Some remarks concerning a class of nonlinear evolution equations in Hilbert spaces

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1. Introduction

Let $H$ be a real Hilbert space and let $X$, $Y$ be two orthogonal subspaces of $H$ such that $H = X \oplus Y$. Let $\lambda$ be a real normed space and let $T > 0$. In this paper we consider evolution problems of the form

\begin{align}
\dot{y}(t) &= F(\lambda(t), x(t), y(t), z(t)) \quad \text{for all } t \in [0,T], \\
x(0) &= x_0, \quad y(0) = y_0
\end{align}

in which the unknowns are the functions $x : [0,T] \to X$ and $y : [0,T] \to Y$, $F : \Lambda \times X \times Y \times H \to H$ is a nonlinear operator and $\lambda : [0,T] \to \Lambda$ is a parameter function (in (1.1) and everywhere in this paper the dot above represents the derivative with respect to the time variable $t$). Such type of problems arise in the study of quasistatic processes for semilinear rate-type materials (see for example [1] - [3]). In this case the unknowns $x$ and $y$ are the small deformation tensor and the stress tensor and $F$ is an operator involving the constitutive law of the material; the parameter $\lambda$ may be interpreted as the absolute temperature or an internal state variable.

For particular forms of $F$ existence and uniqueness of the solution and error estimates of a numerical method for problems of the form (1.1), (1.2) were already given in [3], [4].

In this paper we prove the existence and uniqueness of the solution for problem (1.1), (1.2) using a technique based on the equivalence between (1.1), (1.2) and a Cauchy problem for an ordinary differential equation in the product Hilbert space $X \times Y$ (section 2). We also study the dependence of the solution with respect to the parameter $\lambda$ and the initial data (section 3). In some applications (see
for example [5] the function $\lambda$ in (1.1) is needed to be considered as an unknown function whose evolution is given by

$$\lambda(t) = G(\lambda(t), x(t), y(t)) \quad \text{for all } t \in [0, T]$$

(1.4)

$$\lambda(0) = \lambda_0$$

where $G : \Lambda \times X \times Y \rightarrow \Lambda$ is a nonlinear operator. For this reason we also consider problem (1.1)-(1.4) for which we prove the existence and uniqueness of the solution (section 4). Let us finally notice that the results presented here complete and generalize some results of [2] and may be applied in the study of some evolution problems for rate-type materials (see [1]-[5]).

2. An existence and uniqueness result

Everywhere in this paper if $V$ is a real normed space we utilise the following notations: $|| \cdot ||_V$ - the norm of $V$; $0_V$ - the zero element of $V$; $C^0(0, T, V)$ - the space of continuous functions on $[0, T]$ with values in $V$; $C^1(0, T, V)$ - the space of derivable functions with continuous derivative on $[0, T]$ with values in $V$; $|| \cdot ||_{0, T, V}$ - the norm on the space $C^0(0, T, V)$ i.e. $||z||_{0, T, V} = \max_{t \in [0, T]} ||z(t)||_V$ for all $z \in C^0(0, T, V)$; $||\cdot||_{1, T, V}$ - the norm on the space $C^1(0, T, V)$ i.e. $||z||_{1, T, V} = ||z||_{0, T, V} + ||\dot{z}||_{0, T, V}$ for all $z \in C^1(0, T, V)$. If moreover $V$ is a real Hilbert space we denote by $\langle \cdot, \cdot \rangle_V$ the inner product of $V$. Finally, if $V_1$ and $V_2$ are real Hilbert spaces we denote by $V_1 \times V_2$ the product space endowed with the canonical inner product and by $v = (v_1, v_2)$ the elements of $V_1 \times V_2$.

Let us consider the following assumptions:

$$m > 0 \text{ such that } \langle F(\lambda, x_1, y_1, \alpha_1^1) - F(\lambda, x_1, y_1, \alpha_1^2), \alpha_1^1 - \alpha_1^2 \rangle_H \geq m \|x_1 - y_1\|^2_H \text{ for all } \lambda \in \Lambda, x, y \in X, \alpha_1 \in X.$$  

(2.1)

$$M > 0 \text{ such that } \|F(\lambda, x_1^i, y_1^i, \alpha_1^1) - F(\lambda, x_2^i, y_2^i, \alpha_1^2)\|_H \leq M \left( \|x_1^i - y_1^i\|_H^2 + \|y_1^i - y_2^i\|_H^2 + \|\alpha_1^1 - \alpha_1^2\|_H^2 \right) \text{ for all } \lambda \in \Lambda, x_1^i, x_2^i \in X, y_1^i, y_2^i \in Y, \alpha_1^1, \alpha_1^2 \in X.$$  

(2.2)
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(2.3) \( \lambda \mapsto F(\lambda, x, y, z) : \Lambda \rightarrow H \) is a continuous operator, for all \( x \in X, y \in Y \) and \( z \in H \).

(2.4) \( \lambda \in C^0(0, T, H) \)

(2.5) \( x_0 \in X, \ y_0 \in Y \)

The main result of this section is the following:

Theorem 2.1. Let (2.1)-(2.5) hold. Then problem (1.1), (1.2) has a unique solution \( x \in C^1(0, T, X), \ y \in C^1(0, T, Y) \).

In order to prove theorem 2.1 let us denote by \( Z \) the product Hilbert space \( Z = X \times Y \) (which in fact is isomorph with \( H \)). We have:

Lemma 2.1. Let \( \lambda \in \Lambda, \ x \in X \) and \( y \in Y \): then there exists a unique element \( z = (u, v) \in Z \) such that \( v = F(\lambda, x, y, u) \).

Proof. The uniqueness part is a consequence of (2.1); indeed, if the elements \( z = (u, v), \ \tilde{z} = (\tilde{u}, \tilde{v}) \in Z \) are such that \( v = F(\lambda, x, y, u), \ \tilde{v} = F(\lambda, x, y, \tilde{u}) \), using (2.1) we have \( \langle v - \tilde{v}, u - \tilde{u} \rangle_H > m \|u - \tilde{u}\|_H^2 \) hence by the orthogonality in \( H \) of \( v - \tilde{v} \) and \( u - \tilde{u} \) we deduce \( u = \tilde{u} \) which implies \( v = \tilde{v} \).

For the existence part let us denote by \( P_1 : H \rightarrow X \) the projector map on \( X \). Using (2.1) and (2.2) we get that the operator \( P_1 F(\lambda, x, y, \cdot) : X \rightarrow X \) is a strongly monotone and Lipschitz continuous operator hence by Browder's surjectivity theorem we get that there exists \( u \in X \) such that \( P_1 F(\lambda, x, y, u) = 0_X \). It results that the element \( F(\lambda, x, y, u) \) belongs to \( Y \) and we finish the proof taking \( z = (u, v) \) where \( v = F(\lambda, x, y, u) \).

Lemma 2.1 allows us to consider the operator \( B : \Lambda \times Z \rightarrow Z \) defined by

(2.6) \( B(\lambda, \omega) = z \) iff \( \omega = (x, y), \ z = (u, v) \) and \( v = F(\lambda, x, y, u) \).

Moreover, we have:

Lemma 2.2. \( B \) is a continuous operator and there exists \( L > 0 \) such that

(2.7) \( \|B(\lambda, \omega_1) - B(\lambda, \omega_2)\|_Z \leq L \|\omega_1 - \omega_2\|_Z \) for all \( \lambda \in \Lambda, \ \omega_1, \omega_2 \in Z \).
Proof. Let \( \lambda_i \in \Lambda \), \( u_i = (x_i, y_i, u_i) \in Z \) and \( z_i = (u_i, v_i) = B(\lambda_i, u_i) \), \( i = 1, 2 \). Using (2.6) we get:

\[
v_i = F(\lambda_i, x_i, y_i, u_i), \quad i = 1, 2
\]

which implies

\[
P_i F(\lambda_i, x_i, y_i, u_i) = 0, \quad i = 1, 2
\]

From (2.1) and (2.9) we get

\[
m \| u_1 - u_2 \|_H \leq \delta \left< F(\lambda_1, x_1, y_1, u_1) - F(\lambda_1, x_1, y_1, u_2), u_1 - u_2 \right>_H =
\]

\[
\leq \| P_i F(\lambda_2, x_2, y_2, u_2) - P_i F(\lambda_1, x_1, y_1, u_1), u_1 - u_2 \|_H \leq \| F(\lambda_2, x_2, y_2, u_2) - F(\lambda_1, x_1, y_1, u_1), u_1 - u_2 \|_H \| u_1 - u_2 \|_H
\]

which implies

\[
\| u_1 - u_2 \|_H \leq \delta \left( \frac{1}{m} \right) \| F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2) \|_H
\]

Using now (2.8) and (2.2) we get

\[
\| v_I - v_2 \|_H \leq M \| u_1 - u_2 \|_H + \| F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2) \|_H
\]

hence by (2.10) it results

\[
\| v_I - v_2 \|_H \leq \{ \frac{M}{m} + 1 \} \| F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2) \|_H
\]

Using again (2.2) we get

\[
\| F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2) \|_H < M \| x_1 - x_2 \|_H + \| y_1 - y_2 \|_H
\]

hence by (2.3) we obtain

\[
\| F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2) \|_H \to 0 \quad \text{when} \quad \lambda_1 \to \lambda_2
\]

in \( \Lambda \), \( x_1 \to x_2 \) in \( X \) and \( y_1 \to y_2 \) in \( Y \). Using now (2.10) and (2.11) we get the continuity of \( B \) and taking \( \lambda_1 = \lambda_2 \) from (2.10)-(2.12) we get (2.7).
Proof of theorem 2.1. Let \( A : [0,T] \times Z \rightarrow Z \) and \( z_0 \) be defined by

\[
A(t,z) = B(\lambda(t),z) \quad \text{for all } t \in [0,T] \text{ and } z \in Z.
\]

(2.13)\[ z_0 = (x_0, y_0). \]

Using (2.6) we get that \( z \in C^1(0,T,X) \) and \( y \in C^1(0,T,Y) \) is a solution of (1.1), (1.2) iff \( z = (x, y) \in C^1(0,T,Z) \) is a solution of the problem

\[
\dot{z}(t) = A(t, z(t)) \quad \text{for all } t \in [0,T] \]

(2.15)\[ z(0) = z_0. \]

In order to study (2.15), (2.16) let us remark that by lemma 2.2 and (2.4) we get that \( A \) is a continuous operator and

\[
||A(t,z_1) - A(t,z_2)||_Z \leq L ||z_1 - z_2||_Z \quad \text{for all } t \in [0,T] \text{ and } z_1, z_2 \in Z.
\]

Moreover, by (2.5), (2.14) we get \( z_0 \in Z \). Theorem 2.1 follows now from the classical Cauchy-Lipschitz theorem applied to (2.15), (2.16).

3. The continuous dependence of the solution with respect to the data

Let us now replace (2.2), (2.3) by a stronger assumption namely

\[
\text{there exists } M > 0 \text{ such that } ||F(\lambda_1, x_1, y_1, z_1) - F(\lambda_2, x_2, y_2, z_2)||_H \leq M(||\lambda_1 - \lambda_2||_\Lambda^+ ||x_1 - x_2||_H^+ ||y_1 - y_2||_H^+ ||z_1 - z_2||_H^+) \quad \text{for all } \lambda \in \Lambda, x, y \in X, z \in H, i = 1, 2.
\]

(3.1)

We have the following result:

**Theorem 3.1.** Let (2.1), (3.1) hold and let \( x_i \in C^1(0,T,X), y_i \in C^1(0,T,Y) \) be the solution of (1.1), (1.2) for the data \( \lambda_i, x_{0i}, y_{0i} \) satisfying (2.4), (2.5), \( i = 1, 2 \). Then there exists \( C > 0 \) such that

\[
||x_1 - x_2||_{1,T,H} + ||y_1 - y_2||_{1,T,H} \leq C(||\lambda_1 - \lambda_2||_{0,T,\Lambda^+} ||x_0 - x_0||_H^+ ||y_0 - y_0||_H^+).
\]

(3.2)
Remark 3.1. In (3.1) and everywhere in this section $C$ are strictly positive generic constants which depend only on $F$ and $T$.

Proof of theorem 3.1. Let $x_i = (x_{i1}, x_{i2})$ and $z_{0i} = (z_{01}, z_{02})$, $i = 1, 2$. As it results from the proof of theorem 2.1 we have

\begin{align}
\dot{x}_i(t) &= A_i(t, z_i(t)) \quad \text{for all } t \in [0, T] \\
\dot{z}_i(0) &= z_{0i}
\end{align}

where the operators $A_i$ are defined by (2.13) replacing $\lambda$ by $\lambda_i$, $i = 1, 2$. Since (3.1) implies that $B : \Lambda \times Z \rightarrow Z$ is a Lipschitz continuous operator (see the proof of lemma 2.2), from (2.13) we get that there exists $L > 0$ such that

\begin{equation}
||A_i(t, z_1(t)) - A_i(t, z_2(t))||_Z \leq L(||\lambda_1(t) - \lambda_2(t)||_\Lambda + ||z_1(t) - z_2(t)||_Z)
\end{equation}

for all $t \in [0, T]$. Using now (3.3) and (3.5) we get

\begin{equation}
<\dot{z}_1(t) - \dot{z}_2(t), z_1(t) - z_2(t)>_Z \leq L(||\lambda_1(t) - \lambda_2(t)||_\Lambda + ||z_1(t) - z_2(t)||_Z)
\end{equation}

for all $t \in [0, T]$. Hence by (3.4) and a Gronwall-type lemma we deduce

\begin{equation}
||z_1(s) - z_2(s)||_Z \leq C\left(\int_0^s ||\lambda_1(t) - \lambda_2(t)||_\Lambda + ||z_1(t) - z_2(t)||_Z\right) dt \quad \text{for all } s \in [0, T]
\end{equation}

which implies

\begin{equation}
||z_{1} - z_{2}||_{0, T, Z} \leq C (||\lambda_1 - \lambda_2||_{0, T, \Lambda} + ||z_{01} - z_{02}||_Z).
\end{equation}

Using again (3.3) and (3.5) we have

\begin{equation}
||\dot{z}_1(t) - \dot{z}_2(t)||_Z \leq C(||\lambda_1(t) - \lambda_2(t)||_\Lambda + ||z_1(t) - z_2(t)||_Z)
\end{equation}

for all $t \in [0, T]$. And by (3.6) it results

\begin{equation}
||\dot{z}_1 - \dot{z}_2||_{0, T, Z} \leq C (||\lambda_1 - \lambda_2||_{0, T, \Lambda} + ||z_{01} - z_{02}||_Z).
\end{equation}

From (3.6) and (3.7) we get

\begin{equation}
||z_{1} - z_{2}||_{1, T, Z} \leq C (||\lambda_1 - \lambda_2||_{0, T, \Lambda} + ||z_{01} - z_{02}||_Z)
\end{equation}

which implies (3.2).

Remark 3.2. From (3.2) we deduce in particular the continuous dependence of the
solution with respect the initial data i.e. the finite-time stability of every
solution of (1.1), (1.2) (for definitions in the field see for instance [6] chap.5).

4. A second existence and uniqueness result

In this section we suppose that $\Lambda$ is a real Hilbert space. We consider the
operator $G : \Lambda \times X \times Y \rightarrow \Lambda$ and the element $\lambda_0$ such that

$$\|G(\lambda_1, x_1, y_1) - G(\lambda_2, x_2, y_2)\|_{\Lambda} \leq L(\|\lambda_1 - \lambda_2\|_{\Lambda} + \|x_1 - x_2\|_H + \|y_1 - y_2\|_H)$$
for all $\lambda_i \in \Lambda$, $x_i \in X$, $y_i \in Y$, $i = 1, 2$ ($L > 0$)

$$\lambda_0 \in \Lambda.$$  

We have the following existence and uniqueness result:

**Theorem 4.1.** Let (2.1), (2.5), (3.1), (4.1), (4.2) hold. Then problem (1.1)-(1.4)
has a unique solution $x \in C^1(0, T; X)$, $y \in C^1(0, T; Y)$, $\lambda \in C^1(0, T; \Lambda)$.

**Proof.** Let us consider the product Hilbert spaces $H = H \times \Lambda$, $X = X \times \{\lambda_0\}$,
$Y = Y \times \Lambda$ and let $F : X \times Y \times H \rightarrow H$ be the operator defined by

$$F(x, y, z) = (F(\lambda, x, y, z), G(\lambda, x, y))$$ for all $x = (x, \lambda_0) \in X$, $y = (y, \lambda) \in Y$,
$z = (z, u) \in H$.

Let us also denote

$$x_0 = (x_0, \lambda_0), \quad y_0 = (y_0, \lambda_0).$$

From (2.1), (3.1) and (4.1) we deduce

$$\langle F(x, y, z_1) - F(x, y, z_2), z_1 - z_2 \rangle_H \geq m \|z_1 - z_2\|_H^2$$
for all $x \in X$, $y \in Y$, $z_1, z_2 \in X$

$$\|F(x, y, z_1) - F(x, y, z_2)\|_H \leq L(\|x_1 - x_2\|_H + \|y_1 - y_2\|_H + \|z_1 - z_2\|_H)$$
for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$, $z_1, z_2 \in H$ ($L > 0$)

and from (4.4), (2.5), (4.2) we obtain

$$x_0 \in X, \quad y_0 \in Y.$$
Since (4.5)-(4.7) are fulfilled we may apply theorem 2.1 and we obtain the existence and the uniqueness of $x = (x, 0, \lambda) \in C^1(0,T, X)$, $y = (y, \lambda) \in C^1(0,T, Y)$ such that

\begin{align*}
(4.8) & \quad \dot{y}(t) = F(x(t), y(t), \dot{x}(t)) \quad \text{for all } t \in [0,T] \\
(4.9) & \quad x(0) = x_0, \quad y(0) = y_0.
\end{align*}

Theorem 4.1 follows now from (4.3) and (4.4).

Remark 4.1. As in the case of the problem (1.1)-(1.4), applying theorem 3.1 to (4.8), (4.9) we deduce the finite-time stability of every solution of (1.1)-(1.4).

References