Some Steinhaus type theorems over valued fields


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1. Preliminaries:

In this paper \( K \) denotes \( R \) (the field of real numbers) or \( C \) (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field as will be explicitly stated depending on the context.

In the sequel, infinite matrices \( A = (a_nk), n, k = 1, 2, \ldots \) and sequences \( x = \{x_k\} \), \( k = 1, 2, \ldots \) have their entries in \( K \). If \( X, Y \) are two classes of sequences, we write \((X, Y)\) to denote the class of all infinite matrices \( A = (a_nk), n, k = 1, 2, \ldots \) for which

\[
Ax = \{(Ax)_n\} \in Y \quad \text{whenever} \quad x = \{x_k\} \in X,
\]

where \( (Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k, n = 1, 2, \ldots \), it being assumed that the series on the right converge. The sequence \( Ax = \{(Ax)_n\} \) is called the A-transform of \( x = \{x_k\} \). The sequence spaces \( \ell_p, p \geq 1 \), \( \ell_{\infty} \), \( c, c_0 \) are defined as usual i.e.,

\[
\ell_p = \{x = \{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}, p \geq 1;
\]

\[
\ell_{\infty} = \{x = \{x_k\} : \sup_{k \geq 1}|x_k| < \infty\};
\]

\[
c = \{x = \{x_k\} : \lim_{k \to \infty} x_k = s \text{ for some } s \in K\};
\]

\[
c_0 = \{x = \{x_k\} : \lim_{k \to \infty} x_k = 0\}.
\]

Note that \( \ell_p \subset c_0 \subset c \subset \ell_{\infty} \) where \( p \geq 1 \). For convenience we write \( \ell_1 = \ell \). \((\ell, c; P')\) denotes the class of all infinite matrices \( A \in (\ell, c) \) such that \( \lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \) whenever \( x = \{x_k\} \in \ell \).
2. The case $K = R$ or $C$

When $K = R$ or $C$, it is known ([11]), p. 4,17) that $A = (a_{nk}) \in (\ell, c)$ if and only if

\[(1) \quad \sup_{n,k} |a_{nk}| < \infty ;\]

and

\[(2) \quad \lim_{n \to \infty} a_{nk} = \delta_k \text{ exists, } k = 1, 2, ....\]

We now prove the following

**THEOREM 2.1:**

*When $K = R$ or $C, A \in (\ell, c; P')$ if and only if (1) holds and (2) holds with

\[(3) \quad \delta_k = 1, k = 1, 2, ....\]"

*Proof.*

Let $A \in (\ell, c ; P')$. Let $e_k$ be the sequence in which 1 occurs in the $k^{th}$ place and 0 elsewhere, $k = 1, 2, ...$ i.e.,

$$e_k = \left\{ x_i^{(k)} \right\}_{i=1}^\infty$$

where

$$x_i^{(k)} = 1, \text{ if } i = k ;$$

$$= 0, \text{ otherwise.}$$

Then $e_k \in \ell, k = 1, 2, ..., \sum_{i=1}^\infty x_i^{(k)} = 1$ and $(Ae_k)_n = a_{nk}$ so that $\lim_{n \to \infty} a_{nk} = 1$, i.e.,

$\delta_k = 1, k = 1, 2, ....$ Thus (1) and (3) are necessary for $A \in (\ell, c ; P')$.

Conversely, let (1) and (3) hold. Let $x = \{x_k\} \in \ell$. In view of (1), $\sum_{k=1}^\infty a_{nk}x_k$ converges, $n = 1, 2, ....$ Now,

$$(Ax)_n = \sum_{k=1}^\infty a_{nk}x_k$$

$$= \sum_{k=1}^\infty (a_{nk} - 1)x_k + \sum_{k=1}^\infty x_k,$$

this being true since $\sum_{k=1}^\infty a_{nk}x_k$ and $\sum_{k=1}^\infty x_k$ both converge.
Since \( \sum_{k=1}^{\infty} |x_k| < \infty \), given \( \varepsilon > 0 \), there exists a positive integer \( N \) such that

\[
(4) \quad \sum_{k=N+1}^{\infty} |x_k| < \frac{\varepsilon}{2A},
\]

where \( A = \sup_{n,k} |a_{nk} - 1| \). Since \( \lim_{n \to \infty} a_{nk} = 1, k = 1, 2, \ldots, N \), we can choose a positive integer \( N' > N \) such that

\[
(5) \quad |a_{nk} - 1| < \frac{\varepsilon}{2NM}, \quad n \geq N', k = 1, 2, \ldots, N,
\]

where \( M > 0 \) is such that \( |x_k| \leq M, k = 1, 2, \ldots \). Now, for \( n \geq N' \),

\[
|\sum_{k=1}^{N} (a_{nk} - 1)x_k| \leq \sum_{k=1}^{N} |a_{nk} - 1||x_k| + \sum_{k=N+1}^{\infty} |a_{nk} - 1||x_k|
\]

\[
< N \cdot \frac{\varepsilon}{2NM} \cdot M + A \cdot \frac{\varepsilon}{2A}, \quad \text{in view of (4) and (5)}
\]

so that \( \lim_{n \to \infty} \sum_{k=1}^{N} (a_{nk} - 1)x_k = 0 \). Thus \( \lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \).

Consequently \( A \in (\ell, c ; P') \) which completes the proof of the theorem.

When \( K = R \) or \( C \), the Steinhaus theorem ([4], p. 187, Theorem 14) can be written conveniently in the form \((c, c; P') \cap (\ell, c) = \emptyset \), where \((c, c; P)\) denotes the class of all infinite matrices \( A \in (c, c) \) such that \( \lim_{n \to \infty} (Ax)_n = \lim_{k \to \infty} x_k \).

We shall call such type of theorems as "Steinhaus type theorems". Such theorems were considered in [2], [3], [8]. Using Theorem 1, we shall deduce one such theorem.

**THEOREM 2.2:**

\((\ell, c ; P') \cap (\ell, c) = \emptyset \) whenever \( p > 1 \).

**Proof:**

Suppose \( A = (a_{nk}) \in (\ell, c ; P') \cap (\ell, c) \) where \( p > 1 \). It is known ([11], p. 4, 16) that \( A \in (\ell, p, c) \) whenever \( p > 1 \), if and only if (2) holds and

\[
(6) \quad \sup_{n} \sum_{k=1}^{\infty} |a_{nk}|^q < \infty,
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). It now follows that \( \sum_{k=1}^{\infty} |\delta_k|^q < \infty \), which contradicts the fact that
δ_k = 1, k = 1, 2, ..., since \( A \in (\ell, c ; P') \) and consequently \( \sum_{k=1}^{\infty} |\delta_k|^q \) diverges. This establishes our claim.

**Remark 2.3.**
Since \((\ell_\infty, c) \subset (c, c) \subset (c_0, c) \subset (\ell_p, c)\) where \( p > 1 \), we have \((\ell, c ; P') \cap (X, c) = \emptyset\), when \( X = \ell_\infty, c, c_0, \ell_p \) where \( p > 1 \).

3. The case when \( K \) is a complete, non-trivially valued, non-archimedean field.

For concepts and results in Analysis over complete, non-trivially valued, non-archimedean fields, we refer to [1]. In this case, Steinhaus type theorems were considered in [6], [7], [8], [10].

When \( K \) is a complete, non-trivially valued, non-archimedean field, one can prove that Theorem 2.1 continues to hold. In this case, if \( A = (a_{nk}) \in (\ell, c ; P') \cap (\ell_\infty, c) \), then \( \limsup_{n \to \infty, k \geq 1} |a_{nk} - 1| = 0 \) (see [6], Theorem 2). So for any \( \varepsilon, 0 < \varepsilon < 1 \), there exists a positive integer \( N \) such that
\[
|a_{nk} - 1| < \varepsilon, n \geq N, k = 1, 2, ....
\]
In particular, \( |a_{Nk} - 1| < \varepsilon, k = 1, 2, .... \)
Thus \( \lim_{k \to \infty} |a_{Nk} - 1| \leq \varepsilon \) i.e., \( |0 - 1| \leq \varepsilon \) (since \( A \in (\ell_\infty, c) \), \( \lim_{k \to \infty} a_{nk} = 0, n = 1, 2, ..., \) by Theorem 2 of [6]) i.e., \( 1, \leq \varepsilon \), a contradiction on the choice of \( \varepsilon \). Consequently we have:

**Theorem 3.1**

*When \( K \) is a complete, non-trivially valued, non-archimedean field, \((\ell, c ; P') \cap (\ell_\infty, c) = \emptyset\).*

**Remark 3.2 :**
However, \((\ell, c ; P') \cap (c, c) \neq \emptyset\) when \( K \) is a complete, non-trivially valued, non-archimedean field, as the following example illustrates.

Consider the infinite matrix
\[
A = (a_{nk}) = 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
1 & 1 & -2 & 0 & 0 & 0 & \cdots \\
1 & 1 & 1 & -3 & 0 & 0 & \cdots \\
1 & 1 & 1 & 1 & -4 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]
Then \( \sup_{n,k} |a_{nk}| \leq 1 < \infty \), \( \lim_{n \to \infty} a_{nk} = 1, k = 1, 2, \ldots \) and \( \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 0 \) so that \( A \in (\ell, c; P') \cap (c, c) \) (for criterion for \( A \in (c, c) \), see \([5], [9]\)). Since \((c, c) \subset (c_0, c) \subset (\ell_p, c)\) where \( p > 1 \), it follows that \((\ell, c; P') \cap (X, c) \neq \emptyset\), when \( X = c, c_0, \ell_p \) where \( p > 1 \). This indicates a violent departure in when \( K \) is a non-archimedean valued field from the case \( K = R \) or \( C \).

\((c_0, c; P')\) denotes the class of all infinite matrices \( A \in (c_0, c) \) such that \( \lim_{n \to \infty} (Ax)_n = \sum_{k=1}^{\infty} x_k \) whenever \( x = \{x_k\} \in c_0 \). In this context it is worthwhile to note that \( \sum_{k=1}^{\infty} x_k \) converges if and only if \( \{x_k\} \in c_0 \).

**Remark 3.3:**

\((c_0, c; P') = (\ell, c; P')\).

**Proof.**

Adapting the proof of Theorem 2.1, with suitable modifications for the non-archimedean case, we have, \( A \in (c_0, c; P') \) if and only if \( (1) \) and \( (3) \) hold. The result now follows.

### 4. General remarks

It is to be noted that \( \ell_p, p \geq 1, c_0, c, \ell_\infty \) are linear spaces with respect to coordinatewise addition and scalar multiplication irrespective of how \( K \) is chosen. When \( K = R \) or \( C \), \( c_0, c, \ell_\infty \) are Banach spaces with respect to the norm \( \|x\| = \sup_{k \geq 1} |x_k| \) where \( x = \{x_k\} \in c_0, c \) or \( \ell_\infty \), while they are non-archimedean Banach spaces under the above norm when \( K \) is a complete, non-trivially valued, non-archimedean field.

Whatever be \( K \), \( \ell_p \) is a Banach space with respect to the norm

\[
\|x\| = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, x = \{x_k\} \in \ell_p.
\]

Whatever be \( K \), if \( A = (a_{nk}) \in (\ell, c; P') \), then \( A \) is bounded and \( \|A\| = \sup_{n,k} |a_{nk}| \).

However, \((\ell, c; P')\) is not a subspace of \( BL(\ell, c) \), i.e., the space of all bounded linear mappings of \( \ell \) into \( c \), since \( \lim_{n \to \infty} 2a_{nk} = 2, k = 1, 2, \ldots \) and consequently \( 2A \notin (\ell, c; P') \) when \( A \in (\ell, c; P') \).

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