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Properties of quasi-invariant measures on topological groups and associated algebras.

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Abstract

Properties of quasi-invariant measures relative to dense subgroups are considered on topological groups. Mainly non-locally compact groups are considered such as

- (i) a group of diffeomorphisms $Diff(t, M)$ of real or non-Archimedean manifold M in cases of locally compact and nonlocally compact M , where t is a class of smoothness,
- (ii) a Banach-Lie group over a classical or non-Archimedean field,
- (iii) loop groups of real and non-Archimedean manifolds.

Recently quasi-invariant measures on a group of diffeomorphisms were constructed for real locally compact M in [8, 24] and for non-locally compact real or non-Archimedean manifolds M in [10, 12, 14, 18, 20, 21]. Such groups are also Banach manifolds or strict inductive limits of their sequences. Then on a real and non-Archimedean loop groups and semigroups of families of mappings from one manifold into another they were elaborated in [13, 15, 16, 17]. On real Banach-Lie groups quasi-invariant measures were constructed in [3].

This article is devoted to the investigation of properties of quasi-invariant measures that are important for analysis on topological groups and for construction irreducible representations [8, 23]. The following properties are investigated:

- (1) convolutions of measures and functions,

- (2) continuity of functions of measures,
 (3) non-associative algebras generated with the help of quasi-invariant measures. The theorems given below show that many differences appear to be between locally compact and non-locally compact groups. The groups considered below are supposed to have structure of Banach manifolds over the corresponding fields.

1. Definitions. (a). Let G be a Hausdorff separable topological group. A real (or complex) Radon measure μ on $Af(G, \mu)$ is called left-quasi-invariant (or right) relative to a dense subgroup H of G , if μ_h (or μ^h) is equivalent to μ for each $h \in H$, where $Bf(G)$ is the Borel σ -field of G , $Af(G, \mu)$ is its completion by μ , $\mu_h(A) := \mu(h^{-1}A)$, $\mu^h(A) := \mu(Ah^{-1})$ for each $A \in Af(G, \mu)$, $d_\mu(h, g) := \mu_h(dg)/\mu(dg)$ (or $\bar{d}_\mu(h, g) := \mu^h(dg)/\mu(dg)$) denote a left (or right) quasi-invariance factor. We assume that the uniformity τ_G on G is such that $\tau_G|_H \subset \tau_H$, (G, τ_G) and (H, τ_H) are complete. We suppose also that there exists an open base in $e \in H$ such that their closures in G are compact (such pairs exist for loop groups and groups of diffeomorphisms and Banach-Lie groups). We denote by $M_l(G, H)$ (or $M_r(G, H)$) the set of left- (or right) quasi-invariant measures on G relative to H with a finite norm $\|\mu\| := \sup_{A \in Af(G, \mu)} |\mu(A)| < \infty$.

(b). Let $L_H^p(G, \mu, \mathbb{C})$ for $1 \leq p \leq \infty$ denotes the Banach space of functions $f : G \rightarrow \mathbb{C}$ such that $f_h(g) \in L^p(G, \mu, \mathbb{C})$ for each $h \in H$ and

$$\|f\|_{L_H^p(G, \mu, \mathbb{C})} := \sup_{h \in H} \|f_h\|_{L^p(G, \mu, \mathbb{C})} < \infty,$$

where $f_h(g) := f(h^{-1}g)$ for each $g \in G$. For $\mu \in M_l(G, H)$ and $\nu \in M(H)$ let

$$(\nu * \mu)(A) := \int_H \mu_h(A) \nu(dh) \text{ and } (q \tilde{*} f)(g) := \int_H f(hg) q(h) \nu(dh)$$

be convolutions of measures and functions, where $M(H)$ is the space of Radon measures on H with a finite norm, $\nu \in M(H)$ and $q \in L^s(H, \nu, \mathbb{C})$, that is

$$\left(\int_H |q(h)|^s |\nu|(dh) \right)^{1/s} =: \|q\|_{L^s(H, \nu, \mathbb{C})} < \infty \text{ for } 1 \leq s < \infty.$$

2. Lemma. *The convolutions*

$$* : M(H) \times M_l(G, H) \rightarrow M_l(G, H) \text{ and}$$

$$\tilde{*} : L^1(H, \nu, \mathbb{C}) \times L^1_H(G, \mu, \mathbb{C}) \rightarrow L^1_H(G, \mu, \mathbb{C}).$$

are continuous $\mathbb{C}$ -bilinear mappings

Proof. It follows immediately from the definitions, Fubini theorem and because $d_\mu(h, g) \in L^1(H \times G, \nu \times \mu, \mathbb{C})$. In fact one has,

$$\|\nu * \mu\| \leq \|\nu\| \times \|\mu\|, \quad \|q\tilde{*}f\|_{L^1_H(G, \mu, \mathbb{C})} \leq \|q\|_{L^1(H, \nu, \mathbb{C})} \times \|f\|_{L^1_H(G, \mu, \mathbb{C})}.$$

3. Definition. For $\mu \in M(G)$ its involution is given by the following formula: $\mu^*(A) := \overline{\mu(A^{-1})}$, where $\bar{b}$ denotes complex conjugated $b \in \mathbb{C}$, $A \in Af(G, \mu)$.

4. Lemma. Let $\mu \in M_l(G, H)$ and G and H be non-locally compact with structures of Banach manifolds. Then μ^* is not equivalent to μ .

Proof. Let $T : G \rightarrow TG$ be the tangent mapping. Then μ induces quasi-invariant measure λ from an open neighbourhood W of the unit $e \in G$ on a neighbourhood of the zero section V in $T_e G$ and then it has an extension onto the entire $T_e G$. Let at first $T_e G$ be a Hilbert space. Put $Inv(g) = g^{-1}$ then $T \circ Inv \circ T^{-1} =: K$ on V is such that there is not any operator B of trace class on $T_e G$ such that $\tilde{M}_\lambda \subset B^{1/2} T_e G$ and $KT_e G \subset \tilde{M}_\lambda$, where $Re(1 - \theta(z)) \rightarrow 0$ for $(Bz, z) \rightarrow 0$ and $z \in T_e G$, $\theta(z)$ is the characteristic functional of λ , $\tilde{M}_\lambda$ is the set of all $x \in T_e G$ such that λ_x is equivalent to λ (see theorem 19.1 [25]). Then using theorems for induced measures from a Hilbert space on a Banach space [2, 9], we get the statement of lemma 4.

5. Lemma. For $\mu \in M_l(G, H)$ and $1 \leq p < \infty$ the translation map $(q, f) \rightarrow f_q(g)$ is continuous from $H \times L^p_H(G, \mu, \mathbb{C})$ into $L^p_H(G, \mu, \mathbb{C})$.

Proof. For metrizable G in view of the Lusin theorem (2.3.5 in [5]) and definitions of τ_G and τ_H for each $\epsilon > 0$ there are a neighbourhood $V \ni e$ in H and compacts K_1 and K in G such that the closure $cl_G VK_1 =: K_2$ is compact in G with $K_2 \subset K$, the restriction $f|_{K_2}$ is continuous and $(|\tilde{\mu}| + |\mu|)(G \setminus K_2) < \epsilon$, where $\tilde{\mu}(dg) := f(g)\mu(dg)$.

6. Proposition. For a probability measure $\mu \in M(G)$ there exists an approximate unit, that is a sequence of non-negative continuous functions $\psi_i : G \rightarrow \mathbb{R}$ such that $\int_G \psi_i(g)\mu(dg) = 1$ and for each neighbourhood $U \ni e$ in G there exists i_0 such that $\text{supp}(\psi_i) \subset U$ for each $i > i_0$.

Proof follows from the Radon property of μ and the existence of countable base of neighbourhoods in $e \in G$.

7. Proposition. If $(\psi_i : i \in \mathbb{N})$ is an approximate unit in H relative to a probability measure $\nu \in M(H)$, then $\lim_{i \rightarrow \infty} \psi_i * f = f$ in the $L^1_H(G, \mu, \mathbb{C})$ norm, where $\mu \in M_l(G, H)$, $f \in L^1_H(G, \mu, \mathbb{C})$.

Proof follows from lemma III.11.18 [6] and lemmas 2, 5.

8. Lemma. Suppose $g \in L_H^q(G, \mu, \mathbb{C})$ and $(g^x|_H) \in L^q(H, \nu, \mathbb{C})$ for each $x \in G$, $f \in L^p(H, \nu, \mathbb{C})$ with $1 < p < \infty$, $1/p + 1/q = 1$, where $g^x(y) := g(yx)$ for each x and $y \in G$. Let μ and ν be probability measures, $\mu \in M_l(G, H)$, $\nu \in M(H)$. Then $f \tilde{*} g \in L_H^1(G, \mu, \mathbb{C})$ and there exists a function $h : G \rightarrow \mathbb{C}$ such that $h|_H$ is continuous, $h = f \tilde{*} g$ μ -a.e. on G and h vanishes at ∞ on G .

Proof. In view of Fubini theorem and Hölder inequality we have

$$\begin{aligned} \|f \tilde{*} g\|_{L_H^1(G, \mu, \mathbb{C})} &= \sup_{s \in H} \int_G \int_H |f(y)| \times |g(z)| \nu(dy) \mu((ys)^{-1} dz) \leq \\ \sup_{s \in H} \left(\int_G \int_H |g(z)|^q \nu(dy) \mu((ys)^{-1} dz) \right)^{1/q} &\times \left(\int_G \int_H |f(y)|^p \nu(dy) \mu((ys)^{-1} dz) \right)^{1/p} \leq \\ \|f\|_{L^p(H, \nu, \mathbb{C})} \times \|g\|_{L_H^q(G, \mu, \mathbb{C})} &\times \nu(H) \mu(G). \end{aligned}$$

The equation $\alpha_f(\phi) := \int_H f(y) \overline{\phi(y)} \nu(dy)$ defines a continuous linear functional on $L^q(H, \nu, \mathbb{C})$. In view of lemma 5 the function $\alpha_f(g^{(sx)^{-1}}) =: \tilde{h}((sx)^{-1}) =: w(s, x)$ of two variables s and x is continuous on $H \times H$ for $s, x \in H$, since the mapping $(s, x) \mapsto (sx)^{-1}$ is continuous from $H \times H$ into H . By Fubini theorem (see §2.6.2 in [5])

$$\begin{aligned} \int_G h(y) \psi(y) \mu(dy) &= \int_G \int_H f(y) g(yx) \psi(x) \nu(dy) \mu(dx) = \\ \int_H f(y) \left[\int_G g(yx) \psi(x) \mu(dx) \right] \nu(dy) \end{aligned}$$

for each $\psi \in L^p(G, \mu, \mathbb{C})$, since

$$\int_G \int_H |f(y) g(yx) \psi(x)| |\nu|(dy) |\mu|(dx) < \infty,$$

where $|\nu|$ denotes the variation of the real-valued measure ν , $h(y) := \tilde{h}(y^{-1})$. Here ψ is arbitrary in $L^p(G, \mu, \mathbb{C})$, from this it follows, that $\mu(\{y : h(y) \neq (f \tilde{*} g)(y), y \in G\}) = 0$, since h and $(f \tilde{*} g)$ are μ -measurable functions due to Fubini theorem and the continuity of the composition and the inversion in a topological group. In view of Lusin theorem (see §2.3.5 in [5]) for each $\epsilon > 0$ there are compact subsets $C \subset H$ and $D \subset G$ and functions $f' \in L^p(H, \nu, \mathbb{C})$

and $g' \in L_H^q(G, \mu, \mathbb{C})$ with closed supports $\text{supp}(f') \subset C$, $\text{supp}(g') \subset D$ such that $cl_G CD$ is compact in G ,

$$\|f' - f\|_{L^p(H, \nu, \mathbb{C})} < \epsilon \text{ and } \|g' - g\|_{L_H^q(G, \mu, \mathbb{C})} < \epsilon,$$

since by the supposition of §1 the group H has the base B_H of its topology τ_H , such that the closures $cl_G V$ are compact in G for each $V \in B_H$. From the inequality

$$|h'(x) - h(x)| \leq (\|f\|_{L^p(H, \nu, \mathbb{C})} + \epsilon)\epsilon + \epsilon\|g\|_{L_H^q(G, \mu, \mathbb{C})}$$

it follows that for each $\delta > 0$ there exists a compact subset $K \subset G$ with $|h(x)| < \delta$ for each $x \in G \setminus K$, where $h'(x^{-1}) := \alpha_{f'}(g^x)$.

9. Proposition. *Let $A, B \in Af(G, \mu)$, μ and ν be probability measures, $\mu \in M_l(G, H)$, $\nu \in M(H)$. Then the function $\zeta(x) := \mu(A \cap xB)$ is continuous on H and $\nu(yB^{-1} \cap H) \in L^1(H, \nu, \mathbb{C})$. Moreover, if $\mu(A)\mu(B) > 0$, $\mu(\{y \in G : yB^{-1} \cap H \in Af(H, \nu) \text{ and } \nu(yB^{-1} \cap H) > 0\}) > 0$, then $\zeta(x) \neq 0$ on H .*

Proof. Let $g_x(y) := ch_A(y)ch_B(x^{-1}y)$, then $g_x(y) \in L_H^q(G, \mu, \mathbb{C})$ for $1 < q < \infty$, where $ch_A(y)$ is the characteristic function of A . In view of propositions 6 and 7 there exists $\lim_{i \rightarrow \infty} \psi_i * g_x = g_x$ in $L_H^1(G, \mu, \mathbb{C})$. In view of lemma III.11.18 [6] and lemma 8, $\zeta(x)|_H$ is continuous. There is the following inequality:

$$1 \geq \int_H \mu(A \cap xB) \nu(dx) = \int_H \int_G ch_A(y)ch_B(x^{-1}y) \mu(dy) \nu(dx).$$

In view of Fubini theorem there exists

$$\int_H ch_B(x^{-1}y) \nu(dy) = \nu((yB^{-1}) \cap H) \in L^1(G, \mu, \mathbb{C}), \text{ hence}$$

$$\int_H \mu(A \cap xB) \nu(dx) = \int_G \nu(yB^{-1} \cap H) ch_A(y) \mu(dy).$$

10. Corollary. *Let $A, B \in Af(G, \mu)$, $\nu \in M(H)$ and $\mu \in M_l(G, H)$ be probability measures. Then denoting $\text{Int}_H V$ the interior of a subset V of H with respect to τ_H , one has*

(i) $\text{Int}_H(AB) \cap H \neq \emptyset$, when

$$\mu(\{y \in G : \nu(yB \cap H) > 0\}) > 0;$$

(ii) $\text{Int}_H(AA^{-1}) \ni e$, when

$$\mu(\{y \in G : \nu(yA^{-1} \cap H) > 0\}) > 0.$$

Proof. $AB \cap H \supset \{x \in H : \mu(A \cap xB^{-1}) > 0\}$.

11. Corollary. Let $G = H$. If $\mu \in M_l(G, H)$ is a probability measure, then G is a locally compact topological group.

Proof. Let us take $\nu = \mu$ and $A = C \cup C^{-1}$, where C is a compact subset of G with $\mu(C) > 0$, whence $\mu(yA) > 0$ for each $y \in G$ and inevitably $\text{Int}_G(AA^{-1}) \ni e$.

12. Lemma. Let $\mu \in M_l(G, H)$ be a probability measure and G be non-locally compact. Then $\mu(H) = 0$.

Proof. This follows from theorem 19.2 [25] and theorem 3.21 and lemma 3.26 [19] and the proof of lemma 4, since the embedding $T_e H \hookrightarrow T_e G$ is a compact operator in the non-Archimedean case and of trace class in the real case (see also the papers about construction of quasi-invariant measures on the groups considered here [3, 8, 10, 12, 13, 14, 16, 17, 18], [20, 21, 24]). Indeed, the measure μ on G is induced by the corresponding measure ν on a Banach space Z for which there exists a local diffeomorphism $A : W \rightarrow V$, where W is a neighbourhood of e in G and V is a neighbourhood of 0 in Z . The measure μ on G is quasi-invariant relative to H . Therefore, the measure ν on U is quasi-invariant relative to the action of elements $\psi \in W' \subset W \cap H$ due to the local diffeomorphism A , that is, ν_ϕ is equivalent to ν for each $\phi := A\psi A^{-1}$, where $AW'A^{-1}U \subset V$, W' is an open neighbourhood of e in H and U is an open neighbourhood of 0 in Z , $\nu_\phi(E) := \nu(\phi^{-1}E)$, ϕ is an operator on Z such that it may be non-linear. The quasi-invariance factor $\rho_\nu(\phi, v)$ has expression through $|\det(\phi')|$ and the quasi-invariance factor $q_\nu(z, x)$ relative to linear shifts $z \in Z'$ given by theorems from §26 [25] in the real case and theorem 3.28 [19] in the non-Archimedean case:

$$\nu_\phi(dx)/\nu(dx) = |\det\{\phi'(\phi^{-1}(x))\}|q_\nu(x - \phi^{-1}(x), x),$$

where $x \in U$, $\phi = A\psi A^{-1}$, $\psi \in W'$. Then $(A\psi A^{-1}v - v) \in Z'$ for each $v \in V$ and $\psi \in W'$, where ν on Z is quasi-invariant relative to shifts on vectors $z \in Z'$ and there exists a compact operator in the non-Archimedean case and an operator of trace class in the real case of embedding $\theta : Z' \hookrightarrow Z$ such that $\nu(Z') = 0$.

13. Theorem. *Let (G, τ_G) and (H, τ_H) be a pair of topological non-locally compact groups G, H (Banach-Lie, Frechet-Lie or groups of diffeomorphisms or loop groups) with uniformities τ_G, τ_H such that H is dense in (G, τ_G) and there is a probability measure $\mu \in M_1(G, H)$ with continuous $d_\mu(z, g)$ on $H \times G$. Also let X be a Hilbert space over $\mathbb{C}$ and $U(X)$ be the unitary group. Then (1) if $T : G \rightarrow U(X)$ is a weakly continuous representation, then there exists $T' : G \rightarrow U(X)$ equal μ -a.e. to T and $T'|_{(H, \tau_H)}$ is strongly continuous; (2) if $T : G \rightarrow U(X)$ is a weakly measurable representation and X is separable, then there exists $T' : G \rightarrow U(X)$ equal to T μ -a.e. and $T'|_{(H, \tau_H)}$ is strongly continuous.*

Proof. Let $R(G) := (I) \cup L^1(G, \mu)$, where I is the unit operator on L^1 . Then we can define

$$A_{(\lambda e + a)_h} := \lambda I + \int_G a_h(g) [d(h^{-1}, g)] T_g \mu(dg),$$

where $a_h(g) := a(h^{-1}g)$. Then

$$|(A_{(\lambda e + a)_h} - A_{\lambda e + a})\xi, \eta| \leq \int_G |a_h(g)d_\mu(h, g) - a(g)| |(T_g \xi, \eta)| \mu(dg),$$

hence A_{a_h} is strongly continuous with respect to $h \in H$, that is,

$$\lim_{h \rightarrow e} |A_{a_h} \xi - A_a \xi| = 0.$$

Denote $A_{a_h} = T'_h A_a$ as in §29 [22], so $T'_h \xi = A_{a_h} \xi$, where $\xi = A_a \xi_0$, $a \in L^1$. Whence

$$\begin{aligned} (T'_h \xi, T'_h \xi) &= (A_{a_h} \xi_0, A_{a_h} \xi_0) = \\ &= \int_G \bar{a}_h(g) (T_g \xi_0, T_{g'} \xi_0) d_\mu(h^{-1}, g) d_\mu(h^{-1}, g') a_h(g') \mu(dg) \mu(dg') \\ &= \int_G \bar{a}(z) a(z') (U_z \xi_0, U_{z'} \xi_0) \mu(dz) \mu(dz') = (\xi, \xi). \end{aligned}$$

Therefore, T'_h is uniquely extended to a unitary operator in the Hilbert space $X' \subset X$. In view of lemma 12, $\mu(H) = 0$. Hence T' may be considered equal to T μ -a.e. Then the space $\text{span}_{\mathbb{C}}[A_{a_h} : h \in H]$ is evidently dense in X , since

$$\begin{aligned} (A_{a_h} \xi_1, A_{x_q} \xi_0) &= \left(\int_G a_h(g) T_g d(h^{-1}, g) \mu(dg) \xi_1, \int_G x_q(g') T_{g'} d(q^{-1}, g') \mu(dg') \xi_0 \right) = \\ &= (T_h \int_G a(g) T_g \mu(dg) \xi_1, T_q \int_G x(g') T_{g'} \mu(dg')) = (T_{q^{-1}h} A_a \xi_1, A_x \xi_0). \end{aligned}$$

For proving the second statement let

$R := [\xi : A_a \xi = 0 \text{ for each } a \in L^1(G, \mu)]$. If

$$(A_a \xi, \eta) = \int_G a(g)(T_g \xi, \eta) \mu(dg) = \int_G a(g)(T'_g \xi, \eta) \mu(dg)$$

for each $a(g) \in L^1(G, \mu, \mathbb{C})$, then $(T_g \xi, \eta) = (T'_g \xi, \eta)$ for μ -almost all $g \in G$. Suppose that $\{\xi_n : n \in \mathbb{N}\}$ is a complete orthonormal system in X . If $\xi \in X$, then

$$\int_G a(g)(T_g \xi, \xi_m) \mu(dg) = 0$$

for each $g \in G \setminus S_m$, where $\mu(S_m) = 0$. Therefore, $(T_g \xi, \xi_m) = 0$ for each $m \in \mathbb{N}$, if $g \in G \setminus S$, where $S := \bigcup_{m=1}^{\infty} S_m$. Hence $T_g \xi = 0$ for each $g \in G \setminus S$, consequently, $\xi = 0$. Then $(T_g \xi_n, \xi_m) = (T'_g \xi_n, \xi_m)$ for each $g \in G \setminus \gamma_{n,m}$, where $\mu(\gamma_{n,m}) = 0$. Hence $(T_g \xi_n, \xi_m) = (T'_g \xi_n, \xi_m)$ for each $n, m \in \mathbb{N}$ and each $g \in G \setminus \gamma$, where $\gamma := \bigcup_{n,m} \gamma_{n,m}$ and $\mu(\gamma) = 0$. Therefore, $R = 0$.

14. Definition and note. Let $\{G_i : i \in \mathbb{N}_0\}$ be a sequence of topological groups such that $G = G_0$, $G_{i+1} \subset G_i$ and G_{i+1} is dense in G_i for each $i \in \mathbb{N}_0$ and their topologies are denoted τ_i , $\tau_i|_{G_{i+1}} \subset \tau_{i+1}$ for each i , where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. Suppose that these groups are supplied with real probability quasi-invariant measures μ^i on G_i relative to G_{i+1} . For example, such sequences exist for groups of diffeomorphisms or loop groups considered in previous papers [10, 12, 13, 15, 16, 17, 18], [20, 21]. Let $L^2_{G_{i+1}}(G_i, \mu^i, \mathbb{C})$ denotes a subspace of $L^2(G_i, \mu^i, \mathbb{C})$ as in §1(b). Such spaces are Banach and not Hilbert in general. Let $\tilde{L}^2(G_{i+1}, \mu^{i+1}, L^2(G_i, \mu^i, \mathbb{C})) := H_i$ denotes the subspace of $L^2(G_i, \mu^i, \mathbb{C})$ of elements f such that

$$\|f\|_i^2 := [\|f\|_{L^2(G_i, \mu^i, \mathbb{C})}^2 + \|f\|_i'^2]/2 < \infty, \text{ where}$$

$$\|f\|_i'^2 := \int_{G_{i+1}} \int_{G_i} |f(y^{-1}x)|^2 \mu^i(dx) \mu^{i+1}(dy).$$

Evidently H_i are Hilbert spaces due to the parallelogram identity. Let

$$f^{i+1} * f^i(x) := \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}x) \mu^{i+1}(dy)$$

denotes the convolution of $f^i \in H_i$.

15. Lemma. *The convolution $*$: $H_{i+1} \times H_i \rightarrow H_i$ is a continuous bilinear mapping.*

Proof. In view of Fubini theorem and Cauchy inequality:

$$\begin{aligned}
& \int_{G_{i+1}} \int_{G_i} |f^{i+1} * f^i(z^{-1}x)|^2 \mu^i(dx) \mu^{i+1}(dz) = \\
& \int_{G_{i+1}} \int_{G_i} \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}z^{-1}x) \mu^{i+1}(dy) \int_{G_{i+1}} \bar{f}^{i+1}(q) \bar{f}^i(q^{-1}z^{-1}x) \mu^{i+1}(dq) \mu^i(dx) \mu^{i+1}(dz) \\
& \leq \int_{G_i} \int_{G_{i+1}} \left(\int_{G_{i+1}} |f^{i+1}(y)|^2 \mu^{i+1}(dy) \right)^{1/2} \left(\int_{G_{i+1}} |f^{i+1}(q)|^2 \mu^{i+1}(dq) \right)^{1/2} \\
& \left(\int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy) \right)^{1/2} \left(\int_{G_{i+1}} |f^i(q^{-1}z^{-1}x)|^2 \mu^{i+1}(dq) \right)^{1/2} \mu^i(dx) \mu^{i+1}(dz) \leq \\
& \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})}^2 \int_{G_i} \left[\int_{G_{i+1}} \int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy) \mu^{i+1}(dz) \right]^{1/2} \mu^i(dx) \\
& \left[\int_{G_{i+1}} \int_{G_{i+1}} |f^i(q^{-1}z^{-1}x)|^2 \mu^{i+1}(dq) \mu^{i+1}(dz) \right]^{1/2} \mu^i(dx) \leq \\
& \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})}^2 \int_{G_{i+1}} \int_{G_i} \int_{G_{i+1}} |f^i(y^{-1}z^{-1}x)|^2 \mu^{i+1}(dy) \mu^{i+1}(dz) \mu^i(dx) \\
& = \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})}^2 \left(\int_{G_i} \int_{G_{i+1}} \int_{G_{i+1}} |f^i(y^{-1}\gamma)|^2 \mu^{i+1}(dy) \mu^{i+1}(dz) d\mu^i(z^{-1}, \gamma) \mu^i(d\gamma) \right) \leq \\
& \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})}^2 \int_{G_i} \int_{G_{i+1}} |f^i(z^{-1}x)|^2 \mu^{i+1}(dz) \mu^i(dx), \text{ since} \\
& \int_{G_i} \int_{G_{i+1}} d\mu^i(z^{-1}, \gamma) \mu^i(d\gamma) \mu^{i+1}(dz) = \int_{G_{i+1}} \mu^{i+1}(dz) \int_{G_i} \mu^i(zd\gamma) = 1. \text{ Then} \\
& \|f^{i+1} * f^i\|_{L^2(G_i, \mu^i, \mathbb{C})}^2 = \int_{G_i} \left| \int_{G_{i+1}} f^{i+1}(y) f^i(y^{-1}x) \mu^{i+1}(dy) \right|^2 \mu^i(dx) \\
& \leq \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})}^2 \int_{G_i} \int_{G_{i+1}} |f^i(z^{-1}x)|^2 \mu^{i+1}(dz) \mu^i(dx). \text{ Therefore,} \\
& \|f^{i+1} * f^i\|_i \leq \|f^{i+1}\|_{L^2(G_{i+1}, \mu^{i+1}, \mathbb{C})} \|f^i\|_i.
\end{aligned}$$

16. Definition. Let $l_2(\{H_i : i \in \mathbf{N}_0\}) =: H$ be the Hilbert space consisting of elements $f = (f^i : f^i \in H_i, i \in \mathbf{N}_0)$, for which

$$\|f\|^2 := \sum_{i=0}^{\infty} \|f^i\|_i^2 < \infty.$$

For elements f and $g \in H$ their convolution is defined by the formula: $f * g := h$ with $h^i := f^{i+1} * g^i$ for each $i \in \mathbf{N}_0$. Let $*$: $H \rightarrow H$ be an involution

such that $f^* := (\bar{f}^j : j \in \mathbf{N}_0)$, where $f^{j\wedge}(y_j) := f^j(y_j^{-1})$ for each $y_j \in G_j$, $f := (f^j : j \in \mathbf{N}_0)$, $\bar{z}$ denotes the complex conjugated $z \in \mathbb{C}$.

17. Lemma. *H is a non-associative non-commutative Hilbert algebra with involution $*$, that is $*$ is conjugate-linear and $f^{**} = f$ for each $f \in H$.*

Proof. In view of Lemma 15 the convolution $h = f * g$ in the Hilbert space H has the norm $\|h\| \leq \|f\| \|g\|$, hence is a continuous mapping from $H \times H$ into H . From its definition it follows that the convolution is bilinear. It is non-associative as follows from the computation of i -th terms of $(f * g) * q$ and $f * (g * q)$, which are $(f^{i+2} * g^{i+1}) * q^i$ and $f^{i+1} * (g^{i+1} * q^i)$ respectively, where f, g and $q \in H$. It is non-commutative, since there are f and $g \in H$ for which $f^{i+1} * g^i$ are not equal to $g^{i+1} * f^i$. Since $f^{j\wedge}(y_j) = f^j(y_j)$ and $\bar{z} = z$, one has $f^{**} = (f^*)^* = f$.

18. Note. In general $(f * g^*)^* \neq g * f^*$ for f and $g \in H$, since there exist f^j and g^j such that $g^{j+1} * (f^j)^* \neq (f^{j+1} * (g^j)^*)^*$. If $f \in H$ is such that $f^j|_{G_{j-1}} = f^{j+1}$, then

$$((f^{j-1})^* * f^j)(e) = \int_{G_{j+1}} \bar{f}^{j+1}(y^{-1}) f^{j+1}(y) \mu^{j+1}(dy) = \|f^{j+1}\|_{L^2(G_{j+1}, \mu^{j+1}, \mathbb{C})}^2,$$

where $j \in \mathbf{N}_0$.

19. Definition. Let $l_2(\mathbb{C})$ be the standard Hilbert space over the field $\mathbb{C}$ be considered as a Hilbert algebra with the convolution $\alpha * \beta = \gamma$ such that $\gamma^i := \alpha^{i+1} \beta^i$, where $\alpha := (\alpha^i : \alpha^i \in \mathbb{C}, i \in \mathbf{N}_0)$, α, β and $\gamma \in l_2(\mathbb{C})$.

20. Note. The algebra $l_2(\mathbb{C})$ has two-sided ideals $J_i := \{\alpha \in l_2(\mathbb{C}) : \alpha^j = 0 \text{ for each } j > i\}$, where $i \in \mathbf{N}_0$. That is, $J * l_2(\mathbb{C}) \subset J$ and $l_2(\mathbb{C}) * J = J$ and J is the $\mathbb{C}$ -linear subspace of $l_2(\mathbb{C})$, but $J * l_2(\mathbb{C}) \neq J$. There are also right ideals, which are not left ideals: $K_i := \{\alpha \in l_2(\mathbb{C}) : \alpha^j = 0 \text{ for each } j = 0, \dots, i\}$, where $j \in \mathbf{N}_0$. That is, $l_2(\mathbb{C}) * K_i = K_i$, but $K_i * l_2(\mathbb{C}) = K_{i-1}$ for each $i \in \mathbf{N}_0$, where $K_{-1} := l_2(\mathbb{C})$. The algebra $l_2(\mathbb{C})$ is the particular case of H , when $G_j = \{e\}$ for each $j \in \mathbf{N}_0$. We consider further H for non-trivial topological groups outlined above.

21. Theorem. *If F is a maximal proper left or right ideal in H , then H/F is isomorphic as the nonassociative noncommutative algebra over $\mathbb{C}$ with $l_2(\mathbb{C})$.*

Proof. Since F is the ideal, it is the $\mathbb{C}$ -linear subspace of H . Suppose, that there exists $j \in \mathbf{N}_0$ such that $f^j = 0$ for each $f \in F$, then $f^i = 0$ for each $i \in \mathbf{N}_0$, since the space of bounded complex-valued continuous functions $C_b^0(G_\infty, \mathbb{C})$ on $G_\infty := \bigcap_{j=0}^\infty G_j$ is dense in each $H_j := \{f^j : f \in H\}$

and $C_b^0(G_\infty, \mathbb{C}) \cap F_j = \{0\}$ and $C_b^0(G_j, \mathbb{C})|_{G_{j+1}} \supset C_b^0(G_{j+1}, \mathbb{C})$. Therefore, $F_j \neq \{0\}$ for each $j \in \mathbb{N}_0$, consequently, $\mathbb{C} \hookrightarrow F_j$ for each $j \in \mathbb{N}_0$. Since $\mathbb{C}$ is embeddable into each F_j , then there exists the embedding of $l_2(\mathbb{C})$ into F , where $H_j := \{f^j : f \in H\}$, $\pi_j : H \rightarrow H_j$ are the natural projections.

The subalgebra F is closed in H , since H is a topological algebra and F is a maximal proper subalgebra. The space $H_\infty := \bigcap_{j \in \mathbb{N}_0} H_j$ is dense in each H_j and the group $G_\infty := \bigcap_{j \in \mathbb{N}_0} G_j$ is dense in each G_j .

Suppose that $F_i = H_i$ for some $i \in \mathbb{N}_0$, then $F_j = H_j$ for each $j \in \mathbb{N}_0$, since $C_b^0(G_\infty, \mathbb{C})$ is dense in each H_j and $C_b^0(G_j, \mathbb{C})|_{G_{j+1}} \supset C_b^0(G_{j+1}, \mathbb{C})$. The ideal F is proper, consequently, $F_j \neq H_j$ as the $\mathbb{C}$ -linear subspace for each $j \in \mathbb{N}_0$, where $F_j = \pi_j(F)$.

There are linear continuous operators from $l_2(\mathbb{C})$ into $l_2(\mathbb{C})$ given by the following formulas: $x \mapsto (0, \dots, 0, x^0, x^1, x^2, \dots)$ with 0 as n coordinates at the beginning, $x \mapsto (x^n, x^{n+1}, x^{n+2}, \dots)$ for $n \in \mathbb{N}$; $x \mapsto (x^{kl+\sigma_k(i)} : k \in \mathbb{N}_0, i \in (0, 1, \dots, l-1))$, where $\mathbb{N} \ni l \geq 2$, $\sigma_k \in S_l$ are elements of the symmetric group S_l of the set $(0, 1, \dots, l-1)$. Then $f \star (g \star h) + l_2(\mathbb{C})$ and $(f \star g) \star h + l_2(\mathbb{C})$ are considered as the same class, also $f \star g + l_2(\mathbb{C}) = g \star f + l_2(\mathbb{C})$ in $H/l_2(\mathbb{C})$, since $(f + l_2(\mathbb{C})) \star (g + l_2(\mathbb{C})) = f \star g + l_2(\mathbb{C})$ for each f, g and $h \in H$. For each $f, g, h \in F$: $f \star (g \star h) + l_2(\mathbb{C})$ and $(f \star g) \star h + l_2(\mathbb{C})$ are considered as the same class, also $f \star g + l_2(\mathbb{C}) = g \star f + l_2(\mathbb{C})$ in $F/l_2(\mathbb{C})$, since $(f + l_2(\mathbb{C})) \star (g + l_2(\mathbb{C})) = f \star g + l_2(\mathbb{C}) \subset F$ for each f and $g \in F$. Therefore, the quotient algebras $H/l_2(\mathbb{C})$ and $F/l_2(\mathbb{C})$ are associative commutative Banach algebras.

Let us adjoin a unit to $H/l_2(\mathbb{C})$ and $F/l_2(\mathbb{C})$. As a consequence of the Gelfand and Mazur theorem we have, that $(H/l_2(\mathbb{C}))/(\mathbb{C})$ is isomorphic with $\mathbb{C}$ (see theorem V.6.12 [6] and theorem III.11.1 [22]). On the other hand, as it was proved above $F_j \neq H_j$ for each $j \in \mathbb{N}_0$, hence there exists the following embedding $l_2(\mathbb{C}) \hookrightarrow (H/F)$ and $(H/F)/l_2(\mathbb{C})$ is isomorphic with $(H/l_2(\mathbb{C}))/(\mathbb{C})$. Therefore, H/F is isomorphic with $l_2(\mathbb{C})$.

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