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Stefan Czekalski

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The new properties of the theta functions

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Abstract

It is shown, that the function

$$H(x) = \sum_{k=-\infty}^{\infty} e^{-k^2 x}$$

satisfies the relation

$$H(x) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} H^{(n)}(x).$$

The theta function is defined by the following equation:

$$\Theta_0 = \sum_{k=1}^{\infty} e^{-k^2 \pi t} \qquad t > 0. \tag{1.1}$$

For computational purposes, it is convenient to introduce the function

$$\Theta(x) = \Theta_0(\frac{x}{\pi}) = \sum_{k=1}^{\infty} e^{-k^2 x}, \qquad x > 0.$$

Theorem 1.1. *The following formula holds*

$$\sum_{k=1}^{\infty} e^{-k^2 x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x-y^2}}{1 - e^{2(-x+i\sqrt{x}y)}} dy. \tag{1.2}$$

Proof. For $x > 0$ we have $|e^{2(-x+i\sqrt{x}y}| < 1$, therefore

$$\frac{1}{\sqrt{\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-x-y^2}}{1-e^{2(-x+i\sqrt{x}y)}}dy} = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} e^{-x-y^2}\sum_{k=0}^{\infty} e^{2(-x+i\sqrt{x}y)k}dy$$

$$= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} e^{-(2k+1)x}\int\limits_{-\infty}^{\infty} e^{-y^2+2iy\sqrt{x}k}dy$$

$$= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} e^{-(2k+1)x}\Big(\int\limits_{-\infty}^{\infty} e^{-y^2} cos 2y\sqrt{x}k\, dy + i\int\limits_{-\infty}^{\infty} e^{-y^2} sin 2y\sqrt{x}k\, dy\Big)$$

here the first integral is equal $\sqrt{\pi}e^{-k^2x}$; second integral - zero; therefore we have

$$= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty} e^{-(k^2+2k+1)x}\sqrt{\pi} = \sum_{k=1}^{\infty} e^{-k^2x}.$$

□

By inserting $x = 1$ into (1.2), we obtain

$$\sum_{k=1}^{\infty} e^{-k^2} = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\frac{e^{-1-y^2}}{1-e^{2(-1+iy)}}dy\,. \tag{1.3}$$

In the same way we prove the relation

$$\sum_{k=1}^{\infty}(-1)^{k-1}e^{-k^2x} = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\frac{e^{-x-y^2}}{1+e^{2(-x+iy\sqrt{x})}}dy\,. \tag{1.4}$$

When $x = 1$ we have

$$\sum_{k=1}^{\infty}(-1)^{k-1}e^{-k^2} = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\frac{e^{-1-y^2}}{1+e^{2(-1+iy)}}dy\,. \tag{1.5}$$

By virtue of (1.2) we easy find the formula

$$\sum_{k=1}^{\infty} ke^{-k^2 x} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-x-y^2}[2 - e^{2(-x+iy\sqrt{x})}]}{[1 - e^{2(-x+iy\sqrt{x})}]^2} dy\,. \tag{1.6}$$

By taking $x = 1$, we obtain

$$\sum_{k=1}^{\infty} ke^{-k^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-1-y^2}[2 - e^{2(-1+iy)}]}{[1 - e^{2(-1+iy)}]^2} dy\,. \tag{1.7}$$

Let us consider the following formula

$$\sum_{k=0}^{\infty} e^{-k^2 x} e^{-2axk} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{1 - e^{-2ax+2iy\sqrt{x}}} dy \qquad a > 0. \tag{1.8}$$

Let $h = e^{-x}$ and $z = e^{-ax}$ be, then

$$\sum_{k=0}^{\infty} h^{k^2} z^{2k} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{1 - z^2 e^{2iy\sqrt{-\ln h}}} dy\,. \tag{1.9}$$

If we introduce $h = e^{-x}$ and $z = e^{-ax-\frac{1}{2}x}$ in (1.8) then we obtain

$$\sum_{k=0}^{\infty} h^{(k-\frac{1}{2})^2} z^{2k-1} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2} z h^{-\frac{1}{4}}}{1 - z^2 h^{-1} e^{2iy\sqrt{-\ln h}}} dy\,. \tag{1.10}$$

We define the function $H(x) = 2\Theta(x) + 1$.

Theorem 1.2. *$H(x)$ can be expanded in a series*

$$H(x) = e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} H^{(n)}(x)\,. \tag{1.11}$$

Proof. Let us consider the functions

$$\Theta(x) = \sum_{k=1}^{\infty} e^{-k^2 x} \text{ and } v(x) = \sum_{k=1}^{\infty} ke^{-k^2 x}.$$

We evaluate the sums

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \Theta^{(n)}(x) = \sum_{k=1}^{\infty} e^{-k^2 x}[1 + \frac{k^2(2x)^2}{2!} + \frac{k^4(2x)^4}{4!} + \cdots]$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}v^{(n)}(x)=\sum_{k=1}^{\infty}e^{-k^2x}[\frac{k2x}{1!}+\frac{k^3(2x)^3}{3!}+\frac{k^5(2x)^5}{5!}+\cdots].$$

The above two series are absolutely convergent, therefore we can change the order of the terms. Addition and subtraction of both sides of equations yields

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\Theta^{(n)}(x)+\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}v^{(n)}(x)$$
$$=\sum_{k=1}^{\infty}e^{-k^2x}e^{2kx}=e^x\Theta(x)+e^x$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\Theta^{(n)}(x)+\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}v^{(n)}(x)$$
$$=\sum_{k=1}^{\infty}e^{-k^2x}e^{-2kx}=e^x\Theta(x)-1\,.$$

We add and subtract the equations once again. Thus we get

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\Theta^{(n)}(x)=e^x\Theta(x)+\frac{1}{2}e^x-\frac{1}{2}$$
$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}v^{(n)}(x)=\frac{1}{2}e^x+\frac{1}{2}.$$

Introduction $H(x)=2\Theta(x)+1$ and $U(x)=2v(x)$ yields

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}H^{(n)}(x)=e^xH(x) \tag{1.12}$$
$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n+1}}{(2n+1)!}U^{(n)}(x)=e^x+1\,. \tag{1.13}$$

□

The following relations for $H(x)$ and $U(x)$ hold

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} H^{(n+1)}(x) = e^x H(x) \tag{1.14}$$

$$\sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!} U^{(n)}(x) = e^x U(x) + e^x. \tag{1.15}$$

Proof of these relations is identical as (1.12) and (1.13).

We give now a generalization of (1.12). Let us consider the following extension of $\Theta(x)$

$$\Theta_1(x) = \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} \qquad x > 0,\ a \in R.$$

Then the more general version of (1.12) is equal

$$\mathrm{cha} H_1(x) - \mathrm{sha} = e^{-x} \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!} H_1^{(n)}(x) \tag{1.16}$$

where $H_1(x) = 2\Theta_1(x) + 1$.

The proof of (1.16) is patterned after this of the Theorem 1.2.

By virtue of the relation

$$\sum_{k=-\infty}^{\infty} e^{-k^2 x} e^{-ka} = \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} + \sum_{k=1}^{\infty} e^{-k^2 x} e^{ka} + 1$$

it is easy to verify that, $\Psi(x) = \sum\limits_{k=-\infty}^{\infty} e^{-k^2 x} e^{-ka}$ satisfies (1.16) with the member *sha* removed.

Theorem 1.3. *The formula* (1.12) *can be written in the form*

$$H(x) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} H^{(n)}(x)\,. \tag{1.17}$$

To prove theorem 0.3 we need the lemma

Lemma 1.4.

$$[x^{-\frac{1}{2}} H(x)]^{(n)} = (-1)^n \frac{(2n)!}{2^{2n}} \sum_{k=0}^{n} \frac{2^{2k}}{(n-k)!} \frac{x^{-n-k-\frac{1}{2}}}{(2k)!} \frac{d^k}{d(\frac{1}{x})^k} H(\frac{1}{x}). \tag{1.18}$$

Proof. Proof of Lemma 1.4 (by induction) The case $n = 1$ is obvious. We suppose, that (1.18) is true for any n and differentiate it.

$$
\begin{aligned}
&[x^{-\frac{1}{2}}H(x)]^{(n+1)} \\
&= (-1)^{n+1}\frac{(2n)!}{2^{2n}}\sum_{k=0}^{n}\frac{2^{2k}}{(n-k)!}\frac{(n+k+\frac{1}{2})x^{-n-k-\frac{3}{2}}}{(2k)!}\frac{d^k}{d(\frac{1}{x})^k}H(\frac{1}{x}) \\
&\quad + (-1)^{n+1}\frac{(2n)!}{2^{2n}}\sum_{k=0}^{n}\frac{2^{2k}}{(n-k)!}\frac{x^{-n-k-\frac{5}{2}}}{(2k)!}\frac{d^{k+1}}{d(\frac{1}{x})^{k+1}}H(\frac{1}{x}) \\
&= (-1)^{n+1}\frac{(2n)!}{2^{2n}}[\frac{2n+1}{2n!}x^{-n-\frac{3}{2}}H(\frac{1}{x}) \\
&\quad + \sum_{k=1}^{n}\frac{2^{2k}}{(n-k)!}\frac{(n+k+\frac{1}{2})x^{-n-k-\frac{3}{2}}}{(2k)!}\frac{d^k}{d(\frac{1}{x})^k}H(\frac{1}{x})] \\
&\quad + (-1)^{n+1}\frac{(2n)!}{2^{2n}}\sum_{k=0}^{n}\frac{2^{2k-2}}{(n+1-k)!}\frac{x^{-n-k-\frac{3}{2}}}{(2k-2)!}\frac{d^k}{d(\frac{1}{x})^k}H(\frac{1}{x}) \\
&\quad + (-1)^{n+1}x^{-2n-\frac{5}{2}}\frac{d^{n+1}}{d(\frac{1}{x})^{n+1}}H(\frac{1}{x})
\end{aligned}
$$

and consequently

$$
\begin{aligned}
[x^{-\frac{1}{2}}H(x)]^{(n+1)} &= (-1)^{n+1}\frac{(2n+2)!}{2^{2n+2}}\frac{x^{-n-\frac{3}{2}}}{(n+1)!}H(\frac{1}{x}) \\
&\quad + \sum_{k=1}^{n}\frac{2^{2k}}{(n+1-k)!}\frac{x^{-n-k-\frac{3}{2}}}{(2k)!}\frac{d^k}{d(\frac{1}{x})^k}H(\frac{1}{x}) \\
&\quad + (-1)^{n+1}x^{-2n-\frac{5}{2}}\frac{d^{n+1}}{d(\frac{1}{x})^{n+1}}H(\frac{1}{x})
\end{aligned}
$$

therefore

$$
[x^{-\frac{1}{2}}H(x)]^{(n+1)} = (-1)^{n+1}x^{-n-\frac{3}{2}}\frac{(2n+2)!}{2^{2n+2}}\sum_{k=0}^{n+1}\frac{2^{2k}}{(n+1-k)!}\frac{x^{-k}}{(2k)!}\frac{d^k}{d(\frac{1}{x})^k}H(\frac{1}{x}).
$$

□

Proof. Proof of the Theorem 1.3 The known functional relation

$$H(\pi^2 x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} H(\frac{1}{x})$$

we apply to (1.12) and obtain

$$H(\pi^2 x) = \pi^{-\frac{1}{2}} e^{-\pi^2 x} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n} (\pi^2)^n}{(2n)!} [x^{-\frac{1}{2}} H(\frac{1}{x})]^{(n)}$$

We use now (1.18) and have

$$H(\pi^2 x) = \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-\pi^2 x} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(2\pi)^{2k}}{(2k)!} \frac{(\pi^2 x)^{n-k}}{(n-k)!} \frac{d^k}{d(\frac{1}{x})^k} H(\frac{1}{x})$$

and consequently

$$H(\frac{1}{x}) = e^{-\pi^2 x} (\sum_{n=0}^{\infty} \frac{(\pi^2 x)^n}{n!} H(\frac{1}{x}) + \sum_{n=1}^{\infty} \frac{(\pi^2 x)^{n-1}}{(n-1)!} \frac{(2\pi)^2}{2!} \frac{d}{d(\frac{1}{x})} H(\frac{1}{x})$$
$$+ \sum_{n=2}^{\infty} \frac{(\pi^2 x)^{n-1}}{(n-2)!} \frac{(2\pi)^4}{4!} \frac{d^2}{d(\frac{1}{x})^2} H(\frac{1}{x}) + \cdots)$$

therefore

$$H(\frac{1}{x}) = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} \frac{d^n}{d(\frac{1}{x})^n} H(\frac{1}{x}).$$ □

If we replace $\frac{1}{x}$ by x we obtain (1.17).
The following formula for $\Theta_1(x)$ holds

$$\sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n (\frac{1}{e^a - 1})^{(2n)} \qquad a > 0. \tag{1.19}$$

Proof. We transform the right-hand side of (1.19) and have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n (\sum_{k=1}^{\infty} e^{-ka})^{(2n)} = \sum_{k=1}^{\infty} e^{-ka} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (k^2 x)^n$$
$$= \sum_{k=1}^{\infty} e^{-k^2 x} e^{-ka}.$$

Using the formula (1.19) with $a = 2x$, we find following expression for the theta function [1, 2]

$$\Theta(x) = e^{-x}[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\frac{x}{4})^n (\frac{1}{e^{2x} - 1})^{(2n)}]. \tag{1.20}$$

□

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Stefan Czekalski
Ul. Marszalkowska 1 m 80
00-624 Warszawa
Poland