Some remarks concerning a class of nonlinear evolution equations in Hilbert spaces

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1. Introduction

Let $H$ be a real Hilbert space and let $X$, $Y$ be two orthogonal subspaces of $H$ such that $H = X \oplus Y$. Let $\lambda$ be a real normed space and let $T > 0$. In this paper we consider evolution problems of the form

$$\begin{align*}
\dot{y}(t) &= F(\lambda(t), x(t), y(t), \dot{x}(t)) \quad &\text{for all} & \quad t \in [0, T], \\
x(0) &= x_0, & y(0) &= y_0
\end{align*}$$

in which the unknowns are the functions $x : [0, T] \rightarrow X$ and $y : [0, T] \rightarrow Y$, $F : \Lambda \times X \times Y \times H \rightarrow H$ is a nonlinear operator and $\lambda : [0, T] \rightarrow \Lambda$ is a parameter function (in (1.1) and everywhere in this paper the dot above represents the derivative with respect to the time variable $t$). Such type of problems arise in the study of quasistatic processes for semilinear rate-type materials (see for example [1]-[3]). In this case the unknowns $x$ and $y$ are the small deformation tensor and the stress tensor and $F$ is an operator involving the constitutive law of the material; the parameter $\lambda$ may be interpreted as the absolute temperature or an internal state variable.

For particular forms of $F$ existence and uniqueness of the solution and error estimates of a numerical method for problems of the form (1.1), (1.2) were already given in [3], [4].

In this paper we prove the existence and uniqueness of the solution for problem (1.1), (1.2) using a technique based on the equivalence between (1.1), (1.2) and a Cauchy problem for an ordinary differential equation in the product Hilbert space $X \times Y$ (section 2). We also study the dependence of the solution with respect to the parameter $\lambda$ and the initial data (section 3). In some applications (see...
for example [5] the function \( \lambda \) in (1.1) is needed to be considered as an unknown function whose evolution is given by

\begin{align}
(1.3) & \quad \dot{\lambda}(t) = G(\lambda(t), x(t), y(t)) \quad \text{for all} \quad t \in [0, T], \\
(1.4) & \quad \lambda(0) = \lambda_0
\end{align}

where \( G : \Lambda \times X \times Y \to \Lambda \) is a nonlinear operator. For this reason we also consider problem (1.1)-(1.4) for which we prove the existence and uniqueness of the solution (section 4). Let us finally notice that the results presented here complete and generalize some results of [2] and may be applied in the study of some evolution problems for rate-type materials (see [1] - [5]).

2. An existence and uniqueness result

Everywhere in this paper if \( V \) is a real normed space we utilise the following notations:

- \( || \cdot ||_V \) - the norm of \( V \);
- \( 0_V \) - the zero element of \( V \);
- \( C^0(0,T,V) \) - the space of continuous functions on \([0,T]\) with values in \( V \);
- \( C^1(0,T,V) \) - the space of derivable functions with continuous derivative on \([0,T]\) with values in \( V \);
- \( || \cdot ||_{0,T,V} \) - the norm on the space \( C^0(0,T,V) \) i.e. \( || z ||_{0,T,V} = \max_{t \in [0,T]} || z(t) ||_V \) for all \( z \in C^0(0,T,V) \);
- \( || \cdot ||_{1,T,V} \) - the norm on the space \( C^1(0,T,V) \) i.e. \( || z ||_{1,T,V} = || z ||_{0,T,V} + || \dot{z} ||_{0,T,V} \) for all \( z \in C^1(0,T,V) \).

Moreover \( V \) is a real Hilbert space we denote by \( \langle , \rangle_V \) the inner product of \( V \). Finally, if \( V_1 \) and \( V_2 \) are real Hilbert spaces we denote by \( V_1 \times V_2 \) the product space endowed with the canonical inner product and by \( v = (v_1, v_2) \) the elements of \( V_1 \times V_2 \).

Let us consider the following assumptions:

\begin{align}
(2.1) & \quad \text{there exists } m > 0 \text{ such that } \langle F(\lambda, x, y, z_1) - F(\lambda, x, y, z_2), z_1 - z_2 \rangle_H \geq m || z_1 - z_2 ||_H^2 \quad \text{for all } \lambda \in \Lambda, x \in X, y \in Y, z_1, z_2 \in X, \\
(2.2) & \quad \text{there exists } M > 0 \text{ such that } || F(\lambda, x_1, y_1, z_1) - F(\lambda, x_2, y_2, z_2) ||_H \leq M (|| x_1 - x_2 ||_H + || y_1 - y_2 ||_H + || z_1 - z_2 ||_H) \quad \text{for all } \lambda \in \Lambda, x, y \in X, z \in Y, z_1, z_2 \in H, i = 1, 2.
\end{align}
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(2.3) \[ \lambda \mapsto F(\lambda; x, y, z) : \Lambda \to H \] is a continuous operator, for all \( x \in X, \ y \in Y \) and \( z \in H \).

(2.4) \( \lambda \in C^0(0, T, H) \)

(2.5) \( x_0 \in X, \ y_0 \in Y \)

The main result of this section is the following:

Theorem 2.1. Let (2.1)-(2.5) hold. Then problem (1.1), (1.2) has a unique solution \( x \in C^1(0, T, X), \ y \in C^1(0, T, Y) \).

In order to prove theorem 2.1 let us denote by \( Z \) the product Hilbert space \( Z = X \times Y \) (which in fact is isomorph with \( H \)). We have:

Lemma 2.1. Let \( \lambda \in \Lambda, x \in X \) and \( y \in Y \): then there exists a unique element \( z = (u, v) \in Z \) such that \( v = F(\lambda; x, y, u) \).

Proof. The uniqueness part is a consequence of (2.1); indeed, if the elements \( z = (u, v), \bar{z} = (\bar{u}, \bar{v}) \in Z \) are such that \( v = F(\lambda; x, y, u), \bar{v} = F(\lambda; x, y, \bar{u}) \), using (2.1) we have \( \langle v - \bar{v}, u - \bar{u} \rangle_H \geq m ||u - \bar{u}||_H^2 \) hence by the orthogonality in \( H \) of \( v - \bar{v} \) and \( u - \bar{u} \) we deduce \( u = \bar{u} \) which implies \( v = \bar{v} \).

For the existence part let us denote by \( P_j : H \to X \) the projector map on \( X \). Using (2.1) and (2.2) we get that the operator \( P_j F(\lambda; x, y, \cdot) : X \to X \) is a strongly monotone and Lipschitz continuous operator hence by Browder's surjectivity theorem we get that there exists \( u \in X \) such that \( P_j F(\lambda; x, y, u) = 0_X \). It results that the element \( F(\lambda; x, y, u) \) belongs to \( Y \) and we finish the proof taking \( z = (u, v) \) where \( v = F(\lambda; x, y, u) \).

Lemma 2.1 allows us to consider the operator \( B : \Lambda \times Z \to Z \) defined by

(2.6) \[ B(\lambda, \omega) = z \iff \omega = (x, y), \ z = (u, v) \text{ and } v = F(\lambda; x, y, u) \]

Moreover, we have:

Lemma 2.2. \( B \) is a continuous operator and there exists \( L > 0 \) such that

(2.7) \[ ||B(\lambda, \omega_1) - B(\lambda, \omega_2)||_Z \leq L ||\omega_1 - \omega_2||_Z \] for all \( \lambda \in \Lambda, \ \omega_1, \omega_2 \in Z \).
Proof. Let \( \lambda_i \in \Lambda \), \( \omega_i = (x_i, y_i, \mu_i) \in \mathbb{Z} \) and \( z_i = (u_{i1}, u_{i2}) = B(\omega_i) \), \( i = 1, 2 \). Using (2.6) we get:

\[
(2.8) \quad v_i = F(\lambda_i, x_i, y_i, u_i), \quad i = 1, 2
\]

which implies

\[ F_{\lambda}F(\lambda_i, x_i, y_i, u_i) = 0, \quad i = 1, 2 \]

From (2.1) and (2.9) we get

\[
m|u_1-u_2|_H^2 \leq <F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_1, y_1, u_1), u_1-u_2>_H =
\]

\[ = <P_{\lambda}F(\lambda_1, x_1, y_1, u_1) - P_{\lambda}F(\lambda_2, x_1, y_1, u_1), u_1-u_2>_H \leq |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_1, y_1, u_1)|_H |u_1-u_2|_H \]

which implies

\[
(2.10) \quad |u_1-u_2|_H \leq \frac{1}{m} |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_1, y_1, u_2)|_H .
\]

Using now (2.8) and (2.2) we get

\[
|v_1-v_2|_H \leq M |u_1-u_2|_H + |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_2, y_2, u_2)|_H
\]

hence by (2.10) it results

\[
(2.11) \quad |v_1-v_2|_H \leq \left( \frac{M + 1}{m} \right) |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_2, y_2, u_2)|_H .
\]

Using again (2.2) we get

\[
(2.12) \quad |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_2, y_2, u_2)|_H < M(\|x_1-x_2\|_H + \|y_1-y_2\|_H) + |F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_2, y_2, u_2)|_H
\]

hence by (2.3) we obtain

\[
|F(\lambda_1, x_1, y_1, u_1) - F(\lambda_2, x_2, y_2, u_2)|_H \to 0 \quad \text{when} \quad \lambda_1 \to \lambda_2
\]

in \( \Lambda \), \( x_1 \to x_2 \) in \( X \) and \( y_1 \to y_2 \) in \( Y \). Using now (2.10) and (2.11) we get the continuity of \( B \) and taking \( \lambda_1 = \lambda_2 \) from (2.10)-(2.12) we get (2.7).
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Proof of theorem 2.1. Let $A : [0, T] \times Z \longrightarrow Z$ and $x_0$ be defined by

$$A(t, z) = B(\lambda(t), z) \quad \text{for all } t \in [0, T] \text{ and } z \in Z$$

(2.13)

$$x_0 = (x_0, y_0).$$

(2.14)

Using (2.6) we get that $z \in C^1(0, T, X)$ and $y \in C^1(0, T, Y)$ is a solution of (1.1), (1.2) iff $z = (x, y) \in C^1(0, T, Z)$ is a solution of the problem

$$\dot{z}(t) = A(t, z(t)) \quad \text{for all } t \in [0, T]$$

(2.15)

$$z(0) = x_0.$$  \hspace{1cm} (2.16)

In order to study (2.15), (2.16) let us remark that by lemma 2.2 and (2.4) we get that $A$ is a continuous operator and

$$|A(t, z_1) - A(t, z_2)|_Z \leq L |z_1 - z_2|_Z \quad \text{for all } t \in [0, T] \text{ and } z_1, z_2 \in Z.$$

Moreover, by (2.5), (2.14) we get $z_0 \in Z$. Theorem 2.1 follows now from the classical Cauchy-Lipschitz theorem applied to (2.15), (2.16).

3. The continuous dependence of the solution with respect to the data

Let us now replace (2.2), (2.3) by a stronger assumption namely

there exists $M > 0$ such that

$$M(|\lambda_1 - \lambda_2|, |x_1 - x_2|_H, |y_1 - y_2|_H, |z_1 - z_2|_H)$$

(3.1)

for all $\lambda_i \in \Lambda$, $x_i \in X$, $y_i \in Y$, $z_i \in H$, $i = 1, 2$.

We have the following result : 

Theorem 3.1. Let (2.1), (3.1) hold and let $x_i \in C^1(0, T, X)$, $y_i \in C^1(0, T, Y)$ be the solution of (1.1), (1.2) for the data $\lambda_i, x_{0i}, y_{0i}$ satisfying (2.4), (2.5), $i = 1, 2$. Then there exists $C > 0$ such that

$$|x_1 - x_2|_{1, T, H} + |y_1 - y_2|_{1, T, H} \leq C(|\lambda_1 - \lambda_2|, |x_{01} - x_{02}|_H, |y_{01} - y_{02}|_H).$$

(3.2)
Remark 3.1. In (3.2) and everywhere in this section \( C \) are strictly positive generic constants which depend only on \( F \) and \( T \).

**Proof of Theorem 3.1.** Let \( x_i(t, y_i(t)) \) and \( z_0(t, y_0(t)) \), \( i = 1, 2 \). As it results from the proof of Theorem 2.1 we have

\[
(3.3) \quad \dot{x}_i(t) = A_i(t, x_i(t)), \quad \text{for all} \quad t \in [0, T]
\]

\[
(3.4) \quad x_i(0) = z_0_i \]

where the operators \( A_i \) are defined by (2.13) replacing \( \lambda \) by \( \lambda_i \), \( i = 1, 2 \). Since (3.1) implies that \( B : \Lambda \times Z \rightarrow Z \) is a Lipschitz continuous operator (see the proof of Lemma 2.2), from (2.13) we get that there exists \( L > 0 \) such that

\[
(3.5) \quad \| A_i(t, x_i(t)) - A_i(t, z_0(t)) \|_Z \leq L \left( \| \lambda_1(t) - \lambda_2(t) \|_\Lambda + \| z_1(t) - z_2(t) \|_Z \right)
\]

for all \( t \in [0, T] \). Using now (3.3) and (3.5) we get

\[
\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t) \rangle_Z \leq L \left( \| \lambda_1(t) - \lambda_2(t) \|_\Lambda + \| z_1(t) - z_2(t) \|_Z \right)
\]

for all \( t \in [0, T] \). Hence by (3.4) and a Gronwall-type lemma we deduce

\[
\| x_1(s) - z_0(s) \|_Z \leq C \left( \int_0^s \| \lambda_1(t) - \lambda_2(t) \|_\Lambda + \| z_1(t) - z_2(t) \|_Z \right) \quad \text{for all} \ s \in [0, T]
\]

which implies

\[
(3.6) \quad \| x_1 - z_2 \|_{0, T, Z} \leq C \left( \| \lambda_1 - \lambda_2 \|_{0, T, \Lambda} + \| z_0 - z_2 \|_{0, T, Z} \right)
\]

Using again (3.3) and (3.5) we have

\[
\| \dot{x}_i(t) - \dot{z}_i(t) \|_Z \leq C \left( \| \lambda_1(t) - \lambda_2(t) \|_\Lambda + \| z_1(t) - z_2(t) \|_Z \right)
\]

for all \( t \in [0, T] \). By (3.6) it results

\[
(3.7) \quad \| \dot{x}_1 - \dot{z}_2 \|_{0, T, Z} \leq C \left( \| \lambda_1 - \lambda_2 \|_{0, T, \Lambda} + \| z_0 - z_2 \|_Z \right)
\]

From (3.6) and (3.7) we get

\[
\| x_1 - z_2 \|_{0, T, Z} \leq C \left( \| \lambda_1 - \lambda_2 \|_{0, T, \Lambda} + \| z_0 - z_2 \|_Z \right)
\]

which implies (3.2).

**Remark 3.2.** From (3.2) we deduce in particular the continuous dependence of the
solution with respect the initial data i.e. the finite-time stability of every solution of (1.1), (1.2) (for definitions in the field see for instance [6] chap.5).

4. A second existence and uniqueness result

In this section we suppose that Λ is a real Hilbert space. We consider the operator $G: Λ \times X \times Y \rightarrow Λ$ and the element $λ_0$ such that

$$||G(λ_1, x_1, y_1) - G(λ_2, x_2, y_2)||_Λ \leq L(||λ_1 - λ_2||_Λ + ||x_1 - x_2||_H + ||y_1 - y_2||_H)$$

for all $λ_1, λ_2 \in Λ$, $x_1, x_2 \in X$, $y_1, y_2 \in Y$, $i = 1, 2$ ($L > 0$)

$$λ_0 \in Λ.$$

We have the following existence and uniqueness result:

**Theorem 4.1.** Let (2.1), (2.5), (3.1), (4.1), (4.2) hold. Then problem (1.1)-(1.4) has a unique solution $x \in C^1(0, T; X)$, $y \in C^1(0, T; Y)$, $λ \in C^1(0, T; Λ)$.

**Proof.** Let us consider the product Hilbert spaces $H = Λ \times X \times Y \times Λ$, $Y = Y \times Λ$ and let $F: X \times Y \times H \rightarrow H$ be the operator defined by

$$F(x, y, z) = (F(λ, x, y, z), G(λ, x, y))$$

for all $x = (x, 0)_X \in X$, $y = (y, λ) \in Y$, $z = (z, ω) \in H$.

Let us also denote

$$x_0 = (x_0, 0_Λ) \quad y_0 = (y_0, λ_0).$$

From (2.1), (3.1) and (4.1) we deduce

$$||F(x, y, z_1) - F(x, y, z_2)||_H \geq m||z_1 - z_2||_H^2$$

for all $x \in X$, $y \in Y$, $z_1, z_2 \in X$

$$||F(x_1, y_1, z_1) - F(x_2, y_2, z_2)||_H \leq L(||x_1 - x_2||_H + ||y_1 - y_2||_H + ||z_1 - z_2||_H)$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$, $z_1, z_2 \in H$ ($L > 0$)

and from (4.4), (2.5), (4.2) we obtain

$$x_0 \in X \quad y_0 \in Y.$$
Since (4.5) - (4.7) are fulfilled we may apply theorem 2.1 and we obtain the existence and the uniqueness of \( x = (x, 0, \lambda) \in C^1(0, T, X), \ y = (y, \lambda) \in C^1(0, T, Y) \) such that

\[
\begin{align*}
\dot{y}(t) &= F(x(t), y(t), \dot{x}(t)) \quad \text{for all} \quad t \in [0, T] \\
\dot{x}(t) &= x_0, \quad y(t) = y_0.
\end{align*}
\]

Theorem 4.1 follows now from (4.3) and (4.4).

**Remark 4.1.** As in the case of the problem (1.1), (1.2), applying theorem 3.1 to (4.8), (4.9) we deduce the finite-time stability of every solution of (1.1)-(1.4).

**References**


