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REPRESENTATIVE SUBALGEBRA OF A COMPLETE ULTRAMETRIC HOPF ALGEBRA

Bertin Diarra

ABSTRACT. Let (H, m, c, η, σ) be a complete ultrametric Hopf algebra over a complete ultrametric valued field K , e be the unit of H and k the canonical map of K in H . In other words, H is a Banach algebra with multiplication $m : H \hat{\otimes} H \rightarrow H$, coproduct $c : H \rightarrow H \hat{\otimes} H$ a continuous algebra homomorphism, inversion or antipode $\eta : H \rightarrow H$ a continuous linear map and counit $\sigma : H \rightarrow K$ a continuous algebra homomorphism. The coassociativity and counitary axioms hold, and

$$m \circ (\eta \otimes 1_H) \circ c = k \circ \sigma = m \circ (1_H \otimes \eta) \circ c.$$

We define the representative subalgebra $\mathcal{R}(H)$ of H , i.e. the subalgebra of H generated by the coefficient "functions" associated with the finite dimensional left H -comodules. Under some conditions on H , $\mathcal{R}(H)$ is a direct sum of finite dimensional subcoalgebras and is dense in H . But in general, $\mathcal{R}(H)$ is not dense in H . The algebra $\mathcal{R}(H)$ is a generalization of the algebra of representative functions on a group. Notice that when the valuation of K and the norm of H are trivial, one obtains the well known fact that H is equal to its representative subalgebra.

INTRODUCTION.

Let (H, m, c, η, σ) be a complete ultrametric Hopf algebra over the complete ultrametric valued field K . An ultrametric Banach space E over K is said to be a *left Banach H -comodule* if there exists a continuous linear map $\Delta_E : E \rightarrow H \hat{\otimes} E$, called coproduct, such that

- (i) $(c \otimes 1_E) \circ \Delta_E = (1_H \otimes \Delta_E) \circ \Delta_E$
- (ii) $(\sigma \otimes 1_E) \circ \Delta_E = 1_E$

A closed linear subspace E of H is a (left) Banach *subcomodule* of H if $\Delta_E(M) \subset H \hat{\otimes} M$.

Let (E, Δ_E) and (F, Δ_F) be two left Banach comodules. A continuous linear map $u : E \rightarrow F$ is a *Banach comodule morphism* if $\Delta_F \circ u = (1_H \otimes u) \circ \Delta_E$.

It is associated with any left Banach H -comodule (E, Δ_E) the closed linear subspace $R(\Delta_E)$ of H spanned by the coefficient "functions" $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'$, $x \in E$, where E' is the Banach space dual of E . Furthermore, let $\mathcal{R}(H)$ be the linear subspace of H spanned by all the $R(\Delta_E)$ where (E, Δ_E) is a finite dimensional left H -comodule. Then $\mathcal{R}(H)$ is a (non necessary closed) sub-Hopf-algebra of H ; $\mathcal{R}(H)$ is called the *representative subalgebra* of H . In general, $\mathcal{R}(H)$ is not dense in H (cf. [1] or [5], [6]). However, with additional conditions on H it will be shown that $\mathcal{R}(H)$ is dense in H .

If E and F are ultrametric Banach spaces over K , we denote by $E \hat{\otimes} F$ the complete tensor product, that is the completion of $E \otimes F$ with respect to the norm $\|z\| = \inf_{z = \sum x_j \otimes y_j} (\max_j \|x_j\| \|y_j\|)$. In the sequel all Banach spaces are ultrametric.

I - LEFT BANACH COMODULES

I - 1 Tensor products of left Banach comodules

Let (E, Δ_E) and (F, Δ_F) be two left Banach comodules. One has the continuous linear map $\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \rightarrow H \hat{\otimes} E \hat{\otimes} H \hat{\otimes} F \rightarrow H \hat{\otimes} H \hat{\otimes} E \hat{\otimes} F \rightarrow H \hat{\otimes} E \hat{\otimes} F$ where $\Delta_{E \hat{\otimes} F} = (m \otimes 1_E \otimes 1_F) \circ (1_H \otimes \tau_{E \hat{\otimes} F} \otimes 1_F) \circ (\Delta_E \otimes \Delta_F)$ and $\tau_{E \hat{\otimes} F}(x \otimes a) = a \otimes x$.

Proposition 1 : $\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \rightarrow H \hat{\otimes} E \hat{\otimes} F$ is the coproduct of a left Banach H -comodule structure on $E \hat{\otimes} F$.

Proof : Put, for $x \in E$ and $y \in F$, $\Delta_E(x) = \sum_{j \geq 1} a_j \otimes x_j \in H \hat{\otimes} E$ and $\Delta_F(y) = \sum_{\ell \geq 1} b_\ell \otimes y_\ell \in$

$H \hat{\otimes} F$. Therefore, one has $\Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{\ell \geq 1} a_j b_\ell \otimes x_j \otimes y_\ell$.

(i) It follows immediately that $(\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{\ell \geq 1} \sigma(a_j b_\ell) x_j \otimes y_\ell =$

$= \sum_{j \geq 1} \sum_{\ell \geq 1} \sigma(a_j) \sigma(b_\ell) x_j \otimes y_\ell = \sum_{j \geq 1} \sigma(a_j) x_j \otimes \sum_{\ell \geq 1} \sigma(b_\ell) y_\ell = x \otimes y = 1_{E \hat{\otimes} F}(x \otimes y)$. From

what, one deduces $(\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = 1_{E \hat{\otimes} F}$

(ii) Also, one has for $x \in E$, $y \in F$

$\alpha) (c \otimes 1_E) \circ \Delta_E(x) = \sum_{j \geq 1} c(a_j) \otimes x_j = \sum_{j \geq 1} \sum_{s \geq 1} \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 \otimes x_j = (1_H \otimes \Delta_E) \circ \Delta_E(x) =$

$= \sum_{j \geq 1} a_j \otimes \Delta_E(x_j) = \sum_{j \geq 1} \sum_{k \geq 1} a_j \otimes \gamma_{k,j} \otimes x_{k,j}$

and

$$\begin{aligned} (c \otimes 1_F) \circ \Delta_F(y) &= \sum_{\ell \geq 1} c(b_\ell) \otimes y_\ell = \sum_{\ell \geq 1} \sum_{t \geq 1} \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 \otimes y_\ell = (1_H \otimes \Delta_F) \circ \Delta_F(y) = \\ &= \sum_{\ell \geq 1} b_\ell \otimes \Delta_F(y_\ell) = \sum_{\ell \geq 1} \sum_{m \geq 1} b_\ell \otimes \rho_{m,\ell} \otimes y_{m,\ell} \end{aligned}$$

Let $E_x = E[(x_j, j \geq 1) \cup (x_{k,j}, k \geq 1, j \geq 1)]$ be the closed linear subspace of E spanned by $(x_j, j \geq 1) \cup (x_{k,j}, k \geq 1, j \geq 1)$, and $F_y = E[(y_\ell, \ell \geq 1) \cup (y_{m,\ell}, m \geq 1, \ell \geq 1)]$ be the closed linear subspace of F spanned by $(y_\ell, \ell \geq 1) \cup (y_{m,\ell}, m \geq 1, \ell \geq 1)$. It is clear that the Banach spaces E_x and F_y are of countable type. Furthermore, if $x' \in E'_x$ and $y' \in F'_y$ one has

$$\begin{aligned} (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) &= \sum_{j \geq 1} \sum_{s \geq 1} \langle x', x_j \rangle \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 = \\ &= (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) = \sum_{j \geq 1} \sum_{k \geq 1} \langle x', x_{k,j} \rangle a_j \otimes \gamma_{k,j} \end{aligned}$$

and

$$\begin{aligned} (1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(y) &= \sum_{\ell \geq 1} \sum_{t \geq 1} \langle y', y_\ell \rangle \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 = \\ &= (1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_F) \circ \Delta_F(y) = \sum_{\ell \geq 1} \sum_{m \geq 1} \langle y', y_{m,\ell} \rangle b_\ell \otimes \rho_{m,\ell}. \end{aligned}$$

$$\begin{aligned} \beta) \quad \text{On one hand, one has, } (c \otimes 1_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y) &= \sum_{j \geq 1} \sum_{t \geq 1} c(a_j b_t) \otimes x_j \otimes y_t = \\ &= \sum_{j \geq 1} \sum_{t \geq 1} c(a_j) c(b_t) \otimes x_j \otimes y_t = \sum_{j \geq 1} \sum_{t \geq 1} \left(\sum_{s \geq 1} \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 \right) \left(\sum_{t \geq 1} \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 \right) \otimes x_j \otimes y_t. \end{aligned}$$

On the other hand, one has

$$(1_H \otimes \Delta_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{t \geq 1} a_j b_t \otimes \Delta_{E \widehat{\otimes} F}(x_j \otimes y_t) = \sum_{j \geq 1} \sum_{t \geq 1} \sum_{k \geq 1} \sum_{m \geq 1} a_j b_t \otimes$$

$$\gamma_{k,j} \rho_{m,t} \otimes x_{k,j} \otimes y_{m,t}.$$

Hence, if $x' \in E'_x$ and $y' \in F'_y$; first, one has

$$(1_H \otimes 1_H \otimes x' \otimes y') \circ (c \otimes 1_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{s \geq 1} \langle x', x_j \rangle \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 \sum_{t \geq 1} \sum_{t \geq 1} \langle y', y_t \rangle$$

$$\begin{aligned} \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 &= (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(y) = \\ &= (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_F) \circ \Delta_F(y). \end{aligned}$$

And, second, one has

$$(1_H \otimes 1_H \otimes x' \otimes y') \circ (1_H \otimes \Delta_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{k \geq 1} \langle x', x_{k,j} \rangle a_j \otimes \gamma_{k,j} \sum_{t \geq 1} \sum_{m \geq 1} \langle y', y_{m,t} \rangle$$

$$b_t \otimes \rho_{m,t} = (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_F) \circ \Delta_F(y).$$

Therefore, for any $x' \in E'_x$ and any $y' \in F'_y$, we have

$$(a) : (1_H \otimes 1_H \otimes x' \otimes y') [(c \otimes 1_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y) - (1_H \otimes \Delta_{E \widehat{\otimes} F}) \circ \Delta_{E \widehat{\otimes} F}(x \otimes y)] = 0$$

$\gamma)$ Since E_x [resp. F_y] is of countable type, there exist $\alpha_0 > 0, \alpha_1 > 0$ and

$(e_j)_{j \geq 1} \subset E_x$ [resp. $(f_\ell)_{\ell \geq 1} \subset F_y$] such that for $z \in E_x$ [resp. $\zeta \in F_y$] one has
 $z = \sum_{j \geq 1} \lambda_j e_j$ [resp. $\zeta = \sum_{\ell \geq 1} \mu_\ell f_\ell$] with $\alpha_0 \sup_{j \geq 1} |\lambda_j| \leq \|z\| \leq \alpha_1 \sup_{j \geq 1} |\lambda_j|$ [resp. $\alpha_0 \sup_{\ell \geq 1} |\mu_\ell| \leq \|\zeta\| \leq \alpha_1 \sup_{\ell \geq 1} |\mu_\ell|$] (cf. [4]).

Moreover, one has $E_x \hat{\otimes} F_y \simeq c_o(\mathbb{N}^* \times \mathbb{N}^*, K)$ and $(H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y) \simeq c_o(\mathbb{N}^* \times \mathbb{N}^*, H \hat{\otimes} H)$ (cf. [7]); any Z in $(H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y)$ can be written in the unique form $Z = \sum_{j, \ell} A_{j\ell} \otimes e_j \otimes f_\ell$ with $A_{j\ell} \in H \hat{\otimes} H$ and $\alpha_0^2 \sup_{j, \ell} \|A_{j\ell}\| \leq \|Z\| \leq \alpha_1^2 \sup_{j, \ell} \|A_{j\ell}\|$.

Let $e'_j \in E'_x$ [resp. $f'_\ell \in F'_y$] be the continuous linear form defined by $\langle e'_j, e_{j_1} \rangle = \delta_{jj_1}$ [resp. $\langle f'_\ell, f_{\ell_1} \rangle = \delta_{\ell\ell_1}$]. Setting $(c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) - (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = Z_0 = \sum_{j, \ell} A_{j\ell}^0 \otimes e_j \otimes f_\ell \in H \hat{\otimes} H \hat{\otimes} E_x \hat{\otimes} F_y$, for any $j_1 \geq 1$ and any $\ell_1 \geq 1$, by (a), one has $(1_H \otimes 1_H \otimes e'_{j_1} \otimes f'_{\ell_1})(Z_0) = \sum_{j, \ell} A_{j\ell}^0 \delta_{j_1 j} \delta_{\ell_1 \ell} = A_{j_1 \ell_1}^0 = 0$. It follows that $Z_0 = 0$, i.e.
 $(c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y)$. From what, one deduces that
 $(c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}$.

Corollary : Let M [resp. N] be a left Banach H -subcomodule of E [resp. F]. Then $M \hat{\otimes} N$ is a left Banach subcomodule of $E \hat{\otimes} F$.

I - 2 Banach comodule morphisms

I - 2 - 1 Range and kernel

Proposition 2 : Let $u : E \rightarrow F$ be a Banach comodule morphism.

- (i) If V is a Banach subcomodule of F , then $u^{-1}(V)$ is a Banach subcomodule of E .
- (ii) The closure $\overline{u(E)}$ of $u(E)$ is a Banach subcomodule of F

Corollary : Let V and W be Banach subcomodule of the left Banach H -comodule E ; then $V \cap W$ is a Banach subcomodule of E .

Proofs : Rather easy, or see [3].

Note : One can also see [3] for the spaces of comodule morphisms.

Remark 1 : If M is a Banach subcomodule of the left Banach H -comodule E , it is induced on the quotient Banach space E/M a structure of Banach left H -comodule such that the canonical map $E \rightarrow E/M$ is a comodule morphism.

Then, if $u : E \rightarrow F$ is a Banach comodule morphism and if u is strict, the Banach comodule $E/\ker u$ and $u(E)$ are isomorphic. Also, one can define the cokernel of u as being $F/\overline{u(E)}$.

I - 2- 2 Comodule morphisms of E into H associated with $\Delta ; R(\Delta)$

Put $\Delta = \Delta_E$ the coproduct of the left Banach H -comodule E . Obviously, H is a left Banach H -comodule with respect to its coproduct c .

Proposition 3 : For any $x' \in E'$, the linear map $A_{x'} = (1_H \otimes x') \circ \Delta : E \rightarrow H$ is a Banach comodule morphism.

Proof : It is easy to see that $c \circ (1_H \otimes x') = c \otimes x' = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E)$. Therefore $c \circ A_{x'} = c \circ (1_H \otimes x') \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta) \circ \Delta = (1_H \otimes [(1_H \otimes x') \circ \Delta]) \circ \Delta = (1_H \otimes A_{x'}) \circ \Delta$.

Corollary 1 :

- (i) $\ker A_{x'}$ is a closed subcomodule of E .
- (ii) $\overline{A_{x'}(E)}$ is a left Banach subcomodule (= closed left coideal) of H .

Corollary 2 : If E is a space of countable type, one has $\ker A_{x'} \neq E$ for any $x' \in E, x' \neq 0$

Proof : Indeed, if $x' \in E', x' \neq 0$ and $0 < \alpha < 1$, there exists a α -orthogonal base $(e_j)_{j \geq 1} \subset E$ such that $\langle x', e_1 \rangle = 1$ and $\langle x', e_j \rangle = 0, j \geq 2$. Moreover for any $j \geq 1$, $\Delta(e_j) = \sum_{t \geq 1} a_{tj} \otimes e_t$ and $e_j = \sum_{t \geq 1} \sigma(a_{tj}) e_t$; therefore $\sigma(a_{tj}) = \delta_{tj}$ and $A_{x'}(e_1) = (1_H \otimes x') \circ \Delta(e_1) = a_{11} \neq 0$ since $\sigma(a_{11}) = 1$.

Corollary 3 : Assume that H is a pseudo-reflexive Banach space ; i.e. $H \rightarrow H''$ is isometric.

Let E be a simple Banach left H -comodule, i.e. E contains no proper closed subcomodule. Then E is a Banach space of countable type and $A_{x'}$ is injective for each $x' \in E', x' \neq 0$.

Proof : If H is pseudo-reflexive, it is shown in [3] that any simple Banach left H -comodule is a space of countable type. Applying Corollary 2, one sees that $A_{x'}$ is injective for $x' \in E', x' \neq 0$ $\square$

Let $\beta : E \otimes E' \rightarrow K$ be the continuous linear form defined upon $\beta(x \otimes x') = \langle x', x \rangle$. Put $\rho_\Delta = (1_H \otimes \beta) \circ (\Delta \otimes 1_{E'}) \circ \tau : E' \widehat{\otimes} E \rightarrow H$, where $\tau(x' \otimes x) = x \otimes x'$. Then ρ_Δ is linear and continuous with $\|\rho_\Delta\| \leq \|\Delta\|$. Moreover for $x' \in E', x \in E$, one has $\rho_\Delta(x' \otimes x) = (1_H \otimes x') \circ \Delta(x)$.

Put $R(\Delta) = \overline{\rho_\Delta(E' \hat{\otimes} E)}$ the closure of $\rho_\Delta(E' \hat{\otimes} E)$ in H . Obviously, $R(\Delta)$ is the closed linear subspace of H spanned by the elements $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'$, $x \in E$, called the coefficients of the comodule (E, Δ) .

Proposition 4 : $R(\Delta) = \overline{\rho_\Delta(E' \hat{\otimes} E)}$ is a left Banach subcomodule (= closed left coideal) of H .

Proof : Since $c : H \rightarrow H \hat{\otimes} H$ is linear and is a homeomorphism of H onto $c(H)$, one has $c(R(\Delta)) = \overline{c(\rho_\Delta(E' \hat{\otimes} E))}$, a closed linear subspace of H .

It remains to show that if $a = \rho_\Delta(x' \otimes x) = (1_H \otimes x') \circ \Delta(x) = A_{x'}(x)$, $x' \in E'$, $x \in E$; then $c(a) \in H \hat{\otimes} R(\Delta)$. Writing $\Delta(x) = \sum_{j \geq 1} a_j \otimes x_j$; one has $c(a) = c \circ A_{x'}(x) = (1_H \otimes$

$$A_{x'}) \circ \Delta(x) = \sum_{j \geq 1} a_j \otimes A_{x'}(x_j) = \sum_{j \geq 1} a_j \otimes \rho_\Delta(x' \otimes x_j) \in H \hat{\otimes} R(\Delta). \quad \square$$

Proposition 5 : If the left Banach comodules E and E_1 with coproduct respectively Δ and Δ_1 are isomorphic, then $R(\Delta) = R(\Delta_1)$.

Proof : Let $u : E \rightarrow E_1$ be a comodule isomorphism, in other words, u is linear, continuous and bijective with $\Delta_1 \circ u = (1_H \otimes u) \circ \Delta$. Moreover, the reciprocal map u^{-1} of u satisfies $(1_H \otimes u^{-1}) \circ \Delta_1 = \Delta \circ u^{-1}$ and the transpose of u , ${}^t u : E'_1 \rightarrow E'$ is linear, continuous and bijective with $({}^t u)^{-1} = {}^t u^{-1}$.

Set $a = \rho_{\Delta_1}(z_1) \in \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ and $z_1 = \sum_{j \geq 1} y'_j \otimes y_j$, $y'_j \in E'_1$, $y_j \in E_1$, $\lim_j \|y'_j\| \|y_j\| = 0$.

There exist, for $j \geq 1$ unique $x'_j \in E'$ and $x_j \in E$ such that $y'_j = {}^t u^{-1}(x'_j) = x'_j \circ u^{-1}$ and $y_j = u(x_j)$; moreover $\lim_j \|x'_j\| \|x_j\| = 0$. Therefore $a = \rho_{\Delta_1}(z_1) = \sum_{j \geq 1} \rho_{\Delta_1}(y'_j \otimes y_j) =$

$$= \sum_{j \geq 1} (1_H \otimes y'_j) \circ \Delta_1(y_j) = \sum_{j \geq 1} (1_H \otimes x'_j \circ u^{-1}) \circ \Delta_1 \circ u(x_j) = \sum_{j \geq 1} (1_H \otimes x'_j) \circ (1_H \otimes u^{-1}) \circ \Delta_1 \circ$$

$$u(x_j) = \sum_{j \geq 1} (1_H \otimes x'_j) \circ \Delta(x_j) = \sum_{j \geq 1} \rho_\Delta(x'_j \otimes x_j) = \rho_\Delta\left(\sum_{j \geq 1} x'_j \otimes x_j\right). \text{ Hence, } a = \rho_\Delta(z) \in$$

$\rho_\Delta(E' \hat{\otimes} E)$ where $z = \sum_{j \geq 1} x'_j \otimes x_j$; that is $\rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \subset \rho_\Delta(E' \hat{\otimes} E)$. Likewise, one has

$$\rho_\Delta(E' \hat{\otimes} E) \subset \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1).$$

Therefore $\rho_\Delta(E' \hat{\otimes} E) = \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ and $R(\Delta) = R(\Delta_1)$. $\square$

Assume that E is a free Banach space i.e. $E \simeq c_0(I, K) = \{(\lambda_j)_{j \in I} \subset K / \lim_j \lambda_j = 0\}$.

In other words, there exist $(e_j)_{j \in I} \subset E$, $\alpha_0, \alpha_1 \in \mathbb{R}_+^*$ such that any $x \in E$ can be written

in the form $x = \sum_{j \in I} \lambda_j e_j$, $\lambda_j \in K$ and $\alpha_0 \sup_{j \in I} |\lambda_j| \leq \|x\| \leq \alpha_1 \sup_{j \in I} |\lambda_j|$. For any continuous linear form $x' \in E'$, one has $\frac{1}{\alpha_1} \sup_{j \in I} | \langle x', e_j \rangle | \leq \|x'\| \leq \frac{1}{\alpha_0} \sup_{j \in I} | \langle x', e_j \rangle |$. Let e'_j be the element of E' defined by $\langle e'_j, e_\ell \rangle = \delta_{j\ell}$. Put $E'_0 = E[(e'_j)_{j \in I}]$, the closed linear subspace of E' spanned by $(e'_j)_{j \in I}$. Hence each $x' \in E'_0$ can be written in the unique form $x' = \sum_{j \in I} \mu_j e'_j$, $\mu_j \in K$, $\lim_j |\mu_j| = 0$. Moreover, if $v \in E'_0 \widehat{\otimes} E \subset E' \widehat{\otimes} E$, one has $v = \sum_{j, \ell} \mu_{\ell j} e'_\ell \otimes e_j$, $\mu_{\ell j} \in K$, $\lim_{(j, \ell)} \mu_{\ell j} = 0$ and $\frac{\alpha_0}{\alpha_1} \sup_{j, \ell} |\mu_{\ell j}| \leq \|v\| \leq \frac{\alpha_1}{\alpha_0} \sup_{j, \ell} |\mu_{\ell j}|$.

On the other hand, one has $H \widehat{\otimes} E \simeq c_0(I, H) = \{(a_j)_{j \in I} \subset H / \lim_j a_j = 0\}$. For any $z \in H \widehat{\otimes} E$ one has $z = \sum_{j \in I} a_j \otimes e_j$, $a_j \in H$ with $\lim_j \|a_j\| = 0$ and $\alpha_0 \sup_{j \in I} \|a_j\| \leq \|z\| \leq \alpha_1 \sup_{j \in I} \|a_j\|$. Hence, if (E, Δ) is a left Banach H -comodule, for $x \in E$, one has $\Delta(x) = \sum_{j \in I} A_j(x) \otimes e_j$. In particular $\Delta(e_\ell) = \sum_{j \in I} A_j(e_\ell) \otimes e_j = \sum_{j \in I} a_{\ell j} \otimes e_j$ and $(c \otimes 1_E) \circ \Delta(e_\ell) = \sum_{j \in I} c(a_{\ell j}) \otimes e_j = (1_H \otimes \Delta) \circ \Delta(e_\ell) = (1_H \otimes \Delta) \left(\sum_{k \in I} a_{\ell k} \otimes e_k \right) = \sum_{k \in I} a_{\ell k} \otimes \sum_{j \in I} a_{kj} \otimes e_j = \sum_{k \in I} \sum_{j \in I} a_{\ell k} \otimes a_{kj} \otimes e_j$. Thus one obtains

$$(1) \quad c(a_{\ell j}) = \sum_{k \in I} a_{\ell k} \otimes a_{kj} ; \ell, j \in I$$

Also, one has

$$(2) \quad \sigma(a_{\ell j}) = \delta_{\ell j} ; \ell, j \in I$$

$$(3) \quad \sum_{k \in I} a_{\ell k} \eta(a_{kj}) = \delta_{\ell j} \cdot e = \sum_{k \in I} \eta(a_{\ell k}) \otimes a_{kj} ; \ell, j \in I.$$

Proposition 6 : $R_0(\Delta) = \overline{\rho_\Delta(E'_0 \widehat{\otimes} E)}$ is a closed subcoalgebra of H . In other words $c(R_0(\Delta)) \subset R_0(\Delta) \widehat{\otimes} R_0(\Delta)$

Proof : Since $(e'_j \otimes e_\ell)_{(j, \ell) \in I \times I}$ is a total family of $E'_0 \widehat{\otimes} E$ and ρ_Δ is linear and continuous, the family $(\rho_\Delta(e'_j \otimes e_\ell))_{(j, \ell) \in I \times I}$ is total in $\overline{\rho_\Delta(E'_0 \widehat{\otimes} E)} = R_0(\Delta)$ = the closed linear subspace of H spanned by the $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'_0$, $x \in E$.

To see that $c(R_0(\Delta)) \subset R_0(\Delta) \widehat{\otimes} R_0(\Delta)$, it suffices to show that for $\ell, j \in I$ one has $c(\rho_\Delta(e'_j \otimes e_\ell)) \in R_0(\Delta) \widehat{\otimes} R_0(\Delta)$. However, by definition, $\rho_\Delta(e'_j \otimes e_\ell) = (1_H \otimes e'_j) \circ \Delta(e_\ell) = a_{\ell j} \in R_0(\Delta)$. Then, one deduces from (1) that $c(\rho_\Delta(e'_j \otimes e_\ell)) = c(a_{\ell j}) = \sum_{k \in I} a_{\ell k} \otimes a_{kj} \in$

$$R_0(\Delta) \widehat{\otimes} R_0(\Delta).$$

Note : If $v = \sum_{\ell,j} \mu_{\ell j} e'_j \otimes e_\ell \in E'_0 \widehat{\otimes} E$, one has $\rho_\Delta(v) = \sum_{\ell,j} \mu_{\ell j} a_{\ell j}$ and $a \in R_0(\Delta)$ iff there exist $v_n \in E' \widehat{\otimes} E$ such that $a = \lim_{n \rightarrow +\infty} \rho_\Delta(v_n)$.

Remark 2 : Let (E, Δ) and (E_1, Δ_1) be two isomorphic left Banach comodules that are free Banach spaces. If $u : E \rightarrow E_1$ is a comodule isomorphism, $(e_j)_{j \in I}$ a base of E and $(\varepsilon_j)_{j \in I}$ the base of E_1 defined by $\varepsilon_j = u(e_j)$; then, with the above notations, one has $R_0(\Delta) = R_0(\Delta_1)$.

Remark 3 : If $\dim E = n < +\infty$, one has $R(\Delta) = R_0(\Delta) = \rho_\Delta(E' \otimes E)$ and $\dim R(\Delta) \leq n^2$

II - REPRESENTATIVE SUBALGEBRA

II - 1 Conjugate comodule of a finite dimensional comodule

Let (E, Δ) be a (Banach) left H -comodule of finite dimension and $(e_j)_{1 \leq j \leq n}$ a K -base of E . As above, for any $x \in E$, one has $\Delta(x) = \sum_{j=1}^n A_j(x) \otimes e_j$ and $A_j(x) = (1_H \otimes e'_j) \circ$

$\Delta(x) = \rho_\Delta(e'_j \otimes x)$; $A_j = (1_H \otimes e'_j) \circ \Delta \in \mathcal{L}(E, H)$. In particular $\Delta(e_\ell) = \sum_{j=1}^n a_{\ell j} \otimes e_j$ where $a_{\ell j} = A_j(e_\ell) = \rho_\Delta(e'_j \otimes e_\ell)$; and we have the relations (1), (2) and (3), with $I = [1, n]$. The relation (3) means here, that the matrix $A = (a_{\ell j})_{1 \leq \ell, j \leq n} \in \text{Mat}_n(H)$ is invertible with inverse $A^{-1} = (\eta(a_{\ell j}))_{1 \leq \ell, j \leq n}$.

Fix the base $(e_j)_{1 \leq j \leq n}$ of E and define the linear map $\Delta^\vee : E' \rightarrow H \otimes E'$ by setting $\Delta^\vee(e'_j) = \sum_{\ell=1}^n \eta(a_{\ell j}) \otimes e'_\ell$, $1 \leq j \leq n$. Hence for $x' = \sum_{j=1}^n \mu_j e'_j \in E'$, one has $\Delta^\vee(x') = \sum_{\ell=1}^n \sum_{j=1}^n \mu_j \eta(a_{\ell j}) \otimes e'_\ell = \sum_{\ell=1}^n A_\ell^\vee(x') \otimes e'_\ell$.

Lemma 1 : $(E', \Delta^\vee)$ is a left H -comodule.

Proof : One verifies that $\sigma \circ \eta = \sigma$; indeed, if $a \in H$, then $c(a) = \sum_{i \geq 1} a_i^1 \otimes a_i^2$. Hence, one has $m \circ (\eta \otimes 1_H) \circ c(a) = \sum_{i \geq 1} \eta(a_i^1) a_i^2 = \sigma(a) e$ and $a = (1_H \otimes \sigma) \circ c(a) = \sum_{i \geq 1} a_i^1 \sigma(a_i^2)$. It

follows that $\eta(a) = \sum_{t \geq 1} \eta(a_t^1) \sigma(a_t^2)$ and $\sigma \circ \eta(a) = \sum_{t \geq 1} \sigma(\eta(a_t^1)) \sigma(a_t^2) = \sigma\left(\sum_{t \geq 1} \eta(a_t^1) a_t^2\right) = \sigma(\sigma(a)e) = \sigma(a)$.

Since $\sigma(a_{tj}) = \delta_{tj}$, one has $(\sigma \otimes 1_{E'}) \circ \Delta^\vee(e'_j) = \sum_{t=1}^n \sigma \circ \eta(a_{tj}) e'_t = \sum_{t=1}^n \sigma(a_{tj}) e'_t = e'_j$, $1 \leq j \leq n$. It follows, by linearity, that $(\sigma \otimes 1_{E'}) \circ \Delta^\vee = 1_{E'}$.

Let us remember that $c \circ \eta = \tau \circ (\eta \otimes \eta) \circ c$ where $\tau(a \otimes b) = b \otimes a$. Hence, we have $c \circ \eta(a_{tj}) = \sum_{k=1}^n \eta(a_{kj}) \otimes \eta(a_{tk})$. Therefore $(c \otimes 1_{E'}) \circ \Delta^\vee(e'_j) = (c \otimes 1_{E'}) \left(\sum_{t=1}^n \eta(a_{tj}) \otimes e'_t \right) = \sum_{t=1}^n \sum_{k=1}^n \eta(a_{kj}) \otimes \eta(a_{tj}) \otimes e'_t = \sum_{k=1}^n \eta(a_{kj}) \otimes \Delta^\vee(e'_k) = (1_H \otimes \Delta^\vee) \left(\sum_{k=1}^n \eta(a_{kj}) \otimes e'_k \right) = (1_H \otimes \Delta^\vee) \circ \Delta^\vee(e'_j)$, and $(c \otimes 1_{E'}) \circ \Delta^\vee = (1_H \otimes \Delta^\vee) \circ \Delta^\vee$.

Corollary : $R(\Delta^\vee) = \eta(R(\Delta))$.

Proof : Identifying E'' with E , one has $R(\Delta^\vee) = \rho_{\Delta^\vee}(E \otimes E')$. Set $z = \sum_{1 \leq t, j \leq n} \lambda_{tj} e_t \otimes e'_j \in E \otimes E'$; hence $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t, j \leq n} \lambda_{tj} \rho_{\Delta^\vee}(e_t \otimes e'_j)$. However $\rho_{\Delta^\vee}(e_t \otimes e'_j) = (1_H \otimes e_t) \circ \Delta^\vee(e'_j) = \eta(a_{tj}) = \eta(\rho_\Delta(e'_j \otimes e_t))$; therefore $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t, j \leq n} \lambda_{tj} \eta(\rho_\Delta(e'_j \otimes e_t)) = \eta(\rho_\Delta(z_1))$ where $z_1 = \sum_{1 \leq t, j \leq n} \lambda_{tj} e'_j \otimes e_t \in E' \otimes E$. It follows that $R(\Delta^\vee) \subset \eta(R(\Delta))$. The same formulae show that if $a = \rho_\Delta(z_1) \in R(\Delta)$, where $z_1 = \sum_{1 \leq t, j \leq n} \lambda_{tj} e'_j \otimes e_t \in E' \otimes E$, one has $\eta(a) = \rho_{\Delta^\vee}(z)$ where $z = \sum_{1 \leq t, j \leq n} \lambda_{tj} e_t \otimes e'_j \in E \otimes E'$, hence $\eta(R(\Delta)) \subset R(\Delta^\vee)$.

II- 2 Direct sum of Banach comodules

Let $(E_s)_{1 \leq s \leq m}$ be a finite family of left Banach H -comodules with Δ_s the coproduct of E_s . The direct sum $E = \bigoplus_{s=1}^m E_s$ equipped with any norm equivalent to the norm $\left\| \sum_{s=1}^m x_s \right\| = \max_{1 \leq s \leq m} \|x_s\|$ is a Banach space. Put $\Delta = \bigoplus_{s=1}^m \Delta_s$, i.e. $\Delta\left(\sum_{s=1}^m x_s\right) = \bigoplus_{s=1}^m \Delta_s(x_s)$. It is readily seen that (E, Δ) is a left Banach comodule. Moreover, if $p_s : E \rightarrow E_s$ is the projection of E onto E_s , then $1_H \otimes p_s$ is a projection of $H \hat{\otimes} E$ onto $H \hat{\otimes} E_s$ and one has

$H \hat{\otimes} E = \bigoplus_{s=1}^m H \hat{\otimes} E_s$. On the other hand p_s is a comodule morphism i.e. $(1_H \otimes p_s) \circ \Delta = \Delta \circ p_s$; furthermore $(1_H \otimes p_s) \circ \Delta(x_t) = 0$ for $s \neq t$ and $x_t \in E_t$.

Also, we have $E' = \bigoplus_{s=1}^m E'_s$; the projections associated with this direct sum are the ${}^t p_s$, $1 \leq s \leq m$.

Proposition 7 : *With the above notations, one has $\rho_\Delta(E' \hat{\otimes} E) = \sum_{s=1}^m \rho_{\Delta_s}(E'_s \hat{\otimes} E_s)$*

and $R(\Delta)$ is the closure of $\sum_{s=1}^m R(\Delta_s)$ in H .

Proof : If $x'_s \in E'_s$, $x_t \in E_t$ and $s \neq t$, then $(1_H \otimes x'_s) \circ \Delta(x_t) = (1_H \otimes x'_s) \circ \Delta_t(x_t) = 0$.

Set $z = \sum_{j \geq 1} x'_j \otimes x_j \in E' \hat{\otimes} E = \bigoplus_{s=1}^m \bigoplus_{t=1}^m E'_s \hat{\otimes} E_t$, one has $z = \sum_{j \geq 1} \sum_{s=1}^m \sum_{t=1}^m x'_{s,j} \otimes x_{t,j}$. It follows

$$\begin{aligned} \text{that } \rho_\Delta(z) &= \sum_{j \geq 1} \sum_{s=1}^m \sum_{t=1}^m \rho_\Delta(x'_{s,j} \otimes x_{t,j}) = \sum_{j \geq 1} \sum_{s=1}^m \sum_{t=1}^m (1_H \otimes x'_{s,j}) \circ \Delta(x_{t,j}) = \\ &= \sum_{j \geq 1} \sum_{s=1}^m \sum_{t=1}^m \delta_{s,t} (1_H \otimes x'_{s,j}) \circ \Delta_t(x_{t,j}) = \sum_{j \geq 1} \sum_{s=1}^m \rho_{\Delta_s}(x'_{s,j} \otimes x_{s,j}) = \sum_{s=1}^m \rho_{\Delta_s} \left(\sum_{j \geq 1} x'_{s,j} \otimes x_{s,j} \right) = \\ &= \sum_{s=1}^m \rho_{\Delta_s}(z_s). \text{ If } z_s \in E'_s \hat{\otimes} E_s \subset E' \hat{\otimes} E, 1 \leq s \leq m, \text{ one has } \rho_\Delta(z_s) = \rho_{\Delta_s}(z_s). \text{ Therefore,} \end{aligned}$$

on one hand, $\rho_\Delta(E' \hat{\otimes} E) \subset \sum_{s=1}^m \rho_{\Delta_s}(E'_s \hat{\otimes} E_s)$, and on the other hand, $\rho_{\Delta_s}(E'_s \hat{\otimes} E_s) \subset \rho_\Delta(E' \hat{\otimes} E)$. Hence, one has $\rho_\Delta(E' \hat{\otimes} E) = \sum_{s=1}^m \rho_{\Delta_s}(E'_s \hat{\otimes} E_s)$. One verifies readily that $R(\Delta)$

is equal to the closure of $\sum_{s=1}^m R(\Delta_s)$ in H .

Corollary : *If $\dim E_s < +\infty$, $1 \leq s \leq m$, then one has $R(\Delta) = \sum_{s=1}^m R(\Delta_s)$ where*

$$E = \bigoplus_{s=1}^m E_s, \quad \Delta = \bigoplus_{s=1}^m \Delta_s.$$

Remark 3 : *If the comodules (E_s, Δ_s) , $1 \leq s \leq m$, are pairwise isomorphic, then for the comodule (E, Δ) where $E = \bigoplus_{s=1}^m E_s$, $\Delta = \bigoplus_{s=1}^m \Delta_s$, one has $R(\Delta) = R(\Delta_s)$, $1 \leq s \leq m$.*

II - 3 The representative subalgebra of H

Let $\mathcal{S}(H)$ be the set of all elements of the form $a = (1_H \otimes x') \circ \Delta(x)$ of H where (E, Δ) is a finite dimensional left H -comodule and $x' \in E'$, $x \in E$. Let us put $\dim E = \dim \Delta$

Lemma 2 : $\mathcal{S}(H)$ is a multiplicative, unitary submonoid of H .

Proof : Set $a = (1_H \otimes x') \circ \Delta(x)$ and $b = (1_H \otimes y') \circ \Delta_1(y) \in \mathcal{S}(H)$ where (E, Δ) and (E, Δ_1) are left H -comodules of finite dimension and $x' \in E'$, $x \in E$, $y' \in E'_1$, $y \in E_1$.

One has $\Delta(x) = \sum_{j=1}^p a_j \otimes x_j$, $\Delta_1(y) = \sum_{\ell=1}^q b_\ell \otimes y_\ell$ and $\Delta_{E \otimes E_1}(x \otimes y) = \sum_{j=1}^p \sum_{\ell=1}^q a_j b_\ell \otimes x_j \otimes y_\ell$.

Hence, $ab = (1_H \otimes x') \circ \Delta(x) \cdot (1_H \otimes y') \circ \Delta(y) = \sum_{j=1}^p \sum_{\ell=1}^q a_j b_\ell \langle x', x_j \rangle \langle y', b_\ell \rangle = (1_H \otimes x' \otimes y') \circ \Delta_{E \otimes E_1}(x \otimes y) \in \mathcal{S}(H)$.

Since $c(e) = e \otimes e$, $E = K.e$ is a left subcomodule of H of dimension 1, one has $e = (1_H \otimes \sigma) \circ c(e) \in \mathcal{S}(H)$. $\square$

Let $\mathcal{R}(H)$ be the linear subspace of H spanned by $\mathcal{S}(H)$. Then $\mathcal{R}(H)$ is an unitary subalgebra of H . Indeed, if $a = \sum_{j=1}^p \lambda_j a_j$ and $b = \sum_{\ell=1}^q \mu_\ell b_\ell$ are two elements of $\mathcal{R}(H)$, since $a_j b_\ell \in \mathcal{S}(H)$, one has $ab = \sum_{j=1}^p \sum_{\ell=1}^q \lambda_j \mu_\ell a_j b_\ell \in \mathcal{R}(H)$. One says that $\mathcal{R}(H)$ is the representative subalgebra of H .

Note : Put, for the left H -comodule (E, Δ) of finite dimension, $\mathcal{S}(\Delta) = \{a = (1_H \otimes x') \circ \Delta(x) \in H; x' \in E', x \in E\}$. As in Proposition 5, $\mathcal{S}(\Delta)$ depends only of the isomorphism class $\tilde{\Delta}$ of (E, Δ) . Furthermore, one has $\mathcal{S}(H) = \bigcup_{\dim \Delta < +\infty} \mathcal{S}(\Delta)$. $\square$

Also, it is clear that the K -linear vector space $R(\Delta) = \rho_\Delta(E' \otimes E)$ is spanned by $\mathcal{S}(\Delta)$. Hence one has $\mathcal{R}(H) = \bigcup_{\dim \Delta < +\infty} R(\Delta)$. Moreover, if (E_1, Δ_1) and (E_2, Δ_2) are two comodules, then $R(\Delta_1 \oplus \Delta_2) = R(\Delta_1) + R(\Delta_2)$ contains $R(\Delta_1)$ and $R(\Delta_2)$ i.e. the family $(R(\Delta))_\Delta$ ordered by inclusion is directed upward.

Theorem 1 : The representative subalgebra $\mathcal{R}(H)$ of H is such that $c(\mathcal{R}(H)) \subset \mathcal{R}(H) \otimes \mathcal{R}(H)$. Moreover $(\mathcal{R}(H), m, c, \eta, \sigma)$ is a Hopf algebra.

Proof : It follows from Proposition 6 and Remark 3 that if Δ is a coproduct of finite dimension, then $c(R(\Delta)) \subset R(\Delta) \otimes R(\Delta)$: that is $R(\Delta)$ is a coalgebra. Since

$\mathcal{R}(H) = \bigcup_{\dim \Delta < +\infty} R(\Delta)$ is the union of coalgebras, it is a coalgebra. On the other hand, one deduces from the Corollary of Lemma 1 that $\eta(\mathcal{R}(H)) \subset \mathcal{R}(H)$. The Theorem 1 is proved.

II - 4 Simple comodules of finite dimension

Let $(e_j)_{1 \leq j \leq n}$ be a base of the finite dimensional left H -comodule (E, Δ) . Let us remember that $A_j = (1_H \otimes e'_j) \circ \Delta$ is a comodule morphism. One sees that $\bigcap_{1 \leq j \leq n} \ker A_j = (0)$ and $(A_j)_{1 \leq j \leq n}$ is free in $\mathcal{L}(E, H)$. Since $A_j(e_j) = a_{jj}$, one deduces from (2) or from Corollary 2 of Proposition 3 that $e_j \notin \ker A_j$ and $\ker A_j \neq E$.

Put $H_j = A_j(E)$; then H_j is a left subcomodule of H of dimension $\leq n$. Furthermore, with previous notations, one has $R(\Delta) = \rho_\Delta(E' \otimes E) = \sum_{j=1}^n H_j$ and $H_j = \sum_{t=1}^n K \cdot a_{tj}$, also $R(\Delta)$ is a subcoalgebra of dimension $\leq n^2$. One can have $\dim R(\Delta) < n^2$; for example, if $E_q = \bigoplus_{t=1}^q E$ and $\Delta_q = \bigoplus_{t=1}^q \Delta$, $q \geq 2$, one has $R(\Delta_q) = R(\Delta)$ and $\dim R(\Delta_q) = \dim R(\Delta) \leq n^2 < (qn)^2 = (\dim(E_q))^2$.

Definition : A left Banach H -comodule E is called *simple* or *topologically irreducible* if E is not the null space and does not contain any closed subcomodule different from (0) and E .

Let $\text{Hom.com}(E, E_1)$ be the Banach space of the left Banach comodule morphisms of (E, Δ) into (E_1, Δ_1) and $\text{End.com}(E) = \text{Hom.com}(E, E)$, this later is a Banach algebra.

Remark 4 : Schur's Lemma.

Let (E, Δ) and (E_1, Δ_1) be two simple, finite dimensional left H -comodules.

- (i) If E and E_1 are not isomorphic, one has $\text{Hom.com}(E, E_1) = (0)$
- (ii) In the alternative case, any non null comodule morphism of E into E_1 is an isomorphism. In particular, $\text{End.com}(E)$ is a (skew) field of finite dimension $\leq (\dim E)^2$. If K is algebraically closed, then $\text{End.com}(E) = K.1_E$.

Proposition 8 : Let (E, Δ) be a simple Banach left H -comodule of finite dimension n . Let $(e_j)_{1 \leq j \leq n}$ be a base of E and $A_j = (1_H \otimes e_j) \circ \Delta$, $1 \leq j \leq n$.

Then $H_j = A_j(E)$ is a simple left H -comodule of H of finite dimension n . Furthermore, there exists $J \subset [1, n]$ such that $R(\Delta) = \bigoplus_{j \in J} H_j$ (a direct sum of comodules).

Proof : It is the same as in semi-simple module theory. Indeed, since $\ker A_j \neq E$ and E is a simple comodule of finite dimension n , the map $A_j : E \rightarrow H_j = A_j(E)$ is a comodule

isomorphism. Hence H_j is a simple comodule of dimension n with base $(a_{\ell j})_{1 \leq \ell \leq n}$. If $1 \leq j, q \leq n$, one has $H_j \cap H_q = (0)$, or $H_j = H_q$. Changing the order if necessary, we may assume that $(H_1, \dots, H_m)$ is the family of the distinct comodules H_j ; $m \leq n$. Hence

$$R(\Delta) = \sum_{j=1}^m H_j, H_j \neq H_q \text{ for } j \neq q.$$

Since $H_1 \cap H_2 = (0)$, one has the direct sum of comodules $H_1 \oplus H_2$. Let j_0 be the least integer ≥ 3 such that $(H_1 \oplus H_2) \cap H_{j_0} = (0)$. Hence, one has the direct sum $H_1 \oplus H_2 \oplus H_{j_0}$ and for $j < j_0$, one has $(H_1 \oplus H_2) \cap H_j \neq (0)$, therefore $H_j \subset H_1 \oplus H_2$. Hence, by induction, one obtains $J = \{1, 2, j_0, \dots, j_k = m\} \subset [1, m]$ and the direct sum of comodules $\bigoplus_{j \in J} H_j$ such that for $\ell \notin J$, $H_\ell \subset \bigoplus_{j \in J} H_j$. It follows that $R(\Delta) = \bigoplus_{j \in J} H_j$.

Corollary : *Let (E, Δ) and (E_1, Δ_1) be two simple left H -comodules of finite dimension that are not isomorphic; then $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$, a direct sum of comodules.*

Proof : With previous notations, put $R(\Delta) = \bigoplus_{j \in J} H_j$ and $R(\Delta_1) = \bigoplus_{\ell \in L} H_\ell^1$. Let

p_j [resp. p_ℓ^1] be the projection of $R(\Delta)$ [resp. $R(\Delta_1)$] onto H_j [resp. H_ℓ^1]. Suppose that $R(\Delta) \cap R(\Delta_1) \neq (0)$; this finite dimensional comodule must contain at least one simple comodule V . There exists $j \in J$ [resp. $\ell \in L$] such that $p_j(V) \neq (0)$ [resp. $p_\ell^1(V) \neq (0)$]; therefore $p_j(V) = H_j$ [resp. $p_\ell^1(V) = H_\ell^1$]. Since V is simple, $p_j|_V$ [resp. $p_\ell^1|_V$] is an isomorphism of V onto H_j [resp. H_ℓ^1]. It follows that H_j and H_ℓ^1 are isomorphic. Hence E and E_1 are isomorphic; a contradiction. Therefore $R(\Delta) \cap R(\Delta_1) = (0)$ and $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$.

Remark 5 : *Notations and hypothesis as above. If K is algebraically closed, then the $H_j, 1 \leq j \leq n$, are pairwise distinct.*

Proof : Indeed, if $H_j = H_q$ for $j \neq q$, then $u = A_j \circ A_q^{-1}$ is an automorphism of the finite dimensional simple comodule H_j . By Schur's lemma, one has $u = \lambda \cdot 1_{H_j}, \lambda \in K, \lambda \neq 0$. Hence $A_j = \lambda A_q$ and $a_{jj} = A_j(e_j) = \lambda A_q(e_j) = \lambda a_{jq}$. Therefore $\sigma(a_{jj}) = 1 = \lambda \sigma(a_{jq}) = \lambda \delta_{jq} = 0$; a contradiction. $\square$

Let H' be the Banach space dual of H ; if we set for $a', b' \in H', a' * b' = (a' \otimes b') \circ c$, then H' becomes a complete normed algebra with unit σ . If (E, Δ) is a left Banach comodule, setting for $a' \in H'$, and $x \in E, a' \cdot x = (a' \otimes 1_E) \circ \Delta(x)$, one induces on E a complete normed right H' -module structure. Moreover, if H is a pseudo-reflexive Banach space, then any closed right H' -submodule of E is a Banach left H -subcomodule of E and reciprocally (cf. [3]).

Let $(E', \Delta^\vee)$ be the conjugate of the *finite dimensional* left H -comodule (E, Δ) . One has for any $a' \in H', x' \in E'$ and $x \in E, \langle a' \cdot x', x \rangle = \langle x', {}^t\eta(a') \cdot x \rangle$. Therefore, if M is a H' -submodule of E , then $M^\perp = \{x' \in E' / \langle x', x \rangle = 0, x \in M\}$ is a H' -submodule of E' . Reciprocally, if η is *bijective* and if M' is H' -subcomodule of E' , then $M'^\perp$ is a H' -submodule of E .

Proposition 9 : *Let H be a complete ultrametric Hopf algebra that is a pseudo-reflexive Banach space such that η is bijective.*

Then, a finite dimensional left H -comodule (E, Δ) is simple if and only if $(E', \Delta^\vee)$ is simple .

Proof : Indeed, suppose that (E, Δ) is simple ; if M' is a left H -subcomodule of $(E', \Delta^\vee)$ then $M'^\perp$ is a left H -subcomodule of E ; therefore $M'^\perp = (0)$ or $M'^\perp = E$ and $M' = E'$ or $M' = (0)$. By the same way, one shows the reciprocal.

II - 5 When H admits a left integral

II - 5 - 1 Again some general facts

Lemma 3 : *Let (E, Δ) be a finite dimensional left H -comodule and let Δ_c be the restriction of c to $R(\Delta) = \rho_\Delta(E' \otimes E)$; then $R(\Delta_c) = R(\Delta)$.*

Proof : Let $(e_j)_{1 \leq j \leq n}$ be a base of E . One has $\Delta(e_\ell) = \sum_{j=1}^n a_{\ell j} \otimes e_j, 1 \leq \ell \leq n$ and $(a_{\ell j})_{1 \leq \ell, j \leq n}$ spans $R(\Delta)$. Since $\sigma|_{R(\Delta)} \in R(\Delta)'$, one has , according to (1) , $a_{\ell j} = (1_H \otimes \sigma) \circ c(a_{\ell j}) = \rho_{\Delta_c}(\sigma \otimes a_{\ell j}) \in R(\Delta_c)$ and $R(\Delta) \subset R(\Delta_c)$. Reciprocally, if $a' \in R(\Delta)'$ and $a = \sum_{1 \leq \ell, j \leq n} \lambda_{\ell j} a_{\ell j} \in R(\Delta)$, one has $(1_H \otimes a') \circ \Delta_c(a) = \sum_{1 \leq \ell, j \leq n} \sum_{k=1}^n \lambda_{\ell j} \langle a', a_{k j} \rangle a_{\ell k} \in R(\Delta)$ and $R(\Delta_c) \subset R(\Delta)$.

Lemma 4 : *Any finite dimensional left H -subcomodule E of H is contained in the representative subalgebra $\mathcal{R}(H)$ of H .*

Proof : If $(e_j)_{1 \leq j \leq n}$ is a base of $E \subset H$, one has $c(e_j) = \sum_{\ell=1}^n a_{\ell j} \otimes e_\ell$. Let c_E be the restriction of c to E , then $R(c_E)$ is spanned by $(a_{j\ell})_{1 \leq j, \ell \leq n}$. Since $e_j = (1_H \otimes \sigma) \circ c(e_j) = \sum_{\ell=1}^n \sigma(e_\ell) a_{j\ell} \in R(c_E)$, one has $E \subset R(c_E) \subset R(H)$.

Note : If K and H are discrete, one deduces from the above result and from Theorem 1 - (ii) - of [3] that $\mathcal{R}(H) = H$.

II - 5 - 2 Under the hypothesis : H admits a left integral

Let Ω be the family of the isomorphic classes of the *simple, finite dimensional* left H -comodules ; Ω is not empty : it contains the class of the left subcomodule $K.e$ of H . If $\omega \in \Omega$ is the class of (E, Δ) , we set $R(\omega) = R(\Delta)$ that is independent of (E, Δ) . It is readily seen that $\mathcal{R}_s(H) = \sum_{\omega \in \Omega} R(\omega)$ is a subcoalgebra of $\mathcal{R}(H)$. Moreover $\mathcal{R}_s(H) =$

$\bigoplus_{\omega \in \Omega} R(\omega)$, a direct sum of coalgebras. Indeed for any finite subset $(\omega_1, \dots, \omega_m)$ of Ω , one

has $\sum_{t=1}^n R(\omega_t) = \bigoplus_{t=1}^m R(\omega_t)$: see Corollary of Propositions 8 and its proof. Furthermore, if

η is *bijective*, then $\mathcal{R}_s(H)$ is a sub-Hopf-algebra of $\mathcal{R}(H)$. $\square$

By definition, a *left integral* for the complete Hopf algebra H is an element ν of H' such that $\mu * \nu = \langle \mu, e \rangle \nu$ for all $\mu \in H'$. The complete Hopf algebra H is called *supple* if H is a pseudo-reflexive Banach space and $\eta \circ \eta = 1_H$. For H , a supple complete Hopf algebra that admits a left integral ν such that $\langle \nu, e \rangle = 1$, we know that *any simple left Banach H -comodule is finite dimensional* (Theorem 3 - [3]).

Theorem 2 : Let H a supple complete Hopf algebra that admits a left integral ν such that $\langle \nu, e \rangle = 1$. Then

(i) $\mathcal{R}(H) = \bigoplus_{\omega \in \Omega} R(\omega)$ where Ω is the family of the isomorphic classes of simple Banach left H -comodules.

(ii) The Hopf algebra $\mathcal{R}(H)$ is dense in H , that is $H = \overline{\mathcal{R}(H)} = \widehat{\bigoplus_{\omega \in \Omega} R(\omega)}$.

Proof :

(i) One deduces from [2] - Theorem 3 that any finite dimensional H -comodule (E, Δ) is semi-simple i.e. $(E, \Delta) = \bigoplus_{\tau} \bigoplus_t (V_{t,\tau}, \Delta_{t,\tau})$ with $V_{t,\tau} \in \omega_{\tau}$ and $\omega_{\tau} \in \Omega$. Hence

$$R(\Delta) = \sum_{\tau} \sum_t R(\Delta_{t,\tau}) = \sum_{\tau} R(\omega_{\tau}) = \bigoplus_{\tau} R(\omega_{\tau}) \subset \mathcal{R}_s(H). \text{ It follows that } \mathcal{R}(H) = \mathcal{R}_s(H) = \bigoplus_{\omega \in \Omega} R(\omega).$$

(ii) The Hopf algebra H is naturally a Banach left H -comodule with coproduct c . Let $a \in H$, $a \neq 0$; since H is pseudo-reflexive, the Banach left subcomodule $E(a) = \overline{H' \cdot a}$ of H contains a and is a non null Banach space of countable type (cf. [3]).

With the hypothesis, we know that $E(a)$ contains simple left H -subcomodules (finite dimensional) (cf. [3]).

Let $(V_\tau)_{\tau \in T}$ be the family of all simple subcomodules of $E(a)$. Put $W = \sum_{\tau \in T} V_\tau$, there exists $S \subset T$ such that $W = \bigoplus_{\tau \in S} V_\tau$; one has $c(W) \subset H \otimes W$. Since c is a homeomorphism of H onto $c(H)$, setting $E_0 = \overline{W}$, one has $c(E_0) \subset H \hat{\otimes} E_0$, i.e. E_0 is a Banach left subcomodule of $E(a)$. In fact $E_0 = E(a)$. Otherwise, one has a direct sum of Banach comodules $E(a) = E_0 \oplus E_1$ with $E_1 \neq (0)$ (cf. [2]). However E_1 must contain at least one simple comodule V and by definition of W , one has $V \subset W$. Hence $E_0 \cap E_1 \neq (0)$; a contradiction.

Let ω_τ be the isomorphic class of the simple comodule $V_\tau, \tau \in T$. By Lemma 4, $V_\tau \subset R(\omega_\tau), \tau \in T$. Hence, we have $W = \sum_{\tau \in T} V_\tau \subset \sum_{\tau \in T} R(\omega_\tau) \subset \bigoplus_{\omega \in \Omega} R(\omega) = \mathcal{R}(H)$. It follows that $a \in E(a) = E_0 = \overline{W} \subset \overline{\mathcal{R}(H)}$. We have proved that $H = \overline{\mathcal{R}(H)} = \widehat{\bigoplus_{\omega \in \Omega} R(\omega)}$.

Note : The above results are abstract version of some results of representation theory of groups. In particular Theorem 2 is Peter-Weyl Theorem (cf. [3]).

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