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Transformation of gaussian measures

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Transformation of Gaussian measures

Introduction

We shall be, in our lecture, mainly concerned by some particular cases of the following problem :

Let $(X, \mathcal{F}, \mu)$ be a measure space and $T : X \rightarrow X$ measurable. We denote by $T(\mu)$ or $\mu \circ T^{-1}$ the image of μ by T :

$$T(\mu)(A) = \mu \circ T^{-1}(A) = \mu(T^{-1}A), \quad \forall A \in \mathcal{F}.$$

When does $T(\mu) \ll \mu$ and how to compute the density?

Example 1 : Let $X = \mathbb{R}^n$, $\mu = \lambda_n$ (the Lebesgue measure) and $T : X \rightarrow X$ a diffeomorphism. Then from the formula

$$\int f(T(x)) |\det T'(x)| dx = \int f(y) dy,$$

we conclude that $T(\lambda_n)$ is absolutely continuous with respect to λ_n and

$$T(\lambda_n)(dy) = |\det T'(T^{-1}y)|^{-1} dy = |\det (T^{-1})'(y)| dy.$$

Example 2 : Let $(\Omega, \mathcal{F}, P)$ be the classical Wiener space, $\Omega = \mathcal{C}_0([0, 1])$, $\mathcal{F}$ the Borel σ -field, P the Wiener measure. Let $u : [0, 1] \times \Omega \rightarrow \mathbb{R}$ be a measurable and **adapted** stochastic process such that $\int_0^1 u_t^2(\omega) dt < \infty$ almost surely, and let $T : \Omega \rightarrow \Omega$ be defined by :

$$(T\omega)_t = \omega_t + \int_0^t u_s(\omega) ds.$$

Girsanov has proven that

$$T(P) \ll P.$$

On the other hand, let

$$\xi = \exp \left\{ - \int_0^1 u_t d\omega_t - \frac{1}{2} \int_0^1 u_t^2(\omega) dt \right\}$$

then, if $\mathbb{E}(\xi) = 1$. $(T\omega)_t$ is a Brownian motion with respect to $(\Omega, \mathcal{F}, Q)$, where $\frac{dQ}{dP} = \xi$.

That is $Q \circ T^{-1} = P$.

(This fact was first proven by means of the Itô-calculus, but as we shall see, we can obtain this by analytic methods).

This has an application in Statistical Communication Theory :

Suppose we are receiving a signal corrupted by noise, and we wish to determine if there is indeed a signal or if we are just receiving noise.

If $x(t)$ is the received signal, $\xi(t)$ the noise and $s(t)$ the emitted signal :

$$x(t) = s(t) + \xi(t) \quad (A)$$

In general, we make an hypothesis on the noise : it is a *white noise*.

The "integrated" version of (A) is

$$X(t) = \int_0^t s(u) du + W_t = S_t + W_t \quad (A')$$

(W is the standard Wiener process, $X(t) = \int_0^t x(s) ds$ is the cumulative received signal).

Now we ask the question : is there a signal corrupted by noise, or is there just a noise ($s(t) = 0, \forall t$)?

The hypotheses are :

$$H_0 : X_t = W_t$$

$$H_1 : X_t = \int_0^t s(u) du + W_t.$$

We consider the likelihood ratio

$$\frac{d\mu_w}{d\mu_x} = \exp\left(-\int_0^1 s(t) dW_t - \frac{1}{2} \int_0^1 s(t)^2 dt\right)$$

and we fix a threshold level for the type 1-error :

$$\text{if : } \frac{d\mu_w}{d\mu_x}(\omega) \leq \lambda \quad \text{we reject } (H_0)$$

$$\text{if : } \frac{d\mu_w}{d\mu_x}(\omega) \geq \lambda \quad \text{we accept } (H_0).$$

Some general considerations and examples.

$$\text{If } P \ll Q, \text{ then } T(P) \ll T(Q). \quad (a)$$

Therefore, we do not lose very much if we suppose that P and Q are probabilities.

In the case where Q is a probability, we can have an expression of $\frac{dT(P)}{dT(Q)}$ as conditional mathematical expectation.

Remark : From (a) we see that, if there exists a probability Q such that

$$P \ll Q \text{ and } T(Q) = P, \text{ then } T(P) \ll P.$$

The converse is true if moreover $\frac{dT(P)}{dP} > 0$. (The measures are equivalent). Therefore the following properties are equivalent :

$$(i) : T(P) \sim P,$$

$$(ii) : \exists Q \sim P \text{ such that } T(Q) = P.$$

Let us now consider an example which allows us to guess the situation in infinite dimensional space.

Let $\Omega = \mathbb{R}^n$ and $P = \gamma_n$ the canonical Gaussian measure with density :

$$\frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\|x\|^2}{2}\right)$$

and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism, then

$$\begin{aligned} \int_{\mathbb{R}^n} f(y) T(\gamma_n)(dy) &= \int_{\mathbb{R}^n} f(Tx) \gamma_n(dx) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx) \exp\left(-\frac{\|x\|^2}{2}\right) dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx) \exp\left(-\frac{1}{2} \|T^{-1}Tx\|^2\right) dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{f(y)}{|\det T'(T^{-1}y)|} \exp\left(\frac{1}{2} \|y\|^2 - \frac{1}{2} \|T^{-1}y\|^2\right) \exp\left(-\frac{1}{2} \|y\|^2\right) dy. \end{aligned}$$

Therefore :

$$\begin{aligned} \frac{dT(\gamma_n)}{d\gamma_n}(y) &= \frac{1}{|\det T'(T^{-1}y)|} \exp \left(\frac{1}{2} \|y\|^2 - \frac{1}{2} \|T^{-1}y\|^2 \right) \\ &= |\det (T^{-1})'(y)| \exp \left(\frac{1}{2} \|y\|^2 - \frac{1}{2} \|T^{-1}y\|^2 \right). \end{aligned}$$

Now if we write :

$$T^{-1} = (I + S) \text{ with } S \text{ self adjoint,}$$

then :

$$(T^{-1})'(y) = I + S'(y)$$

and we obtain :

$$\frac{d(I + S)^{-1}(\gamma_n)}{d\gamma_n}(y) = |\det (I + S'(y))| \exp \{ -(Sy, y)_{\mathbb{R}^n} - \frac{1}{2} \|S(y)\|^2 \}. \quad (B)$$

This can be written as :

$$|\det (I + S'(y))| \exp (-\text{Trace } S'(y)) \exp \left\{ -(Sy, y)_{\mathbb{R}^n} + \text{Trace } S'(y) - \frac{1}{2} \|S(y)\|^2 \right\},$$

where $|\det (I + S'(y))| \exp (-\text{Trace } S'(y))$ is the Carleman determinant.

General remark : If $T = Id(\Omega)$, it is clear that $TP = P$ for every P . The idea is to perturb the identity operator.

The problem is :

“what does the word *perturbation* mean ?”

CHAPTER ONE

Anticipative stochastic integral

1 - Gaussian measures on Banach spaces

Let E be a (real) separable Banach space, E' its dual. A (Borelian) probability μ on E is said to be “**Gaussian centered**” if for every $x' \in E'$, $\langle \bullet, x' \rangle_{E, E'}$ is a Gaussian centered (real) variable (eventually degenerated) under μ . All what we shall say is true whatever be the dimension of E (finite or infinite).

If $x' \in E'$ we define $A : E' \rightarrow E$ by

$$Ax' = \int_E \langle x, x' \rangle_{E, E'} x \, d\mu(x),$$

(Bochner integral of a vector function). It is the **barycenter** of the measure $\langle \bullet, x' \rangle d\mu$.

A is injective if $\text{Supp } \mu = E$.

Let $x \in A(E')$ so $x = A(u')$ and let $y \in A(E')$ so $y = A(v')$, we shall put on $A(E') \subset E$ the following scalar product :

$$(x, y) \rightsquigarrow (x, y)_\mu := \int_E \langle u', z \rangle \langle v', z \rangle \, d\mu(z)$$

(it does not depend on u' and v').

$A : E' \rightarrow E$ is continuous. (Since $\int_E \|x\|^2 d\mu(x) < \infty$ by Fernique's theorem).

Therefore, if i denotes the canonical injection of $A(E')$ into E :

$$i : (A(E')), \|\bullet\|_\mu \rightarrow (E, \|\bullet\|) \text{ is continuous.}$$

Actually :

$$\begin{aligned} \|Ax'\|_E &= \sup_{\|y'\| \leq 1} \left| \int_E \langle x', x \rangle \langle y', x \rangle \, d\mu(x) \right| \\ &\leq \sup_{\|y'\| \leq 1} \left(\int_E |\langle x', x \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_E |\langle y', x \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \\ &\leq \left(\int_E |\langle x', x \rangle|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int_E \|x\|^2 d\mu(x) \right)^{\frac{1}{2}}; \end{aligned}$$

hence,

$$\|Ax'\|_E \leq C \|Ax'\|_\mu \quad (\text{where } C \text{ is a constant}).$$

Let H_μ be the completion of $A(E')$ with respect to $\|\cdot\|_\mu$. We have $\hat{i}: H_\mu \rightarrow E$. I say that $\hat{i}$ is injective (it will allow us to consider H_μ as a subspace of E).

H_μ is called the “*reproducing kernel Hilbert space*” (r.k.H.s.) of μ .

Example 1 : Finite dimension

$$E = \mathbb{R}^n, \quad \text{Supp } \mu = \mathbb{R}^n :$$

$$Ax' = \int_E \langle x, x' \rangle x d\mu(x),$$

or :

$$\langle Ax', y' \rangle = \int_E \langle x, x' \rangle \langle x, y' \rangle d\mu(x).$$

A is the covariance, it is invertible and

$$(x, y)_\mu = \int_E \langle A^{-1}x, z \rangle \langle A^{-1}y, z \rangle d\mu(z) = \langle x, A^{-1}y \rangle,$$

and therefore :

$$H_\mu = \mathbb{R}^n.$$

Example 2 : Brownian motion, Wiener space.

Let $T > 0$ and $\Omega = E = \mathcal{C}([0, T], \mathbb{R})$ be the space of real continuous functions on $[0, T]$.

There exists a unique centered measure μ such that :

- a) the support of μ is $\mathcal{C}_0([0, T], \mathbb{R})$, the space of the continuous functions vanishing at 0,
- b) $\forall t \in [0, T] : \omega \rightsquigarrow \omega_t$ has the variance t ,
- c) let $0 \leq t_1 < t_2 < \dots < t_n \leq T$, then : $\omega_{t_1}, \omega_{t_2} - \omega_{t_1}, \dots, \omega_{t_n} - \omega_{t_{n-1}}$ are independent.

We shall call μ the Wiener measure on $\mathcal{C}([0, T], \mathbb{R})$; then E' is the space of signed measures ν on $[0, T]$. We shall also denote :

$$\omega_t = B(t, \omega)$$

and call $t \rightsquigarrow B(t, \cdot)$: the “*Brownian motion*” on $[0, T]$.

For $\nu_1, \nu_2 \in E'$ let :

$$\begin{aligned} B(\nu_1, \nu_2) &= E [\langle \nu_1, B \rangle \langle \nu_2, B \rangle] \\ &= \int_{\Omega} \langle \nu_1, \omega \rangle \langle \nu_2, \omega \rangle d\mu(\omega). \end{aligned}$$

We have for $\nu \in E'$

$$\langle \nu, B \rangle = \int_{[0, T]} B(t, \omega) d\nu(t) = \int_0^T \nu([t, T]) dB(t) \text{ (stochastic integral).}$$

This fact can be verified as follows :

- it is true for $\nu = \delta_s$ (by definition of Brownian motion) ,
- by linearity this remains true if $\nu = \sum \alpha_i \delta_{t_i}$,
- then we apply a continuity argument.

Therefore

$$B(\nu_1, \nu_2) = \int_{[0, T]} \nu_1([t, T]) \nu_2([t, T]) dt.$$

Now let ν_1 be a measure on $[0, T]$. We want to find the barycenter m_{ν_1} of the random variable on $\Omega : \omega \rightsquigarrow \langle \omega, \nu_1 \rangle$. (m_{ν_1} is an element of $\Omega = \mathcal{C}([0, T])$). It is defined by

$$\nu \rightsquigarrow \langle m_{\nu_1}, \nu \rangle = \int_{[0, T]} m_{\nu_1}(t) \nu(dt) = B(\nu, \nu_1) = \int_{[0, T]} \nu_1([t, T]) \nu([t, T]) dt.$$

By the generalized integration by parts this is equal to :

$$\int_{[0, T]} J(\nu_1)(t) d\nu(t)$$

where

$$J(\nu_1)(t) = \int_0^t \nu_1([u, T]) du.$$

$J(\nu_1)$ is then absolutely continuous. On the space

$$\left\{ J(\nu_1), \nu_1 \in \mathcal{M}([0, T]) \right\}$$

we put the norm

$$J(\nu_1) \rightsquigarrow \int_0^T \nu_1([t, T])^2 dt.$$

Its completion is the space of functions from $[0, T]$ into $\mathbb{R}$ absolutely continuous, null at zero, whose derivative belongs to $L^2([0, T], dt)$. It is the Cameron-Martin space.

Then the Cameron-Martin space is the reproducing kernel Hilbert space of the Wiener measure μ .

Definition : We call an “*abstract Wiener space*” a triple (H, E, μ) where :

- E is a separable Banach space, and μ is a centered Gaussian measure on E , whose topological support is E .
- H is the r.k.H.s. associated to μ .

Actually H is dense in E . This can be proven as follows :

Let $i : H \rightarrow E$ be the canonical injection and $i^* : E' \rightarrow H$ its transpose (we identify H to its dual).

Suppose that $\langle x', i(x) \rangle_{E, E'} = 0$ for every $x \in H$. This is equivalent in saying that :

$$(x \mid i^*(x'))_H = 0, \text{ for every } x \in H.$$

Therefore

$$i^*(x') = 0.$$

This means that

$$\|i^*(x')\|_H^2 = \int_E |\langle x', y \rangle_{E, E'}|^2 d\mu(y) = 0.$$

Therefore

$$\langle x', y \rangle = 0 \text{ almost surely,}$$

so this holds for all $y \in E$ since $\text{Supp } \mu = E$ and x' is continuous.

The transpose i^* from $i : H \rightarrow E$ is therefore injective and dense and we have :

$$E' \xrightarrow{i^*} H \xrightarrow{i} E \quad (i \text{ is the canonical injection}).$$

Every $x' \in E'$, defines a Gaussian centered random variable on E' , whose variance is

$$\|i^*(x')\|_H^2.$$

Now we give without proof some properties of an abstract Wiener space :

- 1) H is separable, as a Hilbert space. Therefore it is a borelian subset of E ,
- 2) $\mu(H) = 0$ or 1 and $\mu(H) = 0 \Leftrightarrow \dim H = +\infty$ (therefore $\mu(H) = 1 \Leftrightarrow \dim H < \infty$),
- 3) H is the intersection of the family of measurable subspaces of E , whose probability is equal to one,
- 4) the canonical injection $i : H \rightarrow E$ is compact,
- 5) for every Hilbert space K and $u : E \rightarrow K$ linear continuous, $u \circ i : H \rightarrow K$ is Hilbert-Schmidt,
- 6) for every Hilbert space K and $v : K \rightarrow E'$ linear continuous, $i^* \circ v : K \rightarrow H$ is Hilbert-Schmidt.

As a consequence of 5) and 6) we have :

- 7) let K_1, K_2 two Hilbert spaces ; $u_1 : K_1 \rightarrow E'$ and $u_2 : E \rightarrow K_2$ linear continuous then

$$K_1 \xrightarrow{u_1} E' \xrightarrow{i^*} H \xrightarrow{i} E \xrightarrow{u_2} K_2,$$

the composition $u_2 \circ i \circ i^* \circ u_1$ is nuclear (i.e. it possesses a trace).

2 - L^2 -functionals on an abstract Wiener space

Let (H, E, μ) be an abstract Wiener space.

Suppose $(e_j)_{j \geq 1}$ is a sequence of elements of E' such that $(i^*(e_j))_{j \geq 1}$ is an orthonormal basis in H . A function $f : E \rightarrow \mathbb{R}$ is said to be a polynomial in the (e_j) if there exists an integer n and a polynomial function P on $\mathbb{R}^n$ such that

$$f(x) = P(e_1(x), \dots, e_n(x)), \quad \forall x \in E.$$

We denote $\deg f \equiv \deg P$ (P is not defined uniquely but the degree of f is independent of the choice of P).

We denote by $\mathcal{P}((e_j))$ the set of polynomials and by $\mathcal{P}^n((e_j))$ the set of polynomials of degree $\leq n$. It is easy to see that $\mathcal{P}((e_j))$ is contained in each $L^p(E, \mu)$ $0 \leq p < \infty$ (but clearly not in $L^\infty(E, \mu)$). Moreover, $\mathcal{P}((e_j))$ is dense in $L^p(E, \mu)$ for these p . Therefore, $\overline{\mathcal{P}((e_j))}_{L^p}$ is independent of the chosen orthonormal family (e_j) . The same is true for each $\mathcal{P}^n((e_j))$.

Example : If $n = 1$, $\mathcal{P}^1((e_j))$ is the family of affine continuous functions : an element of $\mathcal{P}^1((e_j))$ is a linear continuous function on E plus a constant.

We have :

$$\overline{\mathcal{P}^1}_{L^2(E, \mu)} \equiv H \oplus \mathbb{R} \quad (\text{see infra}).$$

We call $\overline{\mathcal{P}^n}_{L^2}$ the set of **generalized** polynomials of degree at most n ; $\overline{\mathcal{P}^n}_{L^2}$ is a Hilbert space.

Let us now introduce the “**Wiener chaos decomposition**” (or “**Wiener-Itô decomposition**”). Let $\mathcal{C}_0 = \overline{\mathcal{P}^0}_{L^2}$ the vector space of (μ -equivalence classes of) constants. We define $\mathcal{C}_n$ inductively as follows :

$\mathcal{C}_n$ is the orthogonal complement in $\overline{\mathcal{P}^n}_{L^2}$ of $(\mathcal{C}_0 \oplus \dots \oplus \mathcal{C}_{n-1})$.

(In other words $\mathcal{C}_n$ is the set of generalized polynomials of degree n , orthogonal to all generalized polynomials of degree less than n).

It is clear that for every n :

$$\overline{\mathcal{P}^n}_{L^2} = \mathcal{C}_0 \oplus \dots \oplus \mathcal{C}_n$$

and moreover

$$L^2(E, \mu) = \bigoplus_{n=0}^{\infty} \mathcal{C}_n.$$

The $\mathcal{C}_n$ are called the “ **n th chaos**” (or “**chaos of order n** ”). $\mathcal{C}_1$ is isomorphic to H . We have a description of elements of $\mathcal{C}_n$ in term of Hermite polynomials.

We recall that the Hermite polynomials in one variable are defined by :

$$H_n(t) = \frac{(-1)^n}{n!} \exp\left\{\frac{t^2}{2}\right\} \frac{d^n}{dt^n} \left(\exp\left\{-\frac{t^2}{2}\right\}\right), \quad n \in \mathbb{N}.$$

Then they satisfy :

- $\sum_{n=0}^{\infty} \lambda^n H_n(t) = \exp\left\{-\frac{\lambda^2}{2} + \lambda t\right\}$
- $\frac{d}{dt} H_n(t) = H_{n-1}(t)$
- $\int_{\mathbb{R}} H_m(t) H_n(t) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} dt = \frac{1}{n!} \delta_{nm}.$

Let $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $|\alpha| := \sum_{i=1}^{\infty} \alpha_i < \infty$. We set $\alpha! := \prod_{i=1}^{\infty} \alpha_i!$.

Now let $(e_n)_{n \geq 1}$ be a sequence of elements of E' which is an orthonormal basis in H . If $\alpha \in \mathbb{N}^{\mathbb{N}}$ let

$$H_\alpha(x) := \prod_{i=1}^{\infty} H_{\alpha_i}(e_i(x))$$

(This product is well defined). Then :

$\{\sqrt{\alpha!} H_\alpha(x), \alpha \in \mathbb{N}^{\mathbb{N}} \text{ and } |\alpha| < +\infty\}$ is an orthonormal basis in $L^2(E, \mu)$ and :

$\{\sqrt{\alpha!} H_\alpha(x), |\alpha| = n\}$ is an orthonormal basis in $\mathcal{C}_n$.

In the case of the Wiener measure associated to Brownian motion, we have the following characterization of $\mathcal{C}_n$ in terms of multiple stochastic integrals :

$F : \mathcal{C}([0, T], \mathbb{R}) \rightarrow \mathbb{R}$ belongs to $L^2(P)$ where P is the Wiener measure if and only if for each n there exists $f_n \in L^2(\Delta_n, dt)$ where $\Delta_n = \{t \in \mathbb{R}^n, 0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T\}$ such that

$$F = \sum_n \int_{\Delta_n} f_n(t_1, \dots, t_n) dB(t_1) \dots dB(t_n) = \sum_n F_n.$$

Here

$$F_0 = \mathbb{E}(F) \in \mathcal{C}_0 \text{ and } F_n \in \mathcal{C}_n.$$

3 - Measurable linear functionals and linear measurable operators

Let (H, E, μ) be an abstract Wiener space. Without loss of generality, we shall identify H as a subspace of E (i.e., $i(x) = x$).

A linear mapping $f : E \rightarrow \mathbb{R}$ is said to be a “**linear measurable functional**” if there exists a sequence of linear continuous functionals on E , converging to f , μ -almost surely.

If $x \in H$, it defines a linear measurable functional $\tilde{x}(\cdot)$. Actually, if x_n is a sequence of elements of $E' \subset H$ such that $x_n \rightarrow x$ in H , then $x_n(\cdot)$ converges to the random variable $\tilde{x}$ defined by x , in $L^2(E, \mu)$. Therefore, there exists a subsequence converging almost surely to $\tilde{x}$. Moreover,

$$\int_E |\tilde{x}(x)|^2 d\mu(x) < \infty.$$

The converse is true, shown by the following proposition .

If $h \in H$, the random variable $\tilde{h}$ on E will be denoted by

$$x \rightsquigarrow (x, h)_H.$$

Proposition : *Every linear measurable functional, f , has a restriction to H which is continuous (for the Hilbertian topology). If we denote by f_0 this restriction we have*

$$\|f\|_{L^2(E, \mu)} = \|f_0\|_H.$$

The converse is true.

Proof :

We have already noticed that the converse is true. Let $(x_n) \subset E' \subset H$ such that

$$x_n(x) \longrightarrow f(x) \quad \forall x \in A, \text{ where } \mu(A) = 1.$$

Take $\mathcal{E}$ the linear subspace generated by A , we see that the above convergence holds for all $x \in \mathcal{E}$. Since $\mu(\mathcal{E}) = 1$, then $H \subset \mathcal{E}$ and therefore

$$x_n(x) \longrightarrow f(x), \quad \forall x \in H.$$

Therefore the restriction of f to H is uniquely defined.

Now,

$$\int_E \exp \{i(x_n - x_m)(x)\} \mu(dx) = \exp \left\{ -\frac{1}{2} \|x_n - x_m\|_H^2 \right\} \longrightarrow 1.$$

Therefore, (x_n) converges in H , and

$$\int_E |x_n(x) - x_m(x)|^2 \mu(dx) = \|x_n - x_m\|_H^2 \xrightarrow{m, n \rightarrow \infty} 0.$$

Therefore $(x_n(\cdot))$ converges in $L^2(\mu)$. The limit is equal to f almost surely, as we can see immediately.

— Q.E.D. —

Now let K be a Hilbert space. As before we define a linear measurable function from E to K , as the almost sure limit of a sequence of linear continuous functions from E to K .

And, as before, if A is a linear measurable function from E into K , its restriction to H is well defined and continuous from H to K .

Let us remark that if A is a linear measurable function from E to K , we can define its transpose as a linear function from K to H since, for every $\varphi \in K$, $x \rightsquigarrow \langle Ax, \varphi \rangle_K$ is a linear measurable functional on E therefore defined by an element of H . We have

$$\begin{aligned} \langle Ax, \varphi \rangle_K &= (\widetilde{A^* \varphi})(x), \quad \text{almost surely} \\ &= (x, A^* \varphi)_H \end{aligned}$$

where A^* is the conjugate of the restriction of A to H .

Now we can prove the following result :

THEOREM : *If A is a linear measurable function from E to K such that $\int \|Ax\|_K^2 d\mu(x) < \infty$, then its restriction to H is a Hilbert-Schmidt mapping B from H to K . Conversely if B is a Hilbert-Schmidt mapping from H to K , (we shall note $B \in \mathcal{L}^2(H, K)$ or $B \in \mathcal{L}_2(H, K)$), it possesses a linear measurable continuation on E , denoted by A .*

Moreover, we have :

$$\int_E \|Ax\|_K^2 d\mu(x) = \|B\|_{H-S}^2.$$

Proof :

Let (φ_j) be an orthonormal basis of K .

We have :

$$\|Ax\|_K^2 = \sum_j (Ax, \varphi_j)_K^2 \stackrel{a.s.}{=} \sum_j (x, A^* \varphi_j)_H^2.$$

If we integrate term by term these equalities, we obtain :

$$\begin{aligned} \int_E \|Ax\|_K^2 d\mu(x) &= \sum_j \int_E (x, A^* \varphi_j)_H^2 d\mu(x) \\ &= \sum_j \|A^* \varphi_j\|_H^2 = \sum_j \|B^* \varphi_j\|_H^2 = \|B^*\|_{H-S}^2. \end{aligned}$$

Conversely let $B \in \mathcal{L}_2(H, K)$. We have for $x \in H$:

$$\begin{aligned} Bx &= \sum_j (Bx, \varphi_j)_K \varphi_j \\ &= \sum_j (x, B^* \varphi_j)_H \varphi_j. \end{aligned}$$

Now each term in the right-hand member possesses a linear measurable continuation to E , and the series converges in $\mathcal{L}_2(E, \mu, K)$.

We have then defined a linear measurable extension of A to E .

— Q.E.D. —

4 - Derivatives of functionals on a Wiener space

Let (E, H, μ) be an abstract Wiener space and let K be another Hilbert space. Let $f : E \rightarrow K$ be a function.

We say that f possesses a Fréchet derivative in the direction of H , at the point $x_0 \in E$ if there exists an element denoted $f'(x_0)$ or $Df(x_0)$ or $\nabla f(x_0) \in \mathcal{L}(H, K)$ such that $f(x_0 + h) - f(x_0) = f'(x_0) \bullet h + o(\|h\|_H)$, $\forall h \in H$.

Inductively we can define derivatives of all orders.

Example : Let $x_0 \in H \setminus i^*(E')$ and let f be a measurable continuation of $h \rightsquigarrow (x_0, h)_H$ to E . (f is not continuous).

Then f is derivable at every x , and $f'(x_0) \in H$.

This example shows that a discontinuous function may have Fréchet derivatives in the direction of H .

Definition 1 : Let us denote by $\mathcal{C}^{2,1}(E, K)$ the set of functions $f : E \rightarrow K$ possessing the following properties :

- f possesses H -derivatives at every point $x \in E$ and $f'(x)$ is Hilbert-Schmidt for every x ,
- f and f' are continuous from H to K and to $\mathcal{L}_2(H, K)$ respectively,

$$- \|f\|_{2,1}^2 := \int_E \left[\|f(x)\|_K^2 + \|f'(x)\|_{\mathcal{L}_2(H,K)}^2 \right] \mu(dx) < \infty.$$

Then $\mathcal{C}^{2,1}(E, K)$ is a vector space and $\|\cdot\|_{2,1}$ is a Hilbertian norm on this space.

Definition 2 : Let $\mathbb{D}^{2,1}(E, K)$ be the completion of $\mathcal{C}^{2,1}(E, K)$ for the preceding norm; $\mathbb{D}^{2,1}(E, K)$ is then a Hilbert space.

Clearly the elements of $\mathbb{D}^{2,1}(E, K)$ are μ -equivalence classes of functions.

Convention : Often we shall write $\mathbb{D}^{2,1}(K)$ instead of $\mathbb{D}^{2,1}(E, K)$. In the same manner we shall write $\mathbb{D}^{2,1}$ instead of $\mathbb{D}^{2,1}(E, \mathbb{R})$ or $\mathbb{D}^{2,1}(\mathbb{R})$.

Now the map $f \rightsquigarrow f'$ from $\mathcal{C}^{2,1}(E, K)$ into $L^2(E, \mu, \mathcal{L}_2(H, K))$ is clearly continuous ; therefore it possesses a unique continuous extension from $\mathbb{D}^{2,1}(H, K)$ into $L^2(E, \mu, \mathcal{L}_2(H, K))$. This extension is again denoted by f' , or Df , or ∇f .

Example 1 : Let f be a polynomial function on E , with values in $\mathbb{R}$:

$$f(x) = P(\langle f_1, x \rangle_{E', E}, \dots, \langle f_n, x \rangle_{E', E}), \quad f_1, \dots, f_n \in E'.$$

Then $f \in C^{2,1}$ and

$$f'(x) = \sum_{j=1}^n \frac{\partial P}{\partial y_j} (\langle f_1, x \rangle_{E', E}, \dots, \langle f_n, x \rangle_{E', E}) i^*(f_j).$$

The same result is true if P is a $C^1(\mathbb{R}^n)$ -function such that P and the partial derivatives $\frac{\partial P}{\partial y_j}$ have polynomial growth.

In the same manner if f is defined (μ -almost everywhere) as

$$f(\bullet) = P(\tilde{h}_1(\bullet), \dots, \tilde{h}_n(\bullet)), \quad h_j \in H$$

with P a polynomial function (or a $C^1(\mathbb{R}^n)$ -function with polynomial growth together with its derivatives),

$$\nabla f = \sum_{j=1}^n \frac{\partial P}{\partial y_j} (\tilde{h}_1(\bullet), \dots, \tilde{h}_n(\bullet)) h_j.$$

Example 2 : Let $\mu = \gamma_n$ the canonical Gaussian measure on $\mathbb{R}^n$, $\mathbb{D}^{2,1}$ is the Sobolev space $W^{2,1}(\gamma_n)$ of the distributions in $\mathbb{R}^n$ such that :

- $f \in L^2(\mathbb{R}^n, \gamma_n)$,
- the distribution derivatives of f belong to $L^2(\mathbb{R}^n, \gamma_n)$. The norm of $\mathbb{D}^{2,1}$ is the usual Hilbertian norm :

$$f \rightsquigarrow \left(\int_{\mathbb{R}^n} [|f(x)|^2 + \sum_{j=1}^n \left| \frac{\partial f}{\partial x_j}(x) \right|^2] d\gamma_n(x) \right)^{\frac{1}{2}}.$$

Example 3 : If f is a polynomial function with values in K :

$$f(x) = \sum_{j=1}^m P_j(\langle f_1, x \rangle_{E', E}, \dots, \langle f_n, x \rangle_{E', E}) k_j$$

$$(k_j \in K, \quad f_1, \dots, f_n \in E').$$

$$\nabla f(x) = \sum_j \sum_i \frac{\partial P_j}{\partial y_i} (\langle f_1, x \rangle_{E', E}, \dots, \langle f_n, x \rangle_{E', E}) f_i \otimes k_j.$$

(Analogous assertion for generalized polynomials, or “moderate” regular functions P_j).

Example 4 : Characterization of the elements of $\mathbb{D}^{2,1}$ in the case of the Wiener measure.

If $E = C_0([0, T], \mathbb{R})$ and μ is the Wiener measure, we have seen that an element of $L^2(\mu)$ can be written as a series

$$F = \sum_{n=0}^{\infty} \sqrt{n!} \int_{\Delta_n} f_n(t_1, t_2, \dots, t_n) dB_{t_1}, \dots, dB_{t_n}$$

with

$$\sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\Delta_n)}^2 < \infty.$$

Then F belongs to $\mathbb{D}^{2,1}$ if and only if

$$\sum_{n=1}^{\infty} nn! \|f_n\|_{L^2(\Delta_n)}^2 < \infty$$

and in this case

$$\nabla F = \sum_{n=1}^{\infty} nJ(I_{n-1}(f_n^t))$$

where f_n^t is the function defined on $\Delta_{n-1}^t = \{0 \leq t_1 < t_2 < \dots < t_{n-1} < t\}$ by

$$f_n^t(t_1, t_2, \dots, t_{n-1}) = f_n^{SYM}(t_1, t_2, \dots, t_{n-1}, t),$$

f_n^{SYM} being the symetrisation of f_n .

The formula needs an explanation :

In the right member

$$(t, \omega) \rightsquigarrow I_{n-1}(f_n^t)(\omega) = g(t, \omega)$$

belongs to

$$L^2([0, T] \times \Omega, dt \otimes dP),$$

therefore for almost ω ,

$$t \rightsquigarrow g(t, \omega) \text{ is a } L^2([0, T], dt) \text{ function.}$$

$J(I_{n-1}(f_n^t))(\omega)$ is the indefinite integral null at zero of $I_{n-1}(f_n^t)(\omega)$:

$$J(I_{n-1}(f_n^t)) = \int_0^t I_{n-1}(f_n^s) ds.$$

Therefore $\nabla F(\omega)$ is an element of the Cameron-Martin space.

We now give several useful properties of $\mathbb{D}^{2,1}(E, K)$:

- The set of polynomial functions on E , with values in K is dense in $\mathbb{D}^{2,1}(K)$.
- Therefore the algebraic sum of chaos $\sum \mathcal{C}_n$ is dense in $\mathbb{D}^{2,1}$.
- The set of **smooth functions** on E is dense in $\mathcal{C}^{2,1}$ (a function is said to be “**smooth**” if it has the form :

$$x \rightsquigarrow f(\langle f_1, x \rangle_{E', E}, \dots, \langle f_n, x \rangle_{E', E})$$

with f belonging to $\mathcal{C}_b^\infty(\mathbb{R}^n)$; f and its derivatives are bounded).

- Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function in $\mathcal{C}_b^1(\mathbb{R}^n)$ and let $F^1, \dots, F^n \in \mathbb{D}^{2,1}$. Then $\varphi(F^1, \dots, F^n)$ is in $\mathbb{D}^{2,1}$ and

$$\nabla(\varphi(F^1, \dots, F^n)) = \sum_{i=1}^n \frac{\partial \varphi}{\partial y_i}(F^1, \dots, F^n) \nabla F^i.$$

This result is false if the above hypothesis is not satisfied. For instance on $\mathbb{R}$,

$$f = g = e^x \in \mathbb{D}^{2,1}, \text{ but } f \circ g \notin L^2(\mathbb{R}^n, \gamma_n).$$

Remark : The operator ∇ , called the “**stochastic**” gradient, or “**stochastic**” derivative, is very close to the ordinary gradient as we can see. The usual gradient at the point x_0 is an element of E' (if the function takes its values in $\mathbb{R}$). The stochastic gradient is the composite of the ordinary gradient by the application i^* from E' to H .

In an analogous manner if $f : E \rightarrow K$ has an ordinary gradient, this gradient is a linear mapping of E into K ; $f' : E \rightarrow K$.

The transpose ${}^t f'$ is a linear continuous mapping from K into E' . Then the stochastic gradient is equal to $i^*({}^t f') \in \mathcal{L}(K, H)$.

In his lectures at the EIPES in 1989, D. Nualart, in the case of usual Wiener space defined the stochastic derivative of the functional of the form :

$$F = f(W_{t_1}, \dots, W_{t_n}), \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n) \quad (\text{or } f \text{ polynomial})$$

by

$$DF = \sum_{j=1}^n \frac{\partial F}{\partial y_j}(W_{t_1}, \dots, W_{t_n}) 1_{[0, t_j]}.$$

This definition is actually equivalent to ours, up to the notations.

Actually, let $h_j(t) = \int_0^t 1_{[0,t_j]}(s) ds$, h_j belongs to the Cameron-Martin space and

$$W_{t_j} = \tilde{h}_j = \langle h_j, \bullet \rangle_{C-M}$$

The stochastic derivate of F in our notations is therefore

$$\sum_{j=1}^n \frac{\partial F}{\partial y_j} (\tilde{h}_1, \dots, \tilde{h}_n) h_j.$$

There are actually equivalent since the Cameron-Martin space is isomorphic as Hilbert space to $L^2([0, T], dt)$. We shall have to consider ∇ as an operator (densely defined) from $L^2(E, \mu, K)$ into $L^2(E, \mu, \mathcal{L}_2(H, K))$. It is a closed operator, naturally not continuous.

5 - Anticipative stochastic integral

Definition : *The transpose of the operator ∇ is called the “Skorokhod integral”, or the “divergence operator”.*

The definition needs an explanation : on $L^2(E, \mu, K)$ (K : Hilbert space) we have defined the scalar product

$$(f, g) \rightsquigarrow \int_E \langle f(x), g(x) \rangle_K d\mu(x)$$

and on $L^2(E, \mu, \mathcal{L}_2(H, K))$ we have the pairing :

$$\begin{aligned} (F, G) &\rightsquigarrow \int_E \langle F(x), G(x) \rangle_{\mathcal{L}_2(H, K)} d\mu(x) \\ &= \int_E \text{Trace} (G^*(x) \circ F(x)) d\mu(x). \end{aligned}$$

Then $G \in L^2(E, \mu, \mathcal{L}_2(H, K))$ belongs to $\text{dom}(\delta)$ if and only if the linear form on $\mathbb{D}^{2,1}(K)$: $F \rightsquigarrow \int_E \langle DF, G \rangle_{\mathcal{L}_2(H, K)}(x) d\mu(x)$ is continuous for the topology induced by $L^2(E, \mu, K)$.

We denote δ the Skorokhod integral and we have by definition, for every $F \in \mathbb{D}^{2,1}(K)$,

$$\int_E \langle F, \delta G \rangle_K d\mu = \int_E \langle \nabla F, G \rangle_{\mathcal{L}_2(H, K)} d\mu \quad \text{if } \delta(G) \text{ is defined.}$$

Example 1 : Let $a \in H$, and $\varphi \in \mathbb{D}^{2,1}(K)$. Then $G := \varphi \otimes a$ is Skorokhod integrable and

$$\delta(a \otimes \varphi) = \tilde{a}(\bullet) \varphi - \langle \nabla \varphi, a \rangle.$$

In particular, if $G : E \rightarrow H$ is such that $G(x) = a, \forall x :$

$$\delta G = \tilde{a}(\bullet).$$

Example 2 : $E = \mathbb{R}^n, \mu = \gamma_n, G : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Then

$$\begin{aligned} \delta G(x) &= \langle x, G(x) \rangle_{\mathbb{R}^n} - \sum_{j=1}^n \frac{\partial G_j}{\partial x_j}(x) \\ &= \langle x, G \rangle - \operatorname{div} G(x). \end{aligned}$$

This formula can be written in another manner :

$$\delta G = \langle \bullet, G \rangle - \operatorname{Trace} (\nabla G).$$

Example 3 : If $G \in \mathbb{D}^{2,1}(E, \mu, \mathcal{L}^2(H, K))$, then it is δ -integrable, and δ is continuous from $\mathbb{D}^{2,1}(\mathcal{L}_2(H, K))$ in $L^2(E, \mu, K)$.

Example 4 : Let $F \in L^2(E, \mu, H)$ such that for every $h \in H : \nabla (\langle F, h \rangle_H)$ exists. Then for every linear continuous operator $A : H \rightarrow H$ with **finite rank**, $A(F)$ is Skorokhod integrable.

More precisely, if $A = \sum_{j=1}^n \langle \bullet, a_j \rangle_H e_j$ (with a_j and e_j in H , (e_j) being orthonormal)

we have :

$$\begin{aligned} A(F) &= \sum_{j=1}^n \langle F, a_j \rangle_H e_j \\ \delta(A(F)) &= \sum_{j=1}^n \left[\langle F, a_j \rangle \tilde{e}_j - \nabla_{e_j} (\langle F, a_j \rangle) \right]. \end{aligned}$$

(see example 1).

This can be written in another manner :

Let A^* be the transpose of A : $A^* = \sum_{j=1}^n \langle \bullet, e_j \rangle_H a_j$ and let $\tilde{A}^*$ defined as :

$$\tilde{A}^* = \sum_{j=1}^n a_j \tilde{e}_j .$$

Then

$$\delta(A(F)) = \langle F, \tilde{A}^* \rangle_H - \sum_{j=1}^n \nabla_{e_j} (\langle F, a_j \rangle) .$$

If we now suppose that DF exists, we have :

$$\sum_{j=1}^n \nabla_{e_j} (\langle F, a_j \rangle) = \text{Trace} (A \circ DF) .$$

Therefore, we have :

$$\delta(A(F)) = \langle F(\bullet), \tilde{A}^*(\bullet) \rangle_H - \text{Trace} (A \circ DF) .$$

Example 5 : The Skorokhod integral coincides with the ordinary Itô-Integral for adapted processes (see the above mentioned Nualart's Lecture Notes for a precise statement of this fact).

Now we give some properties of the Skorokhod integral :

a) Let $A : K \rightarrow K'$ be a linear continuous operator (K and K' Hilbert spaces) and let $F \in L^2(E, \mu, \mathcal{L}_2(H, K))$. If F is Skorokhod-integrable so is $A \circ F$ and we have

$$\delta(A \circ F) = A(\delta F) .$$

As a consequence we have :

- Let $F \in L^2(E, \mu, \mathcal{L}_2(H, K))$ such that $\delta(F)$ exists, then for every k in K we have $\langle \delta(F), k \rangle = \delta(F^*(k))$.

- Let $F \in L^2\left(E, \mu, \mathcal{L}_2(H, \mathcal{L}_2(H, K))\right)$ such that $\delta(F)$ exists, then

$$\text{for every } h \in H, \delta\left(\overset{\vee}{F}(\bullet)(h)\right) \text{ exists}$$

and

$$\delta(F)(h) = \delta\left(\overset{\vee}{F}(\bullet)(h)\right).$$

If $F \in \mathcal{L}^2(H, \mathcal{L}_2(H, K))$, $\overset{\vee}{F}$ denotes the operator of $\mathcal{L}^2(H, \mathcal{L}_2(H, K))$ such that :

$$\overset{\vee}{F}(h)(h') = F(h')(h), \quad h, h' \in H.$$

b) Let $\varphi \in \mathbb{D}^{2,1}$, $F \in \mathcal{L}^2(E, \mu, H)$ such that F is Skorokhod integrable. Suppose that $\varphi F \in L^2(E, \mu, H)$ and that $\delta(F)\varphi - \langle F, D\varphi \rangle_H$ belongs to $L^2(E, \mu)$, then φF is Skorokhod integrable and

$$\delta(\varphi F) = \delta(F)\varphi - \langle F, D\varphi \rangle_H.$$

c) Let $A_n : H \rightarrow H$ a sequence of linear continuous operators such that $A_n \rightarrow Id_H$ in the simple convergence.

Let $F \in \mathbb{D}^{2,1}(\mathcal{L}_2(H, K))$, then $\delta(F \bullet A_n) \rightarrow \delta(F)$ in $L^2(E, \mu, K)$. In particular, if (e_n) is an orthonormal basis of H , the sequence

$$\left(\sum_{i=1}^n \tilde{e}_i F(e_i) - \nabla_{e_i} F(e_i) \right)$$

converges to $\delta(F)$.

d) Let F, G in $\mathbb{D}^{2,1}(H)$ we have :

$$\begin{aligned} \mathbb{E}(\delta(F)\delta(G)) &= \mathbb{E}\{\langle F, G \rangle_H\} + \mathbb{E}\{\langle DF, (DG)^* \rangle_{\mathcal{L}_2(H, H)}\} \\ &= \mathbb{E}\{\langle F, G \rangle_H\} + \mathbb{E}\{\text{Trace } DG(\bullet) \circ DF(\bullet)\}. \end{aligned}$$

More generally, if F and G belong to $\mathbb{D}^{2,1}(\mathcal{L}_2(H, K))$ we have :

$$\mathbb{E}\{\langle \delta F, \delta G \rangle_K\} = \mathbb{E}\{\langle F, G \rangle_{\mathcal{L}_2(H, K)}\} + \mathbb{E}\{\langle DF, \overset{\vee}{DG} \rangle_{\mathcal{L}_2(H, \mathcal{L}_2(H, K))}\}.$$

e) The operator δ , as an operator densely defined from $L^2(E, \mu, \mathcal{L}_2(H, K))$ into $L^2(\Omega, \mu, K)$ is **closed**.

We now briefly introduce the Ogawa integral.

Let $P : H \rightarrow H$ be an orthogonal projector with finite rank : $P(h) = \sum_{j=1}^n \langle h, e_j \rangle_H e_j$.

We denote $\tilde{P}$ the random variable with values in H :

$$\tilde{P}(\cdot) := \sum_{j=1}^n \tilde{e}_j(\cdot) e_j.$$

Now let $F \in L^0(E, \mu, H)$ be a random variable with values in H . We shall say that F is “*Ogawa integrable*”, if there exists $G \in L^0(E, \mu)$ such that, for every increasing sequence (P_n) of orthogonal projectors converging simply to Id_H , the sequence of real random variables $(\langle F, \tilde{P}_n \rangle_H)_n$ converges to G in probability.

We shall denote by $\overset{\circ}{\delta}(F)$ the Ogawa integral G of F .

If $F \in L^2(E, \mu, H)$ is such that, for every $a \in H$:

$$\langle F, a \rangle_H \tilde{a}(\cdot) \text{ belongs to } L^2(E, \mu),$$

we shall say that F is “*2-Ogawa integrable*” when there exists $G \in L^2(E, \mu)$ such that

$$\langle F, \tilde{P}_n \rangle_H \longrightarrow G \text{ in quadratic mean.}$$

(The P_n being as above).

Example : $(E, \mu) = (\mathbb{R}^n, \gamma_n)$. The Ogawa integral is equal to $\langle \cdot, F(\cdot) \rangle_{\mathbb{R}^n}$.

In this case , we have :

$$\overset{\circ}{\delta}(F) = \delta(F) + \text{Trace} (\nabla F).$$

Remark : There exists elements of $\mathbb{D}^{2,1}(H)$ which do not possess an Ogawa integral (Rosinski).

For instance, in the case of the Brownian motion, the function : $\omega \rightsquigarrow J(B(T - \cdot)(\omega))$ where J denotes the indefinite integral null at zero, belongs to $\mathbb{D}^{2,1}(H)$ but is not Ogawa integrable.

Next we give a necessary and sufficient condition for Ogawa integrability :

Let $F \in \mathbb{D}^{2,1}(H)$; F is Ogawa integrable if and only if, for almost every x :

$$DF \in \mathcal{L}_1(H, H) \quad (\Longleftrightarrow DF \text{ is nuclear})$$

and we have :

$$\overset{\circ}{\delta}(F) = \delta(F) + \text{Trace}(DF).$$

Sketch of the proof :

Suppose $P : H \rightarrow H$ is an orthogonal projector with finite rank. We know that :

$$\delta(PF) = \langle F, \tilde{P} \rangle - \text{Trace}(D(PF)).$$

Let $P_n \uparrow Id$. We know that

$$\delta(P_n F) \longrightarrow \delta(F).$$

It is trivial that :

$$\langle F, \tilde{P}_n \rangle \longrightarrow \overset{\circ}{\delta}(F)$$

(if $\overset{\circ}{\delta}(F)$ exists) and

$$\text{Trace}(D(P_n F)) \longrightarrow \text{Trace}(DF)$$

— Q.E.D. —

6 - Extensions and remarks - Localization

Now we shall consider the case where (E, H, μ) is the Wiener space. If $F \in \mathbb{D}^{2,1}$, then ∇F is a random variable with values in the Cameron-Martin space. Therefore, if $t \in [0, T]$ we can speak of the value of $\nabla F(\omega)$ at t , denoted $\nabla_t F(\omega)$. Analogously, time derivative of $\nabla F(\omega)$ at time t (**defined for almost every t**) makes sense. We shall denote it : $\dot{\nabla}_t F(\omega)$. We have the equality :

$$\|\nabla F(\bullet)\|_{L^2(H)}^2 = \mathbb{E}(\int_0^t |\dot{\nabla}_t F(\omega)|^2 dt).$$

Lemma 1 : Let $F \in \mathbb{D}^{2,1}$. Then $1_{\{F=0\}} \dot{\nabla}_t F = 0$ almost everywhere on $[0, T] \times \Omega$.

For the proof see Nualart-Pardoux.

This results in a localization theorem : if F is null (almost everywhere) on a set, so is its derivative. The derivation is a **“local operator”**.

Definition 1 : A random variable F will be said to belong to $\mathbb{D}_{loc}^{2,1}$ if there exist

- a sequence of measurable sets of E , $E_k \uparrow E$

and

- a sequence $(F_k) \subset \mathbb{D}^{2,1}$ such that $F|_{E_k} = F_k|_{E_k}$ a.s. $\forall k \in \mathbb{N}$.

Thanks to the preceding lemma we can define the derivation operator for an element of $\mathbb{D}_{loc}^{2,1}$.

Definition 2 : Let $F \in \mathbb{D}_{loc}^{2,1}$ localized by the sequence (E_k, F_k) . DF is the unique equivalence class of $dt \times dP$ a.e equal processes such that

$$DF|_{E_k} = DF_k|_{E_k}, \quad \text{for all } k \text{ in } \mathbb{N}.$$

This generalized derivative has the usual properties of composition :

let $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ of the class C^1 ; suppose $F = (F^1, \dots, F^m)$ is a random vector whose components belong to $\mathbb{D}_{loc}^{2,1}$; then

$$\varphi(F) \in \mathbb{D}_{loc}^{2,1}$$

and

$$\nabla \varphi(F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F) \cdot DF^i.$$

In the same manner we define $(\text{Dom } \delta)_{loc}$ as follows :

$F : E \rightarrow H$ belongs to $(\text{Dom } \delta)_{loc}$ if there exists a sequence $E_k \uparrow E$, and a sequence $F_k : E \rightarrow H$ such that $F_k \in (\text{Dom } \delta)$ for every k , such that

- $F = F_k$ on E_k
- $\delta(F_k) = \delta(F_\ell)|_{F_k}$ a.s. if $k < \ell$;

we shall say that F is “**localized**” by (E_k, F_k) .

For sufficiently reasonable integrands on $(\text{Dom } \delta)$ Nualart-Pardoux have shown that δ is local.

Definition 3 : Let $F \in (\text{Dom } \delta)_{loc}$ localized by (E_k, F_k) , $\delta(F)$ is defined as the unique equivalence class on random variables on E such that

$$\delta(F)|_{E_k} := \delta(F_k)|_{E_k}, \quad \text{for all } k \text{ in } \mathbb{N}.$$

(Note that $\delta(F)$ may depend on the localizing sequence).

We shall need another notion of stochastic derivatives and Skorokhod integrals for some functions not necessarily belonging to $\mathbb{D}^{2,1}$, nor Skorokhod integrable, introduced by Buckdahn :

Let $T : E \rightarrow E$ be a measurable mapping of the form :

$$x \rightsquigarrow x + Fx \text{ where } F \in \mathbb{D}^{2,1}(H).$$

Let $\xi \in \mathbb{D}^{2,1}$ and suppose that for every sequence of smooth random variables $(\xi_n) \in \mathbb{D}^{2,1}$ converging to ξ in $\mathbb{D}^{2,1}$, the following limit exists and is independent of the approximating sequence chosen :

$$\lim_{n \rightarrow \infty} \nabla (\xi_n \circ T)$$

where the limit is taken in probability.

Let us remark that $\xi_n \circ T$ belongs to $\mathbb{D}^{2,1}$ since the ξ_n are *smooth*.

The common limit of the above sequences is denoted by $\tilde{\nabla} (\xi \circ T)$.

Lemma 2 : Suppose that $T(\mu) \ll \mu$, then the limit exists and we have, μ -almost surely :

$$\tilde{\nabla} (\xi \circ T) = (I_H + (\nabla F)^*)((\nabla \xi) \circ T) = (I_H + \nabla F)^*((\nabla \xi) \circ T)$$

(where $(\)^*$ denotes the adjoint of the bounded operator).

Moreover, if $\xi \circ T \in \mathbb{D}^{2,1}$: $\tilde{\nabla} (\xi \circ T) = \nabla(\xi \circ T)$.

Proof :

We have, *since the (ξ_n) are smooth* :

$$\nabla(\xi_n \circ T) = (I_H + \nabla F)^*((\nabla \xi_n) \circ T).$$

Moreover, $\nabla \xi_n$ converges in probability, and since $T(\mu)$ is absolutely continuous with respect to μ , $(\nabla \xi_n) \circ T$ converges in probability, so does $\nabla(\xi_n \circ T)$.

It now remains to prove that the limit does not depend upon the approximating sequence (ξ_n) .

Let $\xi_n \rightarrow \xi$ and $\eta_n \rightarrow \xi$ in $\mathbb{D}^{2,1}$. Since the operator ∇ is closed we have :

$$\lim_n \nabla(\xi_n \circ T) = \lim_n \nabla(\eta_n \circ T).$$

Therefore, $\tilde{\nabla}$ is well defined by what precedes. It is obvious that :

$$\tilde{\nabla} = \nabla \text{ if } \xi \circ T \in \mathbb{D}^{2,1}.$$

By duality, we can define a generalized Skorokhod integral of $\xi \circ T$, for $\xi \in D^{2,1}(H)$:

— Lemma 2 is proven. —

Definition : Let $(e_i)_{i \in \mathbb{N}}$ be a fixed orthonormal basis of H . We define

$$\tilde{\delta}(\xi \circ T) := \sum_i (\langle \xi \circ T, e_i \rangle_H \tilde{e}_i - \tilde{\nabla}_{e_i} (\langle \xi \circ T, e_i \rangle_H),$$

if the limit of the right member is taken in probability.

($\tilde{\nabla}_{e_i}$ denotes the generalized derivative in the e_i -direction introduced just above).

Lemma 3 : Suppose $T = I + F$ as above is such that $T(\mu) \ll \mu$. Then $\tilde{\delta}(\xi \circ T)$ exists and satisfies the following identity :

$$(\delta(\xi)) \circ T = \tilde{\delta}(\xi \circ T) + \langle \xi \circ T, F \rangle_H + \text{Trace} ((\nabla \xi) \circ T \bullet \nabla F) \quad \mu\text{-almost surely}.$$

Proof :

Let $\xi^N = \sum_{i=1}^N \langle \xi, e_i \rangle_H e_i$, then

$$\tilde{\delta}(\xi^N \circ T) = \sum_{i=1}^N \langle \xi \circ T, e_i \rangle_H \tilde{e}_i - \sum_{i=1}^N \tilde{\nabla}_{e_i} (\langle \xi \circ T, e_i \rangle_H).$$

But

$$\tilde{e}_i \circ T = \tilde{e}_i + \langle F, e_i \rangle_H,$$

therefore :

$$\begin{aligned} \delta(\xi^N \circ T) &= \sum_{i=1}^N \left\{ \langle \xi \circ T, e_i \rangle_H [\tilde{e}_i \circ T - \langle F, e_i \rangle_H] - \langle (I_H + \nabla F)^* (\nabla (\langle \xi, e_i \rangle_H)) \circ T, e_i \rangle_H \right. \\ &\quad \left. (\text{by the preceding lemma}) \right. \\ &= \sum_{i=1}^N \left\{ \langle \xi \tilde{e}_i, e_i \rangle_H \circ T - \langle \xi \circ T, e_i \rangle_H \langle F, e_i \rangle_H - \langle (I_H + \nabla F)^* (\nabla (\langle \xi, e_i \rangle_H)) \circ T, e_i \rangle_H \right. \\ &= \sum_{i=1}^N [\langle \xi, e_i \rangle_H \tilde{e}_i - \langle \nabla_{e_i} \xi, e_i \rangle_H] \circ T - \langle \xi^N \circ T, F \rangle_H - \text{Trace} (\nabla F^*, (\nabla \xi^N) \circ T). \end{aligned}$$

Now $\xi^N \longrightarrow \xi$ in $\mathbb{D}^{2,1}(H)$; then the right member of this last equality converges in $L^0(E, \mu)$. Hence the sum is convergent in $L^0(E, \mu)$ and

$$\sum_{i=1}^{\infty} \langle \xi \circ T, e_i \rangle_H \tilde{e}_i - \tilde{\nabla}_{e_i} (\langle \xi \circ T, e_i \rangle_H) \quad \text{is convergent in } L^0(E, \mu).$$

— Lemma 3 is proven. —

CHAPTER TWO

Transformation of a Gaussian measure

Given an abstract Wiener space (H, E, μ) and $T : E \rightarrow E$ of the form :

$$Tx = x + F(x), \quad F : E \rightarrow H.$$

We shall examine when $T(\mu) \ll \mu$. We shall consider the following cases :

- F is linear continuous from E into H ,
- F is regular (i.e., possesses stochastic derivatives).

We shall give some expressions for the Radon-Nikodym density $\frac{dT(\mu)}{d\mu}$.

In the following chapter we shall study a family of flows : $T_t = I + F_t$ where $F_t : E \rightarrow H$, $(t \in [0, 1])$ and shall study the work of Cruzeiro, Buckdahn and Ustunel-Zakai on this subject. We shall only give the statements of the results and from time to time sketch of the proofs.

1 - Preliminary results on equivalence and orthogonality of product measures

Let $(E_k, \mathcal{B}_k)_{k \in \mathbb{N}^*}$ be a sequence of measurable spaces and for every k , let μ_k and ν_k be two probabilities on $(E_k, \mathcal{B}_k)$ such that $\mu_k \ll \nu_k$. Let us set $\rho_k = \frac{d\mu_k}{d\nu_k}$. Let us consider the product measures :

$$\mu = \prod_{k=1}^{\infty} \mu_k$$

and

$$\nu = \prod_{k=1}^{\infty} \nu_k$$

and let

$$\alpha_k = \int_{E_k} \sqrt{\rho_k(x_k)} \nu_k(dx_k).$$

These notations having been fixed we have the following result of Kakutani :

THEOREM 1 : *We have the dichotomy :*

$$\mu \ll \nu \quad \text{or} \quad \mu \perp \nu.$$

a) $\mu \ll \nu \iff \prod \alpha_k$ converges ; and in this case the density is equal to $\rho(x) = \prod_1^\infty \rho_k(x_k)$

(convergence in mean).

b) $\mu \perp \nu \iff \prod \alpha_n$ diverges to zero. (We cannot have divergence to infinity since $\alpha_k^2 \leq 1$).

Applications : $E_k = \mathbb{R}$ for every k

$$\begin{aligned} \nu_k(dx_k) &= \frac{1}{\sigma_k \sqrt{2\pi}} \exp\left\{-\frac{(x_k - \gamma_k)^2}{2\sigma_k^2}\right\} dx_k \\ \mu_k(dx_k) &= \frac{1}{\lambda_k \sqrt{2\pi}} \exp\left\{-\frac{(x_k - \beta_k)^2}{2\lambda_k^2}\right\} dx_k. \end{aligned}$$

Then

$$\rho_k(x_k) = \frac{\sigma_k}{\lambda_k} \exp\left\{-\frac{1}{2\sigma_k^2 \lambda_k^2} \left[(x_k - \beta_k)^2 \sigma_k^2 - (x_k - \gamma_k)^2 \lambda_k^2 \right]\right\}$$

and

$$\alpha_k = \int_{\mathbb{R}} \sqrt{\rho_k(x_k)} d\nu_k(x_k) = \sqrt{\frac{2\lambda_k \sigma_k}{\lambda_k^2 + \sigma_k^2}} \exp\left\{-\frac{(\beta_k - \gamma_k)^2}{4(\lambda_k^2 + \sigma_k^2)}\right\}.$$

We now give some particular cases :

- **Same covariance** ($\lambda_k = \sigma_k$ for every k). μ and ν are equivalent if and only if

$$\sum_k \frac{(\beta_k - \gamma_k^2)^2}{\sigma_k^2} < \infty$$

and the density is then equal to

$$\exp \left\{ \sum_{k=1}^{\infty} \frac{x_k(\beta_k - \gamma_k)}{\sigma_k^2} - \frac{\beta_k^2 - \gamma_k^2}{2\sigma_k^2} \right\}.$$

Otherwise, we have orthogonality of measures.

- **Same mean** $\beta_k = \gamma_k = 0$ for every k .

μ and ν are equivalent if and only if :

$$\sum_{k=1}^{\infty} \frac{(\lambda_k - \sigma_k)^2}{\lambda_k \sigma_k} < \infty$$

and in this case the density is equal to :

$$\frac{d\mu}{d\nu}(x) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{\sigma_k}{\lambda_k} \exp \left\{ -\frac{x_k^2}{2} \left(\frac{\sigma_k^2 - \lambda_k^2}{\sigma_k^2 \lambda_k^2} \right) \right\}.$$

If this condition is not satisfied we have orthogonality.

2 - Affine transformations of Gaussian measures

Now let (E, H, μ) be an abstract Wiener space. If (e_n) is an orthonormal basis of H , the random variables $\tilde{e}_n$ are independent Gaussian variables on E , with mean zero and variance one. The law of the sequence $(\tilde{e}_n)$ is therefore a product measure on $\mathbb{R}^{\mathbb{N}}$:

$$\gamma_{\mathbb{N}} = \bigotimes_{n=0}^{\infty} \gamma_n$$

where $\gamma_n = \gamma$ (Gaussian measure on $\mathbb{R}$) for every n .

Now we have a measurable (defined almost everywhere) map θ of E into $\mathbb{R}^{\mathbb{N}}$:

$$x \rightsquigarrow (\tilde{e}_n(x))_n.$$

If the e_n belong to E' , the $\tilde{e}_n$ are everywhere defined and θ is continuous from E into $\mathbb{R}^{\mathbb{N}}$.

It is clear now that the image of μ under θ is equal to $\gamma_{\mathbb{N}}$. We have $\theta(H) = \ell^2$ as we can see immediately (the $\tilde{e}_n(x)$ are defined in a unique way on H).

Proposition 1 : *Let $a \in E$ and $\tau_a(\mu)$ be the translate of μ by a . Then we have the dichotomy :*

$$\tau_a(\mu) \sim \mu \text{ or } \tau_a(\mu) \perp \mu,$$

$$\tau_a(\mu) \sim \mu \text{ if and only if } a \in H \text{ and the density is equal to } \exp \left\{ \tilde{a}(\bullet) - \frac{1}{2} \|a\|_H^2 \right\}.$$

Proof :

$\tau_a(\mu)$ is a Gaussian (non centered if $a \neq 0$) measure with the same covariance than μ .

Let $(e_n) \subset E'$ (orthonormal in H). It suffices to prove the same result for $\theta(\mu)$ and $\theta(\tau_a(\mu))$. But $\theta(\tau_a(\mu))$ is the product of Gaussian measures on $\mathbb{R}$ with variances one and mean $e_n(a)$. Therefore it suffices to apply the result of the previous paragraph.

— Q.E.D.—

Now let $T = I + F$ be a linear continuous transform of E into E . Let us suppose that $F(E) \subset H$. In this case F is continuous for the topology of H by closed graph theorem.

Suppose moreover, that $T|_H = Id_H + F|_H$ is an *invertible operator*. Then $T : E \rightarrow E$ is also invertible and

$$T^{-1} = I - (T|_H)^{-1} \circ F.$$

Proposition 2 : Suppose $T = I + F$ with the above properties and that $F|_H$ is nuclear. Then $T^{-1}(\mu)$ and μ are equivalent and

$$\frac{dT^{-1}(\mu)}{d\mu}(x) = \exp\left\{-(Fx, x)_H - \frac{1}{2} \|Fx\|_H^2\right\} |\det T|.$$

Proof :

Let us explain what this formula means. Indeed, $F|_H$ being nuclear, admits the decomposition : $F|_H(x) = \sum_n \lambda_n (x, e_n)_H f_n$, (e_n, f_n) orthonormal in H and we can define $\langle F(x), x \rangle_H$ on E by $\sum_n \lambda_n \tilde{e}_n(x) \tilde{f}_n(x)$, we set : $\det(I + F) = \prod_n (1 + \lambda_n)$. (This has sense since $\sum_n |\lambda_n| < \infty$).

• Let us suppose first that F is symmetrical :

$$F(x) = \sum_n \lambda_n (x, e_n)_H e_n$$

where e_n is an orthonormal basis composed of eigenvectors of F .

Let $\theta : E \rightarrow \mathbb{R}^{\mathbb{N}}$ associated to these e_n . We have seen that : $\theta(\mu) = \gamma_{\mathbb{N}}$ (product measure).

Now $\theta((I + F)^{-1}\mu)$ is the product of measures with densities :

$$\frac{1}{\sqrt{2\pi}} (1 + \lambda_n) \exp\left\{-\frac{1}{2} (1 + \lambda_n)^2 x_n^2\right\}.$$

We have

$$\frac{d((1 + \lambda_n)^{-1} \tilde{e}_n(\mu))}{d(\tilde{e}_n(\mu))} (x_n) = (1 + \lambda_n) \exp\left\{-\lambda_n x_n^2 - \frac{1}{2} \lambda_n^2 x_n^2\right\}$$

$$\frac{d(\theta((I + F^{-1})(\mu)))}{d\theta(\mu)} (x) = \prod (1 + \lambda_n) \exp\left\{-(Fx, x)_H - \frac{1}{2} \|Fx\|_H^2\right\}.$$

• Now let us consider the general case (F non necessarily symmetrical)

$$H \xrightarrow{i} E \xrightarrow{I+F} H \xrightarrow{i} E$$

$(I + F) \circ i$ is an operator from H into H . There exists a unitary operator $U : H \rightarrow H$ “*diagonalizing*” $F \circ i$, therefore $(I + F) \circ i$. Let $\tilde{U}$ its extension to $E \rightarrow E$. We apply the result for $\tilde{U}(I + F) \tilde{U}^{-1}$.

— Q.E.D. —

Now we shall consider the case where $F|_H$ is not nuclear.

We know that in any case $F|_H$ is Hilbert-Schmidt.

• Suppose at first that rank (F) is finite.

Then the formula of Proposition 2 gives :

$$\prod_{i=1}^n (1 + \lambda_i) \exp\left\{-\sum_{i=1}^n \lambda_i x_i^2 - \frac{1}{2} \sum_{i=1}^n \lambda_i^2 x_i^2\right\}$$

$$= \prod_{i=1}^n (1 + \lambda_i) e^{-\lambda_i} \exp\left\{-\left(\sum_{i=1}^n \lambda_i x_i^2 - \sum_{i=1}^n \lambda_i - \frac{1}{2} \|Fx\|_H^2\right)\right\}.$$

• Now suppose F Hilbert-Schmidt with infinite rank :

$$\prod_i (1 + \lambda_i) e^{-\lambda_i} \text{ converges since } \sum_i |\lambda_i|^2 < \infty.$$

The limit is called the “*Carleman determinant*”.

Now we can prove that

$$\lim_{n \rightarrow \infty} \exp \left\{ - \left(\sum_{i=1}^n \lambda_i x_i^2 - \sum_{i=1}^n \lambda_i \right) - \frac{1}{2} \|Fx\|_H^2 \right\} \text{ exists in } L^1(\mu) \text{ if } F \text{ is } H\text{-S.}$$

We denote it by :

$$\exp \left\{ - \left[(Fx, x)_H - \text{Trace } F \right] - \frac{1}{2} \|Fx\|_H^2 \right\}.$$

Therefore we have the following theorem :

THEOREM 2 : *Let $T : E \rightarrow E$ linear continuous, such that $Tx = x + Fx$ with $F(E) \subset H$. Then $F|_H$ defines a Hilbert-Schmidt operator from H into H . Suppose that $T|_H$ is invertible then $T : E \rightarrow E$ is invertible. Moreover, $T^{-1}(\mu)$ is absolutely continuous with respect to μ and we have*

$$\frac{d(T^{-1}(\mu))}{d\mu}(x) = \tilde{\Delta}(I + F) \exp \left\{ - \left[(Fx, x)_H - \text{Trace } F \right] - \frac{1}{2} \|Fx\|_H^2 \right\}$$

with

$$\tilde{\Delta}(I + F) = \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i},$$

the λ_i being the eigenvalues of F .

We have seen the affine case.

Now we may give the result for the general case announced in the beginning.

THEOREM 3 : *Let $F \in \mathbb{D}^{2,1}(H)$. Suppose that $(I + F)$ is invertible and that for every $x \in E$, the operator $I_H + \nabla F(x)$ from H to H is invertible, then $(I + F)^{-1}(\mu)$ is absolutely continuous with respect to μ and we have :*

$$\frac{d((I + F)^{-1}(\mu))}{d\mu}(x) = \tilde{\Delta}(I_H + \nabla F(x)) \exp \left\{ -\delta(F)(x) - \frac{1}{2} \|Fx\|_H^2 \right\}.$$

CHAPTER THREE

Transformation of Gaussian measures under anticipative flows

Let (Ω, H, P) be an abstract Wiener space and let T be an invertible transformation of Ω into Ω (the only interesting case will be of the form : $T := Id + F$ with $F \in \mathbb{D}^{2,1}(H)$).

Definition : A family of transformations $(T_t)_{t \in [0,1]}$ from Ω to Ω will be called an “*interpolation*” of the invertible transformation T if

- a) $T_0 = Id, \quad T_1 = T,$
- b) each T_t is invertible,
- c) for each $\omega, \quad t \rightsquigarrow T_t \omega$ and $t \rightsquigarrow T_t^{-1} \omega$ are strongly continuous.

Moreover, if

- d) for each $\omega, \quad t \rightsquigarrow T_t \omega$ and $t \rightsquigarrow T_t^{-1} \omega$ are strongly continuously differentiable, the interpolation will be said to be “*smooth*”.

Example 1 : $T_t(\omega) = \omega + tA(\omega)$ where A is a function from Ω to H , such that

$$\omega \rightsquigarrow \omega + tA(\omega) \text{ is invertible for every } t.$$

Example 2 : Suppose $A : \Omega \rightarrow H$ is continuous and suppose that we have defined a family of transformations (T_t) from Ω into Ω by :

$$T_t \omega = \omega + \int_0^t A(T_s \omega) \, ds \quad (\text{time homogeneous case})$$

$$\text{i.e.} \quad \left| \begin{array}{l} \frac{dT_t}{dt}(\omega) = A(T_t \omega) \\ T_0(\omega) = \omega \end{array} \right.$$

we have then :

$$\frac{dT_t}{dt}(T_t^{-1}(\omega)) = A(\omega).$$

Example 3 : $T_t(\omega) = \omega + \int_0^t \sum(s, T_s(\omega)) ds$.

If $\sum(r, \omega)$ is **continuous** on $[0, 1] \times \Omega$ into Ω or into H and satisfies a global Lipschitz condition :

$$|\sum(t, \omega_1) - \sum(t, \omega_2)| \leq L \|\omega_1 - \omega_2\|_\Omega$$

We can consider $T_t(\omega)$ as the solution of the ordinary differential equation

$$\begin{cases} \frac{dT_t}{dt}(\omega) &= \sum(t, T_t(\omega)) \\ T_0(\omega) &= \omega \end{cases}$$

on the Banach space Ω .

If for every $t \in [0, 1]$, $\sum(t, \bullet)$ is Fréchet differentiable, with Fréchet differential denoted by $\partial \sum(t, \omega)$, and if we assume that $\partial \sum(t, \omega)$ is bounded continuous on $[0, 1] \times \Omega$, then the equation

$$T_t\omega = \omega + \int_0^t \sum(r, T_r(\omega)) dr$$

has a unique solution.

Moreover, $\omega \rightsquigarrow T_t(\omega)$ is Fréchet differentiable and $\partial T_t(\omega)$ is continuous, invertible on $[0, 1] \times \Omega$, and satisfies the differential equation :

$$\frac{d}{dt} (\partial T_t \omega) = (\partial \sum(t, \bullet) \circ T_t(\omega)) \bullet \partial T_t(\omega).$$

Its inverse $\partial^{-1} T_t \omega$ satisfies :

$$\frac{d}{dt} (\partial^{-1} T_t \omega) = -\partial^{-1} T_t(\omega) \bullet (\partial \sum(t, \bullet) \circ T_t(\omega)).$$

Consequently, by the global inverse theorem, $T_t(\omega)$ is a C_1 -diffeomorphism. Therefore, we have an interpolation of T defined by

$$T(\omega) = \omega + \int_0^1 \sum(r, T_r \omega) dr.$$

Later on we shall come back to this example. Now let us return to the general situation.

THEOREM 1 : Let T be a transformation from Ω to Ω and $(T_t, t \in [0, 1])$ be an interpolation of T . Let us assume moreover that

$$(a) \quad T_t(P) \ll P, \quad \forall t \in [0, 1] \text{ and let } X_t(\omega) = \frac{dT_t(P)}{dP}(\omega),$$

$$(b) \quad G_t = T_t^{-1} - I \in \mathbb{D}^{2,1}(H) \quad \text{and} \quad \frac{dT_t^{-1}}{dt} \in H,$$

$$(c) \quad \frac{dT_t^{-1}}{dt} \text{ as a function from } [0, 1] \times \Omega \text{ into } H \text{ is almost surely continuous in } (t, \omega)$$

(for $dt \otimes dP$) and $\nabla T_t^{-1}(\omega)$ will be assumed to possess a continuous extension $[0, 1] \times \Omega$,

$$(d) \quad \frac{dT_s^{-1}}{ds} \circ T_s \in \mathbb{D}^{2,1}(H).$$

Then

$$X_t(\omega) = \exp \left\{ - \int_0^t \left(\delta \left[\frac{dT_s^{-1}}{ds} \circ T_s \right] \right) \circ T_s^{-1}(\omega) ds \right\} \quad (1)$$

This implies that the measures $T_t(P), T_t^{-1}(P)$ and P are equivalent.

Moreover

$$\begin{aligned} X_t = \exp \Big\{ & - \int_0^t \tilde{\delta} \left[\frac{dG_s}{ds} \right] ds \\ & - \frac{1}{2} \langle G_t, G_t \rangle_H \\ & - \int_0^t \text{Trace} \left[\left(\nabla \left[\frac{dG_s}{ds} \right] \circ T_s \right) \circ T_s^{-1} \bullet \nabla G_s \right] ds \Big\} \end{aligned} \quad (2)$$

where $\tilde{\delta}$ was defined precendently by :

$$\tilde{\delta} (\xi \circ T) = (\delta \xi) \circ T - \langle \xi \circ T, F \rangle_H - \text{Trace} ((\nabla \xi) \circ T \bullet \nabla F).$$

Moreover, if $\frac{dG_s}{ds}$ and G_s are in $\mathbb{D}^{2,1}(H)$, then the formula (2) becomes :

$$\begin{aligned} X_t = \exp \Big\{ & - \delta(G_t) - \frac{1}{2} \langle G_t, G_t \rangle_H \\ & - \int_0^t \text{Trace} \left[\left(\nabla \left[\frac{dG_s}{ds} \right] \circ T_s \right) \circ T_s^{-1} \bullet \nabla G_s \right] ds \Big\}. \end{aligned} \quad (3)$$

Proof of (1) :

We have :

$$\begin{aligned} 0 &= \frac{1}{\varepsilon} \left[T_{t+\varepsilon}^{-1} \circ T_{t+\varepsilon} - T_t^{-1} \circ T_t \right] \\ &= \frac{1}{\varepsilon} \left[T_{t+\varepsilon}^{-1} \circ T_{t+\varepsilon} - T_{t+\varepsilon}^{-1} \circ T_t \right] + \frac{1}{\varepsilon} \left[T_{t+\varepsilon}^{-1} \circ T_t - T_t^{-1} \circ T_t \right] . \end{aligned}$$

Therefore by (c)

$$\left[(\nabla T_t^{-1}) \circ T_t(\omega) \right] \cdot \frac{dT_t}{dt}(\omega) + \frac{dT_t^{-1}}{dt} \circ T_t \omega = 0 \quad (4)$$

Let now $a : \Omega \rightarrow \mathbb{R}$ smooth and let $h \in H$. By (d) we have :

$$\begin{aligned} \langle (\nabla a) \circ T_t(\omega), h \rangle_H &= \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} a(T_t \omega + \varepsilon h) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \left[(a \circ T_t)(T_t^{-1}(T_t \omega + \varepsilon h)) \right] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \left[(a \circ T_t)(\omega + \varepsilon (\nabla T_t^{-1}) \circ (T_t \omega) \cdot h + o(\varepsilon)) \right] \\ &= \langle \nabla(a \circ T_t), (\nabla T_t^{-1}) \circ T_t(\omega) \cdot h \rangle_H . \end{aligned}$$

Now if we set $h = \frac{d}{dt} T_t(\omega)$, comparing with (4), we obtain :

$$\langle (\nabla a) \circ T_t \omega, \frac{d}{dt} T_t \omega \rangle_H = - \langle \nabla(a \circ T_t)(\omega), \frac{dT_t^{-1}}{dt} \circ T_t(\omega) \rangle_H .$$

But the left-hand member of this equality is equal to $\frac{d}{dt} (a \circ T_t)(\omega)$. Therefore we obtain :

$$\begin{aligned} \mathbb{E}\{a \circ T_t \omega - a(\omega)\} &= \mathbb{E} \left(\int_0^t \frac{d}{ds} (a \circ T_s \omega) ds \right) \\ &= - \mathbb{E} \left(\int_0^t \langle \nabla(a \circ T_s)(\omega), \frac{dT_s^{-1}}{ds} \circ T_s \omega \rangle ds \right) . \end{aligned}$$

But from condition (d) , $\left(\frac{dT_s^{-1}}{ds} \circ T_s \in \mathbb{D}^{2,1}(H) \right)$, and integrating by parts we obtain :

$$\mathbb{E}\{a \circ T_t(\omega) - a(\omega)\} = - \int_0^t \mathbb{E} \left\{ (a \circ T_s \omega) \delta \left[\frac{dT_s^{-1}}{ds} \circ T_s \right](\omega) \right\} ds$$

and

$$\mathbb{E} \{a(\omega) \cdot (X_t(\omega) - 1)\} = -\mathbb{E} \left(\int_0^t a(\omega) X_s(\omega) \left(\delta \left[\frac{dT_s^{-1}}{ds} \circ T_s \right] \right) \circ T_s^{-1} \omega \, ds \right).$$

Since this last inequality is true for smooth functions we have :

$$X_t(\omega) = 1 - \int_0^t X_s(\omega) \left(\delta \left[\frac{dT_s^{-1}}{ds} \circ T_s \right] \right) \circ T_s^{-1} \omega \, ds.$$

Finally, since X_t is P -almost surely positive, $T_t P$ and P are equivalent.

On the other hand, if $a : \Omega \rightarrow \mathbb{R}$ is smooth, then :

$$\mathbb{E} \{a \circ T_t^{-1} X_t\} = \mathbb{E} a.$$

Hence if B is a Borelian subset of Ω , then

$$P(B) = 0 \iff \mathbb{E}\{1_B \circ T_t^{-1} X_t\} = 0 \iff 1_B \circ T_t^{-1} = 0, \text{ a.s.}$$

Therefore, $T_t^{-1}(P)$ and P are equivalent.

Proof of (2) :

We start from

$$(\delta\xi) \circ T = \tilde{\delta}(\xi \circ T) + \langle \xi \circ T, F \rangle_H + \text{Trace}((\nabla\xi) \circ T \bullet \nabla F)$$

with

$$\xi = \frac{dT_s^{-1}}{ds} \circ T_s, \quad T = T_s^{-1}, \quad F = T - Id = G_s$$

and

$$\frac{dG_s}{ds} = \frac{dT_s^{-1}}{ds}.$$

Then

$$\delta \left[\frac{dT_s^{-1}}{ds} \circ T_s \right] \circ T_s^{-1} = \tilde{\delta} \left(\frac{dG_s}{ds} \right) + \left\langle \frac{dG_s}{ds}, G_s \right\rangle + \text{Trace} \left(\left(\nabla \left[\frac{dG_s}{ds} \circ T_s \right] \right) \circ T_s^{-1} \bullet \nabla G_s \right)$$

and we integrate from 0 to t .

Proof of (3) :

It is immediate from (2) since $\tilde{\delta} = \delta$ under this hypothesis.

We have expressed the density X_s in terms of $\frac{dT_s^{-1}}{dt}$. (The next result will give an expression of X_t in terms of $\frac{dT_s}{ds}$).

— Q.E.D. —

Corollary : *Under the assumptions and conditions of the theorem 1 let us replace T , T_t , T_s and X_t by T^{-1} , T_t^{-1} , T_s^{-1} , $\frac{dT_t^{-1}(P)}{dP} = Y_t$. Then we have :*

$$\begin{aligned} X_t(\omega) &= \frac{dT_t(P)}{dP}(\omega) \\ &= \exp \left\{ \int_0^t \left(\delta \left[\frac{dT_s}{ds} \circ T_s^{-1}(\cdot) \right] \right) \circ T_s T_t^{-1}(\omega) ds \right\} \end{aligned}$$

and

$$\begin{aligned} X_t(\omega) &= \exp \left\{ -\delta(G_t)(\omega) - \frac{1}{2} \langle G_t, G_t \rangle_H(\omega) \right. \\ &\quad \left. + \int_0^t \text{Trace} \left[\left(\nabla \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \circ T_s T_t^{-1}(\omega) \right) \bullet \nabla \left(G_t - G_s(T_s T_t^{-1}) \right)(\omega) \right] ds \right\}. \end{aligned}$$

Proof :

By Theorem 1:

$$Y_t(\omega) = \exp \left\{ - \int_0^t \left(\delta \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \right) \circ T_s(\omega) ds \right\}. \quad (A)$$

On the other hand, if a is a smooth functional :

$$\begin{aligned} \mathbb{E} \{ a(\omega) Y_t^{-1}(T_t^{-1}(\omega)) \} &= \mathbb{E} \{ a(T_t T_t^{-1}(\omega)) Y_t^{-1}(T_t^{-1}(\omega)) \} \\ &= \mathbb{E} \{ a(T_t(\omega)) Y_t^{-1}(\omega) Y_t(\omega) \} \\ &= \mathbb{E} \{ a(\omega) X_t(\omega) \}. \end{aligned}$$

Therefore :

$$X_t(\omega) = Y_t^{-1}(T_t^{-1}(\omega)) = \exp \left\{ \int_0^t \left(\delta \left[\frac{dT_s}{ds} \circ T_s^{-1}(\cdot) \right] \right) \circ T_s \circ T_t^{-1}(\omega) ds \right\},$$

— which proves the first formula. —

To prove the second formula let us start from

$$T_s \omega = \omega + F_s(\omega)$$

which implies

$$T_s T_t^{-1} \omega = T_t^{-1} \omega + F_s(T_t^{-1} \omega),$$

and if $s = t$

$$\omega = T_t^{-1} \omega + F_t(T_t^{-1} \omega).$$

Therefore

$$T_s T_t^{-1} \omega = \omega + F_s(T_t^{-1} \omega) - F_t(T_t^{-1} \omega).$$

Now

$$G_t(\omega) = T_t^{-1}(\omega) - \omega = -F_t(T_t^{-1} \omega).$$

Therefore :

$$T_s T_t^{-1} \omega = \omega + G_t(\omega) - G_s(T_s T_t^{-1} \omega).$$

In the formula

$$X_t(\omega) = \exp \left\{ \int_0^t \left(\delta \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \right) \circ T_s T_t^{-1} \omega \, ds \right\},$$

let us apply the formula given δ in terms of $\tilde{\delta}$. We obtain :

$$\begin{aligned} X_t(\omega) = \exp \bigg\{ & \int_0^t \left(\tilde{\delta} \left[\frac{dT_s}{ds} \circ T_t^{-1} \right] (\omega) \right. \\ & + \left\langle \frac{dT_s}{ds} \circ T_t^{-1}(\omega), G_t(\omega) - G_s(T_s T_t^{-1} \omega) \right\rangle_H \\ & \left. + \text{Trace} \left[\left(\nabla \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \circ T_s T_t^{-1}(\omega) \right) \bullet \nabla (G_t - G_s(T_s T_t^{-1}))(\omega) \right] \right) ds \bigg\} \end{aligned}$$

Now we integrate with respect to s , by using :

$$\frac{d}{ds} (T_s \circ T_t^{-1}(\omega)) = -\frac{d}{ds} (G_s(T_s T_t^{-1} \omega)) = \frac{d}{ds} (G_t(\omega) - G_s(T_s T_t^{-1} \omega)).$$

— We obtain the second formula.—

Now we give an integral equation satisfied by X_t .

THEOREM 2 : Let $T : \Omega \rightarrow \Omega$ and $T_t : \Omega \rightarrow \Omega$ ($t \in [0, 1]$) be an interpolation of T . Assume that for each $t \in [0, 1]$, $T_t(P) \ll P$ and that $X_s \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \in \mathbb{D}_{loc}^{2,1}(H)$ (this condition is satisfied if $\frac{dT_s}{ds} \circ T_s^{-1} \in \mathbb{D}^{2,1}(H)$ and $X_s \in \mathbb{D}_{loc}^{2,1}$), then X_t satisfies :

$$X_t = 1 + \int_0^t \delta \left[X_s \frac{dT_s}{ds} \circ T_s^{-1} \right] ds.$$

Proof :

Let a be a smooth functional. Then

$$\begin{aligned} \mathbb{E}\{X_t(\omega)a(\omega)\} &= \mathbb{E}\{a(T_t(\omega))\} \\ &= \mathbb{E}\left\{a(\omega) + \int_0^t \frac{da(T_s(\omega))}{ds} ds\right\} \\ &= \mathbb{E}\left\{a(\omega) + \int_0^t \langle (\nabla a) \circ T_s \omega, \frac{d}{ds} T_s(\omega) \rangle ds\right\} \\ &= \mathbb{E}\{a(\omega)\} + \int_0^t \mathbb{E}\left\{X_s(\omega) \langle \nabla(a)(\omega), \left[\frac{dT_s}{ds} \circ T_s^{-1}(\omega)\right] \rangle\right\} ds \\ &= \mathbb{E}\{a(\omega)\} + \int_0^t \mathbb{E}\left\{a(\omega) \delta \left[X_s \frac{dT_s}{ds} \circ T_s^{-1} \right](\omega) \right\} ds \end{aligned}$$

— Q.E.D. —

Applications of these formulas.

- In the example (1) : $T_t(\omega) = \omega + t A(\omega)$,

$$X_t(\omega) = \exp \left\{ \int_0^t \left(\delta [A(T_s^{-1}(\cdot))] \right) \circ T_s T_t^{-1}(\omega) ds \right\}$$

(this result was obtained by Bell).

- In the example (2) : $T_t(\omega) = \omega + \int_0^t A(T_s(\omega)) ds$

$$\frac{dT_s}{ds} (T_s^{-1}(\omega)) = A(\omega)$$

and

$$X_t(\omega) = \exp \left\{ \int_0^t \left(\delta(A) \right) \circ T_s T_t^{-1}(\omega) \, ds \right\}.$$

• We shall now study the example three :

$$T_t(\omega) = \omega + \int_0^t \sum(r, T_r(\omega)) \, dr. \quad (B)$$

We have given some hypotheses insuring that $T_t\omega$ is a solution of the ODE with values in the Banach space Ω

$$\left| \begin{array}{l} \frac{dT_t}{dt}(\omega) = \sum(t, T_t(\omega)) \\ T_0(\omega) = \omega \end{array} \right.$$

and that $\omega \rightsquigarrow T_t(\omega)$ and $\omega \rightsquigarrow T_t^{-1}(\omega)$ are Fréchet differentiable (in ω). Then :

$$I_H + \nabla \int_0^t \sum(s, T_s\omega) \, ds$$

is invertible and satisfies the hypotheses of Ramer's theorem

As a consequence the probabilities

$$T_t P, \, P \text{ and } T_t^{-1} P \text{ are equivalent.}$$

Now in (B) we replace ω by $T_s^{-1}\omega$:

$$T_t T_s^{-1}(\omega) = T_s^{-1}(\omega) + \int_0^t \sum(r, T_r T_s^{-1}(\omega)) \, dr.$$

Setting : $T_t T_s^{-1}(\omega) = \varphi_{s,t}(\omega)$ and $T_s T_t^{-1}(\omega) = \psi_{s,t}(\omega)$, $t \geq s$, we have :

$$\psi_{s,t} \circ \varphi_{s,t} = \varphi_{s,t} \circ \psi_{s,t} = Id$$

and :

$$\varphi_{s,t}(\omega) = \omega + \int_s^t \sum(r, \varphi_{s,r}(\omega)) \, dr$$

$$\psi_{s,t}(\omega) = \omega - \int_s^t \sum(r, \psi_{r,t}(\omega)) \, dr.$$

Note that $\varphi_{(1-s)t,t}$, $s \in [0, 1]$ is, for t fixed, an interpolation of T_t and naturally $(T_t)_{t \in [0,1]}$ is an interpolation of T_1 : $\varphi_{s,t}$ is a “two-parameter” interpolation of T .

• Now we shall specialize the example in the case $\Omega = \mathcal{C}_0[0, 1]$, with the Wiener measure and we shall use the following notations in this case :

If U , U_1 and U_2 are random functions with values in H ; if H is the Cameron-Martin space, then

$$\begin{aligned} U(\omega) (\bullet) &= \int_0^\bullet \dot{u}(\theta, \omega) d\theta \\ \delta(U) &= \int_0^1 \dot{u}(\theta, \omega) \delta_\theta(W) \\ \langle U_1, U_2 \rangle_H &= \int_0^1 \dot{u}_1(\theta, \omega) \dot{u}_2(\theta, \omega) d\theta. \end{aligned}$$

But if H is the $L^2[0, 1]$ space

$$\begin{aligned} U(\omega) (\bullet) &= u(\bullet, \omega) \\ \delta U &= \int_0^1 u(\theta, \omega) \delta_\theta(W) \\ \langle U_1, U_2 \rangle_H &= \int_0^1 u_1(\theta, \omega) u_2(\theta, \omega) d\theta \end{aligned}$$

$$(T_t \omega) (\bullet) = \omega(\bullet) + \int_0^t \rho(r, \bullet) \sigma(r, T_r \omega) dr \quad (C)$$

where ρ is a smooth function on $[0, 1]^2$ and $\sigma : [0, 1] \times \Omega \rightarrow \mathbb{R}$ is assumed to satisfy Lipschitzian and differentiability conditions.

In terms of $\varphi_{s,t}$ and $\psi_{s,t}$, ($s \leq t$) we have :

$$\begin{aligned} \varphi_{s,t}(\omega) (\bullet) &= \omega(\bullet) + \int_s^t \rho(r, \bullet) \sigma(r, \varphi_{s,r}(\omega)) dr \\ \psi_{s,t}(\omega) (\bullet) &= \omega(\bullet) - \int_s^t \rho(r, \bullet) \sigma(r, \psi_{r,t}(\omega)) dr. \end{aligned}$$

We consider these equations as ODE in Banach space (the first in t with s fixed ; the second in s for t fixed), we have existence and unicity of solutions with

$$\varphi_{s,s}(\omega) = \omega, \quad \psi_{t,t}(\omega) = \omega \quad \text{and} \quad \varphi_{s,t} \circ \psi_{s,t}(\omega) = \omega.$$

Then $\psi_{s,t}(\omega)$ and $\varphi_{s,t}(\omega)$ are Fréchet differentiable in $\omega \in \mathcal{C}_0([0, 1])$.

Consequently, $\partial\varphi_{s,t}$ and $\partial\psi_{s,t}$ restricted to H are invertible, and by Ramer's theorem: $\varphi_{s,t}(P)$, $\psi_{s,t}(P)$ and P are equivalent.

Set

$$L_{s,t}(\omega) = \frac{d\varphi_{s,t}(P)}{dP}$$

and

$$\Lambda_{s,t} = \frac{d\psi_{s,t}(P)}{dP}.$$

Now let us fix t in the equation :

$$T_t\omega(\bullet) = \omega(\bullet) + \int_0^t \rho(r, \bullet) \sigma(r, T_r\omega) dr.$$

Let $s = t - \lambda$ and $\lambda \in [0, t]$ be the interpolation parameters.

Now let us recall that (cf (3))

$$\begin{aligned} X_t = \exp \Big\{ & -\delta(G_t) - \frac{1}{2} \langle G_t, G_t \rangle_H \\ & - \int_0^t \text{Trace} \left(\nabla \left[\frac{dG_s}{ds} \circ T_s \right] \circ T_s^{-1} \bullet \nabla G_s \right) ds \Big\} \end{aligned} \quad (D)$$

where $G_t = T_t^{-1} - Id$, and apply the result for T_t satisfying the relation :

$$T_t\omega(\bullet) = \omega(\bullet) + \int_0^t \rho(r, \bullet) \sigma(r, T_r\omega) dr.$$

Then we obtain an expression for X_t :

$$\begin{aligned} X_t = \exp \Big\{ & \int_0^1 \left[\int_0^t \frac{\partial \rho}{\partial \theta}(r, \theta) \sigma(r, \psi_{0,r}) dr \right] \delta_\theta(W) \\ & - \frac{1}{2} \int_0^1 \left[\int_0^t \frac{\partial \rho(r, \theta)}{\partial \theta} \sigma(r, \psi_{0,r}) dr \right]^2 d\theta \\ & - \int_0^t \int_0^t \int_0^t \left[\int_0^\lambda \frac{\partial \rho(r, \eta)}{\partial \eta} D_\theta \sigma(r, \psi_{0,r}) dr \right] \circ \frac{\partial \rho(\lambda, \theta)}{\partial \theta} (D_\eta \sigma(\lambda, \bullet)) \circ \psi_{0,\lambda} d\lambda d\theta d\eta \Big\}. \end{aligned}$$

We can obtain another formula for the Radon-Nikodym density using the relation :

$$\delta(aU) = a\delta U - \langle \nabla a, U \rangle_H$$

in the expression :

$$X_t(\omega) = \exp \left\{ \int_0^t \left(\delta \left[\frac{dT_s}{ds} \circ T_s^{-1} \right] \right) \circ T_s T_t^{-1}(\omega) ds \right\}.$$

We then obtain :

$$L_{s,t} = \exp \left\{ \int_s^t \sigma(r, \psi_{r,t}) \left[\delta \rho(r, \cdot) - \int_s^r \sigma(u, \psi_{u,t}) \langle \rho(r, \cdot), \rho(u, \cdot) \rangle_H du \right] dr - \int_s^t \langle (\nabla \sigma)(r, \psi_{r,t}), \rho(r, \cdot) \rangle_H dr \right\}.$$

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