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PEDRO JIMENEZ GARIJO

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INTRODUCTION

Once Jacobson [7] introduces in a natural way the concepts of zero divisor and that of inverse in Jordan rings with unit, there arises also in a natural way the question of rings of quotients for Jordan rings : given a Jordan ring A with unit and without zero divisors, is it possible to embed A in a Jordan division ring ? Or more generally, given a Jordan ring A with unit is it possible to embed A in a Jordan ring Q(A) such that every element which is not a zero divisor in A is invertible in Q(A) ?

In his book [7] Jacobson states the so called "common multiple property" (a Jordan ring A is said to satisfy the common multiple property if for all a,s in A, with $a \neq 0$ and s nonzero divisor, there are a', s' in A, with s' nonzero divisor, such that $U_a(s') = U_s(a') \neq 0$) and he conjectures that such a condition could play for Jordan rings a similar role to the Ore's condition for the associative case. However at the present time it is unknown if the common multiple condition is either sufficient or necessary for a Jordan ring with unit to have a ring of quotients. It can be asserted then that up to date there is not still a well-structured general theory for rings of quotients of Jordan rings. Nevertheless there have been recently several important contributions on this topic (see [8,10,12]).

Following the abstract construction of Berberian [3] for the *-regular ring associated to a finite AW*-algebra, we show in [9] that every finite JBW-algebra A is contained in a von Neumann regular Jordan algebra $\hat{A}$ such that $\hat{A}$ has no new idempotents. For the general theory of AW^*-algebras the reader is referred to [2], and for the theory of JB-algebras and JBW-algebras see [5].

In the associative case (AW^*-algebras or more generally Rickart C^*-algebras) the more suggestive characterizations of the constructed superring are obtained when this latter ring is related to ring of quotients of the former one (see [1,4,6,11]). This same direction is followed in [9] for the case of a finite JBW-algebra. The total ring of quotients of a Jordan ring with unit is defined there in the following way. Let A be a Jordan ring with unit. If $\hat{A}$ is a Jordan ring containing A and with the same unit as A, then $\hat{A}$ is said to be the total ring of quotients of A if :

i) Every nonzero divisor s in A is invertible in $\hat{A}$.

ii) Every morphism f from A into a Jordan ring B, having the property that f(s) is invertible in B whenever s is not a zero divisor in A, extends in a unique way to a morphism from $\hat{A}$ into B. It is proved the following result :

<u>Theorem</u>. Let A be a finite JBW-algebra. Let $\hat{A}$ denote the Jordan regular ring associated to A. Then :

i) For every element X in $\hat{A}$ there are elements a,s in A such that $X = U_{s^{-1}}(a)$, s is not a zero divisor and the subalgebra of A generated by a and s is strongly associative.

ii) A has the common multiple property.

iii) $\hat{A}$ is the (unique) total Jordan ring of quotients of A.

In order to obtain a more general (completely algebraic) result, an affirmative answer to the following question would be crucial :

<u>Problem</u>. If x and y are elements in a Jordan algebra J with unit 1, such that

$$1 + \left[U_x(y^2)\right]^2 \quad \text{and} \quad 1 + \left[U_y(x^2)\right]^2$$

are invertible in J, then

$$U_x U_y\left(\left[1 + \left(U_y(x^2)\right)^2\right]^{-1}\right) = \left[1 + \left(U_x(y^2)\right)^2\right]^{-1} . \, U_x(y^2) \ ?$$

It is easily proved that the problem has an affirmative answer when J is a special Jordan algebra. If it is so in general then we can prove the following :

<u>Conjecture</u>. Let A be a Jordan algebra with unit 1. Assume that there exists a Jordan algebra $\hat{A}$ containing A, with the same unit as A, and satisfying the following properties :

1°) If $X \in \hat{A}$, then :
  i) $1 + X^2$ is invertible in $\hat{A}$.
  ii) $(1 + X^2)^{-1}$ lies in A.
  iii) $X(1 + X^2)^{-1}$ lies in A.
2°) If $s \in A$ is not a zero divisor, then s is invertible in $\hat{A}$.
3°) $a^2 = 0$ implies $a = 0$, for a in A.

Then,

I) For every element X in $\hat{A}$ there are elements a,s in A such that $X = U_{s^{-1}}(a)$, s is not a zero divisor in A and the subalgebra of A generated by a and s is strongly associative.

II) $\hat{A}$ has the common multiple property.

III) $\hat{A}$ is the (unique) total Jordan ring of quotients of A.

<u>Remark</u>. The above conjecture is a theorem if $\hat{A}$ is a special Jordan Algebra.

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Departamento de Analisis Matematico
Universidad de Granada
ESPANA