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INTRODUCTION

Once Jacobson [7] introduces in a natural way the concepts of zero divisor and that of inverse in Jordan rings with unit, there arises also in a natural way the question of rings of quotients for Jordan rings: given a Jordan ring $A$ with unit and without zero divisors, is it possible to embed $A$ in a Jordan division ring? Or more generally, given a Jordan ring $A$ with unit is it possible to embed $A$ in a Jordan ring $Q(A)$ such that every element which is not a zero divisor in $A$ is invertible in $Q(A)$?

In his book [7] Jacobson states the so called "common multiple property" (a Jordan ring $A$ is said to satisfy the common multiple property if for all $a,s$ in $A$, with $a \neq 0$ and $s$ nonzero divisor, there are $a', s'$ in $A$, with $s'$ nonzero divisor, such that $U_a(s') = U_s(a') \neq 0$) and he conjectures that such a condition could play for Jordan rings a similar role to the Ore’s condition for the associative case. However at the present time it is unknown if the common multiple condition is either sufficient or necessary for a Jordan ring with unit to have a ring of quotients. It can be asserted then that up to date there is not still a well-structured general theory for rings of quotients of Jordan rings. Nevertheless there have been recently several important contributions on this topic (see [8,10,12]).

Following the abstract construction of Berberian [3] for the *-regular ring associated to a finite AW*-algebra, we show in [9] that every finite JBW-algebra $A$ is contained in a von Neumann regular Jordan algebra $\hat{A}$ such
that \( \hat{A} \) has no new idempotents. For the general theory of \( AW^* \)-algebras the reader is referred to [2], and for the theory of JB-algebras and JBW-algebras see [5].

In the associative case (\( AW^* \)-algebras or more generally Rickart \( C^* \)-algebras) the more suggestive characterizations of the constructed superring are obtained when this latter ring is related to ring of quotients of the former one (see [1,4,6,11]). This same direction is followed in [9] for the case of a finite JBW-algebra. The total ring of quotients of a Jordan ring with unit is defined there in the following way. Let \( A \) be a Jordan ring with unit. If \( \hat{A} \) is a Jordan ring containing \( A \) and with the same unit as \( A \), then \( \hat{A} \) is said to be the total ring of quotients of \( A \) if:

i) Every nonzero divisor \( s \) in \( A \) is invertible in \( \hat{A} \).

ii) Every morphism \( f \) from \( A \) into a Jordan ring \( B \), having the property that \( f(s) \) is invertible in \( B \) whenever \( s \) is not a zero divisor in \( A \), extends in a unique way to a morphism from \( \hat{A} \) into \( B \). It is proved the following result:

**Theorem.** Let \( A \) be a finite JBW-algebra. Let \( \hat{A} \) denote the Jordan regular ring associated to \( A \). Then:

i) For every element \( X \) in \( \hat{A} \) there are elements \( a, s \) in \( A \) such that \( X = U_s^{-1}(a) \), \( s \) is not a zero divisor and the subalgebra of \( A \) generated by \( a \) and \( s \) is strongly associative.

ii) \( A \) has the common multiple property.

iii) \( \hat{A} \) is the (unique) total Jordan ring of quotients of \( A \).

In order to obtain a more general (completely algebraic) result, an affirmative answer to the following question would be crucial:

**Problem.** If \( x \) and \( y \) are elements in a Jordan algebra \( J \) with unit \( 1 \), such that

\[
1 + [U_x(y^2)]^2 \quad \text{and} \quad 1 + [U_y(x^2)]^2
\]

are invertible in \( J \), then
It is easily proved that the problem has an affirmative answer when \( J \) is a special Jordan algebra. If it is so in general then we can prove the following:

**Conjecture.** Let \( A \) be a Jordan algebra with unit \( 1 \). Assume that there exists a Jordan algebra \( \hat{A} \) containing \( A \), with the same unit as \( A \), and satisfying the following properties:

1°) If \( X \in A \), then:
   i) \( 1 + X^2 \) is invertible in \( \hat{A} \).
   ii) \( (1 + X^2)^{-1} \) lies in \( A \).
   iii) \( X(1 + X^2)^{-1} \) lies in \( A \).

2°) If \( s \in A \) is not a zero divisor, then \( s \) is invertible in \( \hat{A} \).

3°) If \( a^2 = 0 \) implies \( a = 0 \), for \( a \) in \( A \).

Then,

I) For every element \( X \) in \( \hat{A} \) there are elements \( a, s \) in \( A \) such that \( X = s^{-1}(a) \), \( s \) is not a zero divisor in \( A \) and the subalgebra of \( A \) generated by \( a \) and \( s \) is strongly associative.

II) \( \hat{A} \) has the common multiple property.

III) \( \hat{A} \) is the (unique) total Jordan ring of quotients of \( A \).

**Remark.** The above conjecture is a theorem if \( \hat{A} \) is a special Jordan Algebra.

**REFERENCES**


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