P.N. Natarajan

Some more Steinhaus type theorems over valued fields

Annales mathématiques Blaise Pascal, tome 6, n° 1 (1999), p. 47-54

<http://www.numdam.org/item?id=AMBP_1999__6_1_47_0>
1. Preliminaries:

Throughout this paper, $K$ denotes $\mathbb{R}$ (the field of real numbers) or $\mathbb{C}$ (the field of complex numbers) or a complete, non-trivially valued, non-archimedean field. In the relevant context, we mention explicitly which field is chosen. Entries of infinite matrices and sequences, which occur in the sequel, are in $K$. If $X, Y$ are sequence spaces over $K$, by $(X,Y)$ we mean the class of all infinite matrices $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ such that $Ax = \{(Ax)_n\} \in Y$ whenever $x = \{x_k\} \in X$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \ n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converges. Whenever there is some notion of limit or sum in $X,Y$, we denote by $(X,Y; P)$ that subclass of $(X,Y)$ consisting of infinite matrices which preserve this limit or sum. Whatever be $K$, the sequence spaces $\ell, \gamma, \ell_0, c, \ell_\infty, \gamma_\infty$ are defined as:
\[
\ell = \{ \{ x_k \} : \sum_{k=0}^{\infty} |x_k| \text{ converges} \} ;
\]
\[
\gamma = \{ \{ x_k \} : \sum_{k=0}^{\infty} x_k \text{ converges} \} ;
\]
\[
c_0 = \{ \{ x_k \} : \lim_{k \to \infty} x_k = 0 \} ;
\]
\[
c = \{ \{ x_k \} : \lim_{k \to \infty} x_k \text{ exists} \} ;
\]
\[
\ell_\infty = \{ \{ x_k \} : \sup_{k \geq 0} |x_k| < \infty \} ;
\]
\[
\gamma_\infty = \{ \{ x_k \} : \{ s_k \} \in \ell_\infty, s_k = \sum_{i=0}^{k} x_i, k = 0, 1, 2, \ldots \} .
\]

We note that \( \ell \subset \gamma \subset c_0 \subset c \subset \ell_\infty \) and \( \gamma_\infty \subset \ell_\infty \).

\((\ell, \gamma; P)\) denotes the class of all infinite matrices \( A = (a_{nk}) \) in \((\ell, \gamma)\) such that \( \sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k \), \( x = \{ x_k \} \in \ell \).

2. The case \( K = \mathbb{R} \) or \( \mathbb{C} \)

When \( K = \mathbb{R} \) or \( \mathbb{C} \), the following result is well-known (see [6], 48, p.7).

**Theorem 2.1** A matrix \( A = (a_{nk}) \) is in \((\ell, \gamma)\) if and only if

\[
\sup_{m,k} \left| \sum_{n=0}^{m} a_{nk} \right| < \infty ;
\]  

(1)

and

\[
\sum_{n=0}^{\infty} a_{nk} \text{ converges, } k = 0, 1, 2, \ldots .
\]  

(2)

We now prove the following result when \( K = \mathbb{R} \) or \( \mathbb{C} \).

**Theorem 2.2** A matrix \( A = (a_{nk}) \) is in \((\ell, \gamma; P)\) if and only if it satisfies (1) and

\[
\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2, \ldots .
\]  

(3)
Proof. If $A$ is in $(\ell, \gamma; P)$ then (1) holds. For $k = 0, 1, 2, \ldots$, each $e_k = \{0, \ldots, 0, 1, 0, \ldots\}$, (1 occurring at the $k$th place), lies in $\ell$ and so $\sum_{n=0}^{\infty} (Ae_k)_n = 1$, i.e., $\sum_{n=0}^{\infty} a_{nk} = 1$, $k = 0, 1, 2, \ldots$ i.e., (3) holds.

Conversely, let (1) and (3) hold. It follows that $A$ is in $(\ell, \gamma)$ in view of Theorem 2.1. Let $B = (b_{mk})$ where

$$b_{mk} = \sum_{n=0}^{m} a_{nk}, \quad m, k = 0, 1, 2, \ldots$$

Using (1) and (3), we have

$$\sup_{m,k} |b_{mk}| < \infty; \quad (4)$$

and

$$\lim_{m \to \infty} b_{mk} = 1, \quad k = 0, 1, 2, \ldots. \quad (5)$$

Thus $B$ is in $(\ell, c; P')$ (see [5]). Let, now, $\{x_k\} \in \ell$. So

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} b_{mk} x_k \text{ exists and is equal to } \sum_{k=0}^{\infty} x_k,$$

i.e.,

$$\lim_{m \to \infty} \sum_{k=0}^{\infty} \left( \sum_{n=0}^{m} a_{nk} \right) x_k = \sum_{k=0}^{\infty} x_k,$$

i.e.,

$$\lim_{m \to \infty} \sum_{n=0}^{m} \left( \sum_{k=0}^{\infty} a_{nk} x_k \right) = \sum_{k=0}^{\infty} x_k,$$

i.e.,

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{nk} x_k \right) = \sum_{k=0}^{\infty} x_k,$$

i.e.,

$$\sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k.$$

In other words, $A$ is in $(\ell, \gamma; P)$, which completes the proof of the theorem.

Maddox [3] proved that $(\gamma, \gamma; P) \cap (\gamma_{\infty}, \gamma) = \emptyset$. In this context, it is worthwhile to note that the identity matrix (i.e., $I = (i_{nk})$ where $i_{nk} = 1$, if $k = n$ and $i_{nk} = 0$, if $k \neq n$) is in $(\ell, \gamma; P) \cap (\gamma_{\infty}, \gamma)$ so that $(\ell, \gamma; P) \cap (\gamma_{\infty}, \gamma) \neq \emptyset$. Since $(\gamma, \gamma) \supset (\gamma_{\infty}, \gamma)$, it follows that $(\ell, \gamma; P) \cap (\gamma, \gamma) \neq \emptyset$. We note that $(\gamma, \gamma; P) \subset (\ell, \gamma; P)$ and $(c_0, \gamma) \subset (\gamma, \gamma)$. Having
"enlarged" the class \((\gamma, \gamma; P)\) to \((\ell, \gamma; P)\), we would like to "contract" the class \((\gamma, \gamma)\) to \((c_0, \gamma)\) and attempt a Steinhaus type theorem involving the classes \((\ell, \gamma; P)\) and \((c_0, \gamma)\).

**Theorem 2.3** \((\ell, \gamma; P) \cap (c_0, \gamma) = \emptyset\).

**Proof.** Let \(A = (a_{nk})\) be in \((\ell, \gamma; P) \cap (c_0, \gamma)\). Since \(A\) is in \((c_0, \gamma)\),

\[
\sup_{m} \sum_{k=0}^{\infty} \left| \sum_{n=0}^{m} a_{nk} \right| \leq M < \infty
\]  
(see [6], p.6).

Now, for \(L = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots,\)

\[
\sum_{k=0}^{L} \sum_{n=0}^{m} a_{nk} \leq \sum_{k=0}^{\infty} \left| \sum_{n=0}^{m} a_{nk} \right| \leq M.
\]

Taking limit as \(m \to \infty\), we have,

\[
\sum_{k=0}^{L} \left| \sum_{n=0}^{\infty} a_{nk} \right| \leq M, \quad L = 0, 1, 2, \ldots.
\]

Taking limit as \(L \to \infty\), we get,

\[
\sum_{k=0}^{\infty} \left| \sum_{n=0}^{\infty} a_{nk} \right| \leq M.
\]

which is contradiction, since \(\sum_{n=0}^{\infty} a_{nk} = 1, k = 0, 1, 2 \ldots\) in view of (3). This establishes our claim.

**Corollary.** Since \(c_0 \subset c \subset \ell_\infty, (\ell_\infty, \gamma) \subset (c, \gamma) \subset (c_0, \gamma)\) so that \((\ell, \gamma; P) \cap (X, \gamma) = \emptyset\) for \(X = c_0, c, \ell_\infty\).

3. The case when \(K\) is a complete, non trivially valued, non-archimedean field.

When \(K\) is a complete, non-trivially valued, non-archimedean field, we note that \(\gamma = c_0\) and \(\gamma_\infty = \ell_\infty\). In this case, it is easy to prove the following results.

**Theorem 3.1** \((\ell, \gamma) = (\ell, c_0) = (c_0, c_0)\). A matrix \(A = (a_{nk})\) is in \((\ell, c_0)\) if and only if it satisfies
Theorem 3.2 \( (\ell, \gamma; P) = (\ell, c_0; P) = (c_0, c_0; P) = (\gamma, \gamma; P) \). A matrix \( A = (a_{nk}) \) is in \((\ell, c_0; P)\) if and only if it satisfies (3), (7) and (8).

Theorem 3.3 A matrix \( A = (a_{nk}) \) is in \((c, c_0)\) if and only if it satisfies (7), (8) and
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{9}
\]

Remark 3.4 Theorem 2.3 fails to hold when \( K \) is a complete, non-trivially valued, non-archimedean field since \((\ell, c_0) = (c_0, c_0)\). We also have
\[
(\ell, c_0; P) \cap (c, c_0) \neq \emptyset,
\]
as the following example illustrates. Consider the infinite matrix
\[
A = (a_{nk}) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 2 & -2 & 0 & 0 & 0 & \ldots \\
0 & 0 & 3 & -3 & 0 & 0 & \ldots \\
0 & 0 & 0 & 4 & -4 & 0 & \ldots \\
& & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\]
i.e., \( a_{nk} = n + 1 \) if \( k = n \);
\[= -(n + 1), \quad \text{if} \quad k = n + 1;\]
\[= 0, \quad \text{otherwise} .\]

Then (3), (7), (8) and (9) hold so that \( A \) is in \((\ell, c_0; P) \cap (c, c_0)\). These remarks point out significant departure from the case \( K = \mathbb{R} \text{ or } \mathbb{C} \).

The following lemma is needed in the sequel.

Lemma 3.5 The following statements are equivalent :
(a) A matrix \( A = (a_{nk}) \) is in \((\ell_\infty, c_0)\) ;
(b) (i) \( \lim_{k \to \infty} a_{nk} = 0 \); \hspace{1cm} (10)

\text{and}

(ii) \( \limsup_{n \to \infty} \sup_{k \geq 0} |a_{nk}| = 0 \); \hspace{1cm} (11)

(c) (i) (8) holds

\text{and}

(ii) \( \lim_{k \to \infty} \sup_{n \geq 0} |a_{nk}| = 0 \). \hspace{1cm} (12)

\textbf{Proof.} For the proof of "(a) is equivalent to (b)", see ([4], 422). We now prove that (b) and (c) are equivalent. Let us suppose that (b) holds. For every fixed \( k = 0, 1, 2, \ldots \),

\[ |a_{nk}| \leq \sup_{k' \geq 0} |a_{nk'}|. \]

Now (8) follows in view of (b) (ii). Again by (b) (ii), given \( \varepsilon > 0 \), we can choose a positive integer \( N \) such that

\[ \sup_{k \geq 0} |a_{nk}| < \varepsilon, \quad n > N. \] \hspace{1cm} (13)

In view of (b) (i), for \( n = 0, 1, 2, \ldots, N \), we can find a positive integer \( L \) such that

\[ |a_{nk}| < \varepsilon, \quad k > L. \] \hspace{1cm} (14)

(13) and (14) imply that

\[ |a_{nk}| < \varepsilon, \quad n = 0, 1, 2, \ldots, k > L. \]

i.e.,

\[ \sup_{n \geq 0} |a_{nk}| < \varepsilon, \quad k > L \]

i.e.,

\[ \lim_{k \to \infty} \sup_{n \geq 0} |a_{nk}| = 0, \]

so that (c) (ii) holds. Similarly we can prove that (c) implies (b). This establishes the lemma.

We now prove the following Steinhaus type result.

\textbf{Theorem 3.6} \hspace{0.5cm} \textit{When} \( K \) \textit{is a complete, non trivially valued, non-archimedean field, then}

\[ (\ell, c_0; P) \cap (\ell_\infty, c_0) = \emptyset. \]

\textbf{Proof.} Let \( A = (a_{nk}) \) be in \( (\ell, c_0; P) \cap (\ell_\infty, c_0) \). In view of (3), we have,
Taking limit as \( k \to \infty \) and using (12), we get \( 1 \leq 0 \), which is absurd. This proves the theorem.

In view of Theorem 3.2 and Theorem 3.6 we have the following.

**Corollary.** \((c_0, c_0; P) \cap (\ell_\infty, c_0) = \emptyset.\)

We shall now take up an application of Theorem 3.6 to analytic functions. For the theory of analytic functions in non-archimedean fields, one can refer to [2]. Consider the space of bounded analytic functions inside the disk \( d(0, 1^-) \) (usually denoted by \( A_b(d(0, 1^-)) \)), provided with the topology of uniform convergence in each disk \( d(0, r), r < 1 \) and the space of analytic elements in the disk \( d(0, 1) \) (usually denoted by \( H(d(0, 1)) \)), provided with the topology of uniform convergence on \( d(0, 1) \). Thanks to Lemma 3.5, one can check that \( (\ell_\infty, c_0) \) represents the space of continuous linear mappings from \( A_b(d(0, 1^-)) \) into \( H(d(0, 1)) \). We now have the following result, which follows from Theorem 3.6.

**Theorem 3.7** There exists no continuous linear mapping \( \phi \) from \( A_b(d(0, 1^-)) \) into \( H(d(0, 1)) \) satisfying \( \phi(f)(1) = f(1) \) for all \( f \in A_b(d(0, 1^-)) \).

I thank the referee for his useful and helpful suggestions which enabled me to present the material in a much better form.
REFERENCES


Department of Mathematics,
Ramakrishna Mission Vivekananda College,
Chennai - 600 004,
INDIA