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ON THE TOPOLOGY OF COMPACTOID CONVERGENCE IN NON-ARCHIMEDEAN SPACES

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Abstract

Some of the properties, of the topology of uniform convergence on the compactoid subsets of a non-Archimedean locally convex space E , are studied. In case E is metrizable, the compactoid convergence topology coincides with the finest locally convex topology which agrees with $\sigma(E', E)$ on equicontinuous sets.

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1 Introduction

In [7] some of the properties of the topology of uniform convergence on the compactoid subsets, of a non-Archimedean locally convex space, are investigated. In the same paper, the authors defined the ϵ -product $E\epsilon F$ of two non-Archimedean locally convex spaces E and F . $E\epsilon F$ is the space of all continuous linear operators of E'_∞ to F equipped with the topology of uniform convergence on the equicontinuous subsets of E' , where E'_∞ is the dual space E' of E endowed with the topology of uniform convergence on the compactoid subsets of E . In this paper, we continue with the investigation of the compactoid convergence topology τ_∞ . Among other things, we show that, for metrizable E , τ_∞ coincides with the topology τ_σ , where τ_σ is the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on equicontinuous

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sets. We also prove that τ_∞ has a base at zero all sets $\overline{W}^{\sigma(E',E)}$, where W is a τ_σ -neighborhood of zero and $\overline{W}^{\sigma(E',E)}$ denotes the $\sigma(E',E)$ -closure of W . If $T : E \mapsto F$ is a nuclear (resp. compactoid) operator, then $T' : F'_\infty \mapsto E'_\infty$ is nuclear (resp. compactoid). Also, if $T_i : E_i \mapsto F_i$, $i = 1, 2$, are nuclear, then

$$T = T_1 \epsilon T_2 : E_1 \epsilon E_2 \mapsto F_1 \epsilon F_2, \quad Tu = T_2 u T'_1,$$

is nuclear. Finally we show that τ_∞ is compatible with the dual pair $\langle E', E \rangle$ iff every closed compactoid subset of E is complete.

2 Preliminaries

Throughout this paper, $\mathbf{K}$ will stand for a complete non-Archimedean valued field, whose valuation is non-trivial, and $\mathbf{N}$ for the set of natural numbers. By a seminorm, on a vector space E over $\mathbf{K}$, we will mean a non-Archimedean seminorm.

Let now E be a locally convex space over $\mathbf{K}$. The collection of all continuous seminorms on E will be denoted by $cs(E)$. The algebraic dual, the topological dual, and the completion of E will be denoted by E^* , E' and $\widehat{E}$ respectively. A seminorm p on E is called polar if

$$p = \sup\{|f| : f \in E^*, |f| \leq p\},$$

where $|f|$ is defined by $|f|(x) = |f(x)|$. The space E is called polar if its topology is generated by a collection of polar seminorms. The edged hull A^e , of an absolutely convex subset A of E , is defined by:

$A^e = A$ if the valuation of $\mathbf{K}$ is discrete and $A^e = \bigcap \{\lambda A : |\lambda| > 1\}$ if the valuation is dense (see [10]). For a subset S of E , we denote by $co(S)$ the absolutely convex hull of S . A subset B of E is called compactoid if, for each neighborhood V of zero in E , there exists a finite subset S of E such that

$$B \subseteq co(S) + V.$$

The space E is said to be of countable type if, for each $p \in cs(E)$, there exists a countable subset S of E , such that the subspace $[S]$ spanned by S is p -dense in E .

A linear map $T : E \mapsto F$ is called:

- 1) compactoid if there exists a neighborhood V of zero in E such that $T(V)$ is a compactoid subset of F .
- 2) compactifying if $T(B)$ is compactoid in F for each bounded subset of E .

3) nuclear if there exist a null sequence (λ_n) in $\mathbb{K}$, a bounded sequence (y_n) in F and an equicontinuous sequence (f_n) in E' such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$$

for all $x \in E$.

We will denote by E'_{∞} the dual space E' of E equipped with the topology of uniform convergence on the compactoid subsets of E . The ϵ -product $E\epsilon F$, of two locally convex spaces E, F is the space of all continuous linear maps from E'_{∞} to F endowed with the topology of uniform convergence on the equicontinuous subsets of E' . For other notions, concerning non-Archimedean locally convex spaces and for related results, we will refer to [10].

We will need the following

Lemma 2.1 ([7, Lemma 2.6]). *Let E, F be Hausdorff polar quasi-complete spaces and let $T : E' \rightarrow F$ be a linear map. If T is continuous with respect to the weak topologies $\sigma(E', E)$ and $\sigma(F, F')$, then $T \in E\epsilon F$ iff T maps equicontinuous subsets of E' into compactoid subsets of F .*

3 The topology τ_{σ}

Let E be a Hausdorff polar space. We will denote by τ_{σ} the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on equicontinuous sets. It is easy to see that τ_{σ} is the locally convex topology which has as a base at zero all absolutely convex subsets W of E' with the following property: For every equicontinuous subset H of E' there exists a finite subset S of E such that $S^0 \cap H \subseteq W$, where S^0 is the polar of S in E' . In case E is a normed space, τ_{σ} coincides with the bounded weak star topology bw' (see [12] or [13]).

Since a linear functional f on E' is τ_{σ} -continuous iff its restriction to every equicontinuous subset of E' is $\sigma(E', E)$ -continuous we have the following

Proposition 3.1 *If E is a Hausdorff polar space, then $(E', \tau_{\sigma})' = \widehat{E}$.*

Proof. See the proof of Theorem 2 in [5].

The following Proposition for normed spaces was proved by Schikhof in [12, Proposition 3.2].

Proposition 3.2 *If E is a metrizable polar space, then (E', τ_{σ}) is of countable type.*

Proof. Let (V_n) be a decreasing sequence of convex neighborhoods of zero in E which is a base for the neighborhoods of zero. Then

$$E' = \bigcup_{n=1}^{\infty} V_n^0.$$

Let now q be a τ_σ -continuous seminorm on E' and set

$$W_m = \{x' \in E' : q(x') \leq 1/m\}.$$

Each V_n^0 is a $\sigma(E', E)$ -compactoid and hence a τ_σ -compactoid since V_n^0 is absolutely convex and $\tau_\sigma = \sigma(E', E)$ on V_n^0 . Thus, for each $m \in \mathbb{N}$, there exists a finite S_{nm} of E' such that

$$V_n^0 \subseteq co(S_{nm}) + W_m.$$

Now, the set $S = \bigcup_{m,n} S_{nm}$ is countable and the space $[S]$ is q -dense in E' . This completes the proof.

Let now E be a Hausdorff polar space and let $j_E : E \rightarrow E''$ the canonical map. In the following Theorem, we will consider E as a vector subspace of E'' identifying E with its image under the canonical map. For a subset A of E'' we will denote by A^0 and A^{00} , respectively, the polar and the bipolar of A with respect to the pair $\langle E'', E' \rangle$. If we consider on E'' the topology of uniform convergence on the equicontinuous subsets of E' , then E will be a topological subspace of E'' . In this case E'' will have as a base at zero all sets V^{00} where V is a convex neighborhood of zero in E .

The proof of the next Proposition is an adaptation of the corresponding proof for normed spaces given by Schikhof in [12, Proposition 3.3].

Proposition 3.3 *Let E be a Hausdorff polar space and consider on E'' the topology of uniform convergence on the equicontinuous subsets of E' . If F is the dual space of (E', τ_σ) then $F \cap E''$ coincides with the closure of E in E'' . Thus, if $F \subseteq E''$ (e.g. if τ_σ is coarser than the topology of the strong dual of E), then $F = \overline{E}$.*

Proof. Let $x'' \in \overline{E}$ and consider the set

$$W = \{x' \in E' : |\langle x', x'' \rangle| \leq 1\}.$$

For each convex neighborhood V of zero in E , there exists $x_\nu \in E$ such that $x'' - x_\nu \in V^{00}$. Indexing the convex neighborhoods of zero in E by inverse inclusion, we get a net (x_ν) in E . Let now V_0 be a convex neighborhood of

zero in E and let $\mu \in \mathbf{K}$, $\mu \neq 0$. If $V \subseteq \mu V_0$, then $x'' - x_\nu \in \mu V^{00}$ and so $|\langle x'' - x_\nu, x' \rangle| \leq |\mu|$ for all $x' \in V^0$.

$$\langle x_\nu, x' \rangle \rightarrow \langle x'', x' \rangle$$

uniformly on V_0^0 . Since each of the functions $x' \mapsto \langle x_\nu, x' \rangle$ is $\sigma(E', E)$ -continuous on V_0^0 , it follows that the restriction of x'' to V_0^0 is $\sigma(E', E)$ -continuous. This clearly proves that x'' is τ_σ -continuous.

On the other hand, let $x'' \in F \cap E''$ and let V be a convex neighborhood of zero in E . Let $|\lambda| > 1$ and set

$$D = \{x' \in E' : |\langle x', x'' \rangle| \leq 1\}.$$

There exists a finite subset S of E such that

$$S^0 \cap V^0 \subseteq \lambda^{-1} D.$$

The set $A = \text{co}(S)$ is a complete metrizable compactoid in $(E'', \sigma(E'', E'))$. Since V^{00} is absolutely convex and $\sigma(E'', E')$ -closed, it follows that $(A + V^{00})^e$ is $\sigma(E'', E')$ -closed by [11, Theorem 1.4]. Since

$$S^0 \cap V^0 = (A + V)^0,$$

we get that

$$\lambda D^0 \subseteq (A + V)^{00} = (A + V^{00})^{00} = \left(\overline{A + V^{00} \sigma(E'', E')} \right)^e = (A + V^{00})^e$$

and so $D^0 \subseteq A + V^{00} \subseteq E + V^{00}$. Since $x'' \in D^0$, it follows that $x'' \in \overline{E}$, which completes the proof.

As we will see in the next section, if E is metrizable, then τ_σ is coarser than the strong topology on E' and so in this case $(E', \tau_\sigma)' = \overline{E}$, a result proved by Schikhof in [12] for normed spaces.

4 The Topology of Compactoid Convergence

For a locally convex space (E, τ) , we will denote by τ_{co} the topology of compactoid convergence, i.e the topology on E' of uniform convergence on the compactoid subsets of E . We will denote (E', τ_{co}) by E'_{co} . By [7, 3.3], every equicontinuous subset of E' is τ_{co} -compactoid.

Proposition 4.1 ([10, Lemma 10.6]) *If E is a Hausdorff polar space, then $\tau_{co} = \sigma(E', E)$ on equicontinuous subsets of E' .*

Proposition 4.2 *If every compactoid subset of E is metrizable, then τ_{co} is the topology of uniform convergence on the null sequences in E .*

Proof. It follows from [10, Proposition 8.2], since for a metrizable compactoid A , there exists a null sequence (x_n) such that $A \subseteq \overline{co}(X)$ where $X = \{x_n : n \in \mathbb{N}\}$.

Corollary 4.3 $\sigma(E', E) \leq \tau_{co} \leq \tau_\sigma$.

Example If $E = c_0$ with the usual norm topology, then $E' = l_\infty$ and τ_{co} is the topology generated by the seminorms p_z , $z = (z_n) \in c_0$ where $p_z(x) = \max_k |z_k x_k|$ for $x = (x_n) \in l_\infty$. This follows from the fact that a subset A of c_0 is compactoid iff

$$A \subseteq \hat{z} = \{x \in c_0 : |x_n| \leq |z_n| \ \forall n\}$$

for some $z \in c_0$.

Notation For a locally convex topology γ on E' , we will denote by $\overline{\gamma}^\sigma$ the locally convex topology on E' which has as a base at zero all sets of the form $\overline{W}^{\sigma(E', E)}$, where W is a γ -neighborhood of zero.

Theorem 4.4 *If (E, τ) is a Hausdorff polar space, then $\tau_{co} = \overline{\tau_\sigma}^\sigma$.*

Proof. Since $\tau_{co} \leq \tau_\sigma$, we have that

$$\tau_{co} = \overline{\tau_{co}}^\sigma \leq \overline{\tau_\sigma}^\sigma.$$

On the other hand, let W be a convex τ_σ -neighborhood of zero. If V is a polar neighborhood of zero in E and $|\lambda| > 1$, then there exists a finite subset S of E such that $S^0 \cap V^0 \subseteq \lambda^{-1}W$. Since $S^0 \cap V^0 = (co(S) + V)^0$, it follows that

$$\lambda W^0 \subseteq (co(S) + V)^{00} = (co(S) + V)^e \subseteq \lambda (co(S) + V)$$

(by [10, Corollary 5.8]). Thus

$$W^0 \subseteq co(S) + V,$$

which shows that W^0 is a compactoid subset of E . Thus W^{00} is a τ_{co} -neighborhood of zero. Since

$$W^{00} = \left(\overline{W}^{\sigma(E', E)} \right)^e \subseteq \lambda \overline{W}^{\sigma(E', E)},$$

and so $\overline{W}^{\sigma(E', E)}$ is a τ_{co} -neighborhood of zero. This completes the proof.

The following is a Banach-Dieudonné type Theorem for non-Archimedean spaces (see [3, Theorem 10.1]).

Theorem 4.5 *If (E, τ) is metrizable polar space, then $\tau_{co} = \tau_\sigma$.*

Proof. Let (V_n) be a decreasing sequence of convex neighborhoods of zero in E which is a base at zero and let D be a convex τ_σ -neighborhood of zero in E' . Since τ_σ is the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on the sets V_n^0 , $n \in \mathbb{N}$, we may assume that there exists (by [4, Theorem 5.2]) a sequence $(S_n)_{n=0}^\infty$ of finite subsets of E such that for $W_n = S_n^0$ we have

$$D = W_0 \cap \left(\bigcap_{n=1}^{\infty} (W_n + V_n^0) \right).$$

Since each $W_n + V_n^0$ is $\sigma(E', E)$ -closed and since W_0 is also $\sigma(E', E)$ -closed, it follows that $D = \overline{D}^{\sigma(E', E)}$. Now since $\tau_{co} = \overline{\tau_\sigma}^\sigma$ it follows that $\tau_\sigma \leq \tau_{co}$. This clearly completes the proof.

Corollary 4.6 *Let E be a Hausdorff polar space and consider on E'' the topology of uniform convergence on the equicontinuous subsets of E' . Then:*

- a) τ_σ is polar and coarser than the strong topology on E' .
- b) $(E', \tau_\sigma)' = \widehat{E} = \overline{E}$, where $\overline{E}$ is the closure of E in E'' .

Open Problems .

- 1) Is τ_σ always a polar topology ?
- 2) Is it always true that $\tau_\sigma = \tau_{co}$?
- 3) Is it always true that $(E', \tau_\sigma)' \subseteq E''$?

The following Theorem gives a necessary and sufficient condition for the topology τ_{co} to be compatible with the pair $\langle E', E \rangle$.

Theorem 4.7 *For a Hausdorff polar space E , the following are equivalent:*

- (1) τ_{co} is compatible with the pair $\langle E', E \rangle$, i.e. $(E', \tau_{co})' = E$.
- (2) Every closed (or equivalently weakly closed) compactoid subset of E is complete.
- (3) Every closed (or equivalently weakly closed) absolutely convex subset of E is weakly complete.

Proof. First of all we observe that a compactoid subset of E is closed iff it is weakly closed and that an absolutely convex compactoid is complete iff it is weakly complete (by [10, Theorem 5.13]).

(1) $\Rightarrow$ (2). Let A be a closed compactoid subset of E . Since τ_{co} is compatible with the pair $\langle E', E \rangle$, it is the topology of uniform convergence on some

special covering (by [12, Proposition 7.4]). Thus, there exists a weakly bounded, weakly complete edged subset B of E such that $B^0 \subseteq A^0$. Thus

$$A \subseteq A^{00} \subseteq B^{00} = B.$$

Since A^{00} is an absolutely convex weakly complete subset of E , it is complete and hence A is complete.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). The proof is included in the proof of [6, Proposition 4.2].

Proposition 4.8 *Let E be a Hausdorff polar space and let G be the dual space of E'_∞ . Then*

$$(1) \quad G = \bigcup_A \overline{A}^{\sigma(E'', E')}$$

where A ranges over the family of all absolutely convex compactoid subsets of E .

(2) *If we consider on G the topology of uniform convergence on the equicontinuous subsets of E' , then E is a dense topological subspace of G .*

Proof. (1) Since the topology of E'_∞ is coarser than the strong topology on E' , G is a vector subspace of E'' . For a subset B of G we denote by B^0 and B^{00} , respectively, the polar and the bipolar of B with respect to the pair $\langle G, E' \rangle$. Let now $x'' \in G$. There exists an absolutely convex compactoid subset A of E such that

$$A^0 \subseteq \{x' \in E : |\langle x', x'' \rangle| \leq 1\}.$$

If $|\lambda| > 1$, then

$$x'' \in A^{00} \subseteq \lambda \overline{A}^{\sigma(E'', E')}.$$

On the other hand, if $x'' \in \overline{A}^{\sigma(E'', E')}$, for some absolutely convex compactoid subset A of E , then $x'' \in A^{00}$ and so $|\langle x', x'' \rangle| \leq 1$ for $x' \in A^0$, which implies that $x'' \in G$.

(2) Since the topology of E'_∞ is finer than the topology $\sigma(E', E)$ and since E is Hausdorff and polar, it follows that E is a topological subspace of G . It only remains to show that E is dense in G . So let $x'' \in G$. By (1), $x'' \in \overline{A}^{\sigma(G, E')}$ for some absolutely convex compactoid subset A of E . Given a convex neighborhood V of zero in E and $|\lambda| > 1$, there exists a finite subset S of E such that

$$A \subseteq co(S) + \lambda^{-1}V \subseteq co(S) + \lambda^{-1}V^{00}.$$

Now

$$x'' \in A^{00} \subseteq (co(S) + \lambda^{-1}V^{00})^{00} = (co(S) + \lambda^{-1}V^{00})^e$$

and so

$$x'' \in \lambda co(S) + V^{00}.$$

This clearly completes the proof.

By [7, 3.1], every equicontinuous subset of E' is a compactoid set in E'_{co} . Also, by Proposition 4.1, the topology of E'_{co} coincides with the topology $\sigma(E', E)$ on equicontinuous sets. We have the following

Proposition 4.9 *Let (E, τ) be a Hausdorff polar space and let γ be a polar locally convex topology on E' for which every equicontinuous subset of E' is a compactoid set. If γ is compatible with the pair $\langle E', E \rangle$, then γ is coarser than τ_{co} .*

Proof. Since $(E', \gamma)' = E$ and every equicontinuous subset H of E' is γ -compactoid, we have that $\gamma = \sigma(E', E)$ on H and so $\gamma \leq \tau_{co}$. Thus

$$\gamma = \overline{\gamma}^{\sigma(E', E)} \leq \overline{\tau_{co}}^{\sigma(E', E)} = \tau_{co}.$$

Proposition 4.10 *Let E, F be polar Hausdorff spaces and let $T : E \rightarrow F$ be a continuous linear map. Then:*

a) *T is compactifying iff the map*

$$T' : F'_{co} \rightarrow E'_b$$

is continuous, where E'_b is the strong dual of E .

b) *If T is compactifying and each closed compactoid subset of F is complete, then $T''(E'') \subseteq F$.*

Proof. a) If T is compactifying and B is a bounded subset of E , then $D = T(B)$ is a compactoid subset of F and $T'(D^0) \subseteq B^0$, which proves that $T' : F'_{co} \rightarrow E'_b$ is continuous. Conversely, let $T' : F'_{co} \rightarrow E'_b$ be continuous and let B be a bounded subset of E . There exists a compactoid subset D of F such that $T'(D^0) \subseteq B^0$. Now $T(B) \subseteq D^{00}$ and so $T(B)$ is compactoid since D^{00} is compactoid by [10, Theorem 5.3].

b) By [1], we have

$$E'' = \bigcup_B \overline{B}^{\sigma(E'', E')}$$

where B ranges over the family of all bounded subsets of E . Let now B be a bounded absolutely convex subset of E . Since T'' is continuous with respect to the topologies $\sigma(E'', E')$ and $\sigma(F'', F')$, we have

$$T'' \left(\overline{B}^{\sigma(E'', E')} \right) \subseteq \overline{T''(B)}^{\sigma(F'', F')} = \overline{T(B)}^{\sigma(F'', F')}.$$

Let $A = \overline{T(B)}$ be the closure of $T(B)$ in F . Since T is compactifying, the set A is compactoid in F and hence A is complete by our hypothesis. Since A is absolutely convex, it is $\sigma(F, F')$ -complete and hence it is $\sigma(F'', F')$ -complete. Thus A is $\sigma(F'', F')$ -closed and so

$$\overline{T(B)}^{\sigma(F'', F')} \subseteq \overline{T(B)} \subseteq F.$$

This clearly completes the proof.

Proposition 4.11 *Let $T : E \rightarrow F$ be a linear operator, where E and F are Hausdorff polar spaces. Then: (1) If T is continuous, then the adjoint map*

$$T' : F'_{co} \rightarrow E'_{co}$$

is continuous.

(2) *If T is compactoid, then*

$$T' : F'_{co} \rightarrow E'_{co}$$

is compactoid.

Proof. (1) If A is a compactoid subset of E , then $B = T(A)$ is compactoid in F and $T'(B^0) \subseteq A^0$.

(2) Assume that T is compactoid and let $p \in cs(E)$ be such that the set $A = T(V_p)$ is compactoid in F where

$$V_p = \{x \in E : p(x) \leq 1\}.$$

We will finish the proof by showing that $T'(A^0)$ is a compactoid subset of E'_{co} . So, let B be a compactoid subset of E . Since E is polar, it has the approximation property (by [9, Theorem 5.4]). Thus there are $g_1, \dots, g_n$ in E' and $e_1, \dots, e_n$ in E such that

$$p\left(x - \sum_{\kappa=1}^n g_{\kappa}(x)e_{\kappa}\right) \leq 1$$

for all $x \in B$. Let $\phi_{\kappa} \in (E'_{co})'$, $\phi_{\kappa}(x') = x'(e_{\kappa})$.

Claim: For all $y' \in A^0$ we have

$$T'y' - \sum_{\kappa=1}^n \phi_{\kappa}(T'y')g_{\kappa} \in B^0.$$

Indeed, let $y' \in A^0$ and $x \in B$. Then

$$x - \sum_{\kappa=1}^n g_{\kappa}(x)e_{\kappa} \in V_p$$

and so

$$Tx - \sum_{\kappa=1}^n g_{\kappa}(x)T(e_{\kappa}) \in A.$$

Thus,

$$\begin{aligned} \langle T'y' - \sum_{\kappa=1}^n \phi_{\kappa}(T'y')g_{\kappa}, x \rangle &= \langle y', Tx \rangle - \sum_{\kappa=1}^n (T'y')(e_{\kappa})g_{\kappa}(x) \\ &= \langle y', Tx \rangle - \sum_{\kappa=1}^n g_{\kappa}(x) \langle y', Te_{\kappa} \rangle = \langle y', Tx - \sum_{\kappa=1}^n g_{\kappa}(x)Te_{\kappa} \rangle \end{aligned}$$

which clearly proves our claim.

Now, there exists $\mu \in K$ such that $e_{\kappa} \in \mu V_p$ for $\kappa = 1, 2, \dots, n$. If $y' \in A^0$, then

$$|\phi_{\kappa}(T'y')| = |\langle y', Te_{\kappa} \rangle| \leq |\mu|.$$

Replacing ϕ_{κ} by $\mu^{-1}\phi_{\kappa}$ and g_{κ} by μg_{κ} , we may assume that $|\phi_{\kappa}(T'y')| \leq 1$ for all $y' \in A^0$ and that

$$\sum_{\kappa=1}^n \phi_{\kappa}(T'y')g_{\kappa} \in co(g_1, \dots, g_n).$$

It follows that

$$T'(A^0) \subseteq co(g_1, \dots, g_n) + B^0$$

which completes the proof.

Proposition 4.12 *If E, F are Hausdorff polar spaces and $T : E \mapsto F$ a nuclear linear operator, then $T' : F'_{\infty} \mapsto E'_{\infty}$ is nuclear.*

Proof. There exist a bounded sequence (y_n) in F , an equicontinuous sequence (f_n) in E' and a null sequence (λ_n) in K such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$$

for all x in E . For $y' \in F'$ and $x \in E$, we have

$$\langle T'y', x \rangle = \langle y', Tx \rangle = \langle y', \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n \rangle = \sum_{n=1}^{\infty} \lambda_n f_n(x) y'(y_n).$$

Let $|\lambda| > 1$ and choose $\mu_n \in \mathbf{K}$ with

$$|\mu_n| \leq \sqrt{|\lambda_n|} \leq |\lambda \mu_n|.$$

Let $\gamma_n \in \mathbf{K}$, where $\gamma_n = 0$ if $\lambda_n = 0$ and $\gamma_n = \lambda_n \mu_n^{-1}$ otherwise. Let

$$\phi_n : F'_{co} \rightarrow \mathbf{K}, \quad \phi_n(y') = \mu_n y'(y_n).$$

Since $A = \{\mu_n y_n : n \in \mathbf{N}\}$ is a compactoid subset of F , it follows that the sequence (ϕ_n) is equicontinuous in $(F'_{co})'$. Also, (f_n) is a bounded sequence in E'_{co} . Indeed, the set

$$V = \{x \in E : |f_n(x)| \leq 1 \ \forall n\}$$

is a neighborhood of zero in E . If A is a compactoid (and hence bounded) subset of E , then $A \subseteq \mu V$ for some μ in $\mathbf{K}$, and so $f_n \in \mu A^0$. Finally,

$$T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y') f_n$$

in E'_{co} . In fact, let $p \in cs(E)$ be such that $|f_n| \leq p$ for all n . Let $|\mu| > \sup\{|\lambda_n y'(y_n)| : n \in \mathbf{N}\}$. Set

$$s_n = \sum_{\kappa=1}^n \gamma_{\kappa} \phi_{\kappa}(y') f_{\kappa}.$$

If $V = \{y \in E : p(y) \leq 1\}$, then $s_n \in \mu V^0$. Moreover $s_n(x) \rightarrow \langle T'y', x \rangle$ for all $x \in E$. Thus $s_n \rightarrow T'y'$ in E'_{co} since the topology of E'_{co} coincides with $\sigma(E', E)$ on μV^0 by proposition 4.1. Thus

$$T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y') f_n$$

in E'_{co} . Since (γ_n) is a null sequence, the result follows.

5 On the ϵ -product

Proposition 5.1 ([10, 5.1]) *If E, F are Hausdorff polar spaces, then $F\epsilon E$ is a Hausdorff polar space.*

As it is shown in [7], the ϵ -product of two polar complete spaces is complete. The following proposition shows that the same is true for quasi-complete spaces.

Proposition 5.2 *Let E, F be Hausdorff polar spaces. If E and F are quasicomplete, then $E\epsilon F$ is quasicomplete.*

Proof. Let (u_α) be a bounded Cauchy net in $E\epsilon F$. For each $f \in E'$, the net $((u_\alpha(f)))$ is bounded and Cauchy in F and thus the limit $\lim u_\alpha(f)$ exists. Define

$$u_0 : E'_\infty \rightarrow F, u_0(f) = \lim u_\alpha(f).$$

Since the map $u \mapsto u'$ is a topological isomorphism between $E\epsilon F$ and $F\epsilon E$ (by [7, Theorem 3.3]), the net (u'_α) is bounded in $F\epsilon E$. Define

$$v_0 : F'_\infty \rightarrow E, v_0(g) = \lim u'_\alpha(g).$$

Claim 1: u_0 is continuous with respect to the weak topologies $\sigma(E', E)$ and $\sigma(F, F')$. Indeed, let S be a finite subset of F' and $T = v_0(S)$. For $f \in E'$ and $g \in F'$, we have

$$\lim \langle u_\alpha(f), g \rangle = \lim \langle f, u'_\alpha(g) \rangle$$

and so

$$\langle u_0(f), g \rangle = \langle f, v_0(g) \rangle.$$

It follows from this that $u_0(T^0) \subseteq S^0$.

Claim 2: For each equicontinuous subset H of E' , $u_0(H)$ is a compactoid subset of F . In fact, let W be a convex neighborhood of zero in F . The set

$$D = \{u \in E\epsilon F : u(H) \subseteq W\}$$

is a zero neighborhood in $E\epsilon F$. Thus, there exists α_0 such that $u_\alpha - u_\beta \in D$ for $\alpha, \beta \geq \alpha_0$. Since W is closed in F , it follows that $u_\alpha(f) - u_0(f) \in W$ for all $f \in H$ and all $\alpha \geq \alpha_0$. Since $u_{\alpha_0}(H)$ is a compactoid subset of F , there exists a finite subset S of F such that

$$u_{\alpha_0}(H) \subseteq co(S) + W.$$

Thus

$$u_0(H) \subseteq co(S) + W.$$

Now by claims 1, 2 and Lemma 2.1, we have that $u_0 \in E\epsilon F$. Finally it is easy to see that $u_\alpha \rightarrow u_0$ in $E\epsilon F$.

For a Hausdorff polar space F , we denote by $\tilde{F}$ the dual space of F'_{co} equipped with the topology of uniform convergence on the equicontinuous subsets of F' . It is easy to see that if $u \in F\epsilon E$, then the adjoint u' belongs to $E\epsilon\tilde{F}$. We will consider F as a topological subspace of $\tilde{F}$.

Proposition 5.3 *Let E, F be Hausdorff polar spaces. Then, the map $u \mapsto u'$, from $F\epsilon E$ to $E\epsilon\tilde{F}$, is linear, continuous and one-to-one.*

Proof. For a convex neighborhood V of F , we will let V^{00} denote the bipolar of V with respect to the dual pair $\langle \tilde{F}, F \rangle$. Sets of the form V^{00} form a base at zero in $\tilde{F}$. Let now W and V be convex neighborhoods of zero in E and F respectively and let

$$D = \{v \in E\epsilon\tilde{F} : v(W^0) \subseteq V^{00}\}.$$

If $u \in F\epsilon E$ is such that $u(V^0) \subseteq W$, then $u' \in D$. This proves that the map $u \mapsto u'$ is continuous. The rest of the proof is trivial.

Proposition 5.4 *Let E, F be Hausdorff polar spaces and let D be a compactoid subset of $F\epsilon E$. Then:*

(1) *For every equicontinuous subset H of F' , the set*

$$D(H) = \bigcup_{u \in D} u(H)$$

is a compactoid subset of E .

(2) *If every closed compactoid subset of F is complete, then D is an equicontinuous subset of $L(F'_{co}, E)$.*

(3) *If in both E and F the closed compactoid subsets are complete, then the closure $\bar{D}$ of D in $F\epsilon E$ is complete.*

Proof. (1) Let H be an equicontinuous subset of F' . For each $u \in F\epsilon E$ the set $u(H)$ is compactoid. Let now W be a convex neighborhood of zero in E . The set

$$U = \{u \in F\epsilon E : u(H) \subseteq W\}$$

is a neighborhood of zero in $F\epsilon E$ and thus

$$D \subseteq co(S) + U$$

for some finite set S . If $T = co(S)$, then $T(H)$ is a compactoid subset E and hence

$$T(H) \subseteq co(B) + W$$

for some finite subset B of E . Now

$$D(H) \subseteq co(B) + W.$$

(2) If every closed compactoid subset of F is complete, then $\tilde{F} = F$ (by Theorem 4.7) and so the set $D' = \{u' : u \in D\}$ is a compactoid subset of $E \epsilon F$ by the preceding Proposition. Given a polar neighborhood W in E , the set W^0 is an equicontinuous subset of E' and so $A = D'(W^0)$ is a compactoid subset of F by the first part of the proof. Moreover, for $u \in D$, we have

$$u(A^0) \subseteq W^{00} = W$$

which completes the proof of (2).

(3) The set $\overline{D}$ is a compactoid subset of $F \in E$. Let (u_α) be a Cauchy net in $\overline{D}$. For each $x' \in F'$, the set $\overline{D}(x')$ is compactoid in E and $(u_\alpha(x'))$ is a Cauchy net. By our hypothesis, the limit $\lim u_\alpha(x')$ exists in E . Define

$$u : F' \mapsto E, u(x') = \lim u_\alpha(x').$$

Claim: $u \in F \epsilon E$. Indeed, u is linear. Also, given a polar neighborhood W of zero in E , the set $B = \overline{D}(W^0)$ is compactoid in F and $\overline{D}(B^0) \subseteq W$. If $x' \in B^0$, there exists α_0 such that $u(x') - u_\alpha(x') \in W$, for $\alpha \succeq \alpha_0$, and so $u(x') \in u_\alpha(x') + W \subseteq W$, which proves that $u \in F \epsilon E$. If H is an equicontinuous subset of F' , then there exists β_0 such that $(u_\alpha - u_\beta)(H) \subset W$ for $\alpha \succeq \beta \succeq \beta_0$, and so $(u_\alpha - u)(H) \subset W$ for $\alpha \succeq \beta_0$. This proves that $u_\alpha \rightarrow u$ in $F \epsilon E$ and the result follows.

Theorem 5.5 *Let E_1, E_2, F_1, F_2 be Hausdorff polar spaces and let $T_i : E_i \mapsto F_i$, $i = 1, 2$, be continuous linear operators. Then: 1) The map*

$$T = T_1 \epsilon T_2 : E_1 \epsilon E_2 \mapsto F_1 \epsilon F_2, Tu = T_2 u T_1'$$

is continuous.

2) *If both T_1 and T_2 are nuclear, then T is nuclear.*

Proof. First of all we notice that, since

$$T_1' : (F_1')_{co} \mapsto (E_1')_{co}$$

is continuous, we have that $Tu \in F_1 \epsilon F_2$ for $u \in E_1 \epsilon E_2$. To show that T is continuous, let W_i be a convex neighborhood in F_i , $i = 1, 2$, and let

$$U = \{w \in F_1 \epsilon F_2 : w(W_1^0) \subseteq W_2\}.$$

Let $V_i = T_i^{-1}(W_i)$, $i = 1, 2$, and set

$$D = \{u \in E_1 \epsilon E_2 : u(V_1^0) \subseteq V_2\}.$$

Then D is a neighborhood of zero in $E_1 \epsilon E_2$ and $T(D) \subseteq U$. This proves that T is continuous.

2) Assume that both T_1 and T_2 are nuclear. There are null sequences $(\lambda_i), (\mu_i)$ in $\mathbf{K}$, bounded sequences (y_i) and (w_i) in F_1, F_2 , respectively, and equicontinuous sequences $(f_i), (g_i)$ in E'_1 and E'_2 such that

$$T_1 x = \sum_i \lambda_i f_i(x) y_i, \quad T_2 z = \sum_j \mu_j g_j(z) w_j.$$

As it is shown in the proof of proposition 4.12, we have

$$T'_1 y' = \sum_i \lambda_i y'(y_i) f_i, \quad y' \in F'_1,$$

where the series converges in $(E'_1)_{co}$. Thus, for $u \in E_1 \in E_2$ and $y' \in F'_1$, we have

$$\begin{aligned} \langle Tu, y' \rangle &= \langle T_2 u, \sum_i \lambda_i y'(y_i) f_i \rangle = \sum_i \lambda_i y'(y_i) T_2(u(f_i)) \\ &= \sum_i \lambda_i y'(y_i) \left(\sum_j \mu_j g_j(u(f_i)) w_j \right). \end{aligned}$$

Let $v_{ij} \in F_1 \epsilon F_2$, $v_{ij}(y') = y'(y_i) w_j$. The double sequence (v_{ij}) is bounded in $F_1 \epsilon F_2$. Indeed, let W and V be convex neighborhoods of zero in F_2 and F_1 respectively. Set

$$D = \{v \in F_1 \epsilon F_2 : v(V^0) \subset W\}.$$

Let $\mu \in \mathbf{K}$ be such that $y_i \in \mu V$ and $w_j \in \mu W$ for all i, j . Now, for $y' \in V^0$, we have

$$v_{ij}(y') = y'(y_i) w_j \in \mu^2 W$$

which proves that $v_{ij} \in \mu^2 D$. Also, let

$$h_{ij} : E_1 \epsilon E_2 \mapsto \mathbf{K}, \quad h_{ij}(u) = g_j(u(f_i)).$$

The double sequence (h_{ij}) is equicontinuous in $(E_1 \epsilon E_2)'$. Indeed, let V_1, W_1 be convex neighborhoods of zero in E_1, E_2 , respectively, such that $f_i \in V_1^0$ and $g_j \in W_1^0$ for all i, j . If

$$D_1 = \{u \in E_1 \epsilon E_2 : u(V_1^0) \subseteq W_1\},$$

then $h_{ij} \in D_1^0$.

Let now $\sigma = \sigma_1 \times \sigma_2 : \mathbf{N} \mapsto \mathbf{N} \times \mathbf{N}$ be any bijection. Set

$$\gamma_n = \lambda_{\sigma_1(n)} \mu_{\sigma_2(n)}, g_n(u) = h_{\sigma_1(n)\sigma_2(n)}, \phi_n = v_{\sigma_1(n)\sigma_2(n)}.$$

We will show that

$$Tu = \sum_{n=1}^{\infty} \gamma_n g_n(u) \phi_n,$$

where the series converges in $F_1 \epsilon F_2$. To this end, we may assume that $|\lambda_i|, |\mu_j| \leq 1$ for all i, j . Let V, W, V_1, W_1, D and μ be as above. For $y' \in V^0$, we have $|y'(y_i)| \leq |\mu|$ for all i . By Proposition 5.4, the set $A = u(V_1^0)$ is compactoid and hence bounded in E_2 . Since $g_j \in W_1^0$ and $f_i \in V_1^0$, there exists $\gamma \in K$ such that $|g_j(u(f_i))| \leq |\gamma|$ for all i, j . It is now clear that there exists n_0 such that if either $i \geq n_0$ or $j \geq n_0$, then

$$\lambda_i y'(y_i) \mu_j g_j(u(f_i)) w_j \in W$$

for all $y' \in V^0$. Since W is closed, we get that $\lambda_i y'(y_i) T_2(u(f_i)) \in W$, for $i > n_0$, and so, for $y' \in V^0$, we have

$$\langle Tu, y' \rangle = \sum_{i=1}^{n_0} \lambda_i y'(y_i) T_2(u(f_i)) + v, v \in W.$$

For an analogous reason, we get that

$$\langle Tu, y' \rangle = \sum_{i=1}^{n_0} \sum_{j=1}^{n_0} \lambda_i y'(y_i) \mu_j g_j(u(f_i)) w_j + v_1$$

with $v_1 \in W$. Let now m_0 be such that $\sigma_1(n) > n_0$ or $\sigma_2(n) > n_0$ if $n \geq m_0$. It is easy to see that for $n \geq m_0$ we have

$$\sum_{\kappa=1}^n \gamma_{\kappa} g_{\kappa}(u) \phi_{\kappa}(y') - \sum_{i=1}^{n_0} \sum_{j=1}^{n_0} \lambda_i \mu_j h_{ij}(u) v_{ij}(y') \in W$$

and so

$$\langle Tu, y' \rangle - \sum_{\kappa=1}^n \gamma_{\kappa} g_{\kappa}(u) \phi_{\kappa}(y') \in W$$

for all $y' \in V^0$, i.e.

$$Tu - \sum_{\kappa=1}^n \gamma_{\kappa} g_{\kappa}(u) \phi_{\kappa} \in D$$

for $n \geq m_0$. This clearly completes the proof.

References

- [1] N. De Grande-De Kimpe, The bidual of a non-Archimedean locally convex space, *Proc. Kon. Ned. Akad. Wet.*, A 92 (2) (1989) 203-312.
- [2] N. De Grande-De Kimpe and J. Martinez-Maurica, Compact like operators between non-Archimedean normed spaces, *Proc. Kon. Ned. Akad. Wet.*, A 85 (1982), 423-429.
- [3] J. Horvath, *Topological Vector Spaces and Distributions*, Addison-Wesley, Reading, Massachusetts. Palo Alto-London. Don mills. Ontario, 1966.
- [4] A.K. Katsaras, Spaces of non-Archimedean valued functions, *Bolletino U.M.I* (6) 5-B (1986), 603-621.
- [5] A.K. Katsaras, The non-Archimedean Grothendieck's completeness Theorem, *Bull. Inst. of Math. Acad. Sinica*, vol. 19, no 1 (1991), 351-354.
- [6] A.K. Katsaras and A. Beloyiannis, Non-Archimedean weighted spaces of continuous functions, *Rendiconti di Matematica*, Ser. VII, vol. 16, Roma(1996) (to appear).
- [7] A.K. Katsaras and A. Beloyiannis, On non-Archimedean weighted spaces of continuous functions, *Proc. Fourth Conf. on p-adic Analysis* (Nijmegen, The Netherlands), Marcel Dekker (to appear).
- [8] A.K. Katsaras, C. Petalas and T. Vidalis, Non-Archimedean sequential spaces and the finest locally convex topology with the same compactoid sets, *Acta Math. Univ. Comenianae*, vol. LXIII, 1 (1994), 55-75.
- [9] C.Pérez-Garcia, "The Hahn-Banach extension Theorem in p-adic Analysis", in: *p-adic Functional Analysis*, Marcel Dekker, New York. Bassel. Hong Kong (1992), 127-140.
- [10] W.H. Schikhof, Locally convex spaces over nonspherically complete valued fields I, II *Bull. Soc. Math. Belg. Sr. B* 38((1986). 187-224.

- [11] W.H. Schikhof, The continuous linear image of a p -adic compactoid
Proc. Kon. Ned. Akad. Wet. A 92 (1989), 119-123.
- [12] W.H. Schikhof, "The p -adic Krein-Šmulian Theorem ", in: p -adic Functional Analysis, Marcel Dekker, New York. Basel. Hong Kong (1992), 177-189.
- [13] W.H. Schikhof, A perfect duality between p -adic Banach spaces and compactoids, Indag. Math. N.S., 6(3) (1995), 325-339.