ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 12, n°2 (2005), p. 331-345.

<http://ambp.cedram.org/item?id=AMBP_2005__12_2_331_0>

Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France

Annales Mathematiques Blaise Pascal **12**, 331-345 (2005)

The Laguerre inequality and the distribution of zeros of entire functions

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Abstract

The Laguerre inequality and the distribution of zeros of real entire functions are investigated with the aid of certain infinite-order differential operators. The paper includes new proofs, problems, conjectures and many illustrative examples and counterexamples.

1 Introduction

Let $S(A)$ denote the closed strip of width $2A$ in the complex plane $\mathbb{C}$ symmetric about the real axis:

$$S(A) = \{z \in \mathbb{C} \mid |\Im(z)| \leq A\},$$

where $A \geq 0$.

Definition 1.1: Let A be such that $0 \leq A < \infty$. We say that a real entire function f belongs to the class S if f is of the form

$$f(z) = Ce^{-az^2+bz} z^m \prod_{k=1}^{\infty} (1 - z/z_k) e^{z/z_k}, \tag{1.1}$$

where $a \geq 0$, $z_k \in S(A) \setminus \{0\}$, $b \in \mathbb{R}$ and $\sum_{k=1}^{\infty} 1/|z_k|^2 < \infty$.

We allow functions in $\mathfrak{S}(A)$ to have only finitely many zeros by letting, as usual, $z_k = \infty$ and $0 = 1/z_k$, $k \geq k_0$, so that the canonical product in (1.1) is a finite product.

Definition 1.2: If a function $f \in \mathfrak{S}(A)$, for some $A \geq 0$, and if f has only real zeros (i.e., if $A = 0$), then f is said to belong to the *Laguerre-Pólya class*, and we write $f \in \mathcal{L}\text{-}\mathcal{P}$. We write $f \in \mathcal{L}\text{-}\mathcal{P}^*$ if $f = pg$, where $g \in \mathcal{L}\text{-}\mathcal{P}$ and

p is a real polynomial. Thus, $f \in \mathcal{L}\text{-}\mathcal{P}^*$ if and only if $f \in \mathfrak{S}(A)$, for some $A \geq 0$, and f has at most finitely many non-real zeros.

In the sequel, the Laguerre expression, $(f')^2 - ff''$, associated with an entire function f, will be denoted by $L[f]$; that is,

$$L[f] := (f')^2 - ff''.$$

Observe that if

$$p(z) = C\prod_{k=1}^{n}(z - a_k)$$

is a non-constant real polynomial with only real zeros, then

$$L[p](x) = -p^2(x)\left(\frac{p'}{p}\right)'(x) = p^2(x)\sum_{k=1}^{n}\frac{1}{(x-a_k)^2} \geq 0, \tag{1.2}$$

for all $x \in \mathbb{R}$ and equality holds in (1.2) if and only if x is a multiple zero of p.

A function f is in $\mathcal{L}\text{-}\mathcal{P}$ if and only if f is the uniform limit on compact subsets of $\mathbb{C}$ of real polynomials having only real zeros (see [9, Chapter VIII], [10, p. 31], or [12, p. 105, Satz I]. Thus, it follows that if $f \in \mathcal{L}\text{-}\mathcal{P}$, then $L[f](x) \geq 0$ for all $x \in \mathbb{R}$. Moreover, a calculation, similar to the one carried out for p in (1.2) (cf. Theorem 2.1 below), shows that if $f \in \mathcal{L}\text{-}\mathcal{P}$ is not of the form Ce^{bx}, where $C, b \in \mathbb{R}$, (a convention we adopt henceforth), then $L[f](x_0) = 0$ if and only if x_0 is a multiple zero of f. The main goal of a recent paper [6] was to investigate possible converses to this observation. The fundamental problem is to find conditions on the Laguerre expression of a function in $\mathfrak{S}(A)$ that imply that the function has only real zeros. A short computation shows that for $f(z) = z(z^2+1) \in \mathfrak{S}(1)$, $L[f](x) = 3x^4 + 1$, and so some hypotheses beyond $L[f](x) > 0$, for all $x \in \mathbb{R}$, are required if one hopes to conclude that f has only real zeros.

This paper is organized as follows. In Section 2 we recall several (real and complex) forms of the Laguerre inequality and we review some known necessary and sufficient conditions for an entire function in $\mathfrak{S}(A)$ to possess only real zeros. These results (see, for example, Theorem 2.2) point to (nay, underscore) the need for other versions of Laguerre inequality, which are more readily applicable in concrete settings. To this end, we prove an extended version of a strict Laguerre inequality (Theorem 3.2) with the aid of a family (depending on a real parameter) of infinite-order differential operators. These considerations lead to several open problems and conjectures (Section 3). In Section 4, we appeal to the Hermite-Biehler theory to give a new proof of a known result (Theorem 4.5). In this section (indeed throughout the paper) we provide a generous array of illustrative examples and counterexamples. Although we are striving for simplicity, some of these examples are rather involved and are best handled with the aid of a computer. While we are unable to prove Conjecture 3.5, in Section 5, we establish a few partial results (Theorem 5.1, Lemma 5.3 and Proposition 5.4) which provide additional, new evidence for the validity of Conjecture 3.5.

2 Real and complex versions of the Laguerre inequality

For ease of reference, we summarize here some necessary and sufficient conditions that a real entire function must satisfy in order that it belong to the Laguerre-Pólya class, $\mathcal{L}$-$\mathcal{P}$. While the fundamental results reviewed here are of great theoretical importance, they are not readily applicable in concrete settings (see Theorem 2.2 and the subsequent comments and examples). We begin with one of the simplest necessary conditions for membership in $\mathcal{L}$-$\mathcal{P}$.

Theorem 2.1: (Laguerre inequality: Real version I, [4], [3], [2])
If $\varphi(x) \in \mathcal{L}$-$\mathcal{P}$, then

$$L[\varphi](x) := (\varphi'(x))^2 - \varphi(x)\varphi''(x) \geq 0 \quad \forall \quad x \in \mathbb{R}. \tag{2.1}$$

Moreover, equality holds if and only if $\varphi(x)$ is of the form Ce^{bx} or x is a multiple zero of $\varphi(x)$.

Proof: If $\varphi(x)$ is of the form Ce^{bx}, then $L[\varphi](x) = 0$ for all $x \in \mathbb{R}$. Otherwise, consider

$$\varphi(x) := Ce^{-ax^2+bx}x^m \prod_{k=1}^{\infty}(1 - x/x_k)e^{x/x_k}, \tag{2.2}$$

where $a \geq 0$, $x_k \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ and $\sum_{k=1}^{\infty} 1/|x_k|^2 < \infty$ (cf. Definition 1.2). Then the negative of the derivative of the logarithmic derivative of $\varphi(x)$ is given by

$$-\frac{d}{dx}\left(\frac{\varphi'(x)}{\varphi(x)}\right) = \frac{L[\varphi](x)}{(\varphi(x))^2} = \frac{m}{x^2} + 2a + \sum_{k=1}^{\infty} \frac{1}{(x - x_k)^2} > 0.$$

Thus, it follows that if $\varphi(x)$ is *not* of the form Ce^{bx}, then the Laguerre expression $L[\varphi](x)$ is zero at a point x if and only if x is a multiple zero of $\varphi(x)$. □

We remark that since the class $\mathcal{L}$-$\mathcal{P}$ is closed under differentiation, it follows from Theorem 2.1 that the Laguerre expression, $L[\varphi^{(k)}](x)$, is non-negative for all real x. A generalization of the above inequality for real entire functions (and in particular for the class $\mathfrak{S}(A)$, cf. Definition 1.1), yields the following necessary and sufficient conditions.

Theorem 2.2: (Laguerre inequality: Real version II, [4, Theorem 2.9])
Let $f \in \mathfrak{S}(A)$. For $n = 0, 1, 2 \ldots$, set

$$L_n[f](x) := \sum_{k=0}^{2n} \frac{(-1)^{n+k}}{(2n)!} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x) \quad (x \in \mathbb{R}). \tag{2.3}$$

Then $f \in \mathcal{L}$-$\mathcal{P}$ if and only if $L_n[f](x) \geq 0$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2 \ldots$.

By way of illustration, we observe that $L_0[f](x) = (f(x))^2$, $L_1[f](x) = L[f](x)$, a notational convention we will adopt henceforth, and $L_2[f](x) = (3\, f''(x)^2 - 4\, f'(x)\, f^{(3)}(x) + f(x)\, f^{(4)}(x))/12$. Simple examples show that $L_1[f](x)$ can be negative for some $x \in \mathbb{R}$, while $L_2[f](x) \geq 0$ for all $x \in \mathbb{R}$. If $f(x) = e^x(x^2+1)$, then $L_1[f](x) = 2e^{2x}(x^2-1)$, while $L_2[f](x) = e^{2x}$ and $L_n[f] = 0$ for all $n \geq 3$. On the other hand, if $f(x) = e^{-x^2}(x^2+1)$, then $L_1[f](x) = 2e^{-2x^2}x^2(x^2+3) \geq 0$, for all x, while $L_2[f](x) = e^{-2x^2}(-1+8x^2+2x^4)$ is negative for $|x|$ sufficiently small. The verification of the inequalities $L_n[f](x) \geq 0$ *for all* $x \in \mathbb{R}$ and *for all* $n \geq 1$ is non-trivial, except in some very special cases.

We next turn to the complex versions of the Laguerre inequality. These are significant not only because they provide necessary and sufficient conditions for membership in the Laguerre-Pólya class, but also on account of the their geometric content.

Theorem 2.3: (Laguerre inequality: Complex version I, [4, Theorem 2.12])
Let $f \in \mathfrak{S}(A)$. Then $f(z) \in \mathcal{L}$-$\mathcal{P}$ if and only if

$$\frac{1}{y}\Im\{-f'(z)f(z)\} \geq 0 \quad \forall z = x+iy \in \mathbb{C},\ y \neq 0. \tag{2.4}$$

Theorem 2.4: (Laguerre inequality: Complex version II, [4, Theorem 2.10], [7]) *Let $f \in \mathfrak{S}(A)$. Then $f \in \mathcal{L}$-$\mathcal{P}$ if and only if*

$$|f'(z)|^2 \geq \Re\{f(z)\overline{f''(z)}\}, \quad z \in \mathbb{C}. \tag{2.5}$$

The proof of Theorem 2.4 is based on the geometric interpretation of inequality (2.5), which is just that $|f(x+iy)|^2$ is a convex function of y.

3 Infinite order differential operators and the simplicity of zeros

The purpose of this section is to extend the Laguerre inequality, (cf. Theorem 2.1), to a family of entire functions, with the aid of certain infinite-order differential operators. Also, here we lay the groundwork for the analysis of the question, when and under what additional hypotheses is the converse of Theorem 2.1 valid.

Throughout this paper $D := \frac{d}{dx}$ will denote differentiation with respect to x. If

$$\varphi(x) = \sum_{k=0}^{\infty} \alpha_k x^k \qquad (\alpha_k \in \mathbb{R})$$

is a real entire function, we define the operator $\varphi(D)$ by

$$\varphi(D)f(x) = \sum_{k=0}^{\infty} \alpha_k f^{(k)}(x), \tag{3.1}$$

whenever the right-hand side of (3.1) represents an entire function. Here, and below, we will confine our attention to the case when f is an entire function. The operator $\varphi(D)$ has been studied by several authors (see, for example, [1, §11], [5], [3], [2], [8], [9, Chapter IX] and [13]). We hasten note that, in general, some restrictions on the growth of φ or f need to be stipulated, in order that the right-hand side of (3.1) represent an entire function (see, for example, [3, Lemma 3.1] and the references cited therein).

In connection with his study of the distribution of zeros of the Riemann ξ-function, Pólya has shown (cf. [12, p. 296, Hilfsatz III], or [11, p. 21, Hilfsatz III]) that if $f(x)$ is a polynomial possessing only real zeros and if $\varphi(x)$ is a transcendental function in $\mathcal{L}$-$\mathcal{P}$, where $\varphi(x)$ is not of the form $p(x)e^{\alpha x}$, where $p(x)$ is a polynomial, then the polynomial $\varphi(D)f(x)$ has only simple real zeros. It seems natural to ask if this result can be extended to the situation when $f(x)$ is a transcendental entire function in the Laguerre-Pólya class.

In [3, Theorems 4.6 and 4.7] the authors were able to prove this whenever the canonical product in the representation of φ has genus zero or there is a bound on the multiplicities of the zeros of $f(x)$. (A separate analysis shows that if $f \in \mathcal{L}\text{-}\mathcal{P}$ has order less than 2, then for each fixed $t > 0$, the entire function $e^{-tD^2} f(x)$ has only *simple* real zeros.) The question whether or not the aforementioned restrictions on $f(x)$ are necessary, has been recently answered by D. A. Cardon and S. A. de Gaston [2]. They proved the following beautiful and useful result.

Theorem 3.1: (D. A. Cardon and S. A. de Gaston[2]) *Let $\varphi, f \in \mathcal{L}\text{-}\mathcal{P}$, where $\varphi(x) := e^{-\alpha x^2}\varphi_1(x)$, $f(x) := e^{-\beta x^2} f_1(x)$, φ_1 and f_1 have genus 0 or 1, and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$ and if φ has an infinite number of zeros, then $\varphi(D)f(x)$ has only simple real zeros.*

In view of the proof of Theorem 2.1 and as a consequence of Theorem 3.1, here we obtain a family of functions, depending on a real parameter, μ, in the Laguerre-Pólya class such that each member of the family satisfies the *strict* Laguerre inequality.

Theorem 3.2: *Let $\varphi, f \in \mathcal{L}\text{-}\mathcal{P}$, where φ and f have order less than 2 and φ has an infinite number of zeros. For each $\mu \in \mathbb{R}$, $\mu \neq 0$, let $f_{\mu,\varphi}(x) := \varphi(\mu D)f(x)$. If $f_{\mu,\varphi}(x)$ is not of the form Ce^{bx}, for some real constants C and b, then the following strict Laguerre inequality holds for each fixed $\mu \in \mathbb{R}$, $\mu \neq 0$,*

$$L[f_{\mu,\varphi}](x) = (f'_{\mu,\varphi}(x))^2 - f_{\mu,\varphi}(x)f''_{\mu,\varphi}(x) > 0 \quad \forall \quad x \in \mathbb{R}. \tag{3.2}$$

We alert the reader that the notation $f_{\mu,\varphi}$ employed here has a different meaning from a similar notation used in [6]. The fact that $f_{\mu,\varphi}(x) \in \mathcal{L}\text{-}\mathcal{P}$ is well known (cf. [3] and the references contained therein).

Our investigations and those in [6] have led us to the following open problem.

Open problem 3.3. For each $\mu \in \mathbb{R}$, $\mu \neq 0$, let $f_{\mu,\varphi}(x) := \varphi(\mu D)f(x)$, where $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ has order less than 2 and $f \in \mathfrak{S}(A)$. Characterize the functions $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ such that if $L[f_{\mu,\varphi}](x) > 0$ for all $x, \mu \in \mathbb{R}$, $\mu \neq 0$, then $f \in \mathcal{L}\text{-}\mathcal{P}$.

If $\varphi \in \mathcal{L}\text{-}\mathcal{P}$ has order 2 and $f \in \mathfrak{S}(A)$, then $\varphi(\mu D)f(x)$ need not be an entire function (see, for example, [3, Lemma 3.1], or [2, Lemma 2]). Thus, some additional stipulations are required in order to formulate an analog of this open problem. To illustrate this point, consider $\varphi(x) := e^{-x^2}$. Then for any $f \in \mathfrak{S}(A)$ of order less than 2, $f_{\mu,\varphi}(x) := \varphi(\mu D)f(x)$ is an entire function, *provided* that we assume $\mu > 0$. With these assumptions at hand, we conjecture that if

$$L[f_{\mu,\varphi}](x) > 0 \quad \text{for all } \mu > 0 \quad \text{and for all x} \in \mathbb{R}, \tag{3.3}$$

then $f \in \mathcal{L}\text{-}\mathcal{P}$. The following concrete example may be instructive.

Example 3.4. Let $\varphi(x) := e^{-x^2}$ and let $f(x) = a + \sin x$, where a is a real number. Then, for $\mu > 0$, $f_{\mu,\varphi}(x) = \varphi(\mu D)f(x) = a + \cosh(\sqrt{\mu})\sin x$ and

$$L[f_{\mu,\varphi}](x) = \frac{1}{2}(1 + \cosh(2\sqrt{\mu}) + 2a\,\cosh(\sqrt{\mu})\sin x).$$

Thus, elementary considerations show that (3.3) holds if and only if $|a| \leq 1$.

The question left unanswered in [6] pertains to the special case of Open Problem 3.3, when $\varphi(x) = \cos x$ (see Conjecture 3.5 below). In the sequel, we will confine our investigation when $\varphi(x) = \cos x$. In this case, inequality (3.2) also follows from inequality (2.4) of Theorem 2.3 (cf. [6, Theorem I)]. In order to expedite our presentation, it will be convenient to adopt the following nomenclature. If $f \in \mathfrak{S}(A)$, then for $\mu \in \mathbb{R}$, we define the function f_μ by the equation

$$f_\mu(x) := 2\cos(\mu D)f(x) \qquad (D = \frac{d}{dx}). \tag{3.4}$$

We note that since $f \in \mathfrak{S}(A)$ and $\cos x$ has order 1, $f_\mu(x)$ is an entire function [3, Lemma 3.1]. In some situations it is convenient to express $f_\mu(x)$ in the following equivalent form. Since $2\cos x = e^{ix} + e^{-ix}$ and $e^{i\mu D}f(x) = f(x+i\mu)$, we deduce, from the Taylor series expansions of $f(x+i\mu)$ and $f(x-i\mu)$, in powers of x, that $f_\mu(x) = f(x+i\mu) + f(x-i\mu)$. In particular, this relation makes also clear that f_μ is a real entire function whenever f is. Thus, with the above notation, we record here a special case of Open Problem 3.3 as follows.

Conjecture 3.5. Let $f \in \mathfrak{S}(A)$ and set $f_\mu(x) := 2\cos(\mu D)f(x)$. If $L[f_\mu](x) > 0$ for all $x, \mu \in \mathbb{R}$, $\mu \neq 0$, then $f \in \mathcal{L}\text{-}\mathcal{P}$.

Now, it is known ([5]) that if $f \in \mathfrak{S}(A)$, for some $A \geq 0$, then $f_\mu \in \mathcal{L}\text{-}\mathcal{P}$ for all real μ such that $|\mu| \geq A$. It should be pointed out, however, that f_μ may be a constant function, even if f is not. For example, if $f(x) = e^x$, then $f_\mu(x) = e^x \cos\mu$ and so $f_{\pi/2} = 0$. Furthermore, it follows from an extension of the classical Hermite–Poulain theorem (cf. [10, p.4] and [12, p.129]) that if $f \in \mathcal{L}\text{-}\mathcal{P}$, then $f_\mu \in \mathcal{L}\text{-}\mathcal{P}$ for all $\mu \in \mathbb{R}$. Hence, if $f \in \mathcal{L}\text{-}\mathcal{P}$, then $L[f_\mu](x) \geq 0$ for all $x \in \mathbb{R}$ and all $\mu \in \mathbb{R}$.

Remark 3.6. It was shown in [6, Corollary I] that if $f \in \mathcal{L}\text{-}\mathcal{P}^*$ (cf. Definition 1.2) and if f_μ is not of the form Ce^{bx}, then $f \in \mathcal{L}\text{-}\mathcal{P}$ if and only if $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$. Moreover, it was observed that the strict inequality $L[f_\mu](x) > 0$ in the above statement is necessary. Indeed, consider the polynomial $f(x) = x(x^2+1)$. Then f belongs to $\mathcal{L}\text{-}\mathcal{P}^*$, but not to $\mathcal{L}\text{-}\mathcal{P}$, since f has non-real zeros. A computation shows that $L[f_\mu](x) = 4(3x^4 + (3\mu^2-1)^2) \geq 0$ for all $x, \mu \in \mathbb{R}$. We also remark that Conjecture 3.5 is also valid in the case when $f \in \mathfrak{S}(A)$ and f has an infinite number real zeros and the zeros satisfy certain density conditions ([6, Theorem III]). But are these density conditions necessary (or are they merely a ploy to render the problem tractable) in order for f to belong to $\mathcal{L}\text{-}\mathcal{P}$?

4 The Hermite-Biehler theorem, scholia and examples

The proofs of the results in [6], involve the study of the level sets of f; that is, the sets $\{z \in \mathbb{C} \mid \Re(e^{i\theta} f(z)) = 0\}$, where $\theta \in \mathbb{R}$. The connections between the Laguerre expression of f_μ, the level set $\Re(f) = 0$ and the zero set of f_μ play a fundamental role throughout [6]. One key observation in [6] is that for a real entire function f, $f_\mu(x) = 0$ if and only if $\Re(f(x+i\mu)) = 0$. In contrast, our approach here will make use of the Hermite–Biehler theory [9, Chapter VII]. We begin with the statement of the classical Hermite-Biehler theorem for polynomials.

Theorem 4.1: (The Hermite-Biehler Theorem for polynomials, [10, p. 13]) *Let*

$$f(z) = p(z) + iq(z) = a_0 \prod_{k=1}^{n} (z - \alpha_k), \quad (0 \neq a_0 \in \mathbb{R}),$$

where $p(z)$ and $q(z)$ are real polynomials of degree ≥ 2.

(1) *If $f(z)$ has all its zeros in $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, then p and q have simple, real zeros which interlace (that is, separate one another) and*

$$q'(x)p(x) - q(x)p'(x) > 0 \quad \forall\, x \in \mathbb{R}. \tag{4.1}$$

(2) (A partial converse.) *Suppose that the zeros of* p *and* q *are real, simple and interlacing. If for some real* x_0, $q'(x_0)p(x_0) - q(x_0)p'(x_0) > 0$, *then* $f(z)$ *has all its zeros in* $\mathbb{H}^+$.

In order to state an extension of Theorem 4.1 to transcendental entire functions, it will be convenient to recall here the following definition due to M. G. Krein (cf. [9, p. 307]).

Definition 4.2: An entire function $\omega(z)$ is said to be a function in *class* HB, if (i) $\omega(z)$ has no zeros in the closed lower-half plane (i.e., $\omega(z) \neq 0$ if $z \in \overline{\mathbb{H}^-} = \{z \in \mathbb{C} \mid \Im(z) \leq 0\}$) and (ii) $|\omega(z)/\overline{\omega}(z)| < 1$ for $z \in \mathbb{H}^+$, where $\overline{\omega}(z)$ denotes the entire function obtained from $\omega(z)$ by replacing the Maclaurin series coefficients of $\omega(z)$ by their conjugates.

We remark that in case $\omega(z)$ is a polynomial, then condition (ii), in the above definition, is superfluous. Indeed, (ii) follows from the fact that all the zeros of the polynomial lie in the open upper-half plane $\mathbb{H}^+$.

In the sequel, we will appeal to the following characterization of functions in the class HB.

Theorem 4.3: (The Hermite-Biehler Theorem, [9, p. 315]) *Let*

$$\omega(z) := P(z) + iQ(z), \tag{4.2}$$

where $P(z)$ *and* $Q(z)$ *are non-constant real entire functions. Then* $\omega(z)$ *is a function of class* HB *if and only if for any real constants* α *and* β, *the entire function* $\alpha P(z) + \beta Q(z)$ *has only real zeros and*

$$Q'(x)P(x) - Q(x)P'(x) > 0 \quad \forall\, x \in \mathbb{R}. \tag{4.3}$$

Remark 4.4: If the entire function $\omega(z)$ (cf. (4.2)) is a function in class HB of the form $\omega(z) = e^{-az^2} f(z)$, where $a \geq 0$ and $f(z)$ is a real entire function of genus 0 or 1, then it follows from Theorem 4.3 that the entire functions $P(z)$ and $Q(z)$ belong to the Laguerre–Pólya class. Moreover, by (4.3), the zeros of $P(z)$ and $Q(z)$ are all simple.

Preliminaries aside, we are now in position to provide a *new* proof of the following known result (cf. [6, Theorem 2.1]).

Theorem 4.5: *Let* $f \in \mathfrak{S}(A)$. *If* f *is not of the form* Ce^{bx}, *then*

$$L[f_\mu](x) > 0 \quad \text{for all } \mu > A \quad \text{and for all } x \in \mathbb{R}. \tag{4.4}$$

Proof: Fix $\mu > A$ and recall that $f_\mu \in \mathcal{L}\text{-}\mathcal{P}$ for all real μ such that $|\mu| \geq A$ (see the discussion preceding Remark 3.6). Since $f_\mu \in \mathcal{L}\text{-}\mathcal{P}$ ($|\mu| \geq A$) and f_μ is not of the form Ce^{bx}, by Theorem 2.1, it suffices to show that f_μ has only simple zeros. To this end, set $\omega(z) := f(z - i\mu)$ and observe that all the zeros of ω lie in the open upper-half plane. Thus, it follows from a well-known theorem of M. G. Krein ([9, p. 318, Theorem 6]) that $\omega(z)$ belongs to the class HB. Set $\omega(z) := P(z) + iQ(z)$, where $P(z)$ and $Q(z)$ are non-constant real entire functions. Then by Theorem 4.3 and Remark 4.4, P, $Q \in \mathcal{L}\text{-}\mathcal{P}$ and both P and Q have only simple real zeros and whence

$$f_\mu(x) = f(x + i\mu) + f(x - i\mu) = \overline{\omega(x)} + \omega(x) = 2P(x)$$

has only simple real zeros. □

We conclude this section with several examples to illustrate, in particular, that $L[f_\mu](x)$ need not be strictly positive for various types of entire functions. While the details of the computations and algebra have been duly suppressed, we caution the reader that some of these are quite involved and lengthy. Thus, for purposes of verification, the use of an appropriate symbolic computer software is advisable.

Example 4.6: (a) In the first place, we have confined our investigation to the class $\mathfrak{S}(A)$, because, in general, for entire functions, f, outside this class, $L[f_\mu](x)$ need not be non-negative, *even if f has only real zeros*! Indeed, set $f(x) := e^{x^2} \cos x$ and note that it does not belong to $\mathfrak{S}(A)$ for any $A \geq 0$. Then an elementary, albeit tedious, calculation shows that

$$f_\mu(x) = 2\, e^{x^2 - \mu^2}\ (\cos x\ \cos(2\,x\,\mu)\ \cosh\mu + \sin x\ \sin(2\,x\,\mu)\ \sinh\mu)\,, \quad \text{and}$$

$$L[f_\mu](x) = -e^{2x^2 - 2\mu^2}(2(1 - 4\mu^2)\cos(2x)\ + (-8\mu^2 + \cos(2x - 4x\mu)$$
$$+ \cos(2x + 4x\mu))\cosh(2\mu) + (8\mu + \cos(2x - 4x\mu) - \cos(2x + 4x\mu))\sinh(2\mu))$$

so that with $\mu = 1/2$, we have

$$L[f_{1/2}](x) = -\left(e^{-\left(\frac{3}{2}\right) + 2\,x^2}\ \left(-3 + 2\,e^2 + \cos(4\,x)\right)\right) < 0 \quad \forall \quad x \in \mathbb{R}.$$

(b) $L[f_\mu](x)$ can be non-constant (as a function of μ), non-positive and can be independent of x. To see this, set $f(x) := \cosh x$. Then $L[f_\mu](x) = -4\cos^2\mu$.

(c) For an entire function, f, outside the class $\mathfrak{S}(A)$, with infinitely many non-real zeros, the Laguerre inequality $L[f_\mu](x) \geq 0$ may hold for all $x, \mu \in \mathbb{R}$. Set $f(x) := \sinh x$. Then $L[f_\mu](x) = 4\cos^2\mu \geq 0$.

5 A characterization of functions in the Laguerre–Pólya class

In the previous sections we have seen a number of results and a plethora of examples which provide supporting evidence that Conjecture 3.5 is true. Here we establish, under some additional hypotheses, conditions which imply that a function in $\mathfrak{S}(A)$ is in fact in $\mathcal{L}$-$\mathcal{P}$.

If a function $f \in \mathfrak{S}(A)$ has a positive local minimum or a negative local maximum, then elementary considerations show that f must possess some non-real zeros. Such local extrema provide one of the simplest (geometric) manifestations of the existence of non-real zeros. Heuristically speaking, if we interpret the Laguerre inequality $L[f](x) > 0$ for all $x \in \mathbb{R}$ in terms of the convexity (concavity) properties of $\log f(x)$ (cf (1.2)), then we see that f cannot have such local extrema. It is this intuitive geometric idea that is the motivation behind the following theorem.

Theorem 5.1: *Let $f \in \mathfrak{S}(A)$ and suppose that $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$. Then f cannot possess a positive local minimum or a negative local maximum.*

Proof: (*Reductio ad absurdum*) Suppose that $L[f_\mu](x) > 0$ for all $\mu \neq 0$ and for all $x \in \mathbb{R}$, but that f has a positive local minimum at point $x_0 \in \mathbb{R}$. Thus, we know from the calculus that $f(x_0) > 0$, $f'(x_0) = 0$ and $f''(x_0) \geq 0$. First, consider the case when $f''(x_0) > 0$. Then

$$L[f](x_0) = (f'(x_0))^2 - f(x_0)f''(x_0) = -f(x_0)f''(x_0) < 0. \tag{5.1}$$

Now, using continuity and (5.1), we have

$$\lim_{\mu\to 0} L[f_\mu](x_0) = 4\lim_{\mu\to 0}\left[(\Re f'(x_0 + i\,\mu))^2 - \Re f(x_0 + i\,\mu)\Re f''(x_0 + i\,\mu)\right]$$

$$= -4f(x_0)f''(x_0). \tag{5.2}$$

Therefore, it follows from (5.2) and a continuity argument, that there is a $\mu \neq 0$, $|\mu|$ sufficiently small, such that $L[f_\mu](x_0) < 0$. This is a contradiction.

Next, suppose that $f(x_0) > 0$ and

$$f'(x_0) = f''(x_0) = \cdots = f^{(m)}(x_0) = 0 \quad \text{but} \quad f^{(m+1)}(x_0) \neq 0.$$

Then the Taylor series expansion of f about $x = x_0$ may be expressed in the form

$$f(x) - f(x_0) = f^{(m+1)}(x_0)\frac{(x-x_0)^{m+1}}{(m+1)!} + O\left((x-x_0)^{m+2}\right) \quad \text{as } x \to x_0. \tag{5.3}$$

Since f has a positive local minimum at x_0, $m+1$ must be an even integer. Moreover, since $f(x) - f(x_0) \geq 0$ near x_0, $f^{(m+1)}(x_0) > 0$. Then, as in the first part of the proof, we find that

$$f(x) = f(x_0) + f^{(m+1)}(x_0)\frac{(x-x_0)^{m+1}}{(m+1)!} + O\left((x-x_0)^{m+1}\right), \quad (x \to x_0) \tag{5.4}$$

$$f'(x) = f^{(m+1)}(x_0)\frac{(x-x_0)^{m}}{m!} + O\left((x-x_0)^{m+1}\right), \quad (x \to x_0) \tag{5.5}$$

$$f''(x) = f^{(m+1)}(x_0)\frac{(x-x_0)^{m-1}}{(m-1)!} + O\left((x-x_0)^{m}\right), \quad (x \to x_0) \tag{5.6}$$

Hence, with the aid of (5.4), (5.5) and (5.6), we find that

$$L[f](x) = -f(x_0)f^{(m+1)}(x_0)\frac{(x-x_0)^{m-1}}{(m-1)!} + O\left((x-x_0)^{m}\right) \quad (x \to x_0). \tag{5.7}$$

Since $\lim_{\mu \to 0} L[f_\mu](x_0) = 4L[f](x_0)$, it follows from (5.7) that there is a $\mu \neq 0$, $|\mu|$ sufficiently small, such that $L[f_\mu](x_0) < 0$. This is again the desired contradiction. Since the proof when x_0 is a negative local maximum is, *mutatis mutandis*, the same as above, the proof of the theorem is complete.

□

We remark that, since the above analysis was based upon the *local* behavior of f, the assumption that $f \in \mathfrak{S}(A)$ in Theorem 5.1 could be considerably weakened. While we prefer the elementary character of the foregoing proof, we could have also appealed in the above argument to the open mapping property of analytic functions.

In order to relate $L[f_\mu]$ to the theorems in Section 2, we state the following elementary, albeit fundamental, lemma. We omit here the proof of the lemma, since it involves merely some explicit calculations and representations of $L[f_\mu]$.

Lemma 5.2: *Let $f \in \mathfrak{S}(A)$. For $x, \mu \in \mathbb{R}$, set*

$$R(x,\mu) := \Re\left[(f'(x+i\mu))^2 - f(x+i\mu)f''(x+i\mu)\right] \tag{5.8}$$

$$I(x,\mu) := (\Im\,[f'(x+i\mu)])^2 - \Im\,[f(x+i\mu)]\,\Im\,[f''(x+i\mu)] \quad \text{and} \tag{5.9}$$

$$C(x,\mu) := |f'(x+i\mu)|^2 - \Re[f(x+i\mu)\overline{f''(x+i\mu)}]. \tag{5.10}$$

Then, for all $x, \mu \in \mathbb{R}$,

$$\frac{1}{2}L[f_\mu](x) = R(x,\mu) + C(x,\mu), \tag{5.11}$$

$$\frac{1}{4}L[f_\mu](x) = C(x,\mu) - I(x,\mu) \quad \text{and hence} \tag{5.12}$$

$$\frac{1}{4}L[f_\mu](x) = R(x,\mu) + I(x,\mu). \tag{5.13}$$

It is instructive to examine the nature of the functions $R(x,\mu)$ and $I(x,\mu)$ in the above representations of $L[f_\mu](x)$. First, we note that for a fixed x, $R(x,\mu)$ can change sign (as function of μ) even if $f \in \mathcal{L}\text{-}\mathcal{P}$. By way of a concrete illustration, set $f(x) := e^{-x^2}$. Then $R(x,\mu) = 2e^{2(t^2-x^2)}\cos(4tx)$. Another, simpler, calculation shows that if $f(x) = x^2 + a$, $(a \in \mathbb{R})$, then $R(x,\mu) = 2(-a - t^2 + x^2)$. In contrast, the function $I(x,\mu)$ is better behaved as the next lemma shows.

Lemma 5.3: *If $f \in \mathcal{L}\text{-}\mathcal{P}$, then $I(x,\mu) \geq 0$ for all $x, \mu \in \mathbb{R}$.*

Proof: An extension of the Hermite–Poulain theorem (cf. [10, p. 4] and [12, p. 129]) implies that if $f \in \mathcal{L}\text{-}\mathcal{P}$, then $\hat{g}_\mu(x) := 2\sin(\mu D)f(x) \in \mathcal{L}\text{-}\mathcal{P}$ for all $\mu \in \mathbb{R}$, where the notation $\hat{g}_\mu$ is used to indicate the action of the operator $2\sin(\mu D)$ on $f \in \mathcal{L}\text{-}\mathcal{P}$. Hence, by Theorem 2.1, we conclude that $L[\hat{g}_\mu](x) = I(x,\mu) \geq 0$ for all $x \in \mathbb{R}$ and all $\mu \in \mathbb{R}$. □

We remark that $I(x, \mu)$ can be non-negative, for all $x, \mu \in \mathbb{R}$, even if f has some non-real zeros. Indeed, let $f(x) = a + \sin x$, where a is an arbitrary real number. Then, $I(x, \mu) = \frac{1}{2}(\cosh 2\mu - 1) \geq 0$ for all $\mu \in \mathbb{R}$.

As an immediate consequence of Lemma 5.2, Lemma 5.3 and Theorem 2.4 we obtain the following proposition which generalizes Theorem 3.6 in [6].

Proposition 5.4: *Let $f \in \mathfrak{S}(A)$. Then $f \in \mathcal{L}\text{-}\mathcal{P}$ if and only if $L[f_\mu](x) \geq 0$ and $I(x, \mu) \geq 0$ for all $x, \mu \in \mathbb{R}$, $\mu \neq 0$.*

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