), and the extensive version of theorems 3 and 4, and the analogons of theorems 5, 6 and 7 can be found with proofs similar to the proofs of the theorems in this paper.

REFERENCES