

ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

BRUNO IOCHUM

GUY LOUPIAS

Remarks on the bidual of Banach algebras (the C^* case)

Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 97, série *Mathématiques*, n° 27 (1991), p. 107-118

http://www.numdam.org/item?id=ASCFM_1991__97_27_107_0

© Université de Clermont-Ferrand 2, 1991, tous droits réservés.

L'accès aux archives de la revue « Annales scientifiques de l'Université de Clermont-Ferrand 2 » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

**REMARKS ON THE BIDUAL OF BANACH ALGEBRAS
(THE C^* CASE)**

Bruno IOCHUM

and

Guy LOUPIAS

1. INTRODUCTION

Let E be a Banach space. If E carries an algebraic structure, it is natural to ask for a similar structure on the bidual E^{**} which extends the original one. Arens introduced in [1], [2] two different products on a Banach algebra and called it regular if they coincide. As noticed in the review [7] the main effort in subsequent years has been directed at the problem of Arens regularity.

Examples : A C^* -algebra is regular (Sherman, Takeda, Tomita). This is also the case for a JB-algebra (see [9] 4.4.3). Remark that these results were proved some years after the definition of the algebraic structure. Another example can be found in the theory of JB^* -triples. This notion was introduced by Kaup in [13] in connection with bounded symmetric domains in Banach spaces. Recently Dineen [6] proved that the bidual of a JB^* -triple is again a JB^* -triple. His argument uses two facts. First, this category is stable by projection of norm one and second, the bidual of a Banach space is the image by a projection of norm one of an ultraproduct of this space.

Actually this ultrapower technique gives rise to a natural candidate for an alternative product on the bidual of a Banach algebra.

On the other hand there exist Banach binary products on a Banach space which are not bilinear but behave sufficiently well to induce nevertheless "almost-algebraic" structures. This is the case for a Banach-power associative system which is not a Jordan-Banach algebra : that is to say a real Banach space E with an even map denoted by $\circ : x \in E \rightarrow x^2$ (a square map) which induces a binary product by $x \circ y = 2^{-1}[(x+y)^2 - x^2 - y^2]$ and a n^{th} power by $x^n = x^{n-1} \circ x$, $n > 2$, the following axioms being satisfied :

$$\begin{aligned} x^2 \circ x^2 &= x^4 \\ x^m \circ (-x^n) &= -(x^m \circ x^n) \quad m, n \in \mathbb{N}^* \end{aligned}$$

$$\|x^2\| = \|x\|^2$$

$$\|x^2 \circ y^2\| \leq \max(\|x\|^2, \|y\|^2)$$

The square map is continuous on the closed subspace generated by an element and its n^{th} powers.

It is proved in [12] that if \circ is bilinear then E is a JB-algebra. In this setting the Arens construction fails from the beginning due to the lack of linearity. However, the ultrapower-induced product provides an extension which behaves rather well. Of course, a problem similar to the Arens regularity arises but we prove here that the bidual of $C^*(JB^*)$ -algebras are also $C^*(JB^*)$ -algebras for that product.

Let us fix the notations (see [10]).

Let E be a real or complex Banach space, I an index set and u an ultrafilter on I . Define

$$\ell_{\infty}(I, E) = \left\{ (x_i)_{i \in I} / \|(x_i)\| = \sup_I \|x_i\|_E < \infty, x_i \in E \right\}$$

$$N_u = \left\{ (x_i)_{i \in I} \in \ell_{\infty}(I, E) / \lim_u \|x_i\| = 0 \right\}$$

$$(E^I)_u = \ell_{\infty}(I, E) / N_u$$

Clearly $(E^I)_u$ is a Banach space for the canonical quotient norm. If the

equivalence class of $(x_i) \in \ell_\infty(I, E)$ is denoted by $(x_i)_u$ then

$$\|(x_i)_u\| = \lim_u \|x_i\|.$$

We need the following result due to Henson-Moore and Stern (see [10]) :

Theorem 1. Let E be a Banach space. There exist an index set I , an ultrafilter u on I and a linear isometry $J : E^{**} \rightarrow (E^I)_u$ with the properties

- 1°) The restriction of J to E is the canonical embedding of E into $(E^I)_u$;
- 2°) $J(E^{**})$ is the image of $(E^I)_u$ under a projection P of norm one.

Actually I is the set of triples (M, N, ϵ) where M is a finite dimensional subspace of E^{**} , N is a finite dimensional subspace of E^* and ϵ is a positive number. If we order this set by $(M, N, \epsilon) \leq (M', N', \epsilon')$ if $M \subset M'$, $N \subset N'$ and $\epsilon' \leq \epsilon$, then u can be any ultrafilter finer than the section filter $\mathcal{J}(I)$. Moreover $P = JQ$ where

$$Q(x_i)_u = w^* - \lim_u x_i$$

(the limit exists by the w^* -compactness of the unit ball of E^{**} where w^* means $\sigma(E^{**}, E^*)$). Finally J is defined by the principle of local reflexivity

$$J(a) = (a_i)_u \quad \text{where} \quad a_i = \begin{cases} T_i a & \text{if } a \in M_i \\ 0 & \text{if not} \end{cases}$$

and T_i is an operator from M_i to E such that

$$\begin{aligned} T_i / M_i \cap E &= \text{id}_{M_i \cap E} \\ \|T_i\| \|T_i^{-1}\| &\leq 1 + \epsilon_i \\ T_i^* \rho &= \rho \quad \text{for } \rho \in N_i \end{aligned}$$

In the following we will use freely these notations and will consider E to be canonically embedded in E^{**} . Remark that for $a \in E^{**}$

$$(1) \quad a = QJa = w^* \lim_u a_i$$

$$(2) \quad \|a\| = \|Ja\| = \left\| (a_i)_u \right\| = \lim_u \|a_i\|$$

Since, for $a \in E^{**}$, $\|a\|$ is a supremum of w^* -continuous maps, the norm is w^* -lower semicontinuous : for any w^* -converging net (a_α) in E^{**} ,

$$(3) \quad \left\| w^* \lim_\alpha a_\alpha \right\| \leq \lim_\alpha \|a_\alpha\|$$

2. A MOTIVATIONAL LEMMA

Let f be a map between the two Banach spaces E and F . It is easy to lift f from E^{**} to F^{**} : for example by defining

$$\tilde{f}(a) = w^* \lim_\alpha f(x_\alpha) \quad \text{if } (x_\alpha) \text{ is a net in } E \text{ } w^* \text{-converging to } a \in E^{**}.$$

But this implies that f must be bounded on bounded sets and that $\tilde{f}(a)$ is independent of the choice of the approximating net (x_α) . Here we are interested in continuity properties of \tilde{f} . For instance we get the next

Lemma 2. Let f be a uniformly continuous map between the Banach spaces E and F . Then f has a uniformly continuous extension \tilde{f} from E^{**} to F^{**} .

Proof. The map $\tilde{f} : (x_i)_u \in (E^I)_u \rightarrow w^* \lim_u f(x_i) \in F^{**}$ is well defined :

first remark that the limit exists by the w^* -compactness of the unit ball of F^{**} and the fact that f is bounded on bounded sets. Second we need $\tilde{f} \left((x_i)_u \right)$ independent of the choice $(x_i)_{i \in I}$ in $(x_i)_u$: if $(x_i)_u = (x'_i)_u$ then $\lim_u \|x_i - x'_i\| = 0$ and this means that $I_\eta = \left\{ i \in I / \|x_i - x'_i\| \leq \eta \right\}$ is in u for each $\eta > 0$. Because of (3) we need only to prove that $I^\epsilon = \left\{ i \in I / \|f(x_i) - f(x'_i)\| \leq \epsilon \right\}$ is in u for each ϵ . By uniform continuity, for each ϵ there exists $\eta(\epsilon)$ such that $I_{\eta(\epsilon)} \subset I^\epsilon$. Hence $I^\epsilon \in u$.

Now define $\tilde{f} = \bar{f}.J$ on E^{**} . Clearly \tilde{f} is a uniformly continuous extension of f .

Remark 3. If f is Lipschitz continuous, so is \tilde{f} (cf. [11]).

In the following we are interested in the w^* -continuity of \tilde{f} in special cases but it would be interesting to know general conditions on f which imply the w^* -continuity of its extension.

III. THE CASE OF ALGEBRAS

For convenience we introduce the following

Definition. Let E be a Banach space. A binary product on E is a map from $E \times E$ to E which is bounded on bounded sets and denoted

$$(x,y) \in E \times E \rightarrow xoy \in E$$

Note that this product is not necessarily bilinear. For instance Banach power-associative systems can carry such a product. A Banach product is a binary product such that $\|xoy\| \leq \alpha \|x\| \|y\|$. It is tempting to adapt the previous lemma to define a product in the bidual E^{**} via J and not necessarily on the whole of $(E^I)_u$. Similar arguments as in Lemma 2 yield the following.

Proposition 4. Let \circ be a binary product on E . For a, b in E^{**} define

$$(4) \quad a \circ_u b = w^* \text{-} \lim_u a_i \circ b_i$$

Then \circ_u is a binary product on E^{**} which extends \circ . If \circ is a bilinear Banach product, so is \circ_u .

Now the problem is to know what are the extendable properties of a given binary product \circ . For instance if \circ is commutative so is \circ_u but the extension of associativity can fail.

Another problem is to know the dependance of \circ_u upon the ultrafilter u .

Before looking at these problems, namely to put on the bidual the same algebraic structure as the original one, we remark that it is not so easy to weaken the previous result to get norm continuity of \circ_u without the bilinearity of \circ . For instance we are not able to answer the following question : let E be a Banach power-associative system with a continuous square which induces a non bilinear product. For $a \in E^{**}$

$$(5) \quad a \circ_u a = \lim_u a_i^2$$

is well defined. Is the map : $a \in E^{**} \rightarrow a \circ_u a$ norm continuous ? However, the following result shows that norm properties of the induced binary product can easily be extended.

Proposition 5. Let E be a Banach power associative system with a not necessarily bilinear induced product \circ . Then \circ_u defined by (5) has the following properties for $a, b \in E^{**}$:

- i) $\|a \circ_u a\| = \|a\|^2$
- ii) $\|a \circ_u a - b \circ_u b\| \leq \max(\|a\|^2, \|b\|^2)$
- iii) $\|a \circ_u b\| \leq \|a\| \|b\|$ if the same inequality holds for \circ .

Proof. We can add a unit 1 to E such that $(E, E_+, 1)$ is an order-unit space with $E_+ = \{x^2 \mid x \in E\}$ ([12]). Denote by $S(E)$ its state space. Then $(E^{**}, E_+^{**}, 1)$ is a complete order-unit space for

$$E_+^{**} = \{a \in E^{**} \mid a(\rho) \geq 0, \rho \in S(E)\}.$$

Since $\|a\| = \sup_{\rho \in S(E)} |a(\rho)|$ for a in E^{**} ([9] 1.2.7), we get for ρ in $S(E)$

$$\|a \circ_u a\| \geq \lim_u a_i \circ a_i(\rho) \geq \lim_u (a_i(\rho))^2 = (a(\rho))^2$$

and so $\|a \circ_u a\| \geq \|a\|^2$.

Conversely $\|a \circ_u a\| \leq \lim_u \|a_i \circ a_i\| = \lim_u \|a_i\|^2 = \|a\|^2$ where we have used (2) and (3). Thus i) is proved

$$\begin{aligned} \text{ii) } \|a \circ_u a - b \circ_u b\| &\leq \lim_u \|a_i^2 - b_i^2\| \\ &\leq \lim_u \max \{ \|a_i\|^2, \|b_i\|^2 \} \\ &\leq \max \lim_u \{ \|a_i\|^2, \|b_i\|^2 \} \\ &\leq \max \{ \|a\|^2, \|b\|^2 \} \end{aligned}$$

where once again we have used (2) and (3).

iii) is immediate.

Remark 6. It is elementary to check that the above proof can be extended to an arbitrary involutive Banach algebra E with unit 1 such that $(E, E_+, 1)$ is an order-unit space for $E_+ = \{x^2 \in E \mid x = x^*\}$. Note that the involution on E has a natural w^* -continuous extension on the Banach space E^{**} , thus according to (1), $a^* = w^* - \lim_u a_i^*$ for a in E^{**} and this extension satisfies

$(a \circ_u b)^* = b^* \circ_u a^*$ for a, b in E^{**} . In particular, this covers the case of C^* and JB^* algebras. Hence if E is a C^* algebra (resp. a JB^* algebra), (E^{**}, \circ_u) will be a C^* algebra if \circ_u is associative (resp. \circ_u is power-associative, ([12] Corollary V.2)). Actually these conditions are satisfied as shown later because \circ_u is separately w^* -continuous and independent of the ultrafilter u .

Let us recall that the bidual of a C^* (resp. a JB^*)-algebra is also a C^* (resp. a JB^*)-algebra. Two proofs are available. The first is based on an algebraic approach (see for instance [5]) which uses explicitly the Arens product while the second is based on the order structure (cf. [9], 4.43).

Recall the definition of the Arens product. We will use the following notations :

$$\begin{aligned}
 & a, b \in E^{**} \\
 & x, y \in E \\
 & \{x_\alpha\}, \{y_\beta\} \text{ } w^* \text{-converge to } a, b \\
 & \rho \in E^* \\
 (6) \quad & \rho_x(y) = \rho(x \circ y), \rho^x(y) = \rho(y \circ x) \\
 & \rho_a(y) = a(\rho_y), \rho^a(y) = a(\rho^y) \\
 & (a \circ_1 b)(\rho) = a(\rho_b), (a \circ_2 b)(\rho) = b(\rho^a)
 \end{aligned}$$

π will be the canonical injection of E into E^{**} . Note that the Arens products \circ_1 and \circ_2 are not symmetrically defined while \circ_u is. On the other hand, the Arens products are always w^* -continuous at least in one variable while the w^* -continuity of \circ_u is not so clear. We only get the following

Lemma 7. Let E be a Banach space with a bilinear (not necessarily commutative nor associative) Banach product \circ . Then \circ_u is w^* continuous in each variable separately in E^{**} if the other variable is in E . Moreover

$$\begin{aligned}
 (7) \quad & a \circ_u \pi(x) = R_x^{**} a = a \circ_1 \pi(x) \\
 & \pi(x) \circ_u b = L_x^{**} b = \pi(x) \circ_2 b
 \end{aligned} \quad a, b \in E^{**}, x \in E$$

where R_x (resp. L_x) is the right (resp. left) multiplication by x in E .

Proof. Note that for x in $E, J(x) = (x_i)_u$ with $x_i = x$ for all i . Since $R_x^* \rho$ is in E^* for ρ in E^*

$$(a \circ_u \pi(x))(\rho) = \lim_u \rho(a_i \circ x_i) = \lim_u (R_x^* \rho)(a_i) = a(R_x^* \rho) = R_x^{**} a(\rho)$$

On the other hand

$$\begin{aligned}
 (a \circ_1 \pi(x))(\rho) &= a(\rho_{\pi(x)}) = \lim_u \rho_{\pi(x)}(a_i) = \lim_u \pi(x) (\rho_{a_i}) \\
 &= \lim_u \rho(a_i \circ x)
 \end{aligned}$$

Thus $a \circ_u \pi(x) = a \circ_1 \pi(x)$. By symmetry we get the second equality and so the proof is complete because R_x^{**} and L_x^{**} are w^* -continuous.

For non associative algebras we refer to [18]. We are now ready to prove

Theorem 8. If E is a C^* (resp. non necessarily commutative JB^*)-algebra then (E^{**}, \circ_u) is a C^* (resp. JB^*)-algebra and \circ_u is independent of u . The ultrafilter u contains $\mathcal{J}(I)$ so this means that the w^* -convergence in (4) is along the section filter $\mathcal{J}(I)$ of the index set I .

Proof. Suppose E is a C^* -algebra. Without loss of generality we can suppose E has a unit 1 which is also a unit for \circ_u by proposition 7. We need only to prove that \circ_u is w^* -separately continuous. If A is a non necessarily associative complex Banach algebra with unit, then for $a \in A$ define

$$\begin{aligned} D(A,1) &= \{\rho \in A^* / \rho(1) = 1 = \|\rho\|\} \\ V(A,a) &= \{\rho(a) / \rho \in D(A,1)\} \\ H(A) &= \{a \in A / V(A,a) \subset \mathbb{R}\} \end{aligned}$$

Since every unit-preserving linear isometry between (non necessarily associative) Banach algebras preserves the numerical range, we have, by theorem 6 of [19] (or corollary 2 cf. [15])

$$V(A,a) = V(B(A),L_a) = V(B(A),R_a)$$

where $B(A)$ is the Banach algebra of bounded linear operators on A and $L_a(R_a)$ is the left (right) multiplication by a . Thus a is hermitian iff L_a (or R_a) is hermitian.

Now apply this to $A = (E^{**}, \circ_u)$. Note E^* is the unique predual of E^{**} (which is a von Neumann algebra, see [17] or, better for this purpose, [9]). Hence $\exp(itL_a)$, $\exp(itR_a)$ being isometries for all real t and all a in $H(E^{**})$ ([3] page 46) are w^* -continuous by ([8] proposition 8). Then \circ_u will be separately w^* -continuous if

$$(8) \quad E^{**} = H(E^{**}) \oplus i H(E^{**})$$

In that case \circ_u will be an associative product and (E^{**}, \circ_u) will be a C^* -algebra. To achieve the proof, note that (8) is an immediate consequence of ([3] page 47).

Suppose now E is a non commutative JB^* -algebra. (E is Arens regular,

([14] corollary 1.8)). E^{**} has a unique predual ([14] page 18) and (8) is satisfied by ([16] corollary 13). Thus the previous arguments imply that L_a and R_a are w^* -continuous.

Since \circ_u is separately w^* -continuous,

$$a \circ_u b = w^*\text{-}\lim_{\alpha} x_{\alpha} \circ_u b = w^*\text{-}\lim_{\alpha} L_{x_{\alpha}}^{**}(b),$$

the last term is independent of the ultrafilter u and the convergence is along the section filter ([4] Chap. I, § 7, prop. 2).

Remark 9. i) For improving the previous result to a larger class of algebras, let us notice that the w^* -continuity of L_a for an hermitian a is satisfied in every Banach space E such that E^* is the unique predual of E^{**} . For instance it is sufficient that E^* is "well framed" (see [8]).

ii) A (non necessarily associative) unital norm complete algebra satisfying (8) is called a V -algebra. The class of V -algebras coincide with the one of unital (non necessarily commutative) JB^* -algebras (Vidav-Palmer theorem : [16] corollary 13).

iii) The existence of a unit and an order might seem important in this setting since there exists a non unital associative Banach algebra with an involution and with no nonzero positive functional.

However the following example shows that a unit can fail. Recall that there need be no unit in a JB^* -triple, e.g., the three by two matrices over \mathbb{C} . In [6] Dineen introduced the following product on a JB^* -triple E : for a, b, c in E^{**} , $\{a, b, c\}_u = w^*\text{-}\lim_{\alpha} \{a_{\alpha}, b_{\alpha}, c_{\alpha}\}$ and he proved $(\cdot, \cdot)_u$ is a JB^* -triple product. Actually this product doesn't depend on u : the identity between the two JB^* -triples $(E^{**}, (\cdot, \cdot)_u)$ and $(E^{**}, (\cdot, \cdot)_{u'})$ is a linear isometric bijection. Thus by [13] it is a JB^* -triple isomorphism and $\{a, b, c\}_u = \text{id}\{a, b, c\}_{u'} = \{\text{id}(a), \text{id}(b), \text{id}(c)\}_{u'} = \{a, b, c\}_{u'}$. Thus the $w^*\text{-}\lim_{\alpha} \{a_{\alpha}, b_{\alpha}, c_{\alpha}\}_u$ exists for each ultrafilter u containing the section filter $\mathcal{J}(I)$ and the convergence is along this filter.

iv). We will analyse the relations between the product \circ_u and the two Arens products in a forthcoming paper containing different examples :

- Associative Banach algebras where all these products are different.
- Associative Banach algebras E with E^* unique predual of E^{**}

and non $^{\circ}_u$ regular.

REFERENCES

1. R. ARENS, The adjoint of a bilinear operation, Proc. Amer. Math. Soc. 2 (1951), 839-848.
2. R. ARENS, Operations induced in function classes, Monatshefte für Math. 55 (1951), 839-848.
3. F.F. BONSAALL and J. DUNCAN, Numerical ranges of operators on normed spaces and of elements of normed algebras, Cambridge University Press, (1971).
4. N. BOURBAKI, Topologie générale, Herman (1971).
5. P. CIVIN and B. YOOD, The second conjugate space of a Banach algebra as an algebra, Pac. J. Math 11 (1961), 847-870.
6. S. DINEEN, The second dual of a JB^* -triple system, Complex analysis, functional analysis and approximation theory (Campinas, 1984) 67-69, North-Holland Math. Stud., 125, North-Holland, Amsterdam - New York, 1986.
7. J. DUNCAN and S.A.R. HOSSEINIUM, The second dual of a Banach algebra Proc. Royal Soc. Edinburgh, A 84 (1979), 309-325.
8. G. GODEFROY, Parties admissibles d'un espace de Banach. Applications, Ann. Scient. Ecole Norm. Sup. 16 (1983), 109-122.
9. H. HANCHE-OLSEN and E. STORMER, Jordan operator algebras, Pitman (1984).
10. S. HEINRICH, Ultraproducts in Banach space theory, J. Reine. Ang. Math. 313 (1980), 72-104.
11. S. HEINRICH and P. MANKIEWICZ, Applications of ultraproducts to the uniform and Lipschitz classification of Banach spaces, Studia Math. 73

(1982), 225-251.

12. B. IOCHUM and G. LOUPIAS, Banach power-associative algebras and Jordan algebras, Ann. Institut H. Poincaré. 43 (1985), 211-225.

13. W. KAUP, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. 183 (1983), 503-529.

14. R. PAYA, J. PEREZ and A. RODRIGUEZ, Non commutative Jordan C^* algebras, Manuscripta Math. 37 (1982), 87-120.

15. A. RODRIGUEZ-PALACIOS, A Vidav-Palmer theorem for Jordan C^* algebras and related topics, J. London Math. Soc. 22 (1980), 318-332.

16. A. RODRIGUEZ-PALACIOS, Non associative normed algebras spanned by hermitian elements, Proc. London Math. Soc. 47 (1983), 258-279.

17. S. SAKAI, C^* algebras and W^* algebras, Erg. der Math. 60, Springer-Verlag (1971).

18. R.D. SCHAFER, An introduction to non associative algebras, Academic Press, New York (1966).

19. M.A. YOUNGSON, Hermitian operators on Banach Jordan algebras, Proc. Edinburgh Math. Soc. 22 (1979), 169-180.

Bruno IOCHUM

Université de Provence Marseille, Centre de Physique Théorique
13288 MARSEILLE Cedex 9 (France)

Guy LOUPIAS

Laboratoire de Physique Mathématique, Université Montpellier II
34095 MONTPELLIER Cedex 5 (France)