ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 95, série *Mathématiques*, n° 26 (1990), p. 13-20

<http://www.numdam.org/item?id=ASCFM_1990__95_26_13_0>

Ann. Sci. Univ. "Blaise Pascal", Clermont II, Sér. Math., Fasc. 26ème, pp. 13-20

SOME REMARKS CONCERNING A CLASS OF NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACES

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1. Introduction

Let H be a real Hilbert space and let X, Y be two orthogonal subspaces of H such that $H = X \oplus Y$. Let Λ be a real normed space and let $T > 0$. In this paper we consider evolution problems of the form

$$\dot{y}(t) = F(\lambda(t), x(t), y(t), \dot{x}(t)) \qquad \text{for all} \quad t \in [0,T], \tag{1.1}$$

$$x(0) = x_0 \quad , \quad y(0) = y_0 \tag{1.2}$$

in which the unknowns are the functions $x : [0,T] \longrightarrow X$ and $y : [0,T] \longrightarrow Y$, $F : \Lambda \times X \times Y \times H \longrightarrow H$ is a nonlinear operator and $\lambda : [0,T] \longrightarrow \Lambda$ is a parameter function (in (1.1) and everywhere in this paper the dot above represents the derivative with respect to the time variable t). Such type of problems arise in the study of quasistatic processes for semilinear rate-type materials (see for example [1] - [3]). In this case the unknowns x and y are the small deformation tensor and the stress tensor and F is an operator involving the constitutive law of the material ; the paramater λ may be interpreted as the absolute temperature or an internal state variable.

For particular forms of F existence and uniqueness of the solution and error estimates of a numerical method for problems of the form (1.1), (1.2) were already given in [3], [4].

In this paper we prove the existence and uniqueness of the solution for problem (1.1), (1.2) using a technique based on the equivalence between (1.1), (1.2) and a Cauchy problem for an ordinary differential equation in the product Hilbert space $X \times Y$ (section 2). We also study the dependence of the solution with respect to the parameter λ and the initial data (section 3). In some applications (see for example [5]) the function λ in (1.1) is needed to be considered as an unknown function whose evolution is given by

(1.3) $\dot{\lambda}(t) = G(\lambda(t), x(t), y(t))$ for all $t \in [0,T]$,

(1.4) $\lambda(0) = \lambda_0$

where $G : \Lambda \times X \times Y \longrightarrow \Lambda$ is a nonlinear operator. For this reason we also consider problem (1.1)-(1.4) for which we prove the existence and uniqueness of the solution (section 4). Let us finally notice that the results presented here complete and generalize some results of [2] and may be applied in the study of some evolution problems for rate-type materials (see [1] - [5]).

2. An existence and uniqueness result

Everywhere in this paper if V is a real normed space we utilise the following notations : $||\cdot||_V$ - the norm of V ; 0_V - the zero element of V ; $C^0(0,T,V)$ - the space of continuous functions on $[0,T]$ with values in V ; $C^1(0,T,V)$ - the space of derivable functions with continuous derivative on $[0,T]$ with values in V ; $||\cdot||_{0,T,V}$ - the norm on the space $C^0(0,T,V)$ i.e. $||z||_{0,T,V} = \max_{t\in[0,T]} ||z(t)||_V$ for all $z \in C^0(0,T,V)$; $||\cdot||_{1,T,V}$ - the norm on the space $C^1(0,T,V)$ i.e. $||z||_{1,T,V} = ||z||_{0,T,V} + ||\dot{z}||_{0,T,V}$ for all $z \in C^1(0,T,V)$. If moreover V is a real Hilbert space we denote by $< , >_V$ the inner product of V . Finally, if V_1 and V_2 are real Hilbert spaces we denote by $V_1 \times V_2$ the product space endowed with the cannonical inner product and by $v = (v_1, v_2)$ the elements of $V_1 \times V_2$.

Let us consider the following assumptions :

(2.1) there exists $m>0$ such that $<F(\lambda,x,y,z_1)-F(\lambda,x,y,z_2), z_1-z_2>_H \geq$ $\geq m\, ||z_1-z_2||_H^2$ for all $\lambda \in \Lambda$, $x \in X$, $y \in Y$, $z_1, z_2 \in X$;

(2.2) there exists $M>0$ such that $||F(\lambda,x_1,y_1,z_1) - F(\lambda,x_2,y_2,z_2)||_H \leq$ $\leq M(||x_1-x_2||_H + ||y_1-y_2||_H + ||z_1-z_2||_H)$ for all $\lambda \in \Lambda$, $x_i \in X$, $y_i \in Y$, $z_i \in H$, $i=1,2$;

(2.3) $\lambda \longrightarrow F(\lambda,x,y,z) : \Lambda \longrightarrow H$ *is an continuous operator , for all* $x \in X$, $y \in Y$ *and* $z \in H$.

(2.4) $\lambda \in C^0(0,T,\Lambda)$

(2.5) $x_0 \in X$, $y_0 \in Y$.

The main result of this section is the following :

Theorem 2.1. Let (2.1)-(2.5) hold. Then problem (1.1),(1.2) has a unique solution $x \in C^1(0,T,X)$, $y \in C^1(0,T,Y)$.

In order to prove theorem 2.1 let us denote by Z the product Hilbert space $Z = X \times Y$ (which in fact is isomorph with H).We have :

Lemma 2.1. Let $\lambda \in \Lambda$, $x \in X$ *and* $y \in Y$ *: then there exists a unique element* $z = (u,v) \in Z$ *such that* $v = F(\lambda,x,y,u)$.

Proof. The uniqueness part is a consequence of (2.1) ; indeed, if the elements $z = (u,v)$, $\tilde{z} = (\tilde{u},\tilde{v}) \in Z$ are such that $v = F(\lambda,x,y,u)$, $\tilde{v} = F(\lambda,x,y,\tilde{u})$, using (2.1) we have $\langle v-\tilde{v}, u-\tilde{u}\rangle_H \geqslant m||u-u||_H^2$ hence by the orthogonality in H of $v-\tilde{v}$ and $u-\tilde{u}$ we deduce $u = \tilde{u}$ which implies $v = \tilde{v}$.

For the existence part let us denote by $P_1 : H \longrightarrow X$ the projector map on X. Using (2.1) and (2.2) we get that the operator $P_1F(\lambda,x,y,\cdot) : X \longrightarrow X$ is a strongly monotone and Lipschitz continuous operator hence bv Browder's surjectivity theorem we get that there exists $u \in X$ such that $P_1F(\lambda,x,y,u) = 0_X$. Tt results that the element $F(\lambda,x,y,u)$ belongs to Y and we finish the proof taking $z=(u,v)$ where $v = F(\lambda,x,y,u)$.

Lemma 2.1 allows us to consider the operator $B : \Lambda \times Z \longrightarrow Z$ defined by

(2.6) $B(\lambda,w) = z$ *iff* $w = (x,y)$, $z = (u,v)$ *and* $v = F(\lambda,x,y,u)$.

Moreover, we have :

Lemma 2.2. B is a continuous operator and there exists $L > 0$ *such that*

(2.7) $||B(\lambda,w_1) - B(\lambda,w_2)||_Z \leqslant L\, ||w_1 - w_2||_Z$ *for all* $\lambda \in \Lambda$, $w_1,w_2 \in Z$.

Proof. Let $\lambda_i \in \Lambda$, $w_i = (x_i, y_i) \in Z$ and $z_i = (u_i, v_i) = B(\lambda_i, w_i)$, $i = 1,2$. Using (2.6) we get :

(2.8) $$v_i = F(\lambda_i, x_i, y_i, u_i) \quad , \quad i = 1,2$$

which implies

(2.9) $$P_1 F(\lambda_i, x_i, y_i, u_i) = 0_X \quad , \quad i = 1,2$$

From (2.1) and (2.9) we get

$$m||u_1-u_2||_H^2 \leqslant \langle F(\lambda_1, x_1, y_1, u_1) - F(\lambda_1, x_1, y_1, u_2), u_1-u_2 \rangle_H =$$

$$= \langle P_1 F(\lambda_2, x_2, y_2, u_2) - P_1 F(\lambda_1, x_1, y_1, u_2), u_1-u_2 \rangle_H \leqslant ||F(\lambda_2, x_2, y_2, u_2) - F(\lambda_1, x_1, y_1, u_2)||_H ||u_1-u_2||_H$$

which implies

(2.10) $$||u_1-u_2||_H \leqslant \frac{1}{m} \; || F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H \quad .$$

Using now (2.8) and (2.2) we get

$$||v_1-v_2||_H \leqslant M \; ||u_1-u_2||_H + ||F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H$$

hence by (2.10) it results

(2.11) $$||v_1-v_2||_H \leqslant \left(\frac{M}{m} + 1\right) ||F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H \quad .$$

Using again (2.2) we get

(2.12) $$||F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H < M(||x_1-x_2||_H + ||y_1-y_2||_H) +$$
$$+ ||F(\lambda_1, x_2, y_2, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H$$

hence by (2.3) we obtain $||F(\lambda_1, x_1, y_1, u_2) - F(\lambda_2, x_2, y_2, u_2)||_H \longrightarrow 0$ when $\lambda_1 \longrightarrow \lambda_2$ in Λ , $x_1 \longrightarrow x_2$ in X and $y_1 \longrightarrow y_2$ in Y . Using now (2.10) and (2.11) we get the continuity of B and taking $\lambda_1 = \lambda_2$ from (2.10)-(2.12) we get (2.7).

Proof of theorem 2.1. Let $A : [0,T] \times Z \longrightarrow Z$ and z_0 be defined by

(2.13) $$A(t,z) = B(\lambda(t),z) \qquad \text{for all} \quad t \in [0,T] \quad \text{and} \quad z \in Z$$

(2.14) $$z_0 = (x_0,y_0) \quad .$$

Using (2.6) we get that $x \in C^1(0,T,X)$ and $y \in C^1(0,T,Y)$ is a solution of (1.1),(1.2) iff $z = (x,y) \in C^1(0,T,Z)$ is a solution of the problem

(2.15) $$\dot{z}(t) = A(t,z(t)) \qquad \text{for all} \quad t \in [0,T]$$

(2.16) $$z(0) = z_0 \quad .$$

In order to study (2.15),(2.16) let us remark that by lemma 2.2 and (2.4) we get that A is a continuous operator and

$$||A(t,z_1)-A(t,z_2)||_Z \leqslant L\ ||z_1-z_2||_Z \qquad \text{for all} \quad t \in [0,T] \text{ and } z_1,z_2 \in Z \ .$$

Moreover, by (2.5),(2.14) we get $z_0 \in Z$. Theorem 2.1 follows now from the classical Cauchy-Lipschitz theorem applied to (2.15),(2.16).

3. The continuous dependence of the solution with respect to the data

Let us now replace (2.2) ,(2.3) by a stronger assumption namely

(3.1) *there exists* $M > 0$ *such that* $||F(\lambda_1,x_1,y_1,z_1)-F(\lambda_2,x_2,y_2,z_2)||_H \leqslant$ $\leqslant M(||\lambda_1-\lambda_2||_\Lambda+||x_1-x_2||_H+||y_1-y_2||_H+||z_1-z_2||_H)$ *for all* $\lambda_i \in \Lambda$, $x_i \in X$, $y_i \in Y$, $z_i \in H$, $i = 1,2$.

We have the following result :

Theorem 3.1. Let (2.1),(3.1) hold and let $x_i \in C^1(0,T,X)$, $y_i \in C^1(0,T,Y)$ *be the solution of (1.1),(1.2) for the data* λ_i, x_{0i}, y_{0i} *satisfying (2.4),(2.5),* $i = 1,2$ *. Then there exists* $C > 0$ *such that*

(3.2) $$||x_1-x_2||_{1,T,H} + ||y_1-y_2||_{1,T,H} \leqslant C(||\lambda_1-\lambda_2||_{0,T,\Lambda}+||x_{01}-x_{02}||_H+||y_{01}-y_{02}||_H).$$

Remark 3.1. *In (3.2) and everywhere in this section C are strictely positive generic constants which depend only on F and T.*

Proof of theorem 3.1. Let $z_i=(x_i, y_i)$ and $z_{0i}=(x_{0i}, y_{0i})$, $i=1,2$. As it results from the proof of theorem 2.1 we have

$$\dot{z}_i(t) = A_i(t, z_i(t)) \qquad \text{for all} \quad t \in [0,T] \tag{3.3}$$

$$z_i(0) = z_{0i} \tag{3.4}$$

where the operators A_i are defined by (2.13) replacing λ by λ_i, $i=1,2$. Since (3.1) implies that $B : \Lambda \times Z \longrightarrow Z$ is a Lipschitz continuous operator (see the proof of lemma 2.2), from (2.13) we get that there exists $L > 0$ such that

$$\|A_1(t, z_1(t)) - A_2(t, z_2(t))\|_Z \leq L(\|\lambda_1(t)-\lambda_2(t)\|_\Lambda + \|z_1(t)-z_2(t)\|_Z) \tag{3.5}$$
$$\text{for all} \quad t \in [0,T]$$

Using now (3.3) and (3.5) we get

$$<\dot{z}_1(t)-\dot{z}_2(t), z_1(t)-z_2(t)>_Z \leq L(\|\lambda_1(t)-\lambda_2(t)\|_\Lambda + \|z_1(t)-z_2(t)\|_Z)\|z_1(t)-z_2(t)\|_Z$$

for all $t \in [0,T]$ hence by (3.4) and a Gronwall-type lemma we deduce

$$\|z_1(s)-z_2(s)\|_Z \leq C\left(\int_0^s \|\lambda_1(t)-\lambda_2(t)\|_\Lambda dt + \|z_{01}-z_{02}\|_Z\right) \qquad \text{for all} \quad s \in [0,T]$$

which implies

$$\|z_1-z_2\|_{0,T,Z} \leq C\,(\|\lambda_1-\lambda_2\|_{0,T,\Lambda} + \|z_{01}-z_{02}\|_Z) \quad . \tag{3.6}$$

Using again (3.3) and (3.5) we have

$$\|\dot{z}_1(t)-\dot{z}_2(t)\|_Z \leq C(\|\lambda_1(t)-\lambda_2(t)\|_\Lambda + \|z_1(t)-z_2(t)\|_Z) \qquad \text{for all} \quad t \in [0,T]$$

and by (3.6) it results

$$\|\dot{z}_1-\dot{z}_2\|_{0,T,Z} \leq C\,(\|\lambda_1-\lambda_2\|_{0,T,\Lambda} + \|z_{01}-z_{02}\|_Z) \quad . \tag{3.7}$$

From (3.6) and (3.7) we get

$$\|z_1-z_2\|_{1,T,Z} \leq C\,(\|\lambda_1-\lambda_2\|_{0,T,\Lambda} + \|z_{01}-z_{02}\|_Z)$$

which implies (3.2).

Remark 3.2. *From (3.2) we deduce in particular the continuous dependence of the* *solution with respect the initial data i.e. the finite-time stability of every solution of (1.1),(1.2) (for definitions in the field see for instance* [6] *chap.5).*

4. A second existence and uniqueness result

In this section we suppose that Λ is a real Hilbert space. We consider the operator $G : \Lambda \times X \times Y \longrightarrow \Lambda$ and the element λ_0 such that

(4.1) $$||G(\lambda_1,x_1,y_1)-G(\lambda_2,x_2,y_2)||_\Lambda \leq L(||\lambda_1-\lambda_2||_\Lambda+||x_1-x_2||_H+||y_1-y_2||_H)$$
$$\text{for all } \lambda_i \in \Lambda,\ x_i \in X,\ y_i \in Y,\ i=1,2 \quad (L>0)$$

(4.2) $$\lambda_0 \in \Lambda .$$

We have the following existence and uniqueness result :

Theorem 4.1. *Let (2.1),(2.5),(3.1),(4.1),(4.2) hold. Then problem (1.1)-(1.4) has a unique solution* $x \in C^1(0,T,X)$, $y \in C^1(0,T,Y)$, $\lambda \in C^1(0,T,\Lambda)$.

Proof. Let us consider the product Hilbert spaces $\mathrm{H} = H \times \Lambda$, $\mathrm{X} = X \times \{0_\Lambda\}$, $\mathrm{Y} = Y \times \Lambda$ and let $\mathrm{F} : \mathrm{X} \times \mathrm{Y} \times \mathrm{H} \longrightarrow \mathrm{H}$ be the operator defined by

(4.3) $$\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{z}) = (F(\lambda,x,y,z),G(\lambda,x,y)) \quad \text{for all } \mathrm{x}=(x,0_\Lambda)\in \mathrm{X},\ \mathrm{y}=(y,\lambda)\in \mathrm{Y},\ \mathrm{z}=(z,\mu)\in \mathrm{H} .$$

Let us also denote

(4.4) $$\mathrm{x}_0 = (x_0,0_\Lambda) \quad , \quad \mathrm{y}_0 = (y_0,\lambda_0) .$$

From (2.1),(3.1) and (4.1) we deduce

(4.5) $$<\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{z}_1)-\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{z}_2)>_\mathrm{H} \geq m||\mathrm{z}_1-\mathrm{z}_2||^2_\mathrm{H} \qquad \text{for all } \mathrm{x}\in \mathrm{X},\ \mathrm{y}\in \mathrm{Y},\ \mathrm{z}_1,\mathrm{z}_2\in \mathrm{X}$$

(4.6) $$||\mathrm{F}(\mathrm{x}_1,\mathrm{y}_1,\mathrm{z}_1)-\mathrm{F}(\mathrm{x}_2,\mathrm{y}_2,\mathrm{z}_2)||_\mathrm{H} \leq \mathrm{L}(||\mathrm{x}_1-\mathrm{x}_2||_\mathrm{H} + ||\mathrm{y}_1-\mathrm{y}_2||_\mathrm{H} + ||\mathrm{z}_1-\mathrm{z}_2||_\mathrm{H})$$
$$\text{for all } \mathrm{x}_1,\mathrm{x}_2\in \mathrm{X},\ \mathrm{y}_1,\mathrm{y}_2\in \mathrm{Y},\ \mathrm{z}_1,\mathrm{z}_2\in \mathrm{H} \quad (\mathrm{L}>0)$$

and from (4.4),(2.5),(4.2) we obtain

(4.7) $$\mathrm{x}_0\in \mathrm{X} \quad , \quad \mathrm{y}_0\in \mathrm{Y} .$$

Since (4.5)-(4.7) are fulfilled we may apply theorem 2.1 and we obtain the existence and the uniqueness of $x = (x, 0_{\Lambda}) \in C^1(0,T,X)$, $y = (y,\lambda) \in C^1(0,T,Y)$ such that

$$\dot{y}(t) = F(x(t), y(t), \dot{x}(t)) \qquad \text{for all} \quad t \in [0,T] \tag{4.8}$$

$$x(0) = x_0 \quad , \quad y(0) = y_0 \quad . \tag{4.9}$$

Theorem 4.1 follows now from (4.3) and (4.4).

Remark 4.1. *As in the case of the problem (1.1),(1.2) , applying theorem 3.1 to (4.8),(4.9) we deduce the finite-time stability of every solution of (1.1)-(1.4).*

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Manuscrit reçu en Mai 1990