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A zero-two theorem for a certain class of positive contractions in finite dimensional L^P-spaces $(1 \leqslant p < +\infty)$

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A Zero-Two Theorem for a certain class of positive Contractions in Finite Dimensional L^p-spaces $(1 \leq p < +\infty)$

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Summary

Our goal here is to extend Theorem 1.1 from [2] (which is sometimes called the zero-two law for positive contractions in L^1-spaces) to a class of positive contractions in finite dimensional L^p-spaces $(1 \leq p < +\infty)$.

1. A General Lemma

Let (X,Σ,m) be a measure space and $L^p(X,\Sigma,m)$ $(1 \leq p < +\infty)$ the usual Banach spaces. By a positive contraction $T:L^p(X,\Sigma,m) \to L^p(X,\Sigma,m)$ we mean that T is a linear bounded operator which transforms non-negative functions into non-negative functions and its norm is not more than one.

Lemma 1. Let $1 \leq p < +\infty$ and let $T:L^p(X,\Sigma,m) \to L^p(X,\Sigma,m)$ be a positive contraction. Suppose that there exist $\varepsilon > 0$ and $n_0 \in N \cup \{0\}$ such that $\|T^{n_0+1} - T^{n_0}\|_p^p < 2(1 - \varepsilon)$. Let $f \in L^p(X,\Sigma,m)$ be such that for every $n \in N \cup \{0\}$ $T^n f \cdot T^{n+1} f = 0$. Then $\lim_{n\to+\infty} \|T^n f\|_p = 0$.

Proof. Clearly it is enough to prove the lemma for $\|f\|_p = 1$.

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If f is as above then

$$\|T^{n_o+1}f - T^{n_o}f\|_p^p = \|T^{n_o+1}f\|_p^p + \|T^{n_o}f\|_p^p < 2(1-\varepsilon) .$$

Using the fact that $(\|T^n f\|_p^p)_n$ is a decreasing sequence we obtain that

$$\|T^{n_o+1}f\|_p < (1-\varepsilon)^{1/p} .$$

It follows that if we note $\rho = (1-\varepsilon)^{1/p}$ then

$$\|(T^{n_o+1} - T^{n_o})T^{n_o+1}f\|_p^p = \|T^{n_o+1}T^{n_o+1}f\|_p^p + \|T^{n_o}T^{n_o+1}f\|_p^p < 2(1-\varepsilon)\rho^p$$

and we obtain that

$$\|T^{2(n_o+1)}f\|_p < \rho^2 .$$

By induction it follows that for every $h \in N$ $\|T^{h(n_o+1)}f\|_p < \rho^h$ and using the fact that $(\|T^n f\|_p)_n$ is a decreasing sequence it follows that $\lim_n \|T^n f\|_p = 0$.

<u>Remark</u>. In Lemma 1 we may drop the assumption of T being positive. However we will need this assumption later on.

2. The Finite Dimensional L^p-spaces and the Theorem

We will now consider the following case:

Let $k \in N$, $k \geq 2$ be and we note $X = \{1,2,\dots,k\}$, $\Sigma = \mathcal{P}(X)$. Let $m_1,\dots,m_k$ be k non-zero positive real numbers. We will denote by m the measure generated by $m_1,\dots,m_k$ (that is $m(\{i\}) = m_i$, $i = 1,\dots,k$). We will call the space $L^p(X,\Sigma,m)$ a finite dimensional L^p-space and we will note

$\ell_p(k,m) = L^p(X,\Sigma,m)$. A positive contraction $T:\ell_p(k,m) \to \ell_p(k,m)$ is generated by a matrix $(a_{ij})_{i,j=1,2,\dots,k}$ and the resulting positive contraction $T^n (n \in N)$ is generated by $(a_{ij}^{(n)})_{i,j=1,\dots,k}$.

If $x \in \ell_p(k,m)$, $x = (x_1,\dots,x_k)$ then $Tx = (\sum_{i=1}^{k} x_i a_{ij})_{j=1,\dots,k}$ and $T^n x = (\sum_{i=1}^{k} x_i \, a_{ij}^{(n)})_{j=1,\dots,k}$.

If T is a positive contraction on $\ell_1(k,m)$ we will note

$$\Omega = \{i \in X \mid \text{for every } n \in N \cup \{0\} \text{ and } \ell = 1,\dots,k \quad \sum_{j=1}^{k} a_{ij}^{(n)} m_j = m_i \text{ and } a_{i\ell}^{(n)} \cdot a_{i\ell}^{(n+1)} = 0\} .$$

<u>Lemma 2</u>. Let $T: \ell_1(k,m) \to \ell_1(k,m)$ be a positive contraction. Then the following are equivalent:

a) for every $n \in N \cup \{0\}$, $\|T^{n+1}-T^n\|_1 = 2$

b) $\Omega \neq \phi$.

<u>Proof</u>. a) $\Rightarrow$ b) Suppose $\Omega = \phi$. It follows that for every $i \in \{1,2,\dots,k\}$ there exists $n_i \in N$ such that $\sum_{j=1}^{k} a_{ij}^{(n_i)} m_j < m_i$ or there exists j_0 such that $a_{ij_0}^{(n_i)} \cdot a_{ij_0}^{(n_i+1)} \neq 0$. In other words for every $i \in \{1,2,\dots,k\}$ there exists n_i such that $\|(T^{n_i+1} - T^{n_i})1_{\{i\}}\|_1 < 2m_i = 2\|1_{\{i\}}\|_1$.

If we note $n_0 = \max\{n_1,\dots,n_k\}$ it follows that for every $n \geq n_0$ and for every $i \in \{1,2,\dots,k\}$ $\|(T^{n+1} - T^n)1_{\{i\}}\|_1 < 2m_i$ (as for every $i \in \{1,2,\dots,k\}$

the sequence $(\|(T^{n+1} - T^n)1_{\{i\}}\|_1)_n$ is a decreasing one).

It follows that for $n \geq n_0$ $\|T^{n+1} - T^n\|_1 < 2$.

b) $\Rightarrow$ a) If $\Omega \neq \phi$ then for every $n \in N \cup \{0\}$ and $i \in \Omega$ $\|T^{n+1}1_{\{i\}} - T^n 1_{\{i\}}\|_1 =$
$= 2\|1_{\{i\}}\|_1$ and as T is a positive contraction it follows that for every $n \in N \cup \{0\}$ $\|T^{n+1} - T^n\|_1 = 2$.

Now we are able to prove the desired result:

<u>Theorem 3</u>. Let p be such that $1 \leq p < +\infty$ and let T be simultaneously a positive contraction of $\ell_1(k,m)$ and $\ell_p(k,m)$. If there exists $n_0 \in N \cup \{0\}$ such that $\|T^{n_o+1} - T^{n_o}\|_p < 2^{1/p}$ then $\lim_n \|T^{n+1} - T^n\|_p = 0$.

<u>Proof</u>. If $\lim_n \|T^{n+1} - T^n\|_p \neq 0$ then $\lim_n \|T^{n+1} - T^n\|_1 \neq 0$ (as every two norms in R^{k^2} are equivalent) . Using the zero-two law for positive contractions in L^1-spaces (Theorem 1.1 from [2]) it follows that for every $n \in N \cup \{0\}$ $\|T^{n+1} - T^n\|_1 = 2$ and by Lemma 2 it follows that $\Omega \neq \phi$. If $i \in \Omega$ then the characteristic function $1_{\{i\}}$ satisfies the conditions of Lemma 1 and it follows that $\lim_n \|T^{n+1}1_{\{i\}} - T^n 1_{\{i\}}\|_p = 0$. We obtain that $\lim_n \|T^{n+1}1_{\{i\}} - T^n 1_{\{i\}}\|_1 = 0$ which contradicts the fact that $i \in \Omega$.

References

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