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Representative subalgebra of a complete ultrametric Hopf algebra


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**ABSTRACT.** Let \((H, m, c, \eta, \sigma)\) be a complete ultrametric Hopf algebra over a complete ultrametric valued field \(K\). \(e\) be the unit of \(H\) and \(k\) the canonical map of \(K\) in \(H\). In order words, \(H\) is a Banach algebra with multiplication \(m : H \otimes H \to H\), coproduct \(c : H \to H \otimes H\) a continuous algebra homomorphism, inversion or antipode \(\eta : H \to H\) a continuous linear map and counit \(\sigma : H \to K\) a continuous algebra homomorphism. The coassociativity and countinary axioms hold, and

\[ m \circ (\eta \otimes 1_H) \circ c = k \circ \sigma = m \circ (1_H \otimes \eta) \circ c. \]

We define the representative subalgebra \(\mathcal{R}(H)\) of \(H\), i.e. the subalgebra of \(H\) generated by the coefficient "functions" associated with the finite dimensional left \(H\)-comodules. Under some conditions on \(H\), \(\mathcal{R}(H)\) is a direct sum of finite dimensional subcoalgebras and is dense in \(H\). But in general, \(\mathcal{R}(H)\) is not dense in \(H\). The algebra \(\mathcal{R}(H)\) is a generalization of the algebra of representative functions on a group. Notice that when the valuation of \(K\) and the norm of \(H\) are trivial, one obtains the well known fact that \(H\) is equal to its representative subalgebra.

**INTRODUCTION.**

Let \((H, m, c, \eta, \sigma)\) be a complete ultrametric Hopf algebra over the complete ultrametric valued field \(K\). An ultrametric Banach space \(E\) over \(K\) is said to be a left Banach \(H\)-comodule if there exists a continuous linear map \(\Delta_E : E \to H \otimes E\), called coproduct, such that

(i) \((c \otimes 1_E) \circ \Delta_E = (1_H \otimes \Delta_E) \circ \Delta_E\)

(ii) \((\sigma \otimes 1_E) \circ \Delta_E = 1_E\)

A closed linear subspace \(E\) of \(E\) is a (left) Banach subcomodule of \(E\) if \(\Delta_E(M) \subset H \otimes M\).
Let \((E, \Delta_E)\) and \((F, \Delta_F)\) be two left Banach comodules. A continuous linear map \(u : E \to F\) is a Banach comodule morphism if \(\Delta_F \circ u = (1_H \otimes u) \circ \Delta_E\).

It is associated with any left Banach \(H\)-comodule \((E, \Delta_E)\) the closed linear subspace \(R(\Delta_E)\) of \(H\) spanned by the coefficient "functions" \((1_H \otimes x') \circ \Delta(x)\), \(x' \in E', x \in E\), where \(E'\) if the Banach space dual of \(E\). Furthermore, let \(\mathcal{R}(H)\) be the linear subspace of \(H\) spanned by all the \(R(\Delta_E)\) where \((E, \Delta_E)\) is a finite dimensional left \(H\)-comodule. Then \(\mathcal{R}(H)\) is a (non necessary closed) sub-Hopf-algebra of \(H\); \(\mathcal{R}(H)\) is called the representative subalgebra of \(H\). In general, \(\mathcal{R}(H)\) is not dense in \(H\) (cf. [1] or [5], [6]). However, with additional conditions on \(H\) it will be shown that \(\mathcal{R}(H)\) is dense in \(H\).

If \(E\) and \(F\) are ultrametric Banach spaces over \(K\), we denote by \(E \hat{\otimes} F\) the complete tensor product, that is the completion of \(E \otimes F\) with respect to the norm \(\|z\| = \inf_{\pi \in \Xi_{x_j \otimes y_j}} (\max \|x_j\| \|y_j\|).\) In the sequel all Banach spaces are ultrametric.

I - LEFT BANACH COMODULES

I - 1 Tensor products of left Banach comodules

Let \((E, \Delta_E)\) and \((F, \Delta_F)\) be two left Banach comodules. One has the continuous linear map \(\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \to H \hat{\otimes} E \hat{\otimes} F \to H \hat{\otimes} H \hat{\otimes} E \hat{\otimes} F \to H \hat{\otimes} F\), where

\[
\Delta_{E \hat{\otimes} F} = (m \otimes 1_E \otimes 1_F) \circ (1_H \otimes \tau_{E \hat{\otimes} F} \otimes 1_F) \circ (\Delta_E \otimes \Delta_F) \quad \text{and} \quad \tau_{E \hat{\otimes} F}(x \otimes a) = a \otimes x.
\]

Proposition 1 : \(\Delta_{E \hat{\otimes} F} : E \hat{\otimes} F \to H \hat{\otimes} E \hat{\otimes} F\) is the coproduct of a left Banach \(H\)-comodule structure on \(E \hat{\otimes} F\).

Proof: Put, for \(x \in E\) and \(y \in F\),

\[
\Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{t \geq 1} a_j b_t \otimes x_j \otimes y_t.
\]

(i) It follows immediately that \((\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = \sum_{j \geq 1} \sum_{t \geq 1} \sigma(a_j b_t) x_j \otimes y_t = \sum_{j \geq 1} \sigma(a_j) x_j \otimes \sum_{t \geq 1} \sigma(b_t) y_t = x \otimes y = 1_{E \hat{\otimes} F}(x \otimes y).\) From what, one deduces \((\sigma \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = 1_{E \hat{\otimes} F}\)

(ii) Also, one has for \(x \in E\), \(y \in F\)

\[
\alpha) \quad (c \otimes 1_E) \circ \Delta_E(x) = \sum_{j \geq 1} c(a_j) x_j = \sum_{j \geq 1} \sum_{s \geq 1} a^1_{s,j} \otimes a^2_{s,j} \otimes x_j = (1_H \otimes \Delta_E) \circ \Delta_E(x) = \sum_{j \geq 1} a_j \otimes \Delta_E(x_j) = \sum_{j \geq 1} \sum_{k \geq 1} a_j \otimes \gamma_{k,j} \otimes x_{k,j}.
\]
and
\[(c \otimes 1_F) \circ \Delta_F(y) = \sum_{\ell \geq 1} c(b_{\ell}) \otimes y_{\ell} = \sum_{\ell \geq 1} \sum_{t \geq 1} \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 \otimes y_{\ell} = (1_H \otimes \Delta_F) \circ \Delta_F(y) = \]
\[= \sum_{\ell \geq 1} b_{\ell} \otimes \Delta_F(y_{\ell}) = \sum_{\ell \geq 1} \sum_{m \geq 1} b_{\ell} \otimes \rho_{m,\ell} \otimes y_{m,\ell}.
\]

Let \(E_x = E[(x_j, j \geq 1) \cup (x_{k,j}, k \geq 1, j \geq 1)]\) be the closed linear subspace of \(E\) spanned by \((x_{j}, j \geq 1) \cup (x_{k,j}, k \geq 1, j \geq 1)\), and \(F_y = E[(y_{\ell}, \ell \geq 1) \cup (y_{m,\ell}, m \geq 1, \ell \geq 1)]\) be the closed linear subspace of \(F\) spanned by \((y_{\ell}, \ell \geq 1) \cup (y_{m,\ell}, m \geq 1, \ell \geq 1)\). It is clear that the Banach spaces \(E_x\) and \(F_y\) are of countable type. Furthermore, if \(x' \in E'_x\) and \(y' \in F'_y\) one has
\[(1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) = \sum_{j \geq 1} \sum_{s \geq 1} <x', x_j > \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 = \]
\[(1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_E) \circ \Delta_E(x) = \sum_{j \geq 1} \sum_{k \geq 1} <x', x_{k,j} > a_j \otimes \gamma_{k,j} \]
and
\[(1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(y) = \sum_{\ell \geq 1} \sum_{t \geq 1} <y', y_{\ell, t} > \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2 = \]
\[(1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_F) \circ \Delta_F(y) = \sum_{\ell \geq 1} \sum_{m \geq 1} <y', y_{m,\ell} > b_{\ell} \otimes \rho_{m,\ell}.
\]

\(\beta)\) On one hand, one has, \((c \otimes 1_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) = \sum_{j \geq 1} \sum_{s \geq 1} c(a_j b_{s}) \otimes x_j \otimes y_t = \]
\[= \sum_{j \geq 1} \sum_{s \geq 1} (\sum_{s \geq 1} \alpha_{s,j}^1 \otimes \alpha_{s,j}^2) \left(\sum_{t \geq 1} \beta_{t,\ell}^1 \otimes \beta_{t,\ell}^2\right) \otimes x_j \otimes y_t.
\]

On the other hand, one has
\[(1_H \otimes \Delta_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) = \sum_{j \geq 1} \sum_{t \geq 1} a_j b_{t} \otimes \Delta_{E_{\otimes F}}(x_j \otimes y_t) = \sum_{j \geq 1} \sum_{t \geq 1} \sum_{k \geq 1} \sum_{m \geq 1} a_j b_{t} \otimes \gamma_{k,j} \rho_{m,\ell} \otimes x_{k,t} \otimes y_{m,\ell}.
\]

Hence, if \(x' \in E'_x\) and \(y' \in F'_y\); first, one has
\[(1_H \otimes 1_H \otimes x' \otimes y') \circ (c \otimes 1_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) = \sum_{j \geq 1} \sum_{s \geq 1} <x', x_j > \alpha_{s,j}^1 \otimes \alpha_{s,j}^2 \sum_{t \geq 1} <y', y_{t} > \]
\[= (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta_E(x) \cdot (1_H \otimes 1_H \otimes y') \circ (c \otimes 1_F) \circ \Delta_F(y).
\]

And, second, one has
\[(1_H \otimes 1_H \otimes x' \otimes y') \circ (1_H \otimes \Delta_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) = \sum_{j \geq 1} \sum_{s \geq 1} <x', x_{k,j} > a_j \otimes \gamma_{k,j} \sum_{t \geq 1} <y', y_{m,\ell} > \rho_{m,\ell} \otimes b_{t} \]
\[= (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x) \cdot (1_H \otimes 1_H \otimes y') \circ (1_H \otimes \Delta_{F}) \circ \Delta_F(y).
\]

Therefore, for any \(x' \in E'_x\) and any \(y' \in F'_y\), we have
\[(a) : (1_H \otimes 1_H \otimes x' \otimes y') \circ (c \otimes 1_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) - (1_H \otimes \Delta_{E_{\otimes F}}) \circ \Delta_{E_{\otimes F}}(x \otimes y) = 0 \gamma
\]
Since \(E_x\) [resp. \(F_y\)] is of countable type, there exist \(\alpha_0 > 0, \alpha_1 > 0\) and
(e_j)_{j \geq 1} \subset E_x [\text{resp. } (f_{\ell})_{\ell \geq 1} \subset F_y] \text{ such that for } z \in E_x [\text{resp. } \zeta \in F_y] \text{ one has }
z = \sum_{j \geq 1} \lambda_j e_j [\text{resp. } \zeta = \sum_{\ell \geq 1} \mu\ell f_{\ell}] \text{ with } \alpha_0 \sup_{j \geq 1} |\lambda_j| \leq \|z\| \leq \alpha_1 \sup_{j \geq 1} |\lambda_j| [\text{resp. } \alpha_0 \sup_{\ell \geq 1} |\mu_{\ell}| \leq \|\zeta\| \leq \alpha_1 \sup_{\ell \geq 1} |\mu_{\ell}|] (\text{cf. } [4]) .

Moreover, one has \( E_x \hat{\otimes} F_y \simeq c_0(\mathbb{N}^* \times \mathbb{N}^*, K) \) and \((H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y) \simeq c_0(\mathbb{N}^* \times \mathbb{N}^*, H \hat{\otimes} H)\) (cf. [7]); any \( Z \) in \((H \hat{\otimes} H) \hat{\otimes} (E_x \hat{\otimes} E_y)\) can be written in the unique form \( Z = \sum_{j \geq 1} A_{j,\ell} \otimes e_j \otimes f_{\ell} \text{ with } A_{j,\ell} \in H \hat{\otimes} H \text{ and } \alpha_j^0 \sup_{j,\ell} \|A_{j,\ell}\| \leq \|Z\| \leq \alpha_j^1 \sup_{j,\ell} |A_{j,\ell}|. \)

Let \( e'_j \in E'_x [\text{resp. } f'_{\ell} \in F'_y] \) be the continuous linear form defined by \( < e'_j, e_{j,i} > = \delta_{j,j,i} [\text{resp. } < f'_{\ell}, f_{\ell,i} > = \delta_{\ell,\ell,i}]. \) Setting \((c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) - (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = Z_0 = \sum_{j,\ell} A_{j,\ell}^0 \otimes e_j \otimes f_{\ell} \in H \hat{\otimes} H \hat{\otimes} E_x \hat{\otimes} F_y, \) for any \( j_1 \geq 1 \) and any \( \ell_1 \geq 1, \) by (a), one has \((1_H \otimes 1_H \otimes e'_{j_1} \otimes f'_{\ell_1})(Z_0) = \sum_{j,\ell} A_{j,\ell}^0 \delta_{j_1,j} \delta_{\ell_1,\ell} = A_{j_1,\ell_1}^0 = 0. \) It follows that \( Z_0 = 0, \) i.e. \((c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y) = (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}(x \otimes y). \) From what, one deduces that \((c \otimes 1_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F} = (1_H \otimes \Delta_{E \hat{\otimes} F}) \circ \Delta_{E \hat{\otimes} F}. \)

**Corollary:** Let \( M [\text{resp. } N] \) be a left Banach \( H \)-subcomodule of \( E [\text{resp. } F]. \) Then \( M \hat{\otimes} N \) is a left Banach subcomodule of \( E \hat{\otimes} F. \)

**I - 2** Banach comodule morphisms

**I - 2 - 1** Range and kernel

**Proposition 2:** Let \( u : E \to F \) be a Banach comodule morphism.

(i) If \( V \) is a Banach submodule of \( F, \) then \( u^{-1}(V) \) is a Banach submodule of \( E. \)

(ii) The closure \( \overline{u(E)} \) of \( u(E) \) is a Banach submodule of \( F. \)

**Corollary:** Let \( V \) and \( W \) be Banach submodule of the left Banach \( H \)-comodule \( E; \) then \( V \cap W \) is a Banach submodule of \( E. \)

**Proofs:** Rather easy, or see [3].

**Note:** One can also see [3] for the spaces of comodule morphisms.

**Remark 1:** If \( M \) is a Banach submodule of the left Banach \( H \)-comodule \( E, \) it is induced on the quotient Banach space \( E/M \) a structure of Banach left \( H \)-comodule such that the canonical map \( E \to E/M \) is a comodule morphism.
Then, if \( u : E \to F \) is a Banach comodule morphism and if \( u \) is strict, the Banach comodule \( E/\ker u \) and \( u(E) \) are isomorphic. Also, one can define the cokernel of \( u \) as being \( F/\ker u(E) \).

### I - 2-2 Comodule morphisms of \( E \) into \( H \) associated with \( \Delta ; R(\Delta) \)

Put \( \Delta = \Delta_E \) the coproduct of the left Banach \( H \)-comodule \( E \). Obviously, \( H \) is a left Banach \( H \)-comodule with respect to its coproduct \( c \).

**Proposition 3:** For any \( x' \in E' \), the linear map \( A_{x'} = (1_H \otimes x') \circ \Delta : E \to H \) is a Banach comodule morphism.

**Proof:** It is easy to see that \( \co(1_H \otimes x') = c \otimes x' = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \). Therefore \( \co A_{x'} = \co(1_H \otimes x') \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta = (1_H \otimes 1_H \otimes x') \circ \Delta = (1_H \otimes x') \circ \Delta \).

**Corollary 1:**
(i) \( \ker A_{x'} \) is a closed subcomodule of \( E \).
(ii) \( A_{x'}(E) \) is a left Banach submodule (closed left coideal) of \( H \).

**Corollary 2:** If \( E \) is a space of countable type, one has \( \ker A_{x'} \neq E \) for any \( x' \in E, x' \neq 0 \)

**Proof:** Indeed, if \( x' \in E', x' \neq 0 \) and \( 0 < \alpha < 1 \), there exists a \( \alpha \)-orthogonal base \( (e_j)_{j \geq 1} \subset E \) such that \( < x', e_1 > = 1 \) and \( < x', e_j > = 0, j \geq 2 \). Moreover for any \( j \geq 1 \), \( \Delta(e_j) = \sum_{t \geq 1} a_{tj} \otimes e_t \) and \( e_j = \sum_{t \geq 1} \sigma(a_{tj})e_t \); therefore \( \sigma(a_{tj}) = \delta_{tj} \) and \( A_{x'}(e_1) = (1_H \otimes x') \circ \Delta(e_1) = a_{11} \neq 0 \) since \( \sigma(a_{11}) = 1 \).

**Corollary 3:** Assume that \( H \) is a pseudo-reflexive Banach space; i.e. \( H \to H'' \) is isometric.

Let \( E \) be a simple Banach left \( H \)-comodule, i.e. \( E \) contains no proper closed submodule. Then \( E \) is a Banach space of countable type and \( A_{x'} \) is injective for each \( x' \in E', x' \neq 0 \).

**Proof:** If \( H \) is pseudo-reflexive, it is shown in [3] that any simple Banach left \( H \)-comodule is of a space of countable type. Applying Corollary 2, one sees that \( A_{x'} \) is injective for \( x' \in E' \), \( x' \neq 0 \). □

Let \( \beta : E \otimes E' \to K \) be the continuous linear form defined upon \( \beta(x \otimes x') = < x', x > \). Put \( \rho_{\Delta} = (1_H \otimes \beta) \circ (\Delta \otimes 1_{E'}) \circ \tau : E' \otimes E \to H \), where \( \tau(x' \otimes x) = x \otimes x' \). Then \( \rho_{\Delta} \) is linear and continuous with \( \| \rho_{\Delta} \| \leq \| \Delta \| \). Moreover for \( x' \in E', x \in E \), one has \( \rho_{\Delta}(x' \otimes x) = (1_H \otimes x') \circ \Delta(x) \).
Put $R(\Delta) = \rho_{\Delta}(E' \hat{\otimes} E)$ the closure of $\rho_{\Delta}(E' \hat{\otimes} E)$ in $H$. Obviously, $R(\Delta)$ is the closed linear subspace of $H$ spaned by the elements $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'$, $x \in E$, called the coefficients of the comodule $(E, \Delta)$.

**Proposition 4**: $R(\Delta) = \rho_{\Delta}(E' \hat{\otimes} E)$ is a left Banach subcomodule (= closed left coideal) of $H$.

**Proof**: Since $c : H \rightarrow H \hat{\otimes} H$ is linear and is a homeomorphism of $H$ onto $c(H)$, one has $c(R(\Delta)) = \overline{c(\rho_{\Delta}(E' \hat{\otimes} E))}$, a closed linear subspace of $H$.

It remains to show that if $a = \rho_{\Delta}(x' \otimes x) = (1_H \otimes x') \circ \Delta(x) = A_{x'}(x)$, $x' \in E'$, $x \in E$; then $c(a) \in H \hat{\otimes} R(\Delta)$. Writing $\Delta(x) = \sum_{j \geq 1} a_j \otimes x_j$; one has $c(a) = c \circ A_{x'}(x) = (1_H \otimes A_{x'}) \circ \Delta(x) = \sum_{j \geq 1} a_j \otimes A_{x'}(x_j) \in H \hat{\otimes} R(\Delta)$. \(\square\)

**Proposition 5**: If the left Banach comodules $E$ and $E_1$ with coproduct respectively $\Delta$ and $\Delta_1$ are isomorphic, then $R(\Delta) = R(\Delta_1)$.

**Proof**: Let $u : E \rightarrow E_1$ be a comodule isomorphism, in other words, $u$ is linear, continuous and bijective with $\Delta_1 \circ u = (1_H \otimes u) \circ \Delta$. Moreover, the reciprocal map $u^{-1}$ of $u$ satisfies $(1_H \otimes u^{-1}) \circ \Delta_1 = \Delta \circ u^{-1}$ and the transpose of $u$, $t' u : E'_1 \rightarrow E'$ is linear, continuous and bijective with $(t' u)^{-1} = t u^{-1}$.

Set $a = \rho_{\Delta_1}(z_1) \in \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ and $z_1 = \sum_{j \geq 1} y'_j \otimes y_j$, $y'_j \in E'_1$, $y_j \in E_1$, $\lim_j \|y'_j\| \|y_j\| = 0$. There exist, for $j \geq 1$ unique $x'_j \in E'$ and $x_j \in E$ such that $y'_j = t' u^{-1}(x'_j) = x'_j \circ u^{-1}$ and $y_j = u(x_j)$; moreover $\lim_j \|x'_j\| \|x_j\| = 0$. Therefore $a = \rho_{\Delta_1}(z_1) = \sum_{j \geq 1} \rho_{\Delta_1}(y'_j \otimes y_j) = \sum_{j \geq 1} (1_H \otimes y'_j) \circ \Delta_1(y_j) = \sum_{j \geq 1} (1_H \otimes A_{x'_j} \circ u(x_j)) = \sum_{j \geq 1} (1_H \otimes A_{x'_j} \circ u(x_j)) = \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ where $z = \sum_{j \geq 1} x'_j \otimes x_j$; that is $\rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \subset \rho_{\Delta_1}(E' \hat{\otimes} E)$. Likewise, one has $\rho_{\Delta_1}(E'_1 \hat{\otimes} E_1) \subset \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ and $R(\Delta) = R(\Delta_1)$.

Therefore $\rho_{\Delta}(E' \hat{\otimes} E) = \rho_{\Delta_1}(E'_1 \hat{\otimes} E_1)$ and $R(\Delta) = R(\Delta_1)$.

Assume that $E$ is a free Banach space i.e. $E \simeq c_0(I, K) = \{(\lambda_j)_{j \in I} \subset K / \lim_j \lambda_j = 0\}$. In other words, there exist $(e_j)_{j \in I} \subset E$, $\alpha_0, \alpha_1 \in \mathbb{R}_+^*$ such that any $x \in E$ can be written
in the form $x = \sum_{j \in I} \lambda_j e_j$, $\lambda_j \in K$ and $\alpha_0 \sup_{j \in I} |\lambda_j| \leq \|x\| \leq \alpha_1 \sup_{j \in I} |\lambda_j|$. For any continuous linear form $x' \in E'$, one has $\frac{1}{\alpha_0} \sup_{j \in I} |<x', e_j>| \leq \|x'\| \leq \frac{1}{\alpha_1} \sup_{j \in I} |<x', e_j>|$. Let $e'_j$ be the element of $E'$ defined by $<e'_j, e_\ell> = \delta_{j\ell}$. Put $E'_0 = E[(e'_j)_{j \in I}]$, the closed linear subspace of $E'$ spaned by $(e'_j)_{j \in I}$. Hence each $x' \in E'_0$ can be written in the unique form $x' = \sum_{j \in I} \mu_j e'_j$, $\mu_j \in K$, $\lim_j |\mu_j| = 0$. Moreover, if $v \in E'_0 \otimes E \subset E' \otimes E$, one has $v = \sum_{j \in I} \mu_j e'_j \otimes e_j$, $\mu_j \in K$, $\lim_{(j,\ell)} |\mu_{j\ell}| = 0$ and $\alpha_0 \sup_{j \in I} \|\mu_j\| \leq \alpha_1 \sup_{j \in I} \|\mu_{j\ell}\|$.

On the other hand, one has $H \otimes E \simeq c_0(I, H) = \{(a_j)_{j \in I} \subset H / \lim_j a_j = 0\}$. For any $z \in H \otimes E$ one has $z = \sum_{j \in I} a_j \otimes e_j$, $a_j \in H$ with $\lim_j \|a_j\| = 0$ and $\alpha_0 \sup_{j \in I} \|a_j\| \leq \|z\| \leq \alpha_1 \sup_{j \in I} \|a_j\|$. Hence, if $(E, \Delta)$ is a left Banach $H$-comodule, for $x \in E$, one has $\Delta(x) = \sum_{j \in I} A_j(x) \otimes e_j$. In particular $\Delta(e_\ell) = \sum_{j \in I} A_j(e_\ell) \otimes e_j = \sum_{j \in I} a_{j\ell} \otimes e_j$ and $(c \otimes 1_E) \circ \Delta(e_\ell) = \sum_{j \in I} c(a_{j\ell}) \otimes e_j = (1_H \otimes \Delta)(\left(\sum_{k \in I} a_{k\ell} \otimes e_k\right)) = \sum_{k \in I} a_{k\ell} \otimes \sum_{j \in I} a_{kj} \otimes e_j = \sum_{k \in I} \sum_{j \in I} a_{kj} \otimes a_{kj} \otimes e_j$. Thus one obtains

(1) \[ c(a_{j\ell}) = \sum_{k \in I} a_{kj} \otimes a_{kj} ; \ell, j \in I \]

Also, one has

(2) \[ \sigma(a_{j\ell}) = \delta_{j\ell} ; \ell, j \in I \]

(3) \[ \sum_{k \in I} a_{kj} \eta(a_{kj}) = \delta_{j\ell} \cdot e = \sum_{k \in I} \eta(a_{kj}) \otimes a_{kj} ; \ell, j \in I. \]

Proposition 6 : \[ R_0(\Delta) = \rho_\Delta(E'_0 \otimes E) \] is a closed subcoalgebra of $H$. In other words \[ c(R_0(\Delta)) \subset R_0(\Delta) \otimes R_0(\Delta) \]

Proof : Since $(e'_j \otimes e_\ell)_{(j,\ell) \in I \times I}$ is a total family of $E'_0 \otimes E$ and $\rho_\Delta$ is linear and continuous, the family $(\rho_\Delta(e'_j \otimes e_\ell))_{(j,\ell) \in I \times I}$ is total in $\rho_\Delta(E'_0 \otimes E) = R_0(\Delta) = \rho_\Delta(H \otimes E)$.

To see that $c(R_0(\Delta)) \subset R_0(\Delta) \otimes R_0(\Delta)$, it suffices to show that for $\ell, j \in I$ one has $c(\rho_\Delta(e'_j \otimes e_\ell)) \in R_0(\Delta) \otimes R_0(\Delta)$. However, by definition, $\rho_\Delta(e'_j \otimes e_\ell) = (1_H \otimes \Delta)(e'_j \otimes e_\ell) = a_{j\ell} \in R_0(\Delta)$. Then, one deduces from (1) that $c(\rho_\Delta(e'_j \otimes e_\ell)) = c(a_{j\ell}) = \sum_{k \in I} a_{kj} \otimes a_{kj} \in R_0(\Delta) \otimes R_0(\Delta)$. Therefore, $c(R_0(\Delta)) \subset R_0(\Delta) \otimes R_0(\Delta)$. The proof is complete.
Note: If \( v = \sum_{t,j} \mu_{tj} e'_j \otimes e_t \in E'_0 \otimes E \), one has \( \rho_\Delta(v) = \sum_{t,j} \mu_{tj} a_{tj} \) and \( a \in R_0(\Delta) \) iff there exist \( v_n \in E' \otimes E \) such that \( a = \lim_{n \to +\infty} \rho_\Delta(v_n) \).

**Remark 2**: Let \((E, \Delta)\) and \((E_1, \Delta_1)\) be two isomorphic left Banach comodules that are free Banach spaces. If \( u : E \to E_1 \) is a comodule isomorphism, \((e_j)_{j \in I}\) a base of \( E \) and \((e'_j)_{j \in I}\) the base of \( E_1 \) defined by \( e_j = u(e_j) \); then, with the above notations, one has \( R_0(\Delta) = R_0(\Delta_1) \).

**Remark 3**: If \( \dim E = n < +\infty \), one has \( R(\Delta) = R_0(\Delta) = \rho_\Delta(E' \otimes E) \) and \( \dim R(\Delta) \leq n^2 \)

**II - REPRESENTATIVE SUBALGEBRA**

**II - 1 Conjugate comodule of a finite dimensional comodule**

Let \((E, \Delta)\) be a (Banach) left \( H \)-comodule of finite dimension and \((e_j)_{1 \leq j \leq n}\) a \( K \)-base of \( E \). As above, for any \( x \in E \), one has \( \Delta(x) = \sum_{j=1}^n A_j(x) \otimes e_j \) and \( A_j(x) = (1_H \otimes e'_j) \circ \Delta(x) \).

\( \Delta(x) = \rho_\Delta(e'_j \otimes x) \); \( A_j = (1_H \otimes e'_j) \circ \Delta \in \mathcal{L}(E, H) \). In particular \( \Delta(e_t) = \sum_{j=1}^n a_{tj} \otimes e_j \) where \( a_{tj} = A_j(e_t) = \rho_\Delta(e'_j \otimes e_t) \); and we have the relations (1), (2) and (3), with \( I = [1, n] \).

The relation (3) means here, that the matrix \( A = (a_{tj})_{1 \leq t,j \leq n} \in \text{Mat}_n(H) \) is invertible with inverse \( A^{-1} = (\eta(a_{tj}))_{1 \leq t,j \leq n} \).

Fix the base \((e_j)_{1 \leq j \leq n}\) of \( E \) and define the linear map \( \Delta^\vee : E' \to H \otimes E' \) by setting \( \Delta^\vee(e'_j) = \sum_{t=1}^n \eta(a_{tj}) \otimes e'_t \), \( 1 \leq j \leq n \). Hence for \( x' = \sum_{j=1}^n \mu_j e'_j \in E' \), one has \( \Delta^\vee(x') = \sum_{t=1}^n \sum_{j=1}^n \mu_j \eta(a_{tj}) \otimes e'_j = \sum_{t=1}^n A^\vee(x') \otimes e'_t \).

**Lemma 1**: \((E', \Delta^\vee)\) is a left \( H \)-comodule.

**Proof**: One verifies that \( \sigma \circ \eta = \sigma \); indeed, if \( a \in H \), then \( c(a) = \sum_{t \geq 1} a_t \otimes a_t^2 \). Hence, one has \( m \circ (\eta \otimes 1_H) \circ c(a) = \sum_{t \geq 1} \eta(a_t) a_t^2 = \sigma(a) e \) and \( a = (1_H \otimes \sigma) \circ c(a) = \sum_{t \geq 1} a_t \sigma(a_t^2) \). It
follows that $\eta(a) = \sum_{t \geq 1} \eta(a^1_t)\sigma(a^2_t)$ and $\sigma \circ \eta(a) = \sum_{t \geq 1} \sigma(\eta(a^1_t))\sigma(a^2_t) = \sigma\left(\sum_{t \geq 1} \eta(a^1_t)a^2_t\right) = \sigma(\sigma(a)e) = \sigma(a)$.

Since $\sigma(a^1_t) = \delta_{t,1}$, one has $(\sigma \otimes 1_{E'}) \circ \Delta^\vee(e^1_j) = \sum_{t=1}^n \sigma \circ \eta(a^1_t)e^t_t = \sum_{t=1}^n \sigma(a^1_t)e^1_j = e^1_j$, $1 \leq j \leq n$. It follows, by linearity, that $(\sigma \otimes 1_{E'}) \circ \Delta^\vee = 1_{E'}$.

Let us remember that $c \circ \eta = \tau \circ (\eta \otimes \eta) \circ c$ where $\tau(a \otimes b) = b \otimes a$. Hence, we have $c \circ \eta(a^1_t) = \sum_{k=1}^n \eta(a^1_{kj}) \otimes \eta(a^1_{tk})$. Therefore $(c \otimes 1_{E'}) \circ \Delta^\vee(e^1_j) = (c \otimes 1_{E'})(\sum_{t=1}^n \eta(a^1_t) \otimes e^1_j) = \sum_{t=1}^n \sum_{k=1}^n \eta(a^1_k) \otimes \eta(a^1_j) \otimes e^t_t = \sum_{t=1}^n \eta(a^1_k) \otimes \Delta^\vee(e^1_k) = (1_H \otimes \Delta^\vee)\left(\sum_{k=1}^n \eta(a^1_k) \otimes e^1_k\right) = (1_H \otimes \Delta^\vee) \circ \Delta^\vee(e^1_j)$, and $(c \otimes 1_{E'}) \circ \Delta^\vee = (1_H \otimes \Delta^\vee) \circ \Delta^\vee$.

**Corollary**: $R(\Delta^\vee) = \eta(R(\Delta))$.

**Proof**: Identifying $E''$ with $E$, one has $R(\Delta^\vee) = \rho_{\Delta^\vee}(E \otimes E')$. Set $z = \sum_{1 \leq t, j \leq n} \lambda_{t,j} e_t \otimes e^1_j \in E \otimes E'$; hence $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t, j \leq n} \lambda_{t,j} \rho_{\Delta^\vee}(e_t \otimes e^1_j)$. However $\rho_{\Delta^\vee}(e_t \otimes e^1_j) = (1_H \otimes e_t) \circ \Delta^\vee(e^1_j) = \eta(a^1_t) = \eta(\rho_{\Delta}(e^1_j \otimes e_t))$; therefore $\rho_{\Delta^\vee}(z) = \sum_{1 \leq t, j \leq n} \lambda_{t,j} \eta(\rho_{\Delta}(e^1_j \otimes e_t)) = \eta(\rho_{\Delta}(z_1))$ where $z_1 = \sum_{1 \leq t, j \leq n} \lambda_{t,j} e^1_j \otimes e_t \in E' \otimes E$. It follows that $R(\Delta^\vee) \subset \eta(R(\Delta))$. The same formulae show that if $a = \rho_{\Delta}(z_1) \in R(\Delta)$, where $z_1 = \sum_{1 \leq t, j \leq n} \lambda_{t,j} e^1_j \otimes e_t \in E' \otimes E$, one has $\eta(a) = \rho_{\Delta^\vee}(z)$ where $z = \sum_{1 \leq t, j \leq n} \lambda_{t,j} e_t \otimes e^1_j \in E \otimes E'$, hence $\eta(R(\Delta)) \subset R(\Delta^\vee)$.

II- 2 Direct sum of Banach comodules

Let $(E_s)_{1 \leq s \leq m}$ be a finite family of left Banach $H$-comodules with $\Delta_s$ the coproduct of $E_s$. The direct sum $E = \bigoplus_{s=1}^m E_s$ equipped with any norm equivalent to the norm $\left\| \sum_{s=1}^m x_s \right\| = \max_{1 \leq s \leq m} \|x_s\|$ is a Banach space. Put $\Delta = \bigoplus_{s=1}^m \Delta_s$, i.e. $\Delta\left(\sum_{s=1}^m x_s\right) = \bigoplus_{s=1}^m \Delta_s(x_s)$. It is readily seen that $(E, \Delta)$ is a left Banach comodule. Moreover, if $p_s : E \to E$ is the projection of $E$ onto $E_s$, then $1_H \otimes p_s$ is a projection of $H \otimes E$ onto $H \otimes E_s$ and one has
\( H \hat{\otimes} E = \bigoplus_{s=1}^{m} H \hat{\otimes} E_s \). On the other hand \( p_s \) is a comodule morphism i.e. \( (1_H \otimes p_s) \circ \Delta = \Delta \circ p_s \); furthermore \( (1_H \otimes p_s) \circ \Delta(x_t) = 0 \) for \( s \neq t \) and \( x_t \in E_t \).

Also, we have \( E' = \bigoplus_{s=1}^{m} E'_s \); the projections associated with this direct sum are the \( f'p_s, 1 \leq s \leq m \).

**Proposition 7:** With the above notations, one has 
\[
\rho_{\Delta}(E' \hat{\otimes} E) = \sum_{s=1}^{m} \rho_{\Delta,s}(E'_s \hat{\otimes} E_s)
\]
and \( R(\Delta) \) is the closure of \( \sum_{s=1}^{m} R(\Delta_s) \) in \( H \).

**Proof:** If \( x_s \in E'_s, x_t \in E_t \) and \( s \neq t \), then \( (1_H \otimes x'_s) \circ \Delta(x_t) = (1_H \otimes x'_t) \circ \Delta(x_t) = 0 \).

Set \( z = \sum_{j \geq 1} x'_j \otimes x_j \in E' \hat{\otimes} E = \bigoplus_{s=1}^{m} \bigoplus_{t=1}^{m} E'_s \hat{\otimes} E_t \), one has \( z = \sum_{j \geq 1} \sum_{s=1}^{m} \sum_{t=1}^{m} x'_{s,j} \otimes x_{t,j} \). It follows that 
\[
\rho_{\Delta}(z) = \sum_{j \geq 1} \sum_{s=1}^{m} \sum_{t=1}^{m} \rho_{\Delta}(x'_{s,j} \otimes x_{t,j}) = \sum_{j \geq 1} \sum_{s=1}^{m} (1_H \otimes x'_{s,j}) \circ \Delta(x_{j,t}) = \sum_{s=1}^{m} \sum_{j \geq 1} \sum_{t=1}^{m} \delta_{s,t}(1_H \otimes x'_{s,j}) \circ \Delta(x_{j,t}) = \sum_{j \geq 1} \sum_{s=1}^{m} \rho_{\Delta,s}(x'_{s,j} \otimes x_{s,j}) = \sum_{s=1}^{m} \rho_{\Delta,s}(\sum_{j \geq 1} x'_{s,j} \otimes x_{s,j}) = \sum_{s=1}^{m} \rho_{\Delta,s}(z_s).
\]

If \( z_s \in E'_s \hat{\otimes} E_s \subset E' \hat{\otimes} E, 1 \leq s \leq m \), one has \( \rho_{\Delta}(z_s) = \rho_{\Delta,s}(z_s) \). Therefore, on one hand, \( \rho_{\Delta}(E' \hat{\otimes} E) \subset \sum_{s=1}^{m} \rho_{\Delta,s}(E'_s \hat{\otimes} E_s) \), and on the other hand, \( \rho_{\Delta,s}(E'_s \hat{\otimes} E_s) \subset \rho_{\Delta}(E' \hat{\otimes} E) \). Hence, one has \( \rho_{\Delta}(E' \hat{\otimes} E) = \sum_{s=1}^{m} \rho_{\Delta,s}(E'_s \hat{\otimes} E_s) \). One verifies readily that \( R(\Delta) \) is equal to the closure of \( \sum_{s=1}^{m} R(\Delta_s) \) in \( H \).

**Corollary:** If \( \dim E_s < +\infty, 1 \leq s \leq m \), then one has \( R(\Delta) = \sum_{s=1}^{m} R(\Delta_s) \) where

\[
E = \bigoplus_{s=1}^{m} E_s, \ Delta = \bigoplus_{s=1}^{m} Delta_s.
\]

**Remark 3:** If the comodules \( (E_s, \Delta_s), 1 \leq s \leq m \), are pairwise isomorphic, then for the comodule \( (E, \Delta) \) where \( E = \bigoplus_{s=1}^{m} E_s, \ Delta = \bigoplus_{s=1}^{m} Delta_s \), one has \( R(\Delta) = R(\Delta_s), 1 \leq s \leq m \).
II - 3 The representative subalgebra of $H$

Let $S(H)$ be the set of all elements of the form $a = (1_H \otimes x') \circ \Delta(x)$ of $H$ where $(E, \Delta)$ is a finite dimensional left $H$-comodule and $x' \in E'$, $x \in E$. Let us put $\dim E = \dim \Delta$

Lemma 2: $S(H)$ is a multiplicative, unitary submonoid of $H$.

Proof: Set $a = (1_H \otimes x') \circ \Delta(x)$ and $b = (1_H \otimes y') \circ \Delta_1(y) \in S(H)$ where $(E, \Delta)$ and $(E, \Delta_1)$ are left $H$-comodules of finite dimension and $x' \in E'$, $x \in E$, $y' \in E'_1$, $y \in E_1$. One has $\Delta(x) = \sum_{j=1}^{p} a_j \otimes x_j$, $\Delta_1(y) = \sum_{t=1}^{q} b_t \otimes y_t$ and $\Delta_{E \otimes E_1}(x \otimes y) = \sum_{j=1}^{p} \sum_{t=1}^{q} a_j b_t \otimes x_j \otimes y_t$.

Hence, $ab = (1_H \otimes x') \circ \Delta(x) \cdot (1_H \otimes y') \circ \Delta_1(y) = \sum_{j=1}^{p} \sum_{t=1}^{q} a_j b_t \otimes x'_j \otimes y'_t$.

$(1_H \otimes x' \otimes y') \circ \Delta_{E \otimes E_1}(x \otimes y) \in S(H)$.

Since $c(e) = e \otimes e$, $E = K.e$ is a left subcomodule of $H$ of dimension 1, one has $e = (1_H \otimes \sigma) \circ c(e) \in S(H)$.

Let $R(H)$ be the linear subspace of $H$ spaned by $S(H)$. Then $R(H)$ is an unitary subalgebra of $H$. Indeed, if $a = \sum_{j=1}^{p} \lambda_j a_j$ and $b = \sum_{t=1}^{q} \mu_t b_t$ are two elements of $R(H)$, since $a_j b_t \in S(H)$, one has $ab = \sum_{j=1}^{p} \sum_{t=1}^{q} \lambda_j \mu_t a_j b_t \in R(H)$. One says that $R(H)$ is the representative subalgebra of $H$.

Note: Put, for the left $H$-comodule $(E, \Delta)$ of finite dimension, $S(\Delta) = \{ a = (1_H \otimes x') \circ \Delta(x) \in H; x' \in E', x \in E \}$. As in Proposition 5, $S(\Delta)$ depends only of the isomorphism class $\tilde{\Delta}$ of $(E, \Delta)$. Furthermore, one has $S(H) = \bigcup_{\dim \Delta < +\infty} S(\Delta)$.

Also, it is clear that the $K$-linear vector space $R(\Delta) = \rho_{\Delta}(E' \otimes E)$ is spaned by $S(\Delta)$. Hence one has $R(H) = \bigcup_{\dim \Delta < +\infty} R(\Delta)$. Moreover, if $(E_1, \Delta_1)$ and $(E_2, \Delta_2)$ are two comodules, then $R(\Delta_1 \oplus \Delta_2) = R(\Delta_1) + R(\Delta_2)$ contains $R(\Delta_1)$ and $R(\Delta_2)$ i.e. the family $(R(\Delta))_\Delta$ ordered by inclusion is directed upward.

Theorem 1: The representative subalgebra $R(H)$ of $H$ is such that $c(R(H)) \subset R(H) \otimes R(H)$. Moreover $(R(H), m, c, \eta, \sigma)$ is a Hopf algebra.

Proof: It follows from Proposition 6 and Remark 3 that if $\Delta$ is a coproduct of finite dimension, then $c(R(\Delta)) \subset R(\Delta) \otimes R(\Delta)$: that is $R(\Delta)$ is a coalgebra. Since
\( \mathcal{R}(H) = \bigcup_{\text{dim} \Delta < +\infty} R(\Delta) \) is the union of coalgebras, it is a coalgebra. On the other hand, one deduces from the Corollary of Lemma 1 that \( \eta(\mathcal{R}(H)) \subset \mathcal{R}(H) \). The Theorem 1 is proved.

II - 4 Simple comodules of finite dimension

Let \((e_j)_{1 \leq j \leq n}\) be a base of the finite dimensional left \(H\)-comodule \((E, \Delta)\). Let us remember that \(A_j = (1_H \otimes e_j) \circ \Delta\) is a comodule morphism. One sees that \( \bigcap_{1 \leq j \leq n} \ker A_j = (0) \) and \((A_j)_{1 \leq j \leq n}\) is free in \( \mathcal{L}(E, H) \). Since \(A_j(e_j) = a_{jj}\), one deduces from (2) or from Corollary 2 of Proposition 3 that \(e_j \notin \ker A_j\) and \(\ker A_j \neq E\).

Put \(H_j = A_j(E)\); then \(H_j\) is a left subcomodule of \(H\) of dimension \(\leq n\). Furthermore, with previous notations, one has

\[
R(\Delta) = \rho_\Delta(E' \otimes E) = \sum_{j=1}^{n} H_j \quad \text{and} \quad H_j = \sum_{t=1}^{n} K \cdot a_{tj},
\]

also \(R(\Delta)\) is a subcoalgebra of dimension \(\leq n^2\). One can have \(\text{dim} R(\Delta) < n^2\); for example, if \(E_q = \bigoplus_{t=1}^{q} E\) and \(\Delta_q = \bigoplus_{t=1}^{q} \Delta\), \(q \geq 2\), one has \(R(\Delta_q) = R(\Delta)\) and \(\text{dim} R(\Delta_q) = \text{dim} R(\Delta) \leq n^2 < (qn)^2 = (\text{dim}(E_q))^2\).

**Definition:** A left Banach \(H\)-comodule \(E\) is called simple or topologically irreducible if \(E\) is not the null space and does not contain any closed subcomodule different from \((0)\) and \(E\).

Let \(\text{Hom.com}(E, E_1)\) be the Banach space of the left Banach comodule morphisms of \((E, \Delta)\) into \((E_1, \Delta_1)\) and \(\text{End.com}(E) = \text{Hom.com}(E, E)\), this later is a Banach algebra.

**Remark 4:** Schur's Lemma.

Let \((E, \Delta)\) and \((E_1, \Delta_1)\) be two simple, finite dimensional left \(H\)-comodules.

(i) If \(E\) and \(E_1\) are not isomorphic, one has \(\text{Hom.com}(E, E_1) = (0)\)

(ii) In the alternative case, any non null comodule morphism of \(E\) into \(E_1\) is an isomorphism. In particular, \(\text{End.com}(E)\) is a (skew) field of finite dimension \(\leq (\text{dim}E)^2\). If \(K\) is algebraically closed, then \(\text{End.com}(E) = K.1_E\).

**Proposition 8:** Let \((E, \Delta)\) be a simple Banach left \(H\)-comodule of finite dimension \(n\). Let \((e_j)_{1 \leq j \leq n}\) be a base of \(E\) and \(A_j = (1_H \otimes e_j) \circ \Delta, 1 \leq j \leq n\).

Then \(H_j = A_j(E)\) is a simple left \(H\)-comodule of \(H\) of finite dimension \(n\). Furthermore, there exists \(J \subset [1, n]\) such that \(R(\Delta) = \bigoplus_{j \in J} H_j\) (a direct sum of comodules).

**Proof:** It is the same as in semi-simple module theory. Indeed, since \(\ker A_j \neq E\) and \(E\) is a simple comodule of finite dimension \(n\), the map \(A_j : E \to H_j = A_j(E)\) is a comodule
isomorphism. Hence \( H_j \) is a simple comodule of dimension \( n \) with base \((a_{lj})_{1 \leq l \leq n}\). If \( 1 \leq j, q \leq n \), one has \( H_j \cap H_q = (0) \), or \( H_j = H_q \). Changing the order if necessary, we may assume that \((H_1, \ldots, H_m)\) is the family of the distinct comodules \( H_j ; m \leq n \). Hence

\[
R(\Delta) = \sum_{j=1}^{m} H_j, H_j \neq H_q \text{ for } j \neq q.
\]

Since \( H_1 \cap H_2 = (0) \), one has the direct sum of comodules \( H_1 \oplus H_2 \). Let \( j_0 \) be the least integer \( \geq 3 \) such that \( (H_1 \oplus H_2) \cap H_{j_0} = (0) \). Hence, one has the direct sum \( H_1 \oplus H_2 \oplus H_{j_0} \) and for \( j < j_0 \), one has \( (H_1 \oplus H_2) \cap H_j \neq (0) \), therefore \( H_j \subset H_1 \oplus H_2 \). Hence, by induction, one obtains \( J = \{1, 2, j_0, \ldots, j_k \} \subset [1, m] \) and the direct sum of comodules \( \bigoplus_{j \in J} H_j \) such that for \( \ell \notin J \), \( H_\ell \subset \bigoplus_{j \in J} H_j \). It follows that \( R(\Delta) = \bigoplus_{j \in J} H_j \).

**Corollary:** Let \((E, \Delta)\) and \((E_1, \Delta_1)\) be two simple left \( H\)-comodules of finite dimension that are not isomorphic; then \( R(\Delta + \Delta_1) = R(\Delta) \oplus R(\Delta_1) \), a direct sum of comodules.

**Proof:** With previous notations, put \( R(\Delta) = \bigoplus_{j \in J} H_j \) and \( R(\Delta_1) = \bigoplus_{\ell \in L} H_{\ell} \). Let

\[
p_j \text{ [resp. } p_{\ell} \text{]} \quad \text{be the projection of } R(\Delta) \text{ [resp. } R(\Delta_1) \text{]} \text{ onto } H_j \text{ [resp. } H_{\ell} \text{]}.
\]

Suppose that \( R(\Delta) \cap R(\Delta_1) \neq (0) \); this finite dimensional comodule must contain at least one simple comodule \( V \). There exists \( j \in J \text{ [resp. } \ell \in L \] such that \( p_j(V) \neq (0) \text{ [resp. } p_{\ell}(V) \neq (0) \] ; therefore \( p_j(V) = H_j \text{ [resp. } p_{\ell}(V) = H_{\ell} \] \). Since \( V \) is simple, \( p_j|V \text{ [resp. } p_{\ell}|V \] is an isomorphism of \( V \) onto \( H_j \text{ [resp. } H_{\ell} \] \). It follows that \( H_j \) and \( H_{\ell} \) are isomorphic. Hence \( E \) and \( E_1 \) are isomorphic; a contradiction. Therefore \( R(\Delta) \cap R(\Delta_1) = (0) \) and \( R(\Delta + \Delta_1) = R(\Delta) \oplus R(\Delta_1) \).

**Remark 5:** Notations and hypothesis as above. If \( K \) is algebraically closed, then the \( H_j, 1 \leq j \leq n, \) are pairwise distinct.

**Proof:** Indeed, if \( H_j = H_q \) for \( j \neq q \), then \( u = A_j \circ A_q^{-1} \) is an automorphism of the finite dimensional simple comodule \( H_j \). By Schur's lemma, one has \( u = \lambda \cdot 1_{H_j}, \lambda \in k, \lambda \neq 0 \). Hence \( A_j = \lambda A_q \) and \( a_{jj} = A_j(e_j) = \lambda A_q(e_j) = \lambda a_{jq} \). Therefore \( \sigma(a_{jj}) = 1 = \lambda \sigma(a_{jq}) = \lambda \delta_{jq} = 0 \); a contradiction. \( \square \)

Let \( H' \) be the Banach space dual of \( H \); if we set for \( a', b' \in H', a' \ast b' = (a' \otimes b') \circ c, \) then \( H' \) becomes a complete normed algebra with unit \( \sigma \). If \((E, \Delta)\) is a left Banach comodule, setting for \( a' \in H' \), and \( x \in E, a' \cdot x = (a' \otimes 1_E) \circ \Delta(x) \), one induces on \( E \) a complete normed right \( H'\)-module structure. Moreover, if \( H \) is a pseudo-reflexive Banach space, then any closed right \( H'\)-submodule of \( E \) is a Banach left \( H\)-subcomodule of \( E \) and reciprocally (cf. [3]).
Let \((E', \Delta')\) be the conjugate of the finite dimensional left \(H\)-comodule \((E, \Delta)\). One has for any \(a' \in H', x' \in E'\) and \(x \in E, <a' \cdot x', x> = <x', \eta(a') \cdot x>\). Therefore, if \(M\) is a \(H'\)-submodule of \(E\), then \(M^\perp = \{x' \in E'/ <x', x> = 0, x \in M\}\) is a \(H'\)-submodule of \(E'\). Reciprocally, if \(\eta\) is bijective and if \(M'\) is a \(H'\)-subcomodule of \(E'\), then \(M' \subset \) is a \(H'\)-submodule of \(E\).

**Proposition 9:** Let \(H\) be a complete ultrametric Hopf algebra that is a pseudo-reflexive Banach space such that \(\eta\) is bijective.

Then, a finite dimensional left \(H\)-comodule \((E, \Delta)\) is simple if and only if \((E', \Delta')\) is simple.

**Proof:** Indeed, suppose that \((E, \Delta)\) is simple; if \(M'\) is a left \(H\)-subcomodule of \((E', \Delta')\) then \(M'^\perp\) is a left \(H\)-subcomodule of \(E\); therefore \(M'^\perp = (0)\) or \(M' = E'\) or \(M' = (0)\). By the same way, one shows the reciprocal.

**II - 5** When \(H\) admits a left integral

**II - 5 - 1** Again some general facts

**Lemma 3:** Let \((E, \Delta)\) be a finite dimensional left \(H\)-comodule and let \(\Delta_c\) be the restriction of \(\Delta\) to \(R(\Delta) = \rho(\Delta(E' \otimes E))\); then \(R(\Delta_c) = R(\Delta)\).

**Proof:** Let \((e_j)_{1 \leq j \leq n}\) be a base of \(E\). One has \(\Delta(e_\ell) = \sum_{j=1}^{n} a_{\ell j} \otimes e_j, 1 \leq \ell \leq n\) and \((a_{\ell j})_{1 \leq \ell, j \leq n}\) spans \(R(\Delta)\). Since \(\sigma|_{R(\Delta)} \in R(\Delta)\)', one has , according to (1), \(a_{\ell j} = (1_H \otimes \sigma) \circ c(a_{\ell j}) = \rho_{\Delta_c}(\sigma \otimes a_{\ell j}) \in R(\Delta_c)\) and \(R(\Delta) \subset R(\Delta_c)\). Reciprocally, if \(a' \in R(\Delta)\)' and \(a = \sum_{1 \leq \ell, j \leq n} a_{\ell j} a_{\ell j} \in R(\Delta)\), one has \(1_H \otimes a' \circ \Delta_c(a) = \sum_{1 \leq \ell, j \leq n} \sum_{k=1}^{n} \lambda_{\ell j} <a', a_{k j}> a_{\ell k} \in R(\Delta_c)\) and \(R(\Delta_c) \subset R(\Delta)\).

**Lemma 4:** Any finite dimensional left \(H\)-subcomodule \(E\) of \(H\) is contained in the representative subalgebra \(\mathcal{R}(H)\) of \(H\).

**Proof:** If \((e_j)_{1 \leq j \leq n}\) is a base of \(E \subset H\), one has \(c(e_j) = \sum_{\ell = 1}^{n} a_{\ell j} \otimes e_\ell\). Let \(c_E\) be the restriction of \(c\) to \(E\), then \(R(c_E)\) is spanned by \((a_{j \ell})_{1 \leq j, \ell \leq n}\). Since \(e_j = (1_H \otimes \sigma) \circ c(e_j) = \sum_{\ell=1}^{n} \sigma(e_\ell) a_{j \ell} \in R(c_E)\), one has \(E \subset R(c_E) \subset R(H)\).
Note: If $K$ and $H$ are discrete, one deduces from the above result and from Theorem 1 - (ii) - of [3] that $R(H) = H$.

II - 5 - 2 Under the hypothesis: $H$ admits a left integral

Let $\Omega$ be the family of the isomorphic classes of the simple, finite dimensional left $H$-comodules; $\Omega$ is not empty: its contains the class of the left subcomodule $K.e$ of $H$. If $\omega \in \Omega$ is the class of $(E, \Delta)$, we set $R(\omega) = R(\Delta)$ that is independant of $(E, \Delta)$. It is readily seen that $\mathcal{R}_s(H) = \sum_{\omega \in \Omega} R(\omega)$ is a subcoalgebra of $\mathcal{R}(H)$. Moreover $\mathcal{R}_s(H) = \bigoplus_{\omega \in \Omega} R(\omega)$, a direct sum of coalgebras. Indeed for any finite subset $(\omega_1, \ldots , \omega_m)$ of $\Omega$, one has $\sum_{t=1}^{n} R(\omega_t) = \bigoplus_{t=1}^{m} R(\omega_t)$: see Corollary of Propositions 8 and its proof. Furthermore, if $\eta$ is bijective, then $\mathcal{R}_s(H)$ is a sub-Hopf-algebra of $\mathcal{R}(H)$. □

By definition, a left integral for the complete Hopf algebra $H$ is an element $\nu$ of $H'$ such that $\mu \star \nu = \langle \mu, e \rangle > \nu$ for all $\mu \in H'$. The complete Hopf algebra $H$ is called supple if $H$ is a pseudo-reflexive Banach space and $\eta \circ \eta = 1_H$. For $H$, a supple complete Hopf algebra that admits a left integral $\nu$ such that $\langle \nu, e \rangle > 1$, we know that any simple left Banach $H$-comodule is finite dimensional (Theorem 3 - [3]).

Theorem 2: Let $H$ a supple complete Hopf algebra that admits a left integral $\nu$ such that $\langle \nu, e \rangle > 1$. Then

(i) $R(H) = \bigoplus_{\omega \in \Omega} R(\omega)$ where $\Omega$ is the family of the isomorphic classes of simple Banach left $H$-comodules.

(ii) The Hopf algebra $\mathcal{R}(H)$ is dense in $H$, that is $H = \overline{\mathcal{R}(H)} = \bigoplus_{\omega \in \Omega} R(\omega)$.

Proof:

(i) One deduces from [2] - Theorem 3 that any finite dimensional $H$-comodule $(E, \Delta)$ is semi-simple i.e. $(E, \Delta) = \bigoplus_{t} (V_t, \Delta_{t,r})$ with $V_t, r \in \omega_r$ and $\omega_r \in \Omega$. Hence $R(\Delta) = \sum_{r} \sum_{t} R(\Delta_{t,r}) = \sum_{r} R(\omega_r) = \bigoplus_{r} R(\omega_r) \subset \mathcal{R}_s(H)$. It follows that $\mathcal{R}(H) = \mathcal{R}_s(H) = \bigoplus_{\omega \in \Omega} R(\omega)$.

(ii) The Hopf algebra $H$ is naturally a Banach left $H$-comodule with coproduct $c$. Let $a \in H$, $a \neq 0$; since $H$ is pseudo-reflexive, the Banach left subcomodule $E(a) = \overline{H'.a}$ of $H$ contains $a$ and is a non null Banach space of countable type (cf. [3]).
With the hypothesis, we know that $E(a)$ contains simple left $H$-subcomodules (finite dimensional) (cf. [3]).

Let $(V_r)_{r \in T}$ be the family of all simple subcomodules of $E(a)$. Put $W = \sum_{r \in T} V_r$, there exists $S \subseteq T$ such that $W = \bigoplus_{r \in S} V_r$; one has $c(W) \subseteq H \otimes W$. Since $c$ is a homeomorphism of $H$ onto $c(H)$, setting $E_0 = W$, one has $c(E_0) \subseteq H \hat{\otimes} E_0$, i.e. $E_0$ is a Banach left subcomodule of $E(a)$. In fact $E_0 = E(a)$. Otherwise, one has a direct sum of Banach comodules $E(a) = E_0 \oplus E_1$ with $E_1 \neq (0)$ (cf. [2]). However $E_1$ must contain at least one simple comodule $V$ and by definition of $W$, one has $V \subseteq W$. Hence $E_0 \cap E_1 \neq (0)$; a contradiction.

Let $\omega_r$ be the isomorphic class of the simple comodule $V_r$, $r \in T$. By Lemma 4, $V_r \subseteq R(\omega_r), r \in T$. Hence, we have $W = \sum_{r \in T} V_r \subseteq \sum_{r \in T} R(\omega_r) \subseteq \bigoplus_{\omega \in \Omega} R(\omega) = \mathcal{R}(H)$. It follows that $a \in E(a) = E_0 = \overline{W} \subseteq \overline{\mathcal{R}(H)}$. We have proved that $H = \overline{\mathcal{R}(H)} = \bigoplus_{\omega \in \Omega} R(\omega)$.

Note: The above results are abstract version of some results of representation theory of groups. In particular Theorem 2 is Peter-Weyl Theorem (cf. [3]).

\section*{REFERENCES}


