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INVARIANT SETS OF SOLUTIONS OF NAVIER-STOKES AND RELATED EVOLUTION EQUATIONS - A SURVEY

P. BILER

RESUME.

Une bibliographie sur un sujet en plein développement - les attracteurs dans les équations d'évolution est recueillie. On considère le problème de dimension finie d'ensembles des solutions invariants par le flot et les méthodes des démonstrations sont comparées.

ABSTRACT.

A bibliography on the rapidly developing subject of the attractors for evolution equations is given. The problem of finite dimensionality of the sets of solutions invariant under the phase flow and the different methods for proving this are discussed concisely.

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INTRODUCTION

The evolution Navier-Stokes equations are the examples of physically motivated nonlinear problems in partial differential equations which attract attention of the mathematicians for many years. They constitute a highly nontrivial model of the occurence of turbulent phenomena in the dissipative evolution systems. Such systems "forget almost all" initial conditions and, as O.A. Ladyzhenskaya has stated ([35] , [36]), it is more convenient to study the turbulent properties of the flow associated with the system of equations only in a small part of the phase space (actually infinite dimensional) which is "not forgotten" by the system and is flow-invariant. The invariant set consists of trajectories global in time (i.e. defined for all real t) and one may expect that it is small in metric, topological or measure-theoretic sense.

A considerable number of papers devoted to the problem mentioned above has appeared recently. We shall quote some of them closely related with the finite dimension results for the attractors (and more generally the invariant sets) of evolution systems governed by partial differential equations.

The first version of this paper was written in 1982 and it has contained detailed proofs of the estimates of the dimension of the maximal invariant set of solutions to Navier-Stokes equations in a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with nonhomogeneous boundary conditions. The results announced in [17] and in a note in Doklady AN SSSR preceding [36] have been known at this time but some of the proofs were not published or the results were slightly less general. However the submission of the paper to a journal was delayed and soon I have refused to publish it since many interesting and detailed papers containing also entirely new techniques have appeared in 1983 and afterwards. Now, being encouraged by the Editors, I shall give a survey of new references (retaining only some my remarks from the previous version).

PRELIMINARIES

We consider the following initial-boundary value problem for the Navier-Stokes equations

$$\begin{aligned} &u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = g \quad &&\text{in} \quad \Omega \times (0,\infty) \\ &\nabla\cdot u = 0 && \\ &u|_{t=0} = u_o &&\text{in } \Omega \\ &u = f &&\text{on} \quad \Gamma\times(0,\infty) \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ (n=2, 3) locally located on one side of its boundary $\partial\Omega = \Gamma = \Gamma_1 \dot{\cup} \ldots \dot{\cup} \Gamma_k$ — being (n-1)-dimensional manifold of class (at least) C^2, $\nu > 0$ is the kinematic viscosity, $u = (u_1, \ldots, u_n)$ is the velocity, p is the pressure. The external body forces are represented by $g \in (L^2(\Omega))^n$, the initial condition u_o (after correction by f on Γ) is in H - the closure in L^2 norm $|.|$ of $\{v : v \in (C_o^\infty(\Omega))^n, \ \nabla\cdot v = 0\}$ and the boundary condition f is in $\dot{H}^{3/2}(\Omega)$ - the space of traces on Γ of functions in $(H^2(\Omega))^n$ satisfying $\int_{\Gamma_j} f\cdot\bar{n}\, d\Gamma = 0$, j=1,...,k, $\bar{n}$ - the unit outward normal on Γ ; g, f are independent of t.

The space periodic problem (which is mathematically slightly simpler) where $\Omega=[0,1]^n$ and $u(x+e_j,t)=u(x,t)$, (e_j), $j=1,\dots,n$, the standard basis in $\mathbb{R}^n$, can also be considered.

We recall only the simplest results concerning the solvability of this problem referring to [17] , [48] or [50] for the description of the functional setting and the precise exposition. It is known that there exists a weak solution $u \in C((0,\infty);H_{weak}) \cap L^2_{loc}((0,\infty);V)$, V is the closure of $\{v: v\in (C_0^\infty(\Omega))^n, \nabla\cdot v = 0\}$ in $H^1(\Omega)$ norm $\|.\|$. This solution for n=2 is unique and actually $u(t) \in V\cap (H^2(\Omega))^n$. If $u_0 \in V$ then u is the regular solution $u \in C([0,T]; V)$ for all $T>0$. If n=3, $u_0 \in V$, then u remains regular only for $T<T(u_0)$, $T(u_0)$ is at least of order $\|u_0\|^{-4}$. The general results on the uniqueness of the weak solutions for tridimensional Navier-Stokes equations are not known yet. Under these assumptions one can write $u(t)=S(t)(u_0)$ with the continuous (and compact for $t>0$ as operators in H) mappings S(t) in $V \subseteq H$ forming the semigroup for positive t if n=2 and restricted to $[0,T(u_0))$ if n=3.

The function space H has a special orthonormal basis formed by the eigenfunctions (w_N) N=1,2,... , of the Stokes operator Δ projected from $(L^2(\Omega))^n$ into H. The positive eigenvalues (λ_N) N=1,2,... are of order $\lambda_N \sim cN^{2/n}$.

EXISTENCE OF INVARIANT SETS

In the two dimensional case, following the idea that the invariant set X of the (semi-)flow $(S(t))$ $t\geq 0$ "forgets initial conditions" , one can try to define X as

$\bigcap_{t\geq 0} \overline{S(t)(H)}$. However it is more convenient to work with $X= \bigcap_{t\geq 0} \overline{S(t)(B_R)}$ where $S(t)$ is applied to the ball B_R in H of sufficiently large radius $R=R(g,f)$. One does not lose from consideration any trajectory if $\overline{\lim}_{t\to\infty} |u(t)| < R$ which actually holds for each solution with $u_0 \in H$. The set X constructed above is bounded in V: $\overline{\lim}_{t\to\infty} \|u(t)\| < C(g,f)$, hence compact in H. The invariance of X follows from the representation $X= \bigcap_{t\geq T} \overline{S(t)(B_R)}$ for all $T \geq 0$ since $S(T)(\{ u(s): s\geq t\})=\{u(s): s\geq t+T\}$. Moreover X is the maximal attractor for $(S(t))\ t\geq 0$: $\min_{x\in X} |u(t)-x| \xrightarrow[t\to\infty]{} 0$ for each solution u.

Observe that X consists of all global trajectories $\{u(t): t\in \mathbb{R}\}$ which are bounded - each solution with $u_0 \in X$ can be extended backwards. The semigroup $(S(t))\ t\geq 0$ has the extension to the group of transformations acting on X. In particular X contains all the stationary and the periodic solutions. Note that the sets consisting of the initial conditions with bounded global trajectories can be constructed explicitly as follows $Y= \{u_0 \in B:$ there exists $u_n \in B,\ n\geq 1$ with $S(t)u_n = u_{n-1}\} = \bigcap_{n\geq 0} S(nt)(B)$, $t>0$ fixed, B - a bounded sufficiently large set. Y is then negatively invariant $S(t)(Y) \supseteq Y$ and compact.

A natural result on the regularity of X holds: if the exterior forces g and the boundary Γ are more regular then X is contained in a subspace of V consisting of more regular functions , see e.g. [24] , [22] .

Small invariant sets like $\bigcap_{t\geq 0} \overline{\{u(s): s\geq t\}}$ can also be constructed but the set X described above is the good candidate for studying the asymptotic behaviour of solutions of Navier-Stokes equation and the phenomenon of turbulence.

PROPERTIES OF THE NAVIER-STOKES SEMIGROUP

Let P_N denote the orthogonal projection of H onto the linear space spanned by the functions $w_1,\dots,w_N$ corresponding to the first eigenvalues $\lambda_1,\dots,\lambda_N$ of the Stokes operator in H. Let $Q_N = I-P_N$ be the projection onto the complementary subspace in H. The mapping $S = S(t)$ with fixed $t>0$ appears to be a locally Lipschitz mapping and $Q_N \cdot S$ is a contraction for sufficiently large N. More precisely we have (from [36])

PROPOSITION 1. If $\|u\|$, $\|v\| \leq r$ and $0<t<T_0=T(r)$ if $n=3$ (no restrictions on $t>0$ if $n=2$) then there is $M>0$ such that $|S(u) - S(v)| \leq M|u - v|$
and for every $\varepsilon>0$ there exists N satisfying

$$|Q_N(S(u) - S(v))| \leq \varepsilon|u - v| .$$

REMARK. S has also the similar properties as a mapping in V, but the Lipschitz constant M and the contraction constants ε, N depend in more complicated way on the domain Ω, the viscosity ν and the initial and boundary values.

Some additional properties of essentially finite dimensional asymptotics of solutions were obtained in [35], [36]. For instance: the space $P_N(H)$ for sufficiently large N contains the whole necessary information on the trajectories of the Navier-Stokes flow lying in the invariant set X. Precisely speaking: the global trajectory

$\{u(t): t \in \mathbb{R}\}$ can be reconstructed from its projection $P_N(u(t))$; if $P_N(u(t))$ is T-periodic then u is also T-periodic; the same is true for the quasiperiodicity.

There are several versions of the quasicontraction property of S = S(t) for Navier-Stokes or nonlinear parabolic equations e.g. in [2], [11], [17], [22], [49]. In [17] the inequalities for S, P•S, Q•S ($P=P_N$, $Q=Q_N$ for suitably large N) are formulated as the alternative

either $|S(u) - S(v)| \leq \sqrt{2}\, |P(S(u) - S(v))|$ or

$$|S(u) - S(v)| \leq \delta\, |u - v|, \quad \delta = \delta(N) < 1 .$$

This follows of course from the assumptions in Proposition 1: if $|S(u) - S(v)| \geq \sqrt{2}\, |P(S(u) - S(v))|$ then $|S(u) - S(v)| \leq \sqrt{2}\, |Q(S(u) - S(v))| \leq \sqrt{2}\, \varepsilon\, |u - v|$ and it suffices to assume $\varepsilon \leq \delta/\sqrt{2}$ and find suitable N.

In [1] and subsequent papers there has been assumed that the Hilbert space H is continuously embedded in a Banach space E and, instead of the Lipschitz condition imposed on S in the norm of E, one has $|P(S(u) - S(v))|_H \leq$ $\leq M\, |u - v|_E$.

More precise estimations are based on the idea from [2] of determining a product of an N-dimensional ellipsoide and a ball of codimension N, smaller than the ball of radius Mr, containing the image of the starting ball of radius r in Proposition 1.

This quasicontraction (in other words squeezing or flattening) property of S(t) which is intimately connected with the compactness properties of resolvent operator gives a key for proving that any (e.g. maximal) bounded invariant set of solutions is finite dimensional.

DIMENSION OF THE INVARIANT SET

We shall concentrate on the problem of showing that the negatively invariant sets of some special mappings are finite dimensional. The results applied to the maximal invariant set X described before for two dimensional, or to any invariant set for tridimensional Navier-Stokes equations (which structure is not known yet since it lacks the proof of uniqueness of solutions), show their finite dimension.

There are of course many different results on the thinness of the sets of solutions to Navier-Stokes or related evolution equations, like those in [18] (measure-theoretic thinness of the set $\{u(t): u_0 \in H\}$ for any $t>0$) or in [28], [32] (behaviour of finite dimensional approximation to evolution equations in Galerkin procedure when the order of approximation increases). The paper [14] (and 4.4 in [11]) also gives very interesting facts - especially from the computational point of view - on the other aspects of asymptotically finite dimensional asymptotic behaviour of solutions to Navier-Stokes equation (determining modes, determining sets of points, nodal values).

Now let us recall definitions concerning the Hausdorff dimension (cf [13], [27]).
If $\alpha > 0$ and X is a bounded subset of a metric space $\mathfrak{X}$ then the α-dimensional Hausdorff measure of X is

$$m_\alpha(X) = \lim_{\varepsilon \to 0} m_{\alpha,\varepsilon}(X) = \sup_{\varepsilon > 0} m_{\alpha,\varepsilon}(X)$$

where $m_{\alpha,\varepsilon}(X) = \inf \sum (\text{diam } B_j)^\alpha$, the infimum is taken over all coverings of X by the balls B_j of diameters less than ε. The measure m_α is countably additive on the Borel subsets of $\mathfrak{X}$. Clearly $m_\alpha(X) \in [0,\infty]$ and if $m_{\alpha_0}(X)<\infty$ for some $\alpha_0 \in (0,\infty)$ then $m_\alpha(X)=0$ for all $\alpha > \alpha_0$. In such a case $\inf \{\alpha: m_\alpha(X)<\infty\} = \inf \{\alpha: m_\alpha(X)=0\}$ is called the Hausdorff dimension of X $\dim_H(X)$. A compact set X of finite Hausdorff dimension is homeomorphic to a subset of Euclidean space and when X is a subset of a Banach space E such a homeomorphism can be realized as a projection into $(2[\dim_H X]+1)$-dimensional subspace of E ([41], [43]).

If one uses in this construction of $m_{\alpha,\varepsilon}(X)$ only the coverings consisting of equal balls of diameter ε then one obtains the definition of the, so called, limit capacity or the fractal (or entropy, or Pontriagin-Schnirelman) dimension $\dim_F(X)$ which is plainly greater or equal than $\dim_H(X)$. These notions of dimension are different even in the Euclidean spaces (for the examples see [41]).

We reproduce here (with the original proof from [36]) a fundamental theorem on finite fractal (and thus Hausdorff) dimension of negatively invariant bounded sets for special mappings with properties essentially like S.

THEOREM 1. Let X be a bounded subset of a (separable) Hilbert space H and $S: X \longrightarrow H$ be a continuous mapping such that $S(X) \supseteq X$ and for all $u, v \in X$ $|S(u) - S(v)| \leqslant M|u - v|$, $|Q(S(u) - S(v))| \leqslant \varepsilon|u - v|$ where Q is an orthogonal projection of H onto a subspace H'' of finite codimension (equal to N), $\varepsilon<1$. Then the fractal dimension of X is finite.

The estimates of α will follow from the proof.

PROOF. Let X be covered by a finite number (e.g. one) of closed balls B_r^i each of radius r, so $X = \bigcup_{i=1}^{k} X_r^i$ where $X_r^i = X \cap B_r^i$, diam $X_r^i = d_r^i \leq r$. This covering gives an approximation for α-measure of X $m_{\alpha,r} = \sum_{i=1}^{k} (d_r^i)^{\alpha}$. We consider the projections of $S(X_r^i) \cap X$ onto the subspaces H'', $H' = H \ominus H''$; $P = I - Q: H \longrightarrow H'$. We see that $P(S(X_r^i) \cap X)$ is contained in N-dimensional cube C^i in H' which edge is Md_r^i long and that the diameter of $Q(S(X_r^i) \cap X)$ is less than εd_r^i . We divide C^i into cubes C^{ij} which edges are $aN^{-1/2} d_r^i$ long ($a > 0$ will be fixed later) so their diameters are ad_r^i. The total number of C^{ij}-s is not larger than $K = ([MN^{1/2}/a] + 1)^N$. According to this we obtain a subdivision of X_r^i into "small" sets $X_r^{ij} = P^{-1}(C^{ij}) \cap (S(X_r^i) \cap X)$. The diameter d_r^{ij} of X_r^{ij} is less than $(a^2 + \varepsilon^2)^{1/2} d_r^i$:

$$d_r^{ij} = \sup_{u,v \in X_r^{ij}} (|Pu-Pv|^2 + |Qu-Qv|^2)^{1/2} \leq (a^2 + \varepsilon^2)^{1/2} d_r^i =: \varepsilon_1 d_r^i .$$

This covering of X, $X = \bigcup_{i,j} X_r^{ij}$, gives a new approximation of α-measure of X $m_{\alpha, \varepsilon_1 r} = \sum_{j=1}^{K} \sum_{i=1}^{k} (d_r^{ij})^{\alpha} \leq$ $\leq \varepsilon_1^{\alpha} \cdot K m_{\alpha,r}$. It is obvious that for sufficiently large α $\varepsilon_1^{\alpha} K < 1$. Iterating this procedure of subdivision of the coverings into sets of diameters less than $\varepsilon_1^2 r$, $\varepsilon_1^3 r, \ldots$ we get $m_{\alpha, \varepsilon_1^2 r} \leq (\varepsilon_1^{\alpha} K)^2 m_{\alpha,r}$, $m_{\alpha, \varepsilon_1^3 r} \leq (\varepsilon_1^{\alpha} K)^3 m_{\alpha,r}, \ldots$ and finally $m_{\alpha}(X) = \lim_{l} m_{\alpha, \varepsilon_1^l r} \leq \lim_{l} (\varepsilon_1^{\alpha} K)^l m_{\alpha,r} = 0$.

Concretely we may choose $a = ((1-\varepsilon^2)/2)^{1/2}$,

$\alpha > 2N \cdot \log([M(2N)^{1/2}(1-\varepsilon^2)^{-1/2}]+1)/\log(2(1+\varepsilon^2)^{-1})$;

then $\varepsilon_1 = ((1+\varepsilon^2)/2)^{1/2}$, $\varepsilon_1^{\alpha} K < 1$.

If we spare the elements of the coverings we obtain slightly better bound for α. It suffices e.g. to observe that $P(S(X_r^i)\cap X))$ is contained in a ball of diameter Md_r^i which can be covered by considerably less number of cubes of edges $aN^{-1/2}d_r^i$ than before. For another example of improved value of α see [1].

If there is a priori assumed that X is compact then the conditions on S, $Q \circ S$ should be satisfied only locally and the projection P may vary continuously from point to point $P=P_x$ but $\sup \dim P_x(H)=N$ must be finite.

Now let us compare some assumptions on the mapping S which guarantee that the conclusion of Th.1 holds. The earliest paper on this subject [40] has dealt with smooth mappings. There has been assumed that S is a C^1 transformation on a neighbourhood of a compact negatively invariant set $X \subseteq S(X)$ and DS is a (uniform) contraction on a subspace H'' of finite codimension $|DS(u)|_{H''}| < 1$. In particular if S is compact and C^1 then the theorem follows as DS(u) is compact for all $u \in X$. So there are the subspaces $H''(u)$ on which $|DS(u)|_{H''(u)}| \leq 1/3$ and by continuity of the derivative $|DS(v)|_{H''(u)}| \leq 2/3$ for v in a neighborhood of u. Now by the compactness of X the subspace H'' can be chosen universally. The second assumption of Th.1 follows as $S(u) - S(v) = DS(u)(u-v) + o(|u-v|)$.

The verification of Ladyzhenskaya condition from Th.1 is still easier than the estimates of derivatives which however give more precise information on α in other modifications of the proof of similar theorems ([2], [4]).

We note that if S is only a compact Lipschitz mapping then $Q \circ S$ may not be a contraction for any admissible Q.

EXAMPLE. $S: \ell^2 \to \ell^2$ is defined by $S((x_k)_{k=0}^{\infty}) = (f(2^k x_k)/2^k)_{k=0}^{\infty}$ where $f: \mathbb{R}^+ \to [0,1]$ is a Lipschitz function, $f(0)=0$, $f([0,1])=[0,1]$. We see that $S(\ell^2)$ is the Hilbert cube $\mathbb{Q} = \prod_{k=0}^{\infty}[0,2^{-k}]$ and $S(\mathbb{Q})=\mathbb{Q}$ so S has an infinite dimensional invariant set although S is Lipschitzian:

$$|S(x)-S(y)|^2 = \sum_{k=0}^{\infty} |f(2^k x_k)/2^k - f(2^k y_k)/2^k|^2 \leq$$

$$\leq \mathrm{Lip}(f)^2 \cdot \sum_{k=0}^{\infty} |x_k - y_k|^2 = \mathrm{Lip}(f)^2 |x-y|^2.$$

The Lipschitz constant of f can be chosen arbitrarily close to 1 and even for f smooth S can be differentiable only on a proper (but sometimes dense) subset of $\mathbb{Q} \subseteq \ell^2$.

Next we consider two easy generalizations of the Th.1, the first in the context of Banach spaces and the second concerning metric spaces.

THEOREM 2. Let X be a bounded subset of a Banach space E and $S: X \to E$ be a mapping such that $S(X) \supseteq X$, $|S(u)-S(v)| \leq M|u-v|$. Suppose there exists a projection $P: E \to E'$ onto a finite dimensional subspace E' (in general nonlinear and varying from point to point but with bounded dimension of $P(E)$) such that $|P(u)-P(v)| \leq c|u-v|$ and for $Q=I-P$ $|Q(S(u)-S(v))| \leq \varepsilon |u-v|$ for some $\varepsilon < 1$ and all $u,v \in X$. Then the fractal dimension of X is finite.

Observe that the Euclidean norm in E' and the original norm induced from E are equivalent and the proof of Th.1

can be adapted to this situation with only minor alterations (cf slightly more complicated reasoning in [41]).

Analyzing the proof one can notice that the role of finite dimensional linear subspace H′ or E′ may be played by a space which is "uniformly homogeneous" in the following sense: each ball of radius R is contained in a union of at most K(R/r) suitably chosen balls of radii r - e.g. for $\mathbb{R}^N$ $K(x)=c_N x^N$ is a good estimate.
Therefore we have

THEOREM 3. Let $E = E' \times E''$ be a metric space, E′ be homogeneous in the sense described above and $(e',e'') \in E' \times E''$ is fixed. Assume that the mapping $S: X \to E$ defined on a bounded subset X of E satisfies the following conditions $X \subset S(X)$, $\rho(S(u),S(v)) \leq M\rho(u,v)$, $\rho(Q \circ S(u), Q \circ S(v)) \leq \varepsilon \rho(u,v)$ with $\varepsilon < 1$, where ρ is the metric in E and $Q: E \to \{e'\} \times E''$, $P: E \to E' \times \{e''\}$, $\rho(P(u),P(v)) \leq c\,\rho(u,v)$, $\rho(u,v) \leq \rho(P(u),P(v)) + \rho(Q(u),Q(v))$, are the projections on the factors in E.
In such a situation $\dim_F(X) < \infty$.

REMARK. The assumption on the existence of S-invariant bounded set in these theorems is essential - see e.g. $E=\mathbb{R}^N=E'$, P=I and S a translation $Sx=x+a$, $a \neq 0$.
In some cases such a set however must exist. For instance in [20] it has been shown that a continuous mapping which is contractive in the second coordinate has at least one fixed point - on condition that E′ has the fixed point property (i.e. each continuous mapping of E′ has a fixed point) and E″ is complete.

OPTIMAL RESULTS ON THE DIMENSION OF INVARIANT SETS

The upper bounds for the fractal dimension of (maximal) invariant sets of solutions to Navier-Stokes equations following from the proof of Th.1 are crude and can be considerably improved. The methods which allow to obtain more realistic (and inspired by physical arguments) estimates of this dimension are intimately connected with the study of smoothness and regularity properties of the semigroup S(t). The key observation is that considering the Lyapunov exponents (and Lyapunov numbers) along the trajectories one can better control the anisotropic squeezing property of S(t) than it has been done in [12] , [2] , [30] . The general derivation of the estimates on the Hausdorff and fractal dimensions of attractors in [10] , [11] , [19] , [22] , [37] , [49] is partially based on the ideas of [38] and [21] where the relations between the fractal dimension and Lyapunov numbers were emphasized and elucidated.

The problem was ultimately reduced to the estimates of Lyapunov exponents of the mappings or to the estimates of the variation of the N-dimensional volume element transported by the flow. The final step relies on the ingenious application of the Lieb-Thirring inequalities (for the Schroedinger operators) to the linearized operator S - see [11] , [49] . This last idea was pointed out in [39] which completes [44] and then improved in [46] . Note that these papers deal with the linearized equation (not the linearized solution operator).

The best results available at this moment are given in [49] and [11], namely

- for two dimensional Navier-Stokes equations

$$\dim_F X \leq \frac{c|g|}{\nu^2 \lambda_1}$$

- for tridimensional equations the dimension of any attractor bounded in V is estimated by the time average of the maximum rate of dissipation. These results are likely to be (quasi-)optimal as they are consistent with the physical predictions of the conventional theory of turbulence (the celebrated Kolmogorov law).

It is worth to noting that the functional dimension $df(Y) = \limsup_{\varepsilon \to 0} \frac{\log \log n(Y,\varepsilon)}{\log \log 1/\varepsilon} - 1$ ($n(Y,\varepsilon)$ - the minimal number of balls of radius ε covering Y) for the image $Y = S(t)B$ of any bounded set $B \subseteq V$ is not greater than 3/2 for S(t), t>0, solving the Navier-Stokes equtions in three dimensions. This is demonstrated using the squeezing property of S(t) in 2.3, [11] and for the linearized (Stokes) equations $df(Y)=3/2$.

Relatively little is known on the lower bounds of the dimension of the attractor. In the case of two dimensional equations with periodic boundary data an explicit unstable stationary solution and the considerations on the structure of the maximal invariant set produce an estimate from below of the order of Reynolds number, see [2], [4].

GENERALIZATIONS AND RESULTS FOR OTHER EQUATIONS

Applications of this kind of results for related problems in hydrodynamics: MHD equations, thermohydraulic equations, Navier-Stokes equations on manifolds (in particular geophysical flows) have been described in [46],

[22], [45].

Nonlinear parabolic equations of reaction-diffusion type or from chemical kinetics were treated in [2], [3], [4], [33] and from the other point of view in [34].

Similar results on the finite dimension of invariant sets of solutions hold also for the generalized (pseudodifferential) Korteweg-de Vries-Burgers equation (unpublished). These dissipative-dispersive equation were considered in the homogeneous case in [8].

The Kuramoto-Sivashinsky equation which appears in combustion theory and combines two effects: diffusion governed by Δ^2 (iterated Laplacian) and the negative diffusion (represented by Δ term) was examined in [42]. A more detailed study of the inertial manifolds for this equation (containing the maximal attractor) was begun in [15], [16].

The papers [2], [3], [4], [5], [6], [7], [32], [47] deal with the attractors in damped nonlinear hyperbolic equations. The special attention is paid to regular attractors which appear in generic (w.r.t. force term and boundary conditions in the equation) case. They have relatively simple structure: all the equilibrium points of the equation (finite number) and unstable manifolds corresponding to these stationary points constitute the maximal attractor.
In the general case (without genericity assumptions) of damped hyperbolic equations the dimension of the ma-

ximal attractor was estimated in [23].

The supplementary properties of the maximal attractor are known for equations with a Lyapunov function (among others gradient-like systems), see e.g. [2], [3].

There are some interesting results on the estimating the dimension of attractor in (finite dimensional) classical dynamical systems (pursuit games) in [9], [31] where the methods announced in [28], [29] are used.

The theorems on the structure and the dimension of invariant sets are also applicable in problems involving retarded functional-differential equations on Riemannian manifolds. The standard references are [25] and [26]. We note however that some of proofs can be simplified using our "fully nonlinear" Th.3.

A general scheme for applications of Th.3 can be proposed in the context of bundles over manifolds. In particular one often deals with the vector bundles with a function space as infinite dimensional fibre, over Riemannian manifolds and the flows acting on the base of the bundle as Lipschitz mappings and as compact operators on the fibres.

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