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ON THE STABILITY OF MAPPINGS AND
AN ANSWER TO A PROBLEM OF TH.M.
RASSIAS

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Résumé
Le but principal de cet article est la démonstration d'un théorème concernant la stabilité HYERS-ULAM des applications, qui donne une généralisation pour les résultats [1] et [3]. En plus, il répond à une problème posée par TH. M. RASSIAS en [3].

Abstract
The main purpose of this paper is to prove a theorem concerning the HYERS-ULAM stability of mappings, which gives a generalization of the results from [1] and [3]. It also answers a problem posed by TH. M. RASSIAS.

The question concerning the stability of mappings has been originally raised by S.M. ULAM [4]. The first answer was given in 1941 by D. H. HYERS (see [2] for a research survey of the development of the subject). In this paper we provide a generalization of a theorem of TH. M. RASSIAS [3] concerning the HYERS-ULAM stability of mappings and we also answer a problem that TH. M. RASSIAS posed in [3].

Theorem 1 Let $(G, +)$ be an abelian group, $k$ an integer, $k \geq 2$, $(X, \|\|)$ a Banach space, $\varphi : G \times G \rightarrow [0, \infty)$ a mapping such that

$$\phi_k(x, y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \varphi(k^n x, k^n y) < \infty, \quad \forall x, y \in G$$

and $f : G \rightarrow X$ a mapping with the property

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \quad \forall x, y \in G.$$ (2)

Then there exists a unique additive mapping $T : G \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \sum_{m=1}^{k-1} \phi_k(x, mx), \quad \forall x \in G.$$ (3)
Proof. Setting \( y = x \) and \( y = 2x \) in relation (2) we obtain
\[
\| f(2x) - 2f(x) \| \leq \varphi(x, x), \quad \forall x \in G
\] (4)
and respectively,
\[
\| f(3x) - f(x) - f(2x) \| \leq \varphi(x, 2x), \quad \forall x \in G.
\]

Using the triangle inequality and the last two relations, it follows:
\[
\| f(3x) - 3f(x) \| \leq \| f(3x) - f(x) - f(2x) \| + \| f(2x) - 2f(x) \| \leq
\]
\[
\varphi(x, 2x) + \varphi(x, x) = \sum_{m=1}^{2} \varphi(x, mx)
\]

hence
\[
\| f(3x) - 3f(x) \| \leq \sum_{m=1}^{2} \varphi(x, mx). \quad (5)
\]

We will prove by mathematical induction after \( k \) the following inequality:
\[
\| f(kx) - kf(x) \| \leq \sum_{m=1}^{k-1} \varphi(x, mx). \quad (6)
\]

Indeed, for \( k = 2 \) and \( k = 3 \) we have the relation (4) and, respectively (5). Suppose (6) true for \( k \) and let us prove it for \( k + 1 \). We replace \( y \) by \( kx \) in (2) and we obtain:
\[
\| f((k + 1)x) - f(x) - f(kx) \| \leq \varphi(x, kx).
\]

Hence, it follows
\[
\| f((k + 1)x) - (k + 1)f(x) \| \leq \| f((k + 1)x) - f(x) - f(kx) \| + \| f(kx) - kf(x) \| \leq
\]
\[
\varphi(x, kx) + \sum_{m=1}^{k-1} \varphi(x, mx) = \sum_{m=1}^{k} \varphi(x, mx),
\]

using (6) for the last inequality.
So, relation (6) is true for any \( k \geq 2 \), integer.

Dividing (6) by \( k \) we obtain:
\[
\| \frac{f(kx)}{k} - f(x) \| \leq \sum_{m=1}^{k-1} \frac{1}{k} \varphi(x, mx). \quad (7)
\]

We claim that
\[
\| \frac{f(k^n x)}{k^n} - f(x) \| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x), \quad \forall x \in G. \quad (8)
\]
We see that for \( n = 1 \) we have (7). We suppose (8) true for \( n \) and we will prove it for \( n + 1 \). We replace \( x \) by \( kx \) in (8) and we have

\[
\left\| \frac{f(k^n \cdot kx)}{k^n} - f(kx) \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p \cdot kx, mk^p \cdot kx), \quad \forall x \in G.
\]

Dividing this relation by \( k \), it follows:

\[
\left\| \frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k} \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+2}} \varphi(k^{p+1}x, mk^{p+1}x)
\]

and further,

\[
\left\| \frac{f(k^{n+1}x)}{k^{n+1}} - f(x) \right\| \leq \left\| \frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(kx)}{k} \right\| + \left\| \frac{f(kx)}{k} - f(x) \right\| \leq \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+2}} \varphi(k^{p+1}x, mk^{p+1}x) + \sum_{m=1}^{k-1} \frac{1}{k} \varphi(x, mx) = \sum_{m=1}^{k-1} \left[ \sum_{p=1}^{n} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x) + \frac{1}{k} \varphi(x, mx) \right] = \sum_{m=1}^{k-1} \sum_{p=0}^{n} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x), \quad \forall x \in G,
\]

so (8) is true for each \( n \in \mathbb{N}^* \), by mathematical induction.

Then, for \( 0 < n_1 < n \), we have

\[
\left\| \frac{f(k^n x)}{k^n} - \frac{f(k^{n_1} x)}{k^{n_1}} \right\| = \frac{1}{k^{n_1}} \left\| \frac{f(k^{n-n_1}(k^{n_1} x))}{k^{n-n_1}} - f(k^{n_1} x) \right\| \leq \frac{1}{k^{n_1}} \sum_{m=1}^{k-1} \sum_{p=0}^{n_1-1} \frac{1}{k^{p+1}} \varphi(k^{p+n_1} x, mk^{p+n_1} x) = \sum_{m=1}^{k-1} \sum_{p=n_1}^{n_1-1} \frac{1}{k^{p+1}} \varphi(k^p x, mk^p x) \to 0 \text{ as } n_1 \to \infty.
\]

Therefore, the sequence \( \left\{ \frac{f(k^n x)}{k^n} \right\} \) \( n \in \mathbb{N}^* \) is a fundamental sequence. Because \( X \) is a BANACH space it follows that there exists \( \lim_{n \to \infty} \frac{f(k^n x)}{k^n} \), \( \forall x \in G \), denoted by \( T(x) \), so \( T : G \to X \) and we claim that \( T \) is an additive mapping.
From (2) we have
\[ \| f(k^n x + k^n y) - f(k^n x) - f(k^n y) \| \leq \varphi(k^n x, k^n y), \quad \forall x, y \in G. \]

Hence,
\[ \| \frac{f(k^n(x+y))}{k^n} - f(k^n x) - f(k^n y) \| \leq \frac{1}{k^n} \varphi(k^n x, k^n y), \quad \forall x, y \in G. \]

Taking the limit as \( n \to \infty \) we obtain:
\[ \| T(x + y) - T(x) - T(y) \| \leq \lim_{n \to \infty} \frac{1}{k^n} \varphi(k^n x, k^n y) = 0 \]
using the relation (1). This implies \( T(x + y) = T(x) + T(y), \quad \forall x, y \in G. \)

To prove that (3) holds, we take the limit as \( n \to \infty \) in (8) and we obtain
\[ \lim_{n \to \infty} \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^p x, m k^p x) = \sum_{m=1}^{k-1} \varphi(x, m x). \quad \forall x \in G. \]

Supposing now that there exists another additive mapping \( \tilde{T} : G \to \mathbb{X} \) with the property (3). Then
\[ \| T_1(x) - T(x) \| = \| \frac{T_1(k^n x)}{k^n} - \frac{T(k^n x)}{k^n} \| \leq \]
\[ \leq \frac{1}{k^n} (\| T_1(k^n x) - f(k^n x) \| + \| f(k^n x) - T(k^n x) \|) \leq \]
\[ \leq \frac{2}{k^n} \sum_{m=1}^{k-1} \phi_k(k^m x, m k^m x) = \frac{2}{k^n} \sum_{m=1}^{k-1} \sum_{p=0}^{n-1} \frac{1}{k^{p+1}} \varphi(k^{p+n} x, m k^{p+n} x) = \]
\[ = \sum_{m=1}^{k-1} \sum_{p=0}^{n} \frac{1}{k^{p+1}} \varphi(k^p x, m k^p x). \]

Thus, \( \lim_{n \to \infty} \| T_1(x) - T(x) \| = 0, \) for any \( x \in G, \) which implies \( T_1(x) = T(x), \quad \forall x \in G. \)

Q.E.D.

**REMARKS:**

1. For \( k = 2 \) we obtain for the first relation:
\[ \phi_2(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \varphi(2^n x, 2^n y) < \infty, \quad \forall x, y \in G \]
and for the third relation
\[ \|f(x) - T(x)\| \leq \phi_2(x, x), \quad \forall x \in G \]
which is the main theorem from [1].

2. If we take \( \varphi(x, y) = \theta(\|x\|^p + \|y\|^p) \) with \( \theta \geq 0 \) and \( p \in [0, 1) \) we have
\[ \phi_k(x, y) = \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \theta(k^n(\|x\|^p + \|y\|^p)) = \theta \sum_{n=0}^{\infty} k^{n(p-1)} = \theta \cdot \frac{k}{k - k^p}. \]

Then \( \phi_k(x, mx) = \frac{k\theta}{k - k^p} \cdot \frac{1}{k} \cdot \|x\|^p(1 + m^p) \) and
\[
\sum_{m=1}^{k-1} \phi_k(x, mx) = \frac{k\theta}{k - k^p} \|x\|^p \frac{1}{k} \sum_{m=1}^{k-1} (1 + m^p) = \frac{k\theta}{k - k^p} \|x\|^p \frac{1}{k} (k + \sum_{m=2}^{k-1} m^p) = \frac{k\theta}{k - k^p} \|x\|^p s(k, p).
\]

where \( s(k, p) = 1 + \frac{1}{k} \sum_{m=2}^{k-1} m^p \) which implies the theorem of Th.M.Rassias proved in [3].

We prove that the best possible value of \( k \) is 2. Set
\[ R(p) = \frac{2}{2 - 2^p} \quad \text{and} \quad Q(k, p) = \frac{k \cdot s(k, p)}{k - k^p}, k > 2. \]

We prove that
\[ R(p) < Q(k, p) \quad \text{for all} \quad k \geq 3. \quad (9) \]

The verification of (9) follows by mathematical induction on \( k \).

The case \( k = 3 \) is true, because
\[ Q(3, p) - R(p) = \frac{2 \cdot 3^p - 2^p - 4^p}{(2 - 2^p)(3 - 3^p)} > 0, \]

where we use the JENSEN inequality for the concave function \( f : (0, \infty) \to \mathbb{R}, f(x) = x^p, p \in [0, 1) : \)
\[ \left( \frac{x_1 + x_2}{2} \right)^p > \frac{x_1^p + x_2^p}{2} \quad \text{for} \quad x_1, x_2 \in (0, \infty). \quad (10) \]
with $x_1 = 2, x_2 = 4$.
Assume now that (9) is true and we prove that
\[ Q(k+1, p) > R(p). \] (11)
We have from (9)
\[
Q(k + 1) - R(p) > R(p) \left( \frac{k^p}{k + 1 - (k + 1)^p} + \frac{k^p + 1}{k + 1 - (k + 1)^p} - \frac{2}{2 - 2^p} \right) = \\
= \frac{2(k + 1)^p - 2^p - k^p \cdot 2^p}{(2 - 2^p)(k + 1 - (k + 1)^p)} > 0,
\]
where we use the inequality (10) with $x_1 = 2k, x_2 = 2$.
Thus, (9) is proved.
This last result gives an answer to a problem that was posed by Th. M. Rassias in 1991.

References


