ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Probabilités et applications

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 92, série *Probabilités et applications*, n° 7 (1988), p. 1-9
<http://www.numdam.org/item?id=ASCFPA_1988__92_7_1_0>

ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION PROCESSES IN McSHANE'S STOCHASTIC INTEGRAL EQUATION SYSTEMS

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1. INTRODUCTION

We consider the problem of the existence and uniqueness of the solution processes in stochastic integral equation systems of the type:

$$(1) \qquad X^i(t,\omega) = \alpha^i(t,\omega) + \sum_{\rho=1}^{r} \int_a^t g_\rho^i(s,X(s,\omega),\omega)dz^\rho(s,\omega) + \sum_{\rho,\sigma=1}^{r} \int_a^t h_{\rho\sigma}^i(s,X(s,\omega),\omega)dz^\rho(s,\omega)dz^\sigma(s,\omega) \qquad (i=1,\dots n)$$

($t\in[a,b]\subset\mathbb{R}$; $\omega\in\Omega$, being (Ω,A,P) some probability space), where the integrals are interpreted as McShane's stochastic belated integrals.

This problem has been treated previously by E. J. McShane (1974) himself and by K. D. Elworthy (1982). The study McShane made originally is limited to the special case in which α^i is not depending on the time t and the processes z^ρ are sample continuous. On the other hand, Elworthy does not require those restrictions, but considers hypotheses quite stronger related to g_ρ^i and $h_{\rho\sigma}^i$ functions.

We state in this paper a new result on the problem mentioned above. We consider similar conditions to those of McShane respecting g_ρ^i and $h_{\rho\sigma}^i$ functions, and to those of Elworthy respecting α^i and z^ρ processes (in both cases our requirements are even quite weaker in some aspects). So that the stated aspect carried out by the authors previously mentioned are in some ways included in our study.

The results we have obtained will be used in later works on some extensions on the problem of the existence and uniqueness and some regularity associated problems.

2. DEFINITIONS AND PREVIOUS RESULTS. NOTATION

Let $[a,b]$ be a closed real interval. Let (Ω,A,P) be a probability space and $F=\{F_t;t\in[a,b]\}$ a complete filtration of the measurable space (Ω,A) (that is to say each sub-σ-algebre F_t contains all the P-null subsets of A).

We consider the following types of processes (only the first class is referred to the integrators):

$z\in Z_F^4$ iff z is a defined on $[a,b]$ into $\mathbb{R}$,F-measurable (i. e., adapted to the filtration F) process, satisfying for some positive constant K the inequalities:

$$|E[z(t,\omega)-z(s,\omega)/F_s]| \leq K(t-s)$$

$$E([z(t,\omega)-z(s,\omega)]^p/F_s) \leq K(t-s) \qquad (p=2,4)$$

a. s. (almost surely), whenever $a\leq s\leq t\leq b$.

$f\in H_F^2$ iff f is a defined on $[a,b]$ into $\mathbb{R}$, F-measurable, L_2-continuous a. e. (almost everywhere) in $[a,b]$ and L_2-bounded process.

$x\in H_{F,n}^2$ iff x is a defined on $[a,b]$ into $\mathbb{R}^n$ process, in which each component x^i $(i=1,\dots n)$ belongs to H_F^2.

$\tilde{H}_{F,n}^2$ will represent the quotient space respecting the usual process equivalence relation (two processes, x and y, are equivalents $(x\sim y)$ iff for each t the r. v. (random variables) $x(t,\omega)$ and $y(t,\omega)$ are a. s. equal).

Given a process $x\in H_{F,n}^2$, we will represent by means of $\tilde{x}$ the equivalence class of processes associated to x.

We will use the following definitions of norms (or pseudonorms, according to the space that we consider):

For each r. v. $X(\omega)$ ($\mathbb{R}$- or $\mathbb{R}^n$-valued):

$$\|X\| = \{E[X^2]\}^{\frac{1}{2}}$$

For each process $X(t,\omega)$ ($\mathbb{R}$- or $\mathbb{R}^n$-valued):

$$|||X||| = \sup_{t\in[a,b]} \{\|X(t)\|\}$$

We will define and denote

$$|||X|||(t) = |||X1_{[a,b]}||| = \sup_{s\in[a,b]} \{\|X(s)\|\}$$

where $1_{[a,b]}$ is the indicator function of [a,t] in [a.b].
$(\tilde{H}^2_{F,n}, |||.|||)$ is a Banach space.

Respecting to the basic elements of the McShane's stochastic calculus theory (definition of the integral, existence, estimates, indefinite integral,...) we fundamentally refer to McShane's (1974) and Elworthy's (1982) works.

From its later explicit use, we remind here that if f belongs to H^2_F and z, z^1 and z^2 belong to Z^4_F, then it is known that the belated (or McShane's) integrals

$$\int_a^b fdz \qquad\qquad \int_a^b fdz^1dz^2$$

exist, and the following estimates are true:

$$\|\int_a^b f(t,\omega)dz(t,\omega)\| \leq C\{\int_a^b \|f(t,\omega)\|^2 dt\}^{\frac{1}{2}}$$

$$\|\int_a^b f(t,\omega)dz^1(t,\omega)dz^2(t,\omega)\| \leq C\{\int_a^b \|f(t,\omega)\|^2 dt\}^{\frac{1}{2}}$$

where $C=2K(b-a)^{\frac{1}{2}}+K^{\frac{1}{2}}$.

On the other hand, if the integrators and the integrand are F-measurable processes and the corresponding belated integral exist, then any version of the associated indefinite integral is also an F-measurable process.

Finally, we will denote in abridged form by means of $E[\alpha,g,h]$ a system of the type (1), and by means of $\{g,h:(i,\rho,\sigma)\}$ the set of the functions g^i_ρ and $h^i_{\rho\sigma}$ that appear as integrand in the integrals.

3. EXISTENCE AND UNIQUENESS

In this paragraph we prove the existence and uniqueness of the solution processes inside the class $H^2_{F,n}$ for a system of the type (1), when the initial condition α belongs to that class.

Previously we define the set of the hypotheses we will require for it in relation to the functions that intervene in the integrals in (1).

3.1. Hypotheses

We consider the following set of hypotheses relative to the functions g^i_ρ and $h^i_{\rho\sigma}$, and the integrators z^ρ:

[HE] : "1) (Ω,A,P) is a complete probability space. $[a,b]$is a closed real interval.

2) $F=\{F_t;t\in[a,b]\}$ is a complete filtration of the measurable space (Ω,A).

3) Every process z (with or without affixes) belongs to Z^4_F.

4) Every function $f\in\{g,h:(i,\rho,\sigma)\}$ is defined on $[a,b]\times\mathbb{R}^n\times\Omega$ into $\mathbb{R}$, and there is a constant L such that, a. s. in Ω, for all f in $\{g,h:(i,\rho,\sigma)\}$, all t in [a,b] and all x_1, x_2 in $\mathbb{R}^n$:

$$|f(t,x_1,\omega)-f(t,x_2,\omega)| \leq L|x_1-x_2|$$

$$|f(t,0,\ldots0,\omega)| \leq L$$

5) For each $\mathbb{R}^n$-valued, L_2-integrable r. v. $X(\omega)$, all the processes $f(t,X(\omega),\omega)$are P-continuous a. e. in [a,b].

6) For each $\mathbb{R}^n$-valued, F-measurable process $X(t,\omega)$, all the processes $f(t,X(t,\omega),\omega)$ are F-measurable."

REMARK: The previous hypotheses set is similar to that established by McShane in his book (1974) (this author studies the case in which α is a second-order F_a-measurable r. v.); nevertheless we have simplified here in some ways the hypothese [HE-5]. In fact, McShane (1974) explicitly require this condition for any a. s. sample continuous process $X(t,\omega)$, not only for any r. v. $X(\omega)$. We prove that the McShane's condition is satisfied considering ours and the hypothese [HE-4]together. On the other hand, Elworthy (1982) require in relation to the hypothese [HE-5] the a. s. sample continuity of all the processes $f(t,x,\omega)$ (and, equivalently, of all the processes $f(t,X(\omega),\omega)$, for eaxh r. v. $X(\omega)$).

3.2. Existence and uniqueness theorem

We establish the following result:

THEOREM: " Let $E[\alpha,g,h]$ be a system satisfying the hypotheses set [HE], and being $\alpha\epsilon H^2_{F,n}$. Then, there is only an element $\tilde{x}^*$ in $\tilde{H}^2_{F,n}$ such that every version $x^*\epsilon\tilde{x}^*$ is a solution of $E[\alpha,g,h]$."

To proof it, we apply the Picard's method using an adequate Fomin-Kolmogorov's version of the fixed point theorem, (see Bharucha-Reid (1972)):

Proof: Let T be an operator associating to each process $x\epsilon H^2_{F,n}$ the process

$$((Tx)^1,\ldots(Tx)^n)$$

where $(Tx)^i$, for each i, is defined by the right member of (1), for some arbitrary choice of versions of the corresponding indefinite integrals (so that T is defined except for equivalence).

We will prove the following asserts:

a) T is well-defined.

b) T apply $H^2_{F,n}$ into itself.

c) T^k is a contraction, for some $k\epsilon\mathbb{N}$.

a) T is well-defined:

It suffices to prove that for each $x\epsilon H^2_{F,n}$ and $f\epsilon\{g,h:(i,\rho,\sigma)\}$ the process $f(t,x(t,\omega),\omega)$ belongs to H^2_F.

Let G be the set defined as follows:

$G = \{t\epsilon[a,b]$: x is L_2-continuous in t, and for every $u\epsilon[a,b]\cap Q$ (rationals) the process $f(t,x(u,\omega),\omega)$ is P-continuous in t$\}$

G has Lebesgue-measure equal to 1 in [a,b].

From [HE-4] and our "L_p-continuity lemma" (see Angulo (1984)), for each $u\epsilon[a,b]\cap Q$ the process $f(t,x(t,\omega),\omega)$ is L_2-continuous in G, since for each $v\epsilon[a,b]$, a. s. in Ω

$$|f(t,x(v,\omega),\omega)| \leq L(1+|x(v,\omega)|)$$

for all $t\epsilon[a,b]$.

Then, for each $t_0 \in G$, we can assert that for all $t \in [a,b]$ and all $u \in [a,b] \cap Q$,

$$\|f(t,x(t,\omega),\omega)-f(t_0,x(t_0,\omega),\omega)\| \leq$$

$$\leq \|f(t,x(t,\omega),\omega)-f(t,x(u,\omega),\omega)\| +$$

$$+ \|f(t,x(u,\omega),\omega)-f(t_0,x(u,\omega),\omega)\| +$$

$$+ \|f(t_0,x(u,\omega),\omega)-f(t_0,x(t_0,\omega),\omega)\| \leq$$

$$\leq L\|x(t,\omega)-x(u,\omega)\| + \|f(t,x(u,\omega),\omega)-f(t_0,x(u,\omega),\omega)\| +$$

$$+ L\|x(u,\omega)-x(t_0,\omega)\|$$

Consequently,

$$\lim_{t\to t_0} \|f(t,x(t,\omega),\omega)-f(t_0,x(t_0,\omega),\omega)\| \leq 2L\|x(t_0,\omega)-x(u,\omega)\|$$

As the previous limit is not depending on n, and the process x is L_2-continuous in t_0, this limit must be equal to 0. So, we can assert that the process $f(t,x(t,\omega),\omega)$ is L_2-continuous in G.

Besides, from [HE-4] and [HE-5], respectively, the process $f(t,x(t,\omega),\omega)$ is L_2-bounded and F-measurable.

b) T apply $H^2_{F,n}$ into itself:

It suffices to prove that for each $x \in H^2_{F,n}$ and $f \in \{g,h:(i,\rho,\sigma)\}$ (we will suppose, to simplify, that f is one of the functions g^i_ρ and z is the corresponding integrator z^ρ; in the same way we would proceed in relation to second-order integrals), the process defined (except for equivalence) by the indefinite integral

$$\int_a^t f(s,x(s,\omega),\omega)dz(s,\omega)$$

belongs to H^2_F.

In fact, such process is F-measurable, according to that we have indicated in the pargraph 2. On the other hand, it is also an L_2-continuous in [a,b] (and so, L_2-bounded) process, since from [HE-4]:

$$\|f(s,x(s,\omega),\omega)\| \leq L(1+|||x|||)$$

and so,

$$\| \int_s^t f(\tau,x(\tau,\omega),\omega)dz(\tau,\omega) \| \leq CL(1+|||x|||)$$

for any $s,t\in[a,b]$ $(s\leq t)$.

c) T^k is a contraction, for some $k\in\mathbb{N}$:

We will prove, by recurrence, that for all pair of processes x_1, $x_2\in H^2_{F,n}$ and for all $k\in\mathbb{N}$,

(2) $$|||T^k x_1 - T^k x_2|||^2(t) \leq [n(r+r^2)^2C^2L^2(t-a)]^k \frac{1}{k!} |||x_1-x_2|||^2(t)$$

<u>k=1</u>: For each i and each $s\in[a,b]$:

$$\| (Tx_1)^i(s,\omega) - (Tx_2)^i(s,\omega) \|^2 \leq$$

$$\leq (r+r^2)[\sum_{\rho=1}^{r} \| \int_a^s [g^i_\rho(\tau,x_1(\tau,\omega),\omega) - g^i_\rho(\tau,x_2(\tau,\omega),\omega)]dz^\rho(\tau,\omega) \|^2 +$$

$$+ \sum_{\rho,\sigma=1}^{r} \| \int_a^s [h^i_{\rho\sigma}(\tau,x_1(\tau,\omega),\omega) - h^i_{\rho\sigma}(\tau,x_2(\tau,\omega),\omega)]dz^\rho(\tau,\omega)dz^\sigma(\tau,\omega) \|^2] \leq$$

(3) $$\leq (r+r^2)^2C^2L^2 \int_a^s |||x_1-x_2|||^2(\tau)d\tau \leq$$

$$\leq (r+r^2)^2C^2L^2 \int_a^s |||x_1-x_2|||^2(s)d\tau =$$

$$= (r+r^2)^2C^2L^2 |||x_1-x_2|||^2(s)(s-a)$$

Adding for i=1,...n, and taking supremes on [a,t] (for any fixed $t\in[a,b]$) yields (2), for k=1.

<u>k→k+1</u>: From (3), for each i and for all $s\in[a,b]$,

$$\| (T^{k+1}x_1)^i(s,\omega) - (T^{k+1}x_2)^i(s,\omega) \|^2 \leq$$

$$\leq (r+r^2)^2C^2L^2 \int_a^s |||T^k x_1 - T^k x_2|||^2(\tau)d\tau \leq$$

$$\leq (r+r^2)^2C^2L^2[n(r+r^2)^2C^2L^2]^k \frac{1}{k!} \int_a^s (\tau-a)^k |||x_1-x_2|||^2(\tau)d\tau \leq$$

$$\leq n^k[(r+r^2)^2C^2L^2]^{k+1} \frac{1}{(k+1)!} (s-a)^{k+1} |||x_1-x_2|||^2(s)$$

Adding for i=1,...n, and taking supremes on [a,t] (for any fixed $t\in[a,b]$) yields (2), for k+1 instead of k.

Taking now t=b, and since the term

$$\nu_k = [n(r+r^2)^2C^2L^2(b-a)]^k\frac{1}{k!}$$

converge to 0, when $k\to\infty$, it must exist a number $k_o \in \mathbb{N}$ such that $\nu_{k_o} < 1$, and thus T^{k_o} is a contraction.

Finally, we define the induced operator $\tilde{T}$ by the process equivalence relation:

$$\tilde{T}: \tilde{H}^2_{F,n} \longrightarrow \tilde{H}^2_{F,n}$$

It is easy to prove that the previous a), b) and c) points are true, in adequate terms, for the new operator $\tilde{T}$. The Fomin-Kolmogorov's fixed point theorem yields the conclusion: there is only one element $\tilde{x}^*$ in $\tilde{H}^2_{F,n}$ such that

$$\tilde{T}\tilde{x}^* = \tilde{x}^*$$

being so any version x^* of $\tilde{x}^*$ a solution of $E[\alpha,g,h]$. So that, the proof is concluded.

From the previous proof, we inmediately obtain the following consequence:

<u>COROLLARY</u>: "Under the hypotheses of the previous theorem, if x^* is any version of the solution of $E[\alpha,g,h]$, then the process $x^*-\alpha$ is L_2-continuous in all $[a,b]$, and consequently x^* and α have the same L_2-discontinuities in $[a,b]$".

OBSERVATION: This corollary include the case Elworthy (1982) considers (α any L_2-continuous in $[a,b]$ process) and the particular case McShane (1974) considers (α any $F_a$measurable L_2-integrable r. v.).

4. CONCLUSION

The purpose of this paper is to establish a fundamental result on the existence and uniqueness of the solution processes in McShane's stochastic integral equation systems of the type (1), in order to consider and to study some posterior extensions on this problem and some regularity associated problems.

Specifically, in relation to the second of the mentioned problems, we present together with this another work where we consider the problem of the convergence of the solution processes in McShane's stochastic integral equation systems with coefficients depending on a parameter.

In that referred to the first mentioned problem, we point out here that we have prove the existence and uniqueness of the solution processes when α can be expressed as the product (component to component, for $i=1,\ldots n$) of a process belonging to $H^2_{F,n}$ and an F-measurable, a. s. sample-function bounded and P-continuous a. e. process. In that case, we consider in general, in relation to the hypothese [HE-4], an a. s. finite r. v. $L(\omega)$ instead of L, and we add to the hypothese [HE-5] the following condition: for each x in $\mathbb{R}^n$, all the processes $f(t,x,\omega)$ must be separable.

We also have established in that case conditions that allow to assure the convergence of the solution processes when the coefficients of the system depend on a parameter.

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