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<http://www.numdam.org/item?id=AMBP_1996__3_2_135_0>
ON THE TOPOLOGY OF COMPACTOID CONVERGENCE IN NON-ARCHIMEDEAN SPACES

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Abstract

Some of the properties, of the topology of uniform convergence on the compactoid subsets of a non-Archimedean locally convex space $E$, are studied. In case $E$ is metrizable, the compactoid convergence topology coincides with the finest locally convex topology which agrees with $\sigma(E', E)$ on equicontinuous sets.

1 Introduction

In [7] some of the properties of the topology of uniform convergence on the compactoid subsets, of a non-Archimedean locally convex space, are investigated. In the same paper, the authors defined the $\epsilon$-product $E\epsilon F$ of two non-Archimedean locally convex spaces $E$ and $F$. $E\epsilon F$ is the space of all continuous linear operators of $E'_{\infty}$ to $F$ equipped with the topology of uniform convergence on the equicontinuous subsets of $E'$, where $E'_{\infty}$ is the dual space $E'$ of $E$ endowed with the topology of uniform convergence on the compactoid subsets of $E$. In this paper, we continue with the investigation of the compactoid convergence topology $\tau_{\infty}$. Among other things, we show that, for metrizable $E$, $\tau_{\infty}$ coincides with the topology $\tau_{\sigma}$, where $\tau_{\sigma}$ is the finest locally convex topology on $E'$ which agrees with $\sigma(E', E)$ on equicontinuous

Key words and phrases: compactoid set, $\epsilon$-product, polar space, nuclear operator.
A.M.S. Subject Classification: 46S10
sets. We also prove that $\tau_{co}$ has a base at zero all sets $\overline{W}_{\sigma(E',E)}$, where $W$ is a $\tau_{\sigma}$-neighborhood of zero and $\overline{W}_{\sigma(E',E)}$ denotes the $\sigma(E',E)$-closure of $W$. If $T : E \rightarrow F$ is a nuclear (resp. compactoid) operator, then $T' : F'_{\infty} \hookrightarrow E''_{\infty}$ is nuclear (resp. compactoid). Also, if $T_i : E_i \rightarrow F_i$, $i = 1, 2$, are nuclear, then
\[ T = T_1 \circ T_2 : E_1 \circ E_2 \rightarrow F_1 \circ F_2, \quad Tu = T_2uT_1', \]
is nuclear. Finally we show that $\tau_{co}$ is compatible with the dual pair $<E',E>$ iff every closed compactoid subset of $E$ is complete.

2 Preliminaries

Throughout this paper, $K$ will stand for a complete non-Archimedean valued field, whose valuation is non-trivial, and $N$ for the set of natural numbers. By a seminorm, on a vector space $E$ over $K$, we will mean a non-Archimedean seminorm.

Let now $E$ be a locally convex space over $K$. The collection of all continuous seminorms on $E$ will be denoted by $cs(E)$. The algebraic dual, the topological dual, and the completion of $E$ will be denoted by $E^*$, $E'$ and $\overline{E}$ respectively. A seminorm $p$ on $E$ is called polar if
\[ p = \sup \{|f| : f \in E^*, |f| \leq p\}, \]
where $|f|$ is defined by $|f|(x) = |f(x)|$. The space $E$ is called polar if its topology is generated by a collection of polar seminorms. The edged hull $A^e$, of an absolutely convex subset $A$ of $E$, is defined by:
\[ A^e = A \text{ if the valuation of } K \text{ is discrete and } A^e = \cap \{\lambda A : |\lambda| > 1\} \text{ if the valuation is dense (see [10])}. \]
For a subset $S$ of $E$, we denote by $co(S)$ the absolutely convex hull of $S$. A subset $B$ of $E$ is called compactoid if, for each neighborhood $V$ of zero in $E$, there exists a finite subset $S$ of $E$ such that
\[ B \subseteq co(S) + V. \]
The space $E$ is said to be of countable type if, for each $p \in cs(E)$, there exists a countable subset $S$ of $E$, such that the subspace $[S]$ spanned by $S$ is $p$-dense in $E$.

A linear map $T : E \rightarrow F$ is called:
1) compactoid if there exists a neighborhood $V$ of zero in $E$ such that $T(V)$ is a compactoid subset of $F$.
2) compactifying if $T(B)$ is compactoid in $F$ for each bounded subset of $E$. 
3) nuclear if there exist a null sequence \((\lambda_n)\) in \(K\), a bounded sequence \((y_n)\) in \(F\) and an equicontinuous sequence \((f_n)\) in \(E'\) such that

\[
Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n
\]

for all \(x \in E\).

We will denote by \(E'_\omega\) the dual space \(E'\) of \(E\) equipped with the topology of uniform convergence on the compactoid subsets of \(E\). The \(\epsilon\)-product \(E \epsilon F\), of two locally convex spaces \(E, F\) is the space of all continuous linear maps from \(E'_\omega\) to \(F\) endowed with the topology of uniform convergence on the equicontinuous subsets of \(E'\). For other notions, concerning non-Archimedean locally convex spaces and for related results, we will refer to [10].

We will need the following

**Lemma 2.1** ([7, Lemma 2.6]). Let \(E, F\) be Hausdorff polar quasi-complete spaces and let \(T : E' \rightarrow F\) be a linear map. If \(T\) is continuous with respect to the weak topologies \(\sigma(E', E)\) and \(\sigma(F, F')\), then \(T \in E \epsilon F\) iff \(T\) maps equicontinuous subsets of \(E'\) into compactoid subsets of \(F\).

### 3 The topology \(\tau_\sigma\)

Let \(E\) be a Hausdorff polar space. We will denote by \(\tau_\sigma\) the finest locally convex topology on \(E'\) which agrees with \(\sigma(E', E)\) on equicontinuous sets. It is easy to see that \(\tau_\sigma\) is the locally convex topology which has as a base at zero all absolutely convex subsets \(W\) of \(E'\) with the following property: For every equicontinuous subset \(H\) of \(E'\) there exists a finite subset \(S\) of \(E\) such that \(S^0 \cap H \subseteq W\), where \(S^0\) is the polar of \(S\) in \(E'\). In case \(E\) is a normed space, \(\tau_\sigma\) coincides with the bounded weak star topology \(bw'\) (see [12] or [13]).

Since a linear functional \(f\) on \(E'\) is \(\tau_\sigma\)-continuous iff its restriction to every equicontinuous subset of \(E'\) is \(\sigma(E', E)\)-continuous we have the following

**Proposition 3.1** If \(E\) is a Hausdorff polar space, then \((E', \tau_\sigma)' = \tilde{E}\).

Proof. See the proof of Theorem 2 in [5].

The following Proposition for normed spaces was proved by Schikhof in [12, Proposition 3.2].

**Proposition 3.2** If \(E\) is a metrizable polar space, then \((E', \tau_\sigma)\) is of countable type.
Proof. Let \((V_n)\) be a decreasing sequence of convex neighborhoods of zero in \(E\) which is a base for the neighborhoods of zero. Then
\[
E' = \bigcup_{n=1}^{\infty} V_n^0.
\]

Let now \(q\) be a \(\tau_\sigma\)-continuous seminorm on \(E'\) and set
\[
W_m = \{x' \in E' : q(x') \leq 1/m\}.
\]

Each \(V_n^0\) is a \(\sigma(E',E)\)-compactoid and hence a \(\tau_\sigma\)-compactoid since \(V_n^0\) is absolutely convex and \(\tau_\sigma = \sigma(E',E)\) on \(V_n^0\). Thus, for each \(m \in \mathbb{N}\), there exists a finite \(S_{nm}\) of \(E'\) such that
\[
V_{n}^0 \subseteq co(S_{nm}) + W_m.
\]

Now, the set \(S = \bigcup_{m,n} S_{nm}\) is countable and the space \([S]\) is \(q\)-dense in \(E'\). This completes the proof.

Let now \(E\) be a Hausdorff polar space and let \(j_E : E \mapsto E''\) the canonical map. In the following Theorem, we will consider \(E\) as a vector subspace of \(E''\) identifying \(E\) with its image under the canonical map. For a subset \(A\) of \(E''\) we will denote by \(A^0\) and \(A^{00}\), respectively, the polar and the bipolar of \(A\) with respect to the pair \(<E'' , E'>\). If we consider on \(E''\) the topology of uniform convergence on the equicontinuous subsets of \(E'\), then \(E\) will be a topological subspace of \(E''\). In this case \(E''\) will have as a base at zero all sets \(V^{00}\) where \(V\) is a convex neighborhood of zero in \(E\).

The proof of the next Proposition is an adaptation of the corresponding proof for normed spaces given by Schikhof in [12, Proposition 3.3].

**Proposition 3.3** Let \(E\) be a Hausdorff polar space and consider on \(E''\) the topology of uniform convergence on the equicontinuous subsets of \(E'\). If \(F\) is the dual space of \((E',\tau_\sigma)\) then \(F \cap E''\) coincides with the closure of \(E\) in \(E''\). Thus, if \(F \subseteq E''\) (e.g. if \(\tau_\sigma\) is coarser than the topology of the strong dual of \(E\)), then \(F = \overline{E}\).

Proof. Let \(x'' \in \overline{E}\) and consider the set
\[
W = \{x' \in E' : \langle x', x'' \rangle \leq 1\}.
\]

For each convex neighborhood \(V\) of zero in \(E\), there exists \(x_v \in E\) such that \(x'' - x_v \in V^{00}\). Indexing the convex neighborhoods of zero in \(E\) by inverse inclusion, we get a net \((x_v)\) in \(E\). Let now \(V_0\) be a convex neighborhood of
zero in $E$ and let $\mu \in K$, $\mu \neq 0$. If $V \subseteq \mu V_0$, then $x'' - x_\nu \in \mu V^{00}$ and so $| < x'' - x_\nu, x' > | \leq |\mu|$ for all $x' \in V^0$.

$$< x_\nu, x' > \rightarrow < x''_\nu, x' >$$

uniformly on $V^0_0$. Since each of the functions $x' \mapsto < x_\nu, x' >$ is $\sigma(E', E)$-continuous on $V^0_0$, it follows that the restriction of $x''$ to $V^0_0$ is $\sigma(E', E)$-continuous. This clearly proves that $x''$ is $\tau_\sigma$-continuous.

On the other hand, let $x'' \in F \cap E''$ and let $V$ be a convex neighborhood of zero in $E$. Let $|\lambda| > 1$ and set

$$D = \{x' \in E' : | < x', x'' > | \leq 1 \}.$$  

There exists a finite subset $S$ of $E$ such that

$$S^0 \cap V^0 \subseteq \lambda^{-1} D.$$  

The set $A = \text{co}(S)$ is a complete metrizable compactoid in $(E'', \sigma(E'', E'))$. Since $V^{00}$ is absolutely convex and $\sigma(E'', E')$-closed, it follows that $(A + V^{00})^c$ is $\sigma(E'', E')$-closed by [11, Theorem 1.4]. Since

$$S^0 \cap V^0 = (A + V)^0,$$  

we get that

$$\lambda D^0 \subseteq (A + V)^{00} = (A + V^{00})^{00} = (A + V^{00})^{(E'', E')}^c = (A + V^{00})^c$$  

and so $D^0 \subseteq A + V^{00} \subseteq E + V^{00}$. Since $x'' \in D^0$, it follows that $x'' \in E$, which completes the proof.

As we will see in the next section, if $E$ is metrizable, then $\tau_\sigma$ is coarser than the strong topology on $E'$ and so in this case $(E', \tau_\sigma)' = E$, a result proved by Schikhof in [12] for normed spaces.

4 The Topology of Compactoid Convergence

For a locally convex space $(E, \tau)$, we will denote by $\tau_{co}$ the topology of compactoid convergence, i.e the topology on $E'$ of uniform convergence on the compactoid subsets of $E$. We will denote $(E', \tau_{co})$ by $E'^{co}$. By [7, 3.3], every equicontinuous subset of $E'$ is $\tau_{co}$-compactoid.

**Proposition 4.1** ([10, Lemma 10.6]) *If $E$ is a Hausdorff polar space, then $\tau_{co} = \sigma(E', E)$ on equicontinuous subsets of $E'$.*
Proposition 4.2 If every compactoid subset of $E$ is metrizable, then $\tau_\infty$ is the topology of uniform convergence on the null sequences in $E$.

Proof. It follows from [10, Proposition 8.2], since for a metrizable compactoid $A$, there exists a null sequence $(x_n)$ such that $A \subseteq \overline{B}(X)$ where $X = \{x_n : n \in \mathbb{N}\}$.

Corollary 4.3 $\sigma(E', E) \leq \tau_\infty \leq \tau_\sigma$.

Example If $E = c_0$ with the usual norm topology, then $E' = l_\infty$ and $\tau_\infty$ is the topology generated by the seminorms $p_z$, $z = (z_n) \in c_0$ where $p_z(x) = \max_n |x_n - z_n|$. This follows from the fact that if $A$ is a compactoid subset of $c_0$, then

$$A \subseteq z = \{x \in c_0 : |x_n| \leq |z_n| \land n\}$$

for some $z \in c_0$.

Notation For a locally convex topology $\gamma$ on $E'$, we will denote by $\overline{\gamma}^\sigma$ the locally convex topology on $E'$ which has as a base at zero all sets of the form $W^\sigma(E', E)$, where $W$ is a $\gamma$-neighborhood of zero.

Theorem 4.4 If $(E, \tau)$ is a Hausdorff polar space, then $\tau_\infty = \overline{\tau_\sigma}$.

Proof. Since $\tau_\infty \leq \tau_\sigma$, we have that

$$\tau_\infty = \overline{\tau_\sigma} \leq \overline{\tau_\sigma}.$$

On the other hand, let $W$ be a convex $\tau_\sigma$-neighborhood of zero. If $V$ is a polar neighborhood of zero in $E$ and $|\lambda| > 1$, then there exists a finite subset $S$ of $E$ such that $S^0 \cap V^0 \subseteq \lambda^{-1}W$. Since $S^0 \cap V^0 = (\text{co}(S) + V)^0$, it follows that

$$\lambda W^0 \subseteq (\text{co}(S) + V)^0 = (\text{co}(S) + V)^e \subseteq \lambda(\text{co}(S) + V)$$

(by [10, Corollary 5.8]). Thus

$$W^0 \subseteq \text{co}(S) + V,$$

which shows that $W^0$ is a compactoid subset of $E$. Thus $W^{00}$ is a $\tau_\infty$-neighborhood of zero. Since

$$W^{00} = (W^\sigma(E', E)^e) \subseteq \lambda W^\sigma(E', E),$$

and so $\overline{W}^\sigma(E', E)$ is a $\tau_\infty$-neighborhood of zero. This completes the proof.

The following is a Banach-Dieudonné type Theorem for non-archimedean spaces (see [3, Theorem 10.1]).
**Theorem 4.5** If \((E, \tau)\) is metrizable polar space, then \(\tau_{\infty} = \tau_\sigma\).

Proof. Let \((V_n)\) be a decreasing sequence of convex neighborhoods of zero in \(E\) which is a base at zero and let \(D\) be a convex \(\tau_\sigma\)-neighborhood of zero in \(E'\). Since \(\tau_\sigma\) is the finest locally convex topology on \(E'\) which agrees with \(\sigma(E', E)\) on the sets \(V_n^0, \ n \in \mathbb{N}\), we may assume that there exists (by [4, Theorem 5.2]) a sequence \((S_n)_{n=0}^\infty\) of finite subsets of \(E\) such that for \(W_n = S_n^0\) we have

\[
D = W_0 \cap \left( \bigcap_{n=1}^\infty \left( W_n + V_n^0 \right) \right).
\]

Since each \(W_n + V_n^0\) is \(\sigma(E', E)\)-closed and since \(W_0\) is also \(\sigma(E', E)\)-closed, it follows that \(D = D_{\infty}^{\sigma(E', E)}\). Now since \(\tau_{\infty} = \tau_\sigma\) it follows that \(\tau_\sigma \subseteq \tau_{\infty}\). This clearly completes the proof.

**Corollary 4.6** Let \(E\) be a Hausdorff polar space and consider on \(E''\) the topology of uniform convergence on the equicontinuous subsets of \(E'\). Then:

a) \(\tau_\sigma\) is polar and coarser than the strong topology on \(E'\).

b) \((E', \tau_\sigma)' = \hat{E} = \bar{E}\), where \(\hat{E}\) is the closure of \(E\) in \(E''\).

**Open Problems**

1) Is \(\tau_\sigma\) always a polar topology?

2) Is it always true that \(\tau_\sigma = \tau_{\infty}\)?

3) Is it always true that \((E', \tau_\sigma)' \subseteq E''\)?

The following Theorem gives a necessary and sufficient condition for the topology \(\tau_{\infty}\) to be compatible with the pair \(<E', E>\).

**Theorem 4.7** For a Hausdorff polar space \(E\), the following are equivalent:

1) \(\tau_{\infty}\) is compatible with the pair \(<E', E>\), i.e. \((E', \tau_{\infty})' = E\).

2) Every closed (or equivalently weakly closed) compactoid subset of \(E\) is complete.

3) Every closed (or equivalently weakly closed) absolutely convex subset of \(E\) is weakly complete.

Proof. First of all we observe that a compactoid subset of \(E\) is closed iff it is weakly closed and that an absolutely convex compactoid is complete iff it is weakly complete (by [10, Theorem 5.13]).

1) \(\Rightarrow\) 2). Let \(A\) be a closed compactoid subset of \(E\). Since \(\tau_{\infty}\) is compatible with the pair \(<E', E>\), it is the topology of uniform convergence on some
special covering (by [12, Proposition 7.4]). Thus, there exists a weakly bounded, weakly complete edged subset $B$ of $E$ such that $B^0 \subseteq A^0$. Thus
\[ A \subseteq A^{00} \subseteq B^{00} = B. \]

Since $A^{00}$ is an absolutely convex weakly complete subset of $E$, it is complete and hence $A$ is complete.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). The proof is included in the proof of [6, Proposition 4.2].

**Proposition 4.8** Let $E$ be a Hausdorff polar space and let $G$ be the dual space of $E_{co}$. Then

\[ G = \bigcup_A A^{\sigma(E'',E')} \]

where $A$ ranges over the family of all absolutely convex compactoid subsets of $E$.

(2) If we consider on $G$ the topology of uniform convergence on the equicontinuous subsets of $E'$, then $E$ is a dense topological subspace of $G$.

Proof. (1) Since the topology of $E_{co}'$ is coarser than the strong topology on $E'$, $G$ is a vector subspace of $E''$. For a subset $B$ of $G$ we denote by $B^0$ and $B^{00}$, respectively, the polar and the bipolar of $B$ with respect to the pair $<G, E'>$. Let now $x'' \in G$. There exists an absolutely convex compactoid subset $A$ of $E$ such that
\[ A^0 \subseteq \{x' \in E : |<x', x''| \leq 1\}. \]

If $|\lambda| > 1$, then
\[ x'' \in A^{00} \subseteq \lambda A^{\sigma(E'',E')} . \]

On the other hand, if $x'' \in A^{\sigma(E'',E')}$, for some absolutely convex compactoid subset $A$ of $E$, then $x'' \in A^{00}$ and so $|<x', x''| \leq 1$ for $x' \in A^0$, which implies that $x'' \in G$.

(2) Since the topology of $E_{co}'$ is finer then the topology $\sigma(E', E)$ and since $E$ is Hausdorff and polar, it follows that $E$ is a topological subspace of $G$. It only remains to show that $E$ is dense in $G$. So let $x'' \in G$. By (1), $x'' \in A^{\sigma(G,E')}$ for some absolutely convex compactoid subset $A$ of $E$. Given a convex neighborhood $V$ of zero in $E$ and $|\lambda| > 1$, there exists a finite subset $S$ of $E$ such that
\[ A \subseteq co(S) + \lambda^{-1}V \subseteq co(S) + \lambda^{-1}V^{00} . \]

Now
\[ x'' \in A^{00} \subseteq \left(co(S) + \lambda^{-1}V^{00}\right)^{00} = \left(co(S) + \lambda^{-1}V^{00}\right)^e \]
and so
\[ x'' \in \lambda_{\text{co}}(S) + V^{00}. \]
This clearly completes the proof.

By [7, 3.1], every equicontinuous subset of \( E' \) is a compactoid set in \( E'_{\text{co}} \). Also, by Proposition 4.1, the topology of \( E'_{\text{co}} \) coincides with the topology \( \sigma(E', E) \) on equicontinuous sets. We have the following

**Proposition 4.9** Let \( (E, \tau) \) be a Hausdorff polar space and let \( \gamma \) be a polar locally convex topology on \( E' \) for which every equicontinuous subset of \( E' \) is a compactoid set. If \( \gamma \) is compatible with the pair \( <E', E> \), then \( \gamma \) is coarser than \( \tau_{\text{co}} \).

Proof. Since \( (E', \gamma)' = E \) and every equicontinuous subset \( H \) of \( E' \) is \( \gamma \)-compactoid, we have that \( \gamma = \sigma(E', E) \) on \( H \) and so \( \gamma \leq \tau_{\sigma} \). Thus
\[ \gamma = \pi_{\sigma}(E', E) \leq \overline{\tau}_{\sigma}(E', E) = \tau_{\text{co}}. \]

**Proposition 4.10** Let \( E, F \) be polar Hausdorff spaces and let \( T : E \rightarrow F \) be a continuous linear map. Then:

a) \( T \) is compactifying iff the map
\[ T' : F'_\text{co} \rightarrow E'_b \]
is continuous, where \( E'_b \) is the strong dual of \( E \).

b) If \( T \) is compactifying and each closed compactoid subset of \( F \) is complete, then \( T''(E'') \subseteq F' \).

Proof. a) If \( T \) is compactifying and \( B \) is a bounded subset of \( E \), then \( D = T(B) \) is a compactoid subset of \( F \) and \( T'(D^0) \subseteq B^0 \), which proves that \( T' : F'_\text{co} \rightarrow E'_b \) is continuous. Conversely, let \( T' : F'_\text{co} \rightarrow E'_b \) be continuous and let \( B \) be a bounded subset of \( E \). There exists a compactoid subset \( D \) of \( F \) such that \( T'(D^0) \subseteq B^0 \). Now \( T(B) \subseteq D^{00} \) and so \( T(B) \) is compactoid since \( D^{00} \) is compactoid by [10, Theorem 5.3].

b) By [1], we have
\[ E'' = \bigcup_B B_{\sigma(E'', E')} \]
where \( B \) ranges over the family of all bounded subsets of \( E \). Let now \( B \) be a bounded absolutely convex subset of \( E \). Since \( T'' \) is continuous with respect to the topologies \( \sigma(E'', E') \) and \( \sigma(F'', F') \), we have
\[ T''(B_{\sigma(E'', E')}) \subseteq T''(B)_{\sigma(F'', F')} = T(B)_{\sigma(F'', F')} = T''(E'') \subseteq F''. \]
Let $A = \overline{T(B)}$ be the closure of $T(B)$ in $F$. Since $T$ is compactifying, the set $A$ is compactoid in $F$ and hence $A$ is complete by our hypothesis. Since $A$ is absolutely convex, it is $\sigma(F, F')$-complete and hence it is $\sigma(F'', F')$-complete. Thus $A$ is $\sigma(F'', F')$-closed and so
\[
\overline{T(B)}^{\sigma(F'', F')} \subseteq \overline{T(B)} \subseteq F.
\]
This clearly completes the proof.

**Proposition 4.11** Let $T : E \rightarrow F$ be a linear operator, where $E$ and $F$ are Hausdorff polar spaces. Then:

1. If $T$ is continuous, then the adjoint map
   
   \[
   T' : F'_\infty \rightarrow E'_\infty
   \]
   is continuous.

2. If $T$ is compactoid, then
   
   \[
   T' : F'_\infty \rightarrow E'_\infty
   \]
   is compactoid.

**Proof.** (1) If $A$ is a compactoid subset of $E$, then $B = T(A)$ is compactoid in $F$ and $T'(B^0) \subseteq A^0$.

(2) Assume that $T$ is compactoid and let $p \in cs(E)$ be such that the set $A = T(V_p)$ is compactoid in $F$ where
\[
V_p = \{x \in E : p(x) \leq 1\}.
\]
We will finish the proof by showing that $T'(A^0)$ is a compactoid subset of $E'_\infty$. So, let $B$ be a compactoid subset of $E$. Since $E$ is polar, it has the approximation property (by [9, Theorem 5.4]). Thus there are $g_1, \ldots, g_n$ in $E'$ and $e_1, \ldots, e_n$ in $E$ such that
\[
p \left( x - \sum_{\kappa=1}^{n} g_\kappa(x)e_\kappa \right) \leq 1
\]
for all $x \in B$. Let $\phi_\kappa \in (E'_\infty)'$, $\phi_\kappa(x') = x'(e_\kappa)$.

**Claim:** For all $y' \in A^0$ we have
\[
T'y' - \sum_{\kappa=1}^{n} \phi_\kappa(T'y')g_\kappa \in B^0.
\]
Indeed, let \( y' \in A^0 \) and \( x \in B \). Then

\[
x - \sum_{\kappa=1}^{n} g_{\kappa}(x)e_{\kappa} \in V_p
\]

and so

\[
Tx - \sum_{\kappa=1}^{n} g_{\kappa}(x)T(e_{\kappa}) \in A.
\]

Thus,

\[
<T'y' - \sum_{\kappa=1}^{n} \phi_{\kappa}(T'y')g_{\kappa}, x> = <y',Tx> - \sum_{\kappa=1}^{n} (T'y')(e_{\kappa})g_{\kappa}(x)
\]

\[
= <y',Tx> - \sum_{\kappa=1}^{n} g_{\kappa}(x) <y',Te_{\kappa}> = <y',Tx> - \sum_{\kappa=1}^{n} g_{\kappa}(x)Te_{\kappa}>
\]

which clearly proves our claim.

Now, there exists \( \mu \in K \) such that \( e_{\kappa} \in \mu V_p \) for \( \kappa = 1, 2, \ldots, n \). If \( y' \in A^0 \), then

\[
|\phi_{\kappa}(T'y')| = |<y',Te_{\kappa}>| \leq |\mu|.
\]

Replacing \( \phi_{\kappa} \) by \( \mu^{-1}\phi_{\kappa} \) and \( g_{\kappa} \) by \( \mu g_{\kappa} \), we may assume that \( |\phi_{\kappa}(T'y')| \leq 1 \) for all \( y' \in A^0 \) and that

\[
\sum_{\kappa=1}^{n} \phi_{\kappa}(T'y')g_{\kappa} \in \text{co}(g_1, \ldots, g_n).
\]

It follows that

\[
T'(A^0) \subseteq \text{co}(g_1, \ldots, g_n) + B^0
\]

which completes the proof.

**Proposition 4.12** If \( E, F \) are Hausdorff polar spaces and \( T : E \to F \) a nuclear linear operator, then \( T' : F'_\infty \to E'_\infty \) is nuclear.

**Proof.** There exist a bounded sequence \((y_n)\) in \( F \), an equicontinuous sequence \((f_n)\) in \( E' \) and a null sequence \((\lambda_n)\) in \( K \) such that

\[
Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x)y_n
\]
for all $x$ in $E$. For $y' \in F'$ and $x \in E$, we have
\[
< T'y', x > = < y', Tx > = < y', \sum_{n=1}^{\infty} \lambda_n f_n(x)y_n > = \sum_{n=1}^{\infty} \lambda_n f_n(x)y'(y_n).
\]

Let $|\lambda| > 1$ and choose $\mu_n \in K$ with
\[
|\mu_n| \leq \sqrt{\lambda_n} \leq |\lambda| \mu_n.
\]

Let $\gamma_n \in K$, where $\gamma_n = 0$ if $\lambda_n = 0$ and $\gamma_n = \lambda_n \mu_n^{-1}$ otherwise. Let
\[
\phi_n : F'_\infty \rightarrow K, \quad \phi_n(y') = \mu_n y'(y_n).
\]

Since $A = \{\mu_n y_n : n \in \mathbb{N}\}$ is a compactoid subset of $F$, it follows that the sequence $(\phi_n)$ is equicontinuous in $(F'_\infty)'$. Also, $(f_n)$ is a bounded sequence in $E'_\infty$. Indeed, the set
\[
V = \{x \in E : |f_n(x)| \leq 1 \ \forall \ n\}
\]
is a neighborhood of zero in $E$. If $A$ is a compactoid (and hence bounded) subset of $E$, then $A \subseteq \mu V$ for some $\mu$ in $K$, and so $f_n \in \mu A^0$. Finally,
\[
T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y')f_n
\]
in $E'_\infty$. In fact, let $p \in \text{co}(E)$ be such that $|f_n| \leq p$ for all $n$. Let $|\mu| > \sup\{|\lambda_n y'(y_n)| : n \in \mathbb{N}\}$. Set
\[
s_n = \sum_{n=1}^{N} \gamma_n \phi_n(y')f_n.
\]
If $V = \{y \in E : p(y) \leq 1\}$, then $s_n \in \mu V^0$. Moreover $s_n(x) \rightarrow < T'y', x >$ for all $x \in E$. Thus $s_n \rightarrow T'y'$ in $E'_\infty$ since the topology of $E'_\infty$ coincides with $\sigma(E', E)$ on $\mu V^0$ by proposition 4.1. Thus
\[
T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y')f_n
\]
in $E'_\infty$. Since $(\gamma_n)$ is a null sequence, the result follows.
5 On the $\varepsilon$-product

Proposition 5.1 ([10, 5.1]) If $E, F$ are Hausdorff polar spaces, then $F \varepsilon E$ is a Hausdorff polar space.

As it is shown in [7], the $\varepsilon$-product of two polar complete spaces is complete. The following proposition shows that the same is true for quasicomplete spaces.

Proposition 5.2 Let $E, F$ be Hausdorff polar spaces. If $E$ and $F$ are quasicomplete, then $E \varepsilon F$ is quasicomplete.

Proof. Let $(u_\alpha)$ be a bounded Cauchy net in $E \varepsilon F$. For each $f \in E'$, the net $((u_\alpha(f)))$ is bounded and Cauchy in $F$ and thus the limit $\lim u_\alpha(f)$ exists. Define

$$u_0 : E'_{\varepsilon_0} \mapsto F, \quad u_0(f) = \lim u_\alpha(f).$$

Since the map $u \mapsto u'$ is a topological isomorphism between $E \varepsilon F$ and $F \varepsilon E$ (by [7, Theorem 3.3]), the net $(u'_\alpha)$ is bounded in $F \varepsilon E$. Define

$$v_0 : F'_{\varepsilon_0} \mapsto E, \quad v_0(g) = \lim u'_\alpha(g).$$

Claim 1: $u_0$ is continuous with respect to the weak topologies $\sigma(E', E)$ and $\sigma(F, F')$. Indeed, let $S$ be a finite subset of $F'$ and $T = v_0(S)$. For $f \in E'$ and $g \in F'$, we have

$$\lim < u_\alpha(f), g > = \lim < f, u'_\alpha(g) >$$

and so

$$< u_0(f), g > = < f, v_0(g) > .$$

It follows from this that $u_0(T^0) \subseteq S^0$.

Claim 2: For each equicontinuous subset $H$ of $E'$, $u_0(H)$ is a compactoid subset of $F$. In fact, let $W$ be a convex neighborhood of zero in $F$. The set

$$D = \{ u \in E \varepsilon F : u(H) \subseteq W \}$$

is a zero neighborhood in $E \varepsilon F$. Thus, there exists $\alpha_0$ such that $u_\alpha - u_\beta \in D$ for $\alpha, \beta \geq \alpha_0$. Since $W$ is closed in $F$, it follows that $u_\alpha(f) - u_\beta(f) \in W$ for all $f \in H$ and all $\alpha \geq \alpha_0$. Since $u_{\alpha_0}(H)$ is a compactoid subset of $F$, there exists a finite subset $S$ of $F$ such that

$$u_{\alpha_0}(H) \subseteq co(S) + W.$$
Thus
\[ u_0(H) \subseteq \text{co}(S) + W. \]
Now by claims 1, 2 and Lemma 2.1, we have that \( u_0 \in \mathcal{E}_F \). Finally it is easy to see that \( u_0 \to u_0 \) in \( \mathcal{E}_F \).

For a Hausdorff polar space \( F \), we denote by \( \tilde{F} \) the dual space of \( F' \) equipped with the topology of uniform convergence on the equicontinuous subsets of \( F' \). It is easy to see that if \( u \in \mathcal{E}_F \), then the adjoint \( u' \) belongs to \( \mathcal{E}_\tilde{F} \). We will consider \( F \) as a topological subspace of \( \tilde{F} \).

**Proposition 5.3** Let \( E, F \) be Hausdorff polar spaces. Then, the map \( u \mapsto u' \), from \( \mathcal{E}_F \) to \( \mathcal{E}_\tilde{F} \), is linear, continuous and one-to-one.

**Proof.** For a convex neighborhood \( V \) of \( F \), we will let \( V^{00} \) denote the bipolar of \( V \) with respect to the dual pair \( < F, F^* > \). Sets of the form \( V^{00} \) form a base at zero in \( \tilde{F} \). Let now \( W \) and \( V \) be convex neighborhoods of zero in \( E \) and \( F \) respectively and let
\[
D = \{ v \in \mathcal{E}_\tilde{F} : v(W^{00}) \subseteq V^{00} \}.
\]
If \( u \in \mathcal{E}_F \) is such that \( u(V^{00}) \subseteq W \), then \( u' \in D \). This proves that the map \( u \mapsto u' \) is continuous. The rest of the proof is trivial.

**Proposition 5.4** Let \( E, F \) be Hausdorff polar spaces and let \( D \) be a compactoid subset of \( \mathcal{E}_F \). Then:
1. For every equicontinuous subset \( H \) of \( F' \), the set
\[
D(H) = \bigcup_{u \in D} u(H)
\]
is a compactoid subset of \( E \).
2. If every closed compactoid subset of \( F \) is complete, then \( D \) is an equicontinuous subset of \( L(F'_{\text{co}}, E) \).
3. If in both \( E \) and \( F \) the closed compactoid subsets are complete, then the closure \( \overline{D} \) of \( D \) in \( \mathcal{E}_F \) is complete.

**Proof.** (1) Let \( H \) be an equicontinuous subset of \( F' \). For each \( u \in \mathcal{E}_F \) the set \( u(H) \) is compactoid. Let now \( W \) be a convex neighborhood of zero in \( E \). The set
\[
U = \{ u \in \mathcal{E}_F : u(H) \subseteq W \}
\]
is a neighborhood of zero in \( \mathcal{E}_F \) and thus
\[
D \subseteq \text{co}(S) + U
\]
for some finite set $S$. If $T = \text{co}(S)$, then $T(H)$ is a compactoid subset $E$ and hence

$$T(H) \subseteq \text{co}(B) + W$$

for some finite subset $B$ of $E$. Now

$$D(H) \subseteq \text{co}(B) + W.$$ 

(2) If every closed compactoid subset of $F$ is complete, then $\bar{F} = F$ (by Theorem 4.7) and so the set $D' = \{u': u \in D\}$ is a compactoid subset of $E \subseteq F$ by the preceding Proposition. Given a polar neighborhood $W$ in $E$, the set $W_0$ is an equicontinuous subset of $E'$ and so $A = D'(W_0)$ is a compactoid subset of $F$ by the first part of the proof. Moreover, for $u \in D$, we have

$$u(A^0) \subseteq W^{00} = W$$

which completes the proof of (2).

(3) The set $D$ is a compactoid subset of $F \subseteq E$. Let $(u_\alpha)$ be a Cauchy net in $D$. For each $x' \in F'$, the set $D(x')$ is compactoid in $E$ and $(u_\alpha(x'))$ is a Cauchy net. By our hypothesis, the limit $\lim u_\alpha(x')$ exists in $E$. Define

$$u : F' \mapsto E, u(x') = \lim u_\alpha(x').$$

Claim: $u \in F \subseteq E$. Indeed, $u$ is linear. Also, given a polar neighborhood $W$ of zero in $E$, the set $B = D'(W^0)$ is compactoid in $F$ and $D(B^0) \subseteq W$. If $x' \in B^0$, there exists $\alpha_0$ such that $u(x') - u_\alpha(x') \in W$, for $\alpha \geq \alpha_0$, and so $u(x') \in u_\alpha(x') + W \subseteq W$, which proves that $u \in F \subseteq E$. If $H$ is an equicontinuous subset of $F'$, then there exists $\beta_0$ such that $(u_\alpha - u_\beta)(H) \subseteq W$ for $\alpha \geq \beta \geq \beta_0$, and so $(u_\alpha - u)(H) \subseteq W$ for $\alpha \geq \beta_0$. This proves that $u_\alpha \to u$ in $F \subseteq E$ and the result follows.

**Theorem 5.5** Let $E_1, E_2, F_1, F_2$ be Hausdorff polar spaces and let $T_i : E_i \mapsto F_i$, $i = 1, 2$, be continuous linear operators. Then: 1) The map

$$T = T_1 \circ T_2 : E_1 \times E_2 \mapsto F_1 \times F_2, \quad Tu = T_2uT_1'$$

is continuous.

2) If both $T_1$ and $T_2$ are nuclear, then $T$ is nuclear.

Proof. First of all we notice that, since

$$T_1' : (F_1')_{\text{co}} \mapsto (E_1')_{\text{co}}$$
is continuous, we have that \( Tu \in F_1F_2 \) for \( u \in E_1E_2 \). To show that \( T \) is continuous, let \( W_i \) be a convex neighborhood in \( F_i \), \( i = 1, 2 \), and let

\[
U = \{ w \in F_1E_2 : w(W_i^0) \subseteq W_2 \}.
\]

Let \( V_i = T_i^{-1}(W_i) \), \( i = 1, 2 \), and set

\[
D = \{ u \in E_1E_2 : u(V_i^0) \subseteq V_2 \}.
\]

Then \( D \) is a neighborhood of zero in \( E_1E_2 \) and \( T(D) \subseteq U \). This proves that \( T \) is continuous.

2) Assume that both \( T_1 \) and \( T_2 \) are nuclear. There are null sequences \( (\lambda_i), (\mu_i) \) in \( K \), bounded sequences \( (y_i) \) and \( (w_i) \) in \( F_1, F_2 \), respectively, and equicontinuous sequences \( (f_i), (g_i) \) in \( E'_1 \) and \( E'_2 \) such that

\[
T_1x = \sum_i \lambda_i f_i(x)y_i, \quad T_2z = \sum_j \mu_j g_j(z)w_j.
\]

As it is shown in the proof of proposition 4.12, we have

\[
T_1y' = \sum_i \lambda_i y'(y_i)f_i, \quad y' \in F'_1,
\]

where the series converges in \( (E'_1)_{\alpha} \). Thus, for \( u \in E_1 \subseteq E_2 \) and \( y' \in F'_1 \), we have

\[
<Tu, y'> = <T_2u, \sum_i \lambda_i y'(y_i)f_i> = \sum_i \lambda_i y'(y_i)T_2(u(f_i))
\]

\[
= \sum_i \lambda_i y'(y_i) \left( \sum_j \mu_j g_j(u(f_i))w_j \right).
\]

Let \( v_{ij} \in F_1F_2 \), \( v_{ij}(y') = y'(y_i)w_j \). The double sequence \( (v_{ij}) \) is bounded in \( F_1F_2 \). Indeed, let \( W \) and \( V \) be convex neighborhoods of zero in \( F_2 \) and \( F_1 \) respectively. Set

\[
D = \{ v \in F_1E_2 : v(V^0) \subseteq W \}.
\]

Let \( \mu \in K \) be such that \( y_i \in \mu V \) and \( w_j \in \mu W \) for all \( i, j \). Now, for \( y' \in V_0 \), we have

\[
v_{ij}(y') = y'(y_i)w_j \in \mu^2W
\]

which proves that \( v_{ij} \in \mu^2D \). Also, let

\[
h_{ij} : E_1E_2 \rightarrow K, \quad h_{ij}(u) = g_j(u(f_i)).
\]
The double sequence \((h_{ij})\) is equicontinuous in \((E_1 \varepsilon E_2)'\). Indeed, let \(V_1, W_1\) be convex neighborhoods of zero in \(E_1, E_2\), respectively, such that \(f_i \in V_i^0\) and \(g_j \in W_j^0\) for all \(i, j\). If

\[
D_1 = \{ u \in E_1 \varepsilon E_2 : u(V_1^0) \subseteq W_1 \},
\]

then \(h_{ij} \in D_1^0\).

Let now \(\sigma = \sigma_1 \times \sigma_2 : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}\) be any bijection. Set

\[
\gamma_n = \lambda_{\sigma_1(n)} \mu_{\sigma_2(n)}, \quad g_n(u) = h_{\sigma_1(n) \sigma_2(n)}, \quad \phi_n = u_{\sigma_1(n) \sigma_2(n)}.
\]

We will show that

\[
Tu = \sum_{n=1}^{\infty} \gamma_n g_n(u) \phi_n,
\]

where the series converges in \(F_1 \varepsilon F_2\). To this end, we may assume that \(|\lambda_i|, |\mu_j| \leq 1\) for all \(i, j\). Let \(V, W, V_1, W_1, D\) and \(\mu\) be as above. For \(y' \in V_0^0\), we have \(|y'(y_i)| \leq |\mu|\) for all \(i\). By Proposition 5.4, the set \(A = u(V_1^0)\) is compactoid and hence bounded in \(E_2\). Since \(g_j \in W_j^0\) and \(f_i \in V_i^0\), there exists \(\gamma \in K\) such that \(|g_j(u(f_i))| \leq |\gamma|\) for all \(i, j\). It is now clear that there exists \(n_0\) such that if either \(i > n_0\) or \(j > n_0\), then

\[
\lambda_i y'(y_i) \mu_j g_j(u(f_i)) w_j \in W
\]

for all \(y' \in V^0\). Since \(W\) is closed, we get that \(\lambda_i y'(y_i) T_2(u(f_i)) \in W\), for \(i > n_0\), and so, for \(y' \in V_0^0\), we have

\[
<Tu, y'> = \sum_{i=1}^{n_0} \lambda_i y'(y_i) T_2(u(f_i)) + v, \quad v \in W.
\]

For an analogous reason, we get that

\[
<Tu, y'> = \sum_{i=1}^{n_0} \sum_{j=1}^{n_0} \lambda_i y'(y_i) \mu_j g_j(u(f_i)) w_j + v_1
\]

with \(v_1 \in W\). Let now \(m_0\) be such that \(\sigma_1(n) > n_0\) or \(\sigma_2(n) > n_0\) if \(n \geq m_0\). It is easy to see that for \(n \geq m_0\) we have

\[
\sum_{\kappa=1}^{n} \gamma_{\kappa} g_{\kappa}(u) \phi_{\kappa}(y') - \sum_{i=1}^{n_0} \sum_{j=1}^{n_0} \lambda_i \mu_j h_{ij}(u) v_{ij}(y') \in W
\]

and so

\[
<Tu, y'> = -\sum_{\kappa=1}^{n} \gamma_{\kappa} g_{\kappa}(u) \phi_{\kappa}(y') \in W
\]
for all \( y' \in V^0 \), i.e.

\[
Tu - \sum_{\kappa=1}^{n} \gamma_\kappa g_\kappa(u)\phi_\kappa \in D
\]

for \( n \geq m_0 \). This clearly completes the proof.

References


