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ABSOLUTELY SUMMING OPERATORS AND MEASURE AMARTS

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IN FRECHET SPACES (\*)

D. QUANG LUU

Résumé. Dans cet article on démontre quelques théorèmes de caractérisation pour les opérateurs absolument sommants et on donne diverses applications à la convergence des martingales asymptotiques vectorielles dans les espaces de Fréchet.

Summary. In the paper, we prove some characterization theorems for the absolutely summing operators and we give various applications to convergence of vector-valued asymptotic martingales in Fréchet spaces.

(\*) This paper was partly written during the author's stay at the University of Sciences and Technics in Montpellier 1984.

§ 0. INTRODUCTION.

The Radon-Nikodym property and convergence of vector-valued asymptotic martingales (amarts) in Fréchet spaces have been extensively studied in recent years by many authors, see, [6,7,17,3,12,13,8,9] and etc. The purpose of the paper is to continue these above investigations. Namely, after stating some needed notations and definitions in Section 1, we shall prove in Section 2 some representation theorems for the absolutely summing operators in Fréchet spaces which are different from those, given in [16]. And finally, in Section 3 we shall give some applications of the results in Section 2 to convergence and boundedness problems of vector-valued amarts in Fréchet spaces.

§ 1. NOTATIONS AND DEFINITIONS.

In the paper we shall use the notations and definitions, given in [9] and introduce some other one's concerning measures in Fréchet spaces. Namely, let  $E$  be a Fréchet space,  $U(E)$  a fundamental countable family of closed absolutely convex sets which form a  $0$ -neighborhood base for  $E$ ,  $E'$  the topological dual of  $E$  and  $(\Omega, \mathcal{A}, P)$  a probability space. Given  $U \in U(E)$  the polar  $U^\circ$  and the continuous seminorm  $p_U$ , associated with  $U$  are given by

$$U^\circ = \{e \in E' \mid |\langle x, e \rangle| \leq 1\} ,$$

$$p_U(x) = \inf \{\alpha > 0 \mid \alpha^{-1}x \in U\} , \quad (x \in E) .$$

For a  $\sigma$ -additive measure  $\nu : \mathcal{A} \rightarrow E$  and  $U \in U(E)$  we define the

semivariation (or the total variation, resp.) seminorm  $S_U(\mu)$   
(or  $V_U(\mu)$ , resp.) as follows

$$S_U(\mu) = \sup \{ |\langle \mu, e \rangle| \mid (\Omega) \quad e \in U^\circ \} ,$$

$$V_U(\mu, A) = \sup \left\{ \sum_{j=1}^k p_U(\mu(A_j)) |\langle A_j \rangle|_{j=1}^k \in \Pi(A) \right\} , \quad (A \in \mathcal{A}) ,$$

$$V_U(\mu) = V_U(\mu, \Omega) \quad ,$$

where  $\Pi(A)$  denotes the set of all finite measurable partitions of  
 $A \in \mathcal{A}$  .

By  $S(E) = S(\Omega, \mathcal{A}, P, E)$  (or  $V(E) = V(\Omega, \mathcal{A}, P, E)$ , resp.) we mean  
the space of all  $S$ -equivalence (or  $V$ -equivalence, resp.) classes of  
 $S$ -bounded (or  $V$ -bounded, resp.)  $\sigma$ -additive measures  $\mu : \mathcal{A} \rightarrow E$  . Then  
by using the same argument given in [16] for the spaces  $\mathcal{L}_I^1(E)$  and  
 $\mathcal{L}_I^1\{E\}$  , one can establish easily the following property.

Property 1.1 . Both  $(S(E), S\text{-topology})$  and  $(V(E), V\text{-topology})$   
are Fréchet spaces.

Now, for definition of strong measurability and Bochner  
integrability of vector-valued functions  $f : \Omega \rightarrow E$  , we refer to  
[6,7] and let  $L_1(E) = L_1(\Omega, \mathcal{A}, P, E)$  denote the space of all  $V$ -  
equivalence classes of Bochner integrable functions  $f : \Omega \rightarrow E$  ,  
where  $V_U(f) = \int_{\Omega} p_U(f(\omega)) dP(\omega)$  ( $U \in U(E)$ ) . Then according to [6] ,  
one can regard  $(L_1(E), V\text{-topology})$  as a closed subspace of  $V(E)$  with  
the following identification :  $L_1(E) \ni f \mapsto \mu_f \in V(E) : \mu_f(A) = \int_A f dP \quad (A \in \mathcal{A})$  .

Note that with the above identification,  $L_1(E)$  becomes a (not necessarily closed) subspace of  $S(E)$ . Finally, as in the Banach space case (see, e.g. [4]), the following property remains true.

Property 1.2. Let  $\mu' \in S(E)$ ,  $\mu \in V(E)$ ,  $U \in U(E)$  and  $f \in L_1(\Omega, \mathcal{B}, P, E)$  for some sub  $\sigma$ -field  $\mathcal{B} \subset \mathcal{A}$ . Then

$$(1) \quad S_U(\mu) \leq V_U(\mu) .$$

$$(2) \quad q_U(\mu') \leq S_U(\mu') \leq 4 q_U(\mu') ,$$

where  $q_U^{\mathcal{B}}(\mu) \stackrel{\text{df}}{=} \sup \{p_U(\mu(A)) : A \in \mathcal{A}\}$  and  $q_U(\mu) = q_U^{\mathcal{A}}(\mu)$  ,

$$(3) \quad q_U(f) \stackrel{\text{df}}{=} q_U(\mu_f) \leq S_U(f) \stackrel{\text{df}}{=} S_U(\mu_f) \leq \mathcal{A} q_U^{\mathcal{B}}(\mu_f) .$$

## § 2. ABSOLUTELY SUMMING OPERATORS AND MEASURES AMARTS.

Let  $E$  and  $F$  be Fréchet spaces,  $\ell_N^1(E)$  (or  $\ell_N^1\{E\}$ , resp.) the space of all summable (or absolutely summable, resp.) sequences  $(x_n)$  in  $E$ . Thus the  $\varepsilon$ -topology for  $\ell_N^1(E)$  and the  $\Pi$ -topology for  $\ell_N^1\{E\}$  are defined as in [16]. A linear continuous operator  $T : E \rightarrow F$ , write  $T \in \mathcal{L}(E, F)$ , is said to be absolutely summing if it maps  $\ell_N^1(E)$  into  $\ell_N^1\{F\}$ .

Theorem 2.1. Let  $E, F$  be Fréchet spaces and  $T \in \mathcal{L}(E, F)$ . Then the following conditions are equivalent :

- (1)  $T$  is absolutely summing.

(2) For every probability space  $(\Omega, \mathcal{A}, P)$  , the operator  $T^0 : S(E) \rightarrow V(E)$  , defined by

$$(T^0 \mu)(A) = T(\mu(A)) \quad (\mu \in S(E) , A \in \mathcal{A}) ,$$

is linear continuous.

(3) For every probability space  $(\Omega, \mathcal{A}, P)$  , the operator  $T^1 : (L_1(E) , S\text{-topology}) \rightarrow (L_1(F) , V\text{-topology})$  , defined by

$$(T^1 f)(\omega) = T(f(\omega)) \quad (f \in L_1(E) , \omega \in \Omega) ,$$

is linear continuous.

(4) For only special probability space  $(N, \mathcal{P}(N), \gamma)$  , where  $N$  is the set of all positive integers,  $\mathcal{P}(N)$  the class of all subsets of  $N$  and  $\gamma(\{n\}) = 2^{-n}$  ( $n \in N$ ) ,  $T^1$  is linear continuous.

Proof (1  $\rightarrow$  2) . Let  $E , F$  be Fréchet spaces and  $T \in \mathcal{L}(E, F)$  an absolutely summing operator. We shall show that for each  $C \in U(F)$  there are some  $U \in U(E)$  and  $\beta(C, U) > 0$  such that for all finite sequences  $\langle x_j \rangle_{j=1}^k \subset E$  , we have

$$\sum_{j=1}^k p_C(Tx_j) \leq \beta(C, U) \sup \left\{ \sum_{j=1}^k |\langle x_j, e \rangle| \mid e \in U^0 \right\} . \quad (2.1)$$

Indeed, first of all applying ([16], 2.1.3) to  $T$  , we infer that the operator  $T_N : \mathcal{L}_N^1(E) \rightarrow \mathcal{L}_N^1(F)$  , given by

$$T_N(\langle x_n \rangle) = \langle Tx_n \rangle \quad (\langle x_n \rangle \in \mathcal{L}_N^1(E)) ,$$

is linear continuous. Therefore, by Theorem 1 in ([19], I.6) for each  $C \in U(F)$  there is some  $U \in U(E)$  and  $\beta(C,U) > 0$  such that

$$\|_C(\langle Tx_n \rangle) \leq \beta(C,U) \varepsilon_U(\langle x_n \rangle) \quad (\langle x_n \rangle \in \mathcal{L}_N^1(E)) .$$

Equivalently,

$$\sum_N p_C(Tx_n) \leq \beta(C,U) \sup \{ \sum_N |\langle x_n, e \rangle| \mid e \in U^\circ \} \quad (\langle x_n \rangle \in \mathcal{L}_N^1(E)) .$$

Further, since for every finite sequence  $\langle x_j \rangle_{j=1}^k \subset E$ , the sequence  $\langle \langle x_j \rangle_{j=1}^k, 0, 0, \dots \rangle \in \mathcal{L}_N^1(E)$ , then the last inequality implies (2.1).

Now let  $\mu \in S(E)$ ,  $C \in U(F)$  and  $\langle A_j \rangle_{j=1}^k \in \Pi(\Omega)$ . Applying (2.1) to the finite sequence  $\langle \mu(A_j) \rangle_{j=1}^k \subset E$ , we get

$$\begin{aligned} \sum_{j=1}^k p_C((T^\circ \mu)(A_j)) &\leq \beta(C,U) \sup \{ \sum_{j=1}^k |\langle \mu(A_j), e \rangle| \mid e \in U^\circ \} \\ &\leq \beta(C,U) \sup \{ |\langle \mu, e \rangle|(\Omega) \mid e \in U^\circ \} \\ &= \beta(C,U) S_U(\mu) . \end{aligned}$$

Hence,

$$\begin{aligned} V_C(T^\circ \mu) &= \sup \{ \sum_{j=1}^k p_C((T^\circ \mu)(A_j)) \mid \langle A_j \rangle_{j=1}^k \in \Pi(\Omega) \} \\ &\leq \beta(C,U) S_U(\mu) . \end{aligned} \tag{2.2}$$

Finally, again applying Theorem 1 in ([19], I.6) to the operator  $T^\circ : S(E) \rightarrow V(F)$ , it is clear that  $T^\circ \in \mathcal{L}(S(E), V(F))$ . This proves (2), taking into account that the linearity of  $T^\circ$  is naturally satisfied.

(4 → 1) . Suppose that  $T$  fails to be absolutely summing. Then by definition, there is some  $\langle x_n \rangle \in \mathcal{L}_N^1(E)$  such that  $\langle Tx_n \rangle \notin \mathcal{L}_N^1(F)$  , i.e. there is some  $\mathcal{E} \in U(F)$  such that

$$\sum_N p_C(Tx_n) = \infty .$$

For convenience, we can always suppose that  $p_C(\overline{Tx_n}) \neq 0$  ( $n \in N$ ) . Now, choose a strictly increasing subsequence  $\langle n_k \rangle$  of  $N$  such that

$$\sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k \quad (k \in N)$$

and define  $f_k : N \rightarrow E$  ( $k \in N$ ) , by

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} 2^j x_j l_{\{j\}} \quad (k \in N) ,$$

where  $l_A$  is the characteristic function of  $A \in \mathcal{A}$  .

It is clear that by ([16], 1.3.6) the sequence  $\langle f_k \rangle$  in  $L_1(E)$  is  $S$ -convergent to 0 . On the other hand, as

$$\int_N p_C(T^1 f) d\gamma = \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k \quad (k \in N) ,$$

the sequence  $\langle T^1 f_k \rangle$  in  $L_1(F)$  fails to be  $V$ -convergent. It contradicts (4) . Finally since the implications (2 → 3 → 4) are trivial, the proof of the theorem is completed. Now, by Theorem 4.2.5 in [16], the Fréchet space  $E$  is nuclear if and only if the identification operator is absolutely summing so that the following corollary is an easy consequence of the above theorem.



Corollary 2.2 . For a Fréchet space E , the following conditions are equivalent :

- (1) E is nuclear.
- (2) On V(E) , the S-topology is equivalent to the V-topology.
- (3) On L<sub>1</sub>(E) , the S-topology (the Pettis topology) is equivalent to the V-topology (the Bochner topology).

Remark . (1) Theorem 2.1 was first partly proved by Ghoussoub in [14] for Banach spaces and later completed by Bru-Heinich in [4] , using directly Proposition 2.2.1 in [16] which can be applied to only normed spaces.

(2) Egghe [12] has applied however Proposition 4.1.5 in [16] to obtain the equivalence (1 ↔ 3) in the corollary.

In order to give some probability characterizations of absolutely summing operators in Fréchet spaces we give now some additional notations and definitions. Indeed, hereafter we shall fix an increasing sequence  $\langle A_n \rangle$  of sub $\sigma$ -fields of  $A$  such that  $A = \sigma(\cup_N A_n)$ . Let

$$S(\langle A_n \rangle, E) = \{ \langle \mu_n \rangle \mid \forall n \mu_n \in S^n(E) = S(\Omega, A_n, P, E) \}$$

$$V(\langle A_n \rangle, E) = \{ \langle \mu_n \rangle \mid \forall n \mu_n \in V^n(E) = V(\Omega, A_n, P, E) \}$$

$$L_1(\langle A_n \rangle, E) = \{ \langle f_n \rangle \mid \forall n f_n \in L_1^n(E) = L_1(\Omega, A_n, P, E) \}$$

and  $T$  the set of all bounded stopping times. Given  $\tau \in T$ ,  $\langle \mu_n \rangle \in S(\langle A_n \rangle, E)$  and  $\langle f_n \rangle \in L_1(\langle A_n \rangle, E)$  we define

$$A_\tau = \{A \in \mathcal{A} \mid A \cap \{\tau = n\} \in A_n \quad \forall n\} ,$$

$$\mu_\tau : A_\tau \rightarrow E \quad : \quad \mu_\tau(A) = \sum_N \mu_n(A \cap \{\tau = n\}) \quad (A \in A_\tau)$$

$$f_\tau : \Omega \rightarrow E \quad : \quad f_\tau(\omega) = f_n(\omega) \quad (\omega \in \{\tau = n\} , n \in N) .$$

It is known (cf. [15]) that  $\langle A_\tau ; \tau \in T \rangle$  is increasing family of sub $\sigma$ -fields of  $\mathcal{A}$  ;  $\mu_\tau \in S^\tau(E) = S(\Omega, A_\tau, P, E)$  and  $f_\tau \in L_1^\tau(E) = L_1(\Omega, A_\tau, P, E)$  . Moreover, if  $\langle \mu_n \rangle \in V(\langle A_n \rangle, E)$  then  $\mu_\tau \in V^\tau(E) = V(\Omega, A_\tau, P, E)$  .

Definition 2.1 . Call  $\langle \mu_n \rangle \in S(\langle A_n \rangle, E)$  to be a martingale if  $\mu_{m,n} = \mu_m \mid A_n = \mu_n \quad (m \geq n \in N)$  .

Note that if  $\langle \mu_n \rangle \in S(\langle A_n \rangle, E)$  is a martingale then  $\mu_{\sigma,\tau} = \mu_\sigma \mid A_\tau = \mu_\tau \quad (\sigma \geq \tau \in T)$  . Hence  $\mu_\tau(\Omega)$  does not depend upon the choice of  $\tau \in T$  . Thus  $\langle \mu_n \rangle \in S(\langle A_n \rangle, E)$  is said to be an amart if the net  $\langle \mu_\tau(\Omega), \tau \in T \rangle$  is convergent in  $E$  .

We note that as for the amarts in Banach spaces (see, [5] , [10] , [18] , [4] ) , the following basis lemma is obtained.

Lemma 2.3 . Let  $\langle \mu_n \rangle \in S(\langle A_n \rangle, E)$  . Then the following conditions are equivalent :

- (1)  $\langle \mu_n \rangle$  is an amart.
- (2)  $\mu_n$  has a Riesz decomposition :  $\mu_n = \alpha_n + \beta_n \quad (n \in N)$  , where  $\langle \alpha_n \rangle \in S(\langle A_n \rangle, E)$  is a martingale and  $\langle \beta_n \rangle$  is a potential, i.e.

$$\lim_{\tau \in T} S_U^\tau(\beta_\tau) = 0 \quad (U \in U(E)) ,$$

where  $S_U^\tau(\cdot)$  is defined as  $S_U$  with respect to the probability space  $(\Omega, A_\tau, P)$  .

(3) There is a finitely additive measure  $\mu_\infty : \cup_N A_n \rightarrow E$  such that  $\mu_\infty \upharpoonright A_n = \mu_{\infty, n} \in S^n(E)$  ( $n \in N$ ) and

$$\lim_{\tau \in T} S_U^\tau(\mu_\tau - \mu_{\infty, \tau}) = 0 \quad (U \in U(E)) .$$

We shall call  $\mu_\infty$  the limit measure associated with  $\langle \mu_n \rangle$  .

Definition 2.2 . Call  $\langle \mu_n \rangle \in V(\langle A_n \rangle, E)$  to be a uniform amart if the following condition is satisfied

$$\lim_{\tau \in T} \sup_{\sigma \geq \tau} V_U^\tau(\mu_\sigma - \mu_\tau) = 0 \quad (U \in U(E)) ,$$

where  $V_U^\tau$  is defined as  $V_U$  with respect to  $(\Omega, A_\tau, P)$  .

It is clear that by Lemma 2.3 and Property 1.2, every uniform amart is an amart. Moreover, as for the uniform amarts in Banach spaces (cf. [2], [4]) we get the following.

Lemma 2.4 . Let  $\langle \mu_n \rangle \in V(\langle A_n \rangle, E)$  . Then the following conditions are equivalent :

- (1)  $\langle \mu_n \rangle$  is a uniform amart.
- (2)  $\langle \mu_n \rangle$  has a Riesz decomposition :  $\mu_n = \alpha_n + \beta_n$  ( $n \in N$ ) , where  $\langle \alpha_n \rangle$  is a martingale in  $V(\langle A_n \rangle, E)$  and  $\langle \beta_n \rangle$  a uniform potential, i.e.

$$\lim_{\tau \in T} V_U^T(\beta_\tau) = 0 \quad (U \in U(E)) .$$

(3) There is a finitely additive measure  $\mu_\infty : \bigcup_N A_n \rightarrow E$  such that  
each  $\mu_{\infty, n} \in V^n(E)$  and

$$\lim_{\tau \in T} V_U^T(\mu_\tau - \mu_{\infty, \tau}) = 0 \quad (U \in U(E)) .$$

Note that Property 1.1 is needed in the proofs of Lemmas 2.3 and 2.4 .

Finally, we say that a sequence  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$   
has a property  $(\star)$  if so has the sequence  $\langle \mu_n = \mu_{f_n} \rangle$  , associated  
with  $\langle f_n \rangle$  .

Theorem 2.5 . Let  $E, F$  be Fréchet spaces and  $T \in \mathcal{L}(E, F)$  . Then  
the following conditions are equivalent :

- (1)  $T$  is absolutely summing
- (2)  $T^\circ$  maps amarts in  $S(\langle A_n \rangle, E)$  into uniform amarts in  $V(\langle A_n \rangle, F)$  .
- (3) For each  $S$ -bounded amart  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$  and  $C \in U(F)$  ,  
the sequence  $\langle p_C(T^1 f_n) \rangle$  is a uniform amart of nonnegative real-  
valued functions.
- (4)  $T^1$  maps every  $V$ -convergent amart  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$   
into a sequence  $\langle T^1 f_n \rangle$  of class  $(B)$  , i.e.

$$\sup_{\tau \in T} \int_{\Omega} p_C(g_\tau) dP < \infty \quad (C \in U(F)) ,$$

where  $g_n \stackrel{df}{=} T^1 f_n$   $(n \in N)$  .

Proof. Let  $E, F$  be Fréchet space and  $T \in \mathcal{L}(E, F)$  . Suppose first

$(\alpha_n)$  is a martingale in  $S(\langle A_n \rangle, E)$ . It is clear that  $\langle T^\circ \alpha_n \rangle$  is also a martingale in  $S(\langle A_n \rangle, F)$ . Now if  $T$  is absolutely summing and  $(\beta_n)$  is a potential in  $S(\langle A_n \rangle, E)$ , by (2.2) in the proof of Theorem 2.1 it follows that the sequence  $\langle T^\circ \beta_n \rangle$  is a uniform potential in  $V(\langle A_n \rangle, F)$  and  $\langle T^\circ \alpha_n \rangle \in V(\langle A_n \rangle, F)$ , noting that if  $\gamma_n = T^\circ \beta_n$  ( $n \in N$ ) then  $\gamma_\tau = T^\circ \beta_\tau$  ( $\tau \in T$ ). Therefore, by Lemmas 2.3 and 2.4 we get (1  $\rightarrow$  2).

(1  $\rightarrow$  3). To prove (1  $\rightarrow$  3), we suppose first that  $\langle \gamma_n \rangle$  is a uniform amart in  $V(\langle A_n \rangle, F)$ ,  $\gamma_\infty$  the limit measure associated with  $\langle \gamma_n \rangle$  and  $C \in U(F)$ . Then by Lemma 2.4, it follows that

$$\lim_{\tau \in T} V_C^\tau(\gamma_\tau - \gamma_{\infty, \tau}) = 0.$$

This with properties of seminorms in  $V^\tau(E)$  implies

$$\lim_{\tau \in T} |V_C^\tau(\gamma_\tau) - V_C^\tau(\gamma_{\infty, \tau})| = 0 \quad (2.3)$$

Further, if  $\langle \gamma_n \rangle$  is  $V$ -bounded, it is easily checked that

$$\uparrow \lim_{\tau \in T} V_C^\tau(\gamma_{\infty, \tau}) = V_C^\Sigma(\gamma_\infty) < \infty,$$

where  $\Sigma = \bigcup_N A_n$ ,

$$V_C^\Sigma(\gamma_\infty) = \sup \left\{ \sum_{j=1}^k p_C(\gamma_\infty(A_j)) \mid \langle A_j \rangle_{j=1}^k \in \Pi(\Sigma, \Omega) \right\}$$

and  $\Pi(\Sigma, \Omega)$  is the set of all finite  $\Sigma$ -measurable partitions of  $\Omega$ .

Consequently, by (2.3) we get

$$\begin{aligned} & \lim_{\tau \in T} |V_C^\tau(\gamma_\tau) - V_C^\Sigma(\gamma_\infty)| \\ & \leq \lim_{\tau \in T} \{ |V_C^\tau(\gamma_\tau) - V_C^\tau(\gamma_{\infty, \tau})| + |V_C^\tau(\gamma_{\infty, \tau}) - V_C^\Sigma(\gamma_\infty)| \} = 0 \quad (2.4) \end{aligned}$$

Now we suppose that  $T$  is absolutely summing,  $\langle f_n \rangle$  a  $S$ -bounded amart in  $L_1(\langle A_n \rangle, E)$ ,  $\langle \mu_n \rangle$  the measure amart associated with  $\langle f_n \rangle$  and  $\mu_\sigma$  the limit measure associated with  $\langle \mu_n \rangle$ . It is clear that if we define

$$\gamma_n = T^0 \mu_n \quad (n \in N) ,$$

then by (1  $\rightarrow$  2),  $\langle \gamma_n \rangle$  is a uniform amart in  $V(\langle A_n \rangle, F)$  and by (2.2) in the proof of Theorem 2.1,  $\langle \gamma_n \rangle$  is  $V$ -bounded. Therefore, for any but fixed  $C \in U(F)$ , the uniform amart  $\langle \gamma_n \rangle$  must satisfy (2.4). Moreover, if we define

$$g_n = p_C(T^1 f_n) \quad (n \in N) ,$$

then

$$\int_{\Omega} g_{\tau} dP = V_C^{\tau}(\gamma_{\tau}) \quad (\tau \in T) .$$

This with (2.4) proves that the sequence  $\langle p_C(T^1 f_n) \rangle$  is a uniform amart (of nonnegative real-valued functions), taking into account that in (2.4),  $V_C^{\Sigma}(\gamma_{\infty})$  is a finite number. It completes the proof of (1  $\rightarrow$  3).

(2  $\rightarrow$  4) Suppose that  $\langle f_n \rangle$  is a  $V$ -convergent amart in  $L_1(\langle A_n \rangle, E)$ . Then given  $C \in U(F)$ , the sequence  $\langle p_C(T^1 f_n) \rangle$  must be  $V$ -bounded. Thus as in the proof of (1  $\rightarrow$  3), the  $V$ -boundedness of  $\langle p_C(T^1 f_n) \rangle$  with (2) shows that  $\langle p_C(T^1 f_n) \rangle$  must be a uniform amart. Therefore the  $V$ -boundedness of  $\langle p_C(T^1 f_n) \rangle$  is equivalent to

$$\sup_{\tau \in T} \int_{\Omega} p_C(T^1 f_{\tau}) dP < \infty ,$$

i.e.  $\langle p_C(T^1 f_n) \rangle$  is of class (B). This proves (4).

Because (3 → 4) is similarly established, it remains to prove (4 → 1) .

For this purpose, suppose that  $T$  is not absolutely summing. Returning to the sequence  $\langle x_n \rangle$  in the example, given in the proof of

Theorem 2.1 we take  $(\Omega, \mathcal{A}, P) \equiv (N, \mathcal{P}(N), \gamma)$  and define

$$f_j = 2^j x_j \cdot 1_{\{j\}} \quad (j \in N) ;$$

$$A_j = \sigma(f_1, f_2, \dots, f_n) \quad (j \in N) ;$$

$$\underline{\tau} = \min \{n \in N \mid P(\{\tau = n\}) > 0\} ;$$

$$\bar{\tau} = \max \{n \in N \mid P(\{\tau = n\}) > 0\} \quad (\tau \in T) .$$

Then

$$\int_N f_{\tau} dP = \int_N \sum_{j=\underline{\tau}}^{\bar{\tau}} 2^j x_j \cdot 1_{\{j\}} dP = \sum_{j=\underline{\tau}}^{\bar{\tau}} x_j .$$

Consequently, by Theorem 1.3.6 in [16] , the sequence  $\langle f_j \rangle$  defined above is a potential (hence an amart). Further, since

$$\int_N p_U(f_j) dP = p_U(x_j) \quad (j \in N, U \in U(E))$$

the sequence  $\langle f_j \rangle$  is  $V$ -convergent to 0 . On the other hand, if we put

$$i_k(j) = \begin{cases} j & \text{if } n_k + 1 \leq j \leq n_{k+1} \\ n_{k+1} + 1, & \text{if } j \notin \{n_k + 1, \dots, n_{k+1}\} \end{cases}$$

for all  $k \in N$  , then

$$\int_N p_C(T^1 f_{\tau_k}) dP \geq \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k .$$

Therefore, the  $V$ -convergent (to 0) amart  $\langle f_j \rangle$  cannot be of class (B) which contradicts (4) and completes the proof of the theorem.

Note that since the sequence  $\langle p_C(T^1 f_j) \rangle$  is  $V$ -convergent (hence  $V$ -bounded) and is not of class (B),  $\langle p_C(T^1 f_n) \rangle$  cannot neither be an amart. Further, since a Fréchet space  $E$  is nuclear if and only if the identical operator is absolutely summing, the following corollary is an easy consequence of the theorem.

Corollary 2.6 . For a Fréchet space,  $E$ , the following conditions are equivalent

- (1)  $E$  is nuclear.
- (2) Every amart in  $S(\langle A_n \rangle, E)$  is uniform.
- (3) For every  $S$ -bounded amart  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$  and  $U \in U(E)$ , the sequence  $\langle p_U(f_n) \rangle$  is a uniform amart in  $L_1(\langle A_n \rangle, \mathbb{R})$ .
- (4) For every  $V$ -convergent amart  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$  and  $U \in U(E)$ , the sequence  $\langle p_U(f_n) \rangle$  is an amart in  $L_1(\langle A_n \rangle, \mathbb{R})$ .

Remark. Since every  $L_1$ -bounded real-valued uniform amart must be of class (B), Theorem 2 in [13] is hence easily established from Corollary 2.6. Note that for the proof of Theorem 2 in [13], Egghe has needed the Radon-Nikodym property of nuclear Fréchet spaces. So that his proof cannot be applied to Theorem 2.5.



§ 3. CONVERGENCE AND BOUNDEDNESS OF AMARTS.

In this section, we shall apply the results in Section 2 to convergence and boundedness problems of amarts in Fréchet spaces. We begin with

Theorem 3.1 . Let  $E, F$  be Fréchet spaces and  $T \in \mathcal{L}(E, F)$  . Then the following properties are equivalent :

- (1)  $T$  is absolutely summing.
- (2)  $T$  maps potentials in  $L_1(\langle A_n \rangle, E)$  into  $F$ -valued sequences, strongly convergent to 0 , almost everywhere (a.e.) .
- (3)  $T$  maps potentials in  $L_1(\langle A_n \rangle, E)$  into  $F$ -valued sequences, weakly convergent, to 0 , a.e.
- (4)  $T$  maps  $V$ -convergent potentials in  $L_1(\langle A_n \rangle, E)$  into  $F$ -valued sequences, strongly bounded, a.e.

Proof. (1  $\rightarrow$  2) Suppose that  $T \in \mathcal{L}(E, F)$  is absolutely summing. By Theorem 2.5  $T$  maps potentials in  $L_1(\langle A_n \rangle, E)$  into uniform potentials in  $L_1(\langle A_n \rangle, F)$  . Thus to prove (2) it is sufficient to show that every uniform potential  $\langle g_n \rangle$  in  $S(\langle A_n \rangle, F)$  is strongly convergent, a.e. Indeed,  $\langle g_n \rangle$  is a uniform potential, by definition we get

$$\lim_{\tau \in T} \int_{\Omega} p_C(g_{\tau}) dP = 0 \quad (C \in U(F)) .$$

Hence, also by definition, the sequence  $\langle p_C(g_n) \rangle$  is a uniform potential of real-valued functions. Hence it must be convergent to 0 ,

a.e. (cf. [2], [4]). We note that since  $U(F)$  is countable,  $\langle g_n \rangle$  must converge itself strongly a.e. to 0. This proves (2). Here, it is worth to note that in [13], Egghe has hardly proved that every uniform potential in nuclear Fréchet spaces converges strongly, a.e. to 0. But as we have just shown, this fact is clear even for uniform potentials in general Fréchet spaces  $F$ .

Returning to the proof of the theorem we see that the implications  $(2 \rightarrow 3 \rightarrow 4)$  are easy. Thus we have to prove only  $(4 \rightarrow 1)$ . Suppose that  $T$  fails to be absolutely summing. And let  $\langle x_n \rangle$  and  $C$  be as in the example given in the proof of Theorem 2.1. By  $(\Omega, A, P)$  we mean the Lebesgue probability space on  $[0,1)$ . Since

$$\sum_N p_C(Tx_n) = \infty$$

~~and  $(\Omega, A, P)$  has no atoms~~, we can choose a subsequence  $n < n_2 < \dots < n_k < \dots$  of  $N$  such that

$$\alpha_k = \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k \quad (k \in \mathbb{N}).$$

Next, for each  $k \in \mathbb{N}$ , find  $\langle A_{k,j} \rangle_{j=n_k+1}^{n_{k+1}} \in \Pi(\mathcal{B}_{[0,1)}, [0,1))$  with

$$\alpha_{k,j} = P(A_{k,j}) = \alpha_k^{-1} p_C(Tx_j) \quad (n_k+1 \leq j \leq n_{k+1})$$

and define

$$f_j = \alpha_{k,j}^{-1} x_j \mathbb{1}_{A_{k,j}} \quad (k \in \mathbb{N}, n_k+1 \leq j \leq n_{k+1}),$$

$$A_j = \sigma(f_1, f_2, \dots, f_j) \quad (j \in \mathbb{N}).$$

Finally, given  $k, j \in \mathbb{N}$  we put

$$\delta_{k,j} = \begin{cases} 1 & \text{if } j \in \{n_k+1, n_k+2, \dots, n_{k+1}\} \\ 0 & \text{for otherwise} \end{cases}$$

and for each  $\tau \in \mathcal{T}$ , we define

$$\beta_{k,j} = P(A_{k,j} \cap \{\tau = j\}) \quad ;$$

$$k(\underline{\tau}) = \max \{k \in \mathbb{N} \mid \underline{\tau} \geq n_k+1\} \quad ;$$

$$k(\bar{\tau}) = \min \{k \in \mathbb{N} \mid \bar{\tau} \leq n_{k+1}\} \quad .$$

Therefore, with the above notations, one get

$$\int_0^1 f_\tau dP = \int_0^1 \sum_{j=\underline{\tau}}^{\bar{\tau}} f_j 1_{\{\tau=j\}} dP = \sum_{k=k(\underline{\tau})}^{k(\bar{\tau})} \sum_{j=\underline{\tau}}^{\bar{\tau}} \delta_{k,j} \alpha_{k,j}^{-1} \beta_{k,j} x_j \quad .$$

But note that  $0 \leq \delta_{k,j} \alpha_{k,j}^{-1} \beta_{k,j} \leq 1$  ( $k, j \in \mathbb{N}$ ) and  $(\underline{\tau} \rightarrow \infty)$  implies  $(k(\underline{\tau}) \rightarrow \infty)$ . Consequently, by Theorem 1.3.6 in [16], the summability of  $\langle x_n \rangle$  implies that the net  $\langle \int_0^1 f_\tau dP \rangle_{\tau \in \mathcal{T}}$  converges to 0. It means that

$$(a) \quad \langle f_j \rangle \text{ is a potential in } L_1(\langle A_n \rangle, E) \quad .$$

Next, since

$$\int_0^1 p_U(f_j) dP = p_U(x_j) \quad (j \in \mathbb{N}, U \in \mathcal{U}(E))$$

$$(b) \quad \langle f_j \rangle \text{ is } V\text{-convergent to } 0 \quad .$$

Finally, for each  $\omega \in ]0, 1[$ ,  $k \in \mathbb{N}$ , one can choose some  $j_k$  such

that  $n_{k+1} \leq j_k \leq n_{k+1}$ . This yields

$$p_C(T^1 f_{j_k}(\omega)) = \alpha_{k, j_k}^{-1} p_C(Tx_{j_k}) = \alpha_k.$$

Therefore,

$$\sup_N p_C(Tf_j(\omega)) = \infty \quad (\omega \in ]0, 1[).$$

Consequently, the sequence  $\langle f_j \rangle$  with the properties (a-b-c) contradicts (4) which completes the proof of (4  $\rightarrow$  1) and the theorem.

Now suppose that  $E$  is nuclear and  $\langle f_n \rangle$  a  $S$ -bounded amart in  $L_1(\langle A_n \rangle, E)$ . Then by [6],  $E$  has the Radon-Nikodym property,  $\langle L_1(E), V\text{-topology} \rangle$  is a Fréchet space and by Corollary 2.6  $\langle f_n \rangle$  is a  $V$ -bounded uniform amart. Hence,  $\langle f_n \rangle$  has a more precise Riesz decomposition :  $f_n = g_n + h_n$  ( $n \in \mathbb{N}$ ), where  $\langle g_n \rangle$  is a  $V$ -bounded martingale in  $L_1(\langle A_n \rangle, E)$  and  $\langle h_n \rangle$  a uniform potential. Thus the proof of Theorem 3.1 shows that  $\langle h_n \rangle$  converges strongly a.e. to 0. Further, since every nuclear Fréchet space is a projective limit of a sequence of Hilbert spaces, the martingale limit theorem in Hilbert spaces shows that  $\langle g_n \rangle$  must converge strongly a.e. Therefore, the following corollary is an easy consequence of the theorem.

Corollary 3.2 . For a Fréchet space, the following properties are equivalent :

- (1)  $E$  is nuclear.
- (2) Every  $S$ -bounded amart in  $L_1(\langle A_n \rangle, E)$  is convergent strongly, a.e.

(3) Every S-bounded amart in  $L_1(\langle A_n \rangle, E)$  is convergent weakly, a.e.

(4) Every V-convergent potential in  $L_1(\langle A_n \rangle, E)$  is strongly bounded, a.e.

Remark. The equivalence  $(1 \leftrightarrow 2)$  was first proved by Bellow [1] for Banach spaces. This result has been recently extended to Fréchet spaces by Egghe in [13]. Also  $(1 \leftrightarrow 3)$  has been proved by Edgar-Sucheston in [11] for Banach spaces.

We say that a sequence  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$  is S-uniformly integrable, if  $\langle f_n \rangle$  is S-bounded and for every  $U \in U(F)$ ,

$$\lim_{P(A) \rightarrow 0} \sup \left\{ \int_A |\langle f_n, e \rangle| dP \mid e \in U^\circ, n \in \mathbb{N} \right\} = 0$$

It is clear that if  $E$  is a nuclear Fréchet space then every S-uniformly integrable sequence  $\langle f_n \rangle$  in  $L_1(\langle A_n \rangle, E)$  is V-uniformly integrable, i.e. for each  $U \in U(E)$ , the sequence  $\langle p_U(f_n) \rangle$  is uniformly integrable. Conversely, if  $E$  fails to be nuclear then as in the proof of Theorem 2.1, one can construct a potential  $\langle f_k \rangle$  in  $L_1(\langle A_n \rangle, E)$  such that  $\langle f_k \rangle$  fails to be V-bounded. Therefore, the following corollary can be deduced easily from Corollary 3.2.

Corollary 3.3. For a Fréchet space  $E$ , the following conditions are equivalent :

- (1)  $E$  is nuclear.
- (2) Every S-uniformly integrable amart in  $L_1(\langle A_n \rangle, E)$  is V-convergent.
- (3) Every potential in  $L_1(\langle A_n \rangle, E)$  is V-bounded.

Remark. The equivalence  $(1 \leftrightarrow 2)$  was first proved by Egghe in [12], where he gave a very complicated example in order to prove  $(2 \rightarrow 1)$ . The implication  $(3 \rightarrow 1)$  in the corollary seems to be new even for Banach spaces.

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