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ON THE THICKNESS OF TOPOLOGICAL SPACES

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We recall there are three classical definitions of the topological dimension: the small inductive dimension, denoted by $ind$, the large inductive dimension, denoted by $Ind$ and the covering dimension, denoted by $dim$. (For the definitions, one can see (2).)

In this paper, coming back on an idea of J.P. REVEILLES (7), we give a nonstandard definition of the topological dimension - the thickness, denoted by $ep$ (for épaisseur), and we prove this definition coincides with the classical definitions in the class of separable metric spaces.

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1: Preliminary.

In the sequel, we consider a topological space $X$ and an enlargement $E$ (see, for example, (4)) containing $X$.

1) Definition 1.1:

Let us consider a base $B$ of $X$, a point $a$ of $\ast X$ and put $B_a = \{B \in B : a \in \ast B \}$. (In the special case where $a = \ast x$, $B_a = \{B \in B : x \in B \}$.)

Then, we call halo in base $B$ of $a$, the set $h_B(a) = \bigcap_{B \in B_a} \ast B$.

Remark:

If $B'$ is the base consisting of finite intersections of elements of $B$, we have, for every point $a$ of $\ast X$, $h_{B'}(a) = h_B(a)$, whence the convention: we will call base of $X$ only these bases of $X$ saturated by finite intersections.

Proposition 1.2:

For every base $B$ of $X$ and for every point $a$ of $\ast X$, there exists an element $\Omega$ of $\ast B_a$ such that $\Omega \subseteq h_B(a)$.

Indeed, the relation $\mathcal{R} \subset B_a \times B_a$ defined by $\langle A \mathcal{R} B \rangle \iff A \subseteq B$ is concurrent on $B_a$.

Corollary 1.3:

For every base $B$ of $X$, every subset $A$ of $X$ and every $a \in \ast X$, if $a \in \ast \overline{A}$ (with $\overline{A}$ the closure of $A$ in the space $X$), then $h_B(a) \cap \ast A \neq \emptyset$.

Note that, in the special case where $a = \ast x$, $x \in \overline{A}$ if and only if $h_B(\ast x) \cap \ast A \neq \emptyset$.

2) Definition 1.4:

Let us consider a base $B$ of $X$ and $a$ and $b$ two elements of $\ast X$.

Since $a$ belongs to $h_B(b)$ if and only if $h_B(a)$ is contained in $h_B(b)$, the relation $\leq$ defined by $\langle a \leq b \iff a \in h_B(b) \rangle$ is a preorder on $\ast X$, called the preorder associated to $B$.

Note this relation is not necessarily symmetric.

If we have $a \leq b$ and $b \leq a$, we will say that $a$ and $b$ are equivalent modulo $B$ and we will write $a \equiv b$.

Moreover, we will write $a < b$ if and only if $a \leq b$ but not $b \leq a$. 
Proposition 1.5:

For every base $B$ of $X$ and for every element $a$ of $^*X$, there exists an element $b$ of $^*X$ such that $b \leq a$ and $b$ be minimal for the preorder associated to $B$.

Indeed, the set $I = \{b \in ^*X : b \leq a\}$ is inductive.

Proposition 1.6:

Let $B$ be a base of $X$ and $a$ an element of $^*X$. If there exists $B \in B$ such that $a \in ^*FrB$ (with $FrB$ the boundary of $B$ in the space $X$), then $a$ is not minimal for the preorder associated to $B$.

Since $a \in ^*FrB$, it follows from 1.3 that $h_B(a) \cap ^*B \neq \emptyset$ and $h_B(a) \cap ^*(X \setminus B) \neq \emptyset$.

There exists then an element $b$ of $^*B$ such that $b \leq a$. If $a \equiv b$, we would have $h_B(a) = h_B(b) \subset ^*B$ and consequently, $^*B \cap ^*(X \setminus B) \neq \emptyset$ which is impossible. It follows that we have $b < a$, so that $a$ is not minimal.

2: Thickness of a topological space $X$.

1) Definition 2.1:

Let $x \in X$ and $B$ be a base of $X$. We will call chain of length $p$ ($p \in \mathbb{N}$) of $h_B(x)$ every finite subset $\{a_p, \ldots, a_i\}$ of $h_B(x)$ such that $a_p \leq \ldots \leq a_i < x$ and we will say that:

i) the thickness in $x$ of $B$ is less than $n$ (and we will write $ep(x, B) \leq n$) if and only if, for every chain $\{a_p, \ldots, a_i\}$ of $h_B(x)$, we have $p \leq n$.

ii) the thickness in $x$ of $B$ is equal to $n$ if and only if $ep(x, B) \leq n$ and $ep(x, B) > n-1$.

Note our definition of thickness is the same as the « intended » definition in (7), provided the notion of « consecutive halos » is corrected therein p. 707.

2) Definition 2.2:

Let $B$ be a base of $X$. We will call thickness of $B$, the element of $D = \{n \in \mathbb{Z} : n \geq -1\} \cup \{+\infty\}$, denoted by $epB$, defined by $epB = \sup\{ep(x, B) : x \in X\}$.

Remark:

Note that one can give another definition of the thickness of a base $B$, using the thickness of $B$ in all the points of $^*X$, standard or not. This thickness, denoted $EpB$ ($= \sup\{ep(a, B) : a \in ^*X\}$), is such of course that $epB \leq EpB$ and it might happen that $epB < EpB$. However, one can prove that for the « complemented » bases $B$, one has $epB = EpB$ and that, for every base $B$, there exists a « complemented » base $C$ such that $epC \leq epB$, so that, if necessary, one only
considers « complemented » bases of $X$. All these results will be proved in another paper of the author.

We now discuss some examples.

**Proposition 2.3 :**

Let us suppose $X$ non empty and let $B$ be a base of $X$. Then $ep \ B = 0$ if and only if $B$ consists of open-closed subsets of $X$.

i) Suppose $ep \ B = 0$. Let us consider an element $B$ of $B$ and $x$ an element of $\overline{B}$. Then, we have $h_B(x) \cap B \neq \emptyset$. Let $a \in B$ such that $a \leq x$. Since $ep \ B = 0$, we have $a \equiv x$ and therefore $x \in B$, so that $B$ is closed.

ii) Suppose all the elements of $B$ are open-closed. Let $x \in X$ and $a \leq x$. Let us prove that we have $B \leq a$. Let $B \in B$ such that $a \in B$. Then, we have $h_B(a) \cap B \neq \emptyset$ and therefore $h_B(x) \cap B \neq \emptyset$, so that $x \in B \overline{B}$. Since $B$ is closed, we have $x \in B$ and therefore $B \leq a$.

**Proposition 2.4 :**

i) For every totally ordered space $X$ (totally ordered set $X$ with its order topology), if we denote by $B_0$ the base of $X$ consisting of all open intervals, we have $ep \ B_0 \leq 1$.

ii) In the special case where $X = \mathbb{R}$, we have $ep \ B_0 = 1$.

**Proof :**

i) For every $x \in X$ and every $a \in h_B(x)$, we have $h_{B_0}(a) = h_{B_0}(x)$ or $h_{B_0}(a) = h_{B_0}(x) \cap \leftarrow x$, $\rightarrow$, or $h_{B_0}(a) = h_{B_0}(x) \cap \leftarrow x$.

ii) If $X = \mathbb{R}$, since $B_0$ is not a base consisting of open-closed subsets of $\mathbb{R}$, we have $ep \ B_0 \geq 0$ and therefore $ep \ B_0 = 1$.

**Proposition 2.5 :**

Let $X$ a topological space, $B$ a base of $X$ and $A$ a subset of $X$. If we denote by $C$ the trace of $A$ on $X$, we have $ep \ C \leq ep \ B$.

Indeed, for every couple $(a, b) \in A \times A$, the relations « $a < b \ modulo \ C$ » and « $a < b \ modulo \ B \» $ are equivalent.

**Proposition 2.6 :**

Let $X$ and $Y$ be two topological spaces. For every base $B$ of $X$ and every base $C$ of $Y$, we have $ep \ (B \times C) \leq ep \ B + ep \ C$.

Indeed, for every $(a, b) \in A \times B$, we have $h_{B \times C}(a, b) = h_B(a) \times h_B(b)$.
3) Definition 2.7:
Let $X$ a topological space. We will call thickness of $X$ the element of $D$, denoted by $ep\ X$, defined by $ep\ X = \inf \{ep\ B : B \in B(X)\}$, where $B(X)$ is the set of all bases of $X$.

It follows from this definition and the previous results that:

2.8: 1) If $X$ is nonempty, $ep\ X = 0$ if and only if $X$ has a base consisting of open-closed subsets.
2) For every totally ordered space $X$, we have $ep\ X \leq 1$.
   In particular, since $\mathbb{R}$ is connected, we have $ep\ \mathbb{R} = 1$.
3) For every topological space $X$ and every subset $A$ of $X$, we have $ep\ A \leq ep\ X$.
4) For every topological spaces $X$ and $Y$, we have $ep\ (X \times Y) \leq ep\ X + ep\ Y$.

2.9: Remarks.
1) It follows from 2.8.2) and 2.8.4) that, for every $n \geq 1$, $ep\ \mathbb{R}^n \leq n$. (In the sequel, we will prove that $ep\ \mathbb{R}^n = n$).
2) In contrast to the classical definitions, there is no need for any special hypothesis for 2.8.3) and 2.8.4) to be true: recall, for example, there exist (3) two compact spaces $X$ and $Y$ such that $ind(X \times Y) > indX + indY$.

3: Comparison between thickness and classical dimensions.
1) Theorem 3.1:
For every topological space $X$, we have:

   a) $epX = 0$ if and only if $indX = 0$,
   b) $indX \leq epX$.

Proof:
   a) is immediate since these two assertions are equivalent to « there exists a base of $X$ consisting of open-closed subsets ».
   b) The theorem is obvious if $ep\ X = +\infty$, so that we can suppose $ep\ X < +\infty$.

Let us prove the theorem by induction on $n = ep\ X$.

It follows from a) that the statement holds for $n = 0$.

Suppose it holds for every space $Y$ such that $ep\ Y \leq n - 1$ and let us prove then that $indX \leq n$, i.e., that, for every point $x$ of $X$ and every neighbourhood $V$ of $x$, there exists an open subset $0$ such that $x \in 0 \subset V$ and $ind(Fr\ 0) \leq n - 1$.

Since $ep\ X = n$, there exists a base $B$ of $X$ such that $ep\ B = n$. Let us prove then that, for every $B \in B$, we have $ep\ (Fr\ B) \leq n - 1$, which by the induction
hypothesis, implies \( \text{ind} X \leq n \).
Let \( B \in B \). Put \( F = FrB \) and call \( C \) the trace of \( B \) on \( F \).
Let us prove that \( ep C \leq n - 1 \). Let \( x \in F \) and \( \{a_p, \ldots, a_i\} \) be a chain of \( h_C(x) \).
Since \( h_C(x) = h_B(x) \cap F \), it follows from 1.6 that \( a_p \) is not minimal for the preorder associated to \( B \). Consequently, there exists an element \( a_{p+1} \) of \( X \) such that \( \{a_{p+1}, a_p, \ldots, a_i\} \) is a chain of \( h_B(x) \). Since \( ep B = n \), we have necessarily \( p \leq n - 1 \), which implies \( ep (x, C) \leq n - 1 \) and therefore \( ep C \leq n - 1 \). Since \( ep F \leq ep C \), we conclude \( ep F \leq n - 1 \).

Corollary 3.2 :

For every \( n \geq 1 \), we have \( ep \ \mathbb{R}^n = n \).
Indeed, we know that \( \text{ind} \ \mathbb{R}^n = n \) (see for example (2)) and \( ep \ \mathbb{R}^n \leq n \).

Corollary 3.3 :

For every totally ordered space \( X \), we have \( \text{ind} X = ep X \leq 1 \)
This assertion follows from 3.1 and 2.8.2).

Remark :

In another paper (1), we have proved that, for every totally ordered space \( X \), \( \text{ind} X = Ind X = dim X \leq 1 \).

2) An example of a space \( X \) such that \( \text{ind} X = Ind X < ep X \).

In (3), V.V. FILIPPOV has proved there exists two compact (non metric) spaces \( X_1 \) and \( X_2 \) such that \( \text{ind} X_1 = Ind X_1 = 1, \text{ind} X_2 = Ind X_2 = 2 \) and \( \text{ind}(X_1 \times X_2) = Ind(X_1 \times X_2) \geq 4 \). It follows from this example that \( X_1 \) or \( X_2 \) is such that \( \text{ind} X_i = Ind X_i < ep X_i \). Indeed, if \( \text{ind} X_1 = Ind X_1 = ep X_1 \) and \( \text{ind} X_2 = Ind X_2 = ep X_2 \), we would have, from 2.8.4), \( ep (X_1 \times X_2) \leq 3 \), which is impossible since \( ep (X_1 \times X_2) \geq \text{ind}(X_1 \times X_2) \) and \( \text{ind}(X_1 \times X_2) \geq 4 \).

Note the space we are looking for is the space \( X_2 \). Indeed, it is not the space \( X_1 \) because \( X_1 \) is by definition the quotient of a product of a compact totally disconnected space \( Z^* \) by a long line \( L \). Since \( ep Z^* = \text{ind} Z^* = 0 \) and \( ep L = \text{ind} L = 1 \) (use 3.3), we have \( ep (Z^* \times L) = 1 \) and therefore \( ep X_1 = ind X_1 = 1 \).

Note the description of the space \( X_2 \) is quite complicated so that it will not be reproduced here.
3) An example of space $X$ such that $ep\ X = indX < IndX = dimX$.

In (8), P. ROY has proved there exists a completely metric space $X$ such that $indX = 0$ and $IndX = dimX = 1$. It follows from 3.1 a) that, for this space, $ep\ X = indX = 0$ and $ep\ X < IndX = dimX$.

4) An example of space $X$ such that $dimX < ep\ X$.

In (5), O.V. LOKUCIEVSKII has proved there exists a compact (non metric) space such that $dimX = 1 < 2 = indX = IndX$. For this space, we have $dimX < indX \leq ep\ X$.

4 : The case of metric spaces.

Theorem 4.1 : 

For every metric space $X$, we have $indX \leq ep\ X \leq dimX = IndX$.

Since, for every topological space $Y$, we have $indY \leq ep\ Y$ and, for every metric space $Z$, we have $dimZ = IndZ$ (see for example (2)), it suffices to prove that, for every metric space $X$, we have $ep\ X \leq dimX$.

Notations : Let $\mathcal{F} = (F_i)_{i \in I}$ be an indexed family of subsets of $X$. Let us put, for every element $x$ of $X$, $ord\ (x, \mathcal{F}) = \{i \in I : x \in F_i\} - 1$ (where $|A|$ denotes the cardinal of $A$) and $ord\ \mathcal{F} = sup\{ord\ (x, \mathcal{F}) : x \in X\}$ ($ord\ \mathcal{F}$ is called the order of $\mathcal{F}$).

Lemma 4.1.1 : 

For every base $B$ of $X$, let $\mathcal{F} = (FrB)_{B \in B}$, then $ep\ B \leq ord\ \mathcal{F} + 1$.

Let $x$ be an element of $X$ and $\{a_1, ..., a_p\}$ be a chain of $h_B(*x)$. There exists then $p$ distinct elements of $B$, $B_1, ..., B_p$ such that, for every $i \in \{1, ..., p\}$, $a_j \in *B_i$ if and only if $j \geq i$ and such that $x \in FrB_i$. Consequently, by the definition of $ord\ (x, \mathcal{F})$, we have $p \leq ord\ (x, \mathcal{F}) + 1$, which implies $ep\ (x, B) \leq ord\ (x, \mathcal{F}) + 1$. It follows then, from the definitions of $ep\ B$ and $ord\ \mathcal{F}$, that we have $ep\ B \leq ord\ \mathcal{F} + 1$.

4.1.2. : Proof of 4.1 : 

This assertion is obvious if $dimX = +\infty$.

If $dimX = n$, there exists (see, for example, (2), 4.2.2.) a $\sigma$-locally finite base $B$ of $X$ such that, if we put $\mathcal{F} = (FrB)_{B \in B}$, we have $ord\ \mathcal{F} \leq n - 1$. It follows then, from 4.1.1., that, for this base $B$, we have $ep\ B \leq n$, which implies that $ep\ X \leq n$.

4.2 : Let us note that ROY's space is a metric space such that $indX = ep\ X = 0 < dimX = IndX = 1$. 


4.3 : Coincidence theorem for separable metric spaces.

For every separable metric space $X$, we have $ep X = indX = IndX = dimX$.

This assertion is an immediate consequence of 4.1 and the well-known theorem:
« For every separable metric space $X$, we have $indX = IndX = dimX$. »

4.4. : One can give a direct proof of 4.3. Indeed, let $X$ be a separable metric space such that $indX = n$. Let us denote by $N_{2n+1}^{2n+1}$ NOBELING's space (6), viz, the subspace of $\mathbb{R}^{2n+1}$ consisting of all points which have at most $n$ rational coordinates, and, by $C_{2n+1}^{2n+1}$ the trace on $N_{n}^{2n+1}$ of the base $B_{2n+1}^{2n+1}$ of $\mathbb{R}^{2n+1}$ consisting of all parallelepipeds with rational coordinates. One can prove that $ep C_{n}^{2n+1} \leq n$ which implies, since $ind N_{2n+1}^{2n+1} = n$ (see, for example, (2) 1.8.5), that $ep N_{n}^{2n+1} = n$.

Since $indX = n$ and $N_{n}^{2n+1}$ is universal for the class of separable metric spaces whose dimension is not larger than $n$ (see also (2), 1.11.5), $X$ is homeomorphic to a subspace of $N_{n}^{2n+1}$, which implies, from 2.8.3), that $ep X \leq ep N_{n}^{2n+1}$ and therefore that $ep X = n$.

4.5 : An example of a non separable metric space $X$ such that $ep X = indX = IndX = dimX$.

In (9), E.K. VAN DOUWEN proved there exists a non separable metric space $X$ such that $indX = IndX = dimX = 1$.

This space is therefore such that $ep X = indX = IndX = dimX$.

4.6 : Question : Does there exist a metric space $X$ such that $indX < ep X$ ?

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