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THE THEORY OF BOOLEAN ALGEBRAS WITH A DISTINGUISHED SUBALGEBRA IS UNDECIDABLE

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§ 0. INTRODUCTION

We prove the following theorems:

Theorem 1** Let $T_1$ and $T_2$ be theories in the language $L = \{ \cup, \cap, -, 0, 1 \}$ such that there are infinite Boolean algebras (hereafter denoted by $BA$) $B_1, B_2$ such that $B_i \models T_i$, $i = 1, 2$, let $P$ be a unary predicate and $S = T_1 \cup T_2(P)$, where $T_2(P)$ is the relativization of $T_2$ to $P$, then $S$ is undecidable.

Theorem 2: The theory of 1-dimensional cylindric algebras (denoted by $CA_1$) is undecidable.

Theorems 1 and 2 answer a question of Henkin and Monk in [2] Problem 7; there they also point out that the decidability problems of theorems 1 and 2 are closely related, this relation is formulated in the following proposition:

Proposition: (a) Let $<B,c>$ be a $CA_1$ where $B$ is a $BA$ and $c$ a unary operation on $B$ then $A = \{ b \mid b \in B \text{ and } c(b) = b \}$ is a subalgebra of $B$, and for every $b \in B$, $c(b)$ is the minimum of the set $\{ a \mid b \subseteq a \in A \}$.

(b) Let $B$ be a $BA$ and $A$ be a subalgebra of $B$ suppose that for every $b \in B$, $a_b = \min(\{ a \mid b \subseteq a \in A \})$ exists; define $c(b) = a_b$, then $<B,c>$ is a $CA_1$.

Let $T_C$ be the theory of $CA_1$'s and $T_B$ be the theory of $BA$'s with a distinguished subalgebra $P$, with the additional axiom that for every $b$ there is a minimal $a_b$ such that $P(a_b)$ and $b \subseteq a_b$, then certainly $T_C$ and $T_B$ are bi-interpretable.

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** R. McKenzie proved independently at about the same time, that the theory of Boolean algebras with a distinguished subalgebra is undecidable. The method of his proof is different from ours.
The classical result about the decidability of the theory of BA's appears in Tarski's [5], and in Ershov [1]. Ershov in [1] also proved that the theory of BA's with a distinguished maximal ideal is decidable, Rabin [4] proved the decidability of the theory of countable BA's with quantification over ideals.

Henkin proved that the equational theory of CA_n's is decidable and Tarski proved the undecidability of the equational theory of CA_n's for n \geq 4.

In our construction we interpret the theory of two equivalence relations in a model < B, U, \cap, \cdot, 0, 1, A > but neither B nor A are complete BA's. We do not know the answer to the following question:

Let K = \{ < B, U, \cap, \cdot, 0, 1, A > | B is a BA, A is a subalgebra of B, A and B are complete \} is Th(K) decidable?

We also do not know whether an analogue of theorem 1 for TB holds. For instance let S be TB together with the axioms that say that both the universe and P are atomic BA's is S decidable?
§ 1. THE CONSTRUCTION

∪, ∩, →, 0, 1 denote the operations and constants of a BA and ≤ denotes its partial order. A, B, C denote BA’s; At(B), A(t), Aς(B) denote the set of atoms of B, the set of non-zero, non-maximal atomless elements of B and the set of non-zero, non-maximal atomic elements of B respectively. Let I(B) be the ideal generated by \( \text{At}(B) \cup \text{As}(B) \), \( B^{(1)} = B/I(B) \) and if \( b \in B \) \( b^{(1)} \equiv b/I(B) \). If \( D \subseteq B \) \( c_D(D) \) denotes the subalgebra of B generated by D. B × C denotes the direct product of B and C. \( \prod_j B_j \) denotes the direct product of \( \{ B_j \mid j \in J \} \) and we assume that for every \( j_1 \neq j_2 \), \( B_{j_1} \cap B_{j_2} = \{ 0 \} \), so we can identify the element \( c \in B_0 \) with the element \( \prod_j c_j \in \prod_j B_j \) where \( f_c(j) = 0 \) if \( j \neq j_0 \) and \( f_c(j_0) = c \). We denote by \( 1_B \) the maximal element of B.

Let \( B_T \) be the BA of finite and cofinite subsets of \( ω \) and \( B_1 \) the countable atomless BA.

Let \( B_T \) be the non-principal ultrafilter of \( B_T \) and \( F_2 \) be an ultrafilter in \( B_1 \); let \( B_M \) be the following subalgebra of \( B_T \times B_1 : B_M = \{ (a,b) \mid a \in F_1 \text{ iff } b \in F_2 \} \); notice that \( B^{(1)}_M \approx (0,1) \). For every i let \( B_i \cong B_M \), \( B_i = \prod_{i \in ω} B_i \) and \( B^≤ = c_{≤} \left( \bigcup_{i \in ω} B_i \right) \). We denote \( 1_{B_i} \) by 1i.

Lemma 3 : Let \( E_0 \) and \( E_1 \) be equivalence relations on \( ω \) then there is a model \( M = < B, \cup, \cap, \vee, 0, 1, A > \models T_B \) such that \( < ω, E_0, E_1 > \) is explicitly interpretable in M.

Proof : We denote by \( i/E_ε \) the \( E_ε \)-equivalence class of i and by \( ω/E_ε \) the set of \( E_ε \)-equivalence classes. For every \( i \in ω \) let

\[
\{ b^i_{ε, σ, j} \mid ε \in \{ 0,1 \}, \ σ \in ω/E_ε, j \in ω \} \leq A_≤ (B_i)
\]

be such that

\[
<ε, σ, j > \neq <ε', σ', j'> \Rightarrow b^i_{ε, σ, j} \cap b^{i'}_{ε', σ', j'} = 0 \text{ and for every } b \in A_≤ (B_i)
\]

\[
l \subseteq \{ b \mid \cap b^i_{ε, σ, j} \neq 0 \} \text{ for every } i \in ω \text{ let}
\]

\[
\{ a^i_{ε, σ, j} \mid ε \in \{ 0,1 \}, σ \in ω/E_ε, j \in ω \}
\]

be a set of pairwise disjoint subsets of

\( \text{At}(B_i) \) such that \( \text{At}(B_i) = \bigcup \{ a^i_{ε, σ, j} \mid ε \in \{ 0,1 \}, σ \in ω/E_ε, j \in ω \} \) and

\[
| a^i_{ε, σ, j} | = \begin{cases} 
1 & ε = 0 \text{ and } i \in σ \\
2 & ε = 0 \text{ and } i \notin σ \\
3 & ε = 1 \text{ and } i \in σ \\
4 & ε = 1 \text{ and } i \notin σ 
\end{cases}
\]
For every $e$, $\sigma$ and $j$ as above let $c_{e, \sigma, j} \in B^>$ be
\[
 c_{e, \sigma, j} = \bigcup \{ b_{e, \sigma, j}^{i} \cup \bigcup a_{e, \sigma, j}^{i} \mid i \in \omega \} \text{ where } U D \text{ denotes the supremum of } D \text{ in } B^>.
\]
Let $A = c_{\varepsilon}(c_{e, \sigma, j}) \in \{0,1\} \in \omega/E_e$, $j \in \omega$, $B = c_{\varepsilon}(B \cup A)$ and $M = < B, \cup, \cap, \cdot, 0, 1, A >$. We show that $M \models T_B$. It suffices to show that $a_{b} = \min(\{ a \mid b \subseteq a \in A \})$ exists for elements $b \in B$ of the following forms:

- $b \in At(B_i) \cup At(B_i)$;
- $b \in B_i$ and $b(1) = 1$;
- $b \in B^<$ and $1_i \subseteq b$ for almost all $i \in \omega$;
- this follows from the fact that every $b \in B$ can be represented in the form $\bigcup_{i=1}^{n} (b_i \cap a_i)$ where each $b_i$ is of the above form and $a_i \in A$. In each of the above cases the existence of $a_b$ is easily checked. Thus $M \models T_B$.

We now define formulas $\varphi_U(x)$, $\varphi_{Eq}(x,y)$, $\varphi_e(x,y) \in \{0,1\}$ such that $M \models \varphi_U[a]$ iff for some $i \in \omega$ $a(1) = 1_i$, $M \models \varphi_{Eq}[a,b]$ iff $a(1) = b(1)$ and $M \models \varphi_e[a_1,a_2]$ iff for some $j \in \omega$ $a(1) = 1_j$ and $< i_1, i_2 > \in E_\varepsilon$.

$\varphi_U(x)$ says that $x(1) \in At(B(1))$ and for no $y \in At(A)$ $x(1) = y(1)$.

$\varphi_{Eq}(x,y)$ says that $x(1) = y(1)$. $\varphi_0(x,y)$ says: $\varphi_U(x) \land \varphi_U(y)$ and there are $x_1, y_1$ such that $x(1) = x_1(1)$, $y(1) = y_1(1)$ and for every $z \in At(A)$
\[
|\{ u \mid z \cap x_1 \supseteq u \in At(B) \} | = 1 \text{ iff } |\{ u \mid z \cap y_1 \supseteq u \in At(B) \} | = 1.
\]
$\varphi_0$ is defined similarly. The desired properties of $\varphi_U$, $\varphi_{Eq}$ and $\varphi_e$ are easily checked, and the lemma is proved.

Since the theory of two equivalence relations is undecidable $T_B$ and $T_C$ are undecidable and theorem 2 is proved.

Theorem 1 easily follows from the following lemma.

**Lemma 4:** Let $E_1, E_2$ be equivalence relations on $\omega$ then there are models
\[
M_i = < B_i, \cup, \cap, \cdot, 0, 1, A_i > \quad i = 1,..., 4 \text{ such that } < \omega, E_1, E_2 > \text{ is explicitly interpretable in } M_i \text{ and } B_1, A_1 \text{ are atomic, } B_2, A_2 \text{ are atomless, } B_3 \text{ is atomic } A_3 \text{ is atomless, and } B_4 \text{ is atomless } A_4 \text{ is atomic.}
\]

**Proof:** Let $B_0, A_0, M_0$ denote $B, A$ and $M$ of lemma 3 respectively. For $i = 1, 2, M_i$ can easily be constructed so that $< B_i/H_i, \cup, \cap, \cdot, 0, 1, A_i/H_i > \equiv M_0$ where $H_i = \{ b \mid b \in B_i \text{ and for every } a \subseteq b \text{ a } \in A_i \}$. Since such an $H_i$ is definable in $M_i$ $M_0$ can be interpreted in $M_i$ $i = 1, 2$. 
For $i = 3$ a similar construction works. Let $B$ be an atomic saturated countable BA and $I$ a maximal non-principal ideal of $B$. Let $A$ be an atomless subalgebra of $B$ such that:

(a) for every $b \in B$ which contains infinitely many atoms there is a non-zero $a \in A$ such that $a \subseteq b$;

(b) for every $b \in A_3(B)$ there is an $a \in A$ such that $(a-b) \cup (b-a)$ contains only finitely many atoms of $B$. Let $J = I \cap A$. For every non-zero $a \in B_0$ let $F_a$ be an ultrafilter in $B$ which contains $a$.

and $< A_3, A_3, I_3, J_3 >$ a copy of $< B, A, I, J >$. Let $B_3 = \{ b_a | 0 \neq a \in B_0 \}$ and let $B_3$ be the following subalgebra of $B_1$:

$$B_3 = \{ \mathcal{g}_a(\bigcup \{ I_a \ | \ 0 \neq a \in B_0 \}) \bigcup (g_a | 0 \neq a \in B_0) \}$$

where $g_a(b) = 1_{B_0}$ iff $a \in F_b$ and $g_a(b) = 0$ otherwise. Let $A_3 = \{ I_a \bigcup (f_a | a \in A_0) \}$.

Certainly $B_3$ is atomic and $A_3$ is atomless. Let $I = \{ a \bigcup (f_a | a \in A_0) \}$ and $I$ is definable in $M_3$ by the formula

$$\forall (x) \equiv \forall y (0 \neq y \subseteq x \rightarrow P(y))$$

Let $B_{1, 3} = B_3/I$ and $A_{1, 3} = \{ a/I \bigcup (f_a | a \in A_0) \}$. Then $< B_{1, 3}, 0, 1, A_{1, 3} > \equiv M_2$, so $M_2$ is interpretable in $M_3$ and thus $< \omega, E_1, E_2 >$ is interpretable in $M_3$ as desired.

In order to construct $M_4$ we assume that $B_1$ is a subalgebra of $P(\omega)$ and

$$\text{At}(B_1) = \{ n \bigcup ( f_a | a \in \omega) \}.$$  

Let $B_{1, L} = \{ f_a | a \in \omega \}$. Then $B_{1, L}$ is the following subalgebra of $B_1$:

$$f_a(n) = 1_{B_1}$$

if $n \in a$ and $f_a(n) = 0$ otherwise. Let $A_4 = \{ f_a | a \in A_0 \}$ and $M_4 = < B_4, 0, 1, A_4 >$.

Certainly $B_4$ is atomless and $A_4$ is atomic. Let $B_4 = \{ b \bigcup (f_a | a \in B_4) \}$ and for every

$a \in \text{At}(A_4)$ either $b \Rightarrow a$ or $-b \Rightarrow a$, then $< B_4, 0, 1, A_4 > \equiv M_1$ and $B_4$ is certainly definable in $M_4$, thus $< \omega, E_1, E_2 >$ is definable in $M_4$ and the lemma is proved.

We omit the proof of theorem 1 which follows easily from lemma 4, the fact that every countable BA can be embedded in e.g. $B_L$, and from \[ 6 \] pp. 293-302.
REFERENCES


