ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Probabilités et applications

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 92, série *Probabilités et applications*, n° 7 (1988), p. 83-98

<http://www.numdam.org/item?id=ASCFPA_1988__92_7_83_0>

ASYMPTOTIC COVARIANCES OF EMPIRICAL PROCESSES

C.S. WITHERS

Formulae are given for the asymptotic variance of a sum of random variables (r.v.s.) and the asymptotic covariances of the weighted empirical process, the rank process, and the signed rank process, when the sample consists of non-stationary strongly dependent r.v.s. from R. A central limit theorem (C.L.T.) for such r.v.s. is also given.

§1. INTRODUCTION

When adjacent values within a sample are strongly dependent, nearly every test (including so-called robust tests) will give an incorrect α-level. Generally, if adjacent values are positively (negatively) correlated, then variances will be larger (smaller) than in the independent case and the actual type 1 error will be larger (smaller) than the assigned value. If one is able to specify the nature of the pairwise dependence of r.v.s., the results below will often allow computation of the correct asymptotic variance or covariance, and hence provide (at least asymptotically) the correct α-level.

These results apply, for example, to statistics T_N approximated by $h_N(L_N)$ where $h_N: D \to R$ satisfies $\{\sup_{[0,1]} |x_N - x| \to 0,\ x \text{ continuous}\} \Rightarrow \{h_N(x_N) \to h(x)$ for some h, as $N \to \infty\}$, and where L_N is either a weighted empirical process (e.g., k-sample goodness of fit tests), or a rank process (e.g., linear rank statistics), or a signed rank process (e.g., many tests of symmetry). (See Billingsley [1] for definition of D.)

The restrictions we impose on the covariances are shown to hold when a suitable arrangement of the sample is α-mixing and $\Sigma\alpha(i) < \infty$. We do not assume stationarity of subsamples, nor do we need to consider infinite sequences of r.v.s. For applications to signed and unsigned linear rank statistics see Withers [6].

§5 gives a C.L.T. for mixing r.v.s. on R.

If G is a c.d.f. on R we use G^{-1} to denote its right-continuous inverse (i.e., G_- of Withers [5]).

§2. THE VARIANCE OF WEIGHTED SUMS

We begin with consideration of a weighted sum of r.v.s.

For example this result allows correction of the 2-sample t-test when the sub-samples are either dependent or dependent within themselves. (The special case where the sample is stationary was considered in Lemma 3, p. 172 of Billingsley [1]).

THEOREM 1. Suppose

(1) for $N \geq 1$ $\{m_\ell = m_{\ell N},\ \ell = 1,..,k\}$ are integers such that

$$m_\ell \to \infty,\ m_\ell/n \to \lambda_\ell \qquad \text{as } N \to \infty,\ \ell = 1,..,k, \text{ where } n=n_N=\sum_1^k m_\ell.$$

For $N \geq 1$ let $\{Z_{11N},\dots,Z_{m_1 1N},\ Z_{12N},\dots,\ Z_{m_2 2N},\dots,\ Z_{1kN},\dots\ Z_{m_k kN}\} = \mathcal{D}_N$, say, be r.v.s. with $EZ_{i\ell N} \equiv 0$ such that

(2) $$\sum_{\beta=-\infty}^{\infty} \max_{N,j} |EZ_{j+\beta,\ell N}\, Z_{j\ell' N}| < \infty \text{ for } \ell,\ell' = 1,..,k$$

where the max is taken over all j,N such that $1 \leq j \leq m_{\ell'}$, $1 \leq j + \beta \leq m_\ell$.

(3) Suppose for $\ell,\ell' = 1,\dots,k$ for all fixed β, that

$$n^{-1} \sum_j EZ_{j+\beta,\ell N}\, Z_{j\ell' N} \to \min(\lambda_\ell, \lambda_{\ell'})\, \rho_{\ell\ell'}(\beta) \text{ as } N \to \infty,$$

where j is summed from max $(1,1-\beta)$ to min $(m_\ell-\beta, m_{\ell'})$.

(4) Suppose $w_{\ell N} \to w_\ell$, finite, as $N \to \infty$, $\ell = 1,\dots,k$.

(5) Let $S_N = \sum_{\ell=1}^{k} w_{\ell N} \sum_{i=1}^{m_\ell} Z_{i\ell N}$.

(6) Then as $N \to \infty$, $n^{-1} ES_N^2 \to T = \sum_{\ell=1}^{k} \sum_{\ell'=1}^{k} w_\ell w_{\ell'} \min(\lambda_\ell, \lambda_{\ell'}) \Delta_{\ell\ell'}$,

where

(7) $\Delta_{\ell\ell'} = \sum_{-\infty}^{\infty} \rho_{\ell\ell'}(j)$.

Further, if $m_1 \equiv \ldots \equiv m_k$ then (4) can be weakened to

(8) $\sum_{\beta=-\infty}^{\infty} \max_{N,j} |E\, Z_{j+\beta,\ell N} Z_{j\ell' N} + E\, Z_{j\ell N} Z_{j+\beta,\ell' N}| < \infty$ for $\ell, \ell' = 1, \ldots k$,

provided (7) is replaced by

(7)' $\Delta_{\ell\ell'} = \rho_{\ell\ell'}(0) + \sum_{\beta=1}^{\infty} (\rho_{\ell\ell'}(\beta) + \rho_{\ell\ell'}(-\beta))$.

In either case the expression for $\min(\lambda_\ell, \lambda_{\ell'}) \Delta_{\ell\ell'}$ is absolutely convergent

PROOF. It suffices to prove that for $Q_\ell = \sum_{i=1}^{m_\ell} Z_{i\ell N}$

(9) $E\, Q_1 Q_2$ converges absolutely to $\lambda_1 \Delta_{12}$ when $\lambda_2 \geq \lambda_1$.

If $\lambda_2 > \lambda_1$ then $m_2 \geq m_1$ for N large. If $\lambda_2 = \lambda_1$ then the subsequence for which $\{m_2 < m_1\}$ can be treated in the same way as that for which $\{m_2 \geq m_1\}$. Hence we may assume $m_2 \geq m_1$.

Let $\rho_{ij} = \begin{cases} E\, Z_{i1N} Z_{j2N} & \text{if } 1 \leq i \leq m_1,\ 1 \leq j \leq m_2 \\ \rho_{12}(i-j), & \text{otherwise} \end{cases}$

and $T_{1\beta} = (m_1 - \beta)^{-1} \sum_{j=1}^{m_1-\beta} \rho_{j+\beta,j}$. Then $EQ_1Q_2 = \sum_{i=1}^{m_1} \sum_{\beta=1-i}^{m_2-i} \rho_{i,i+\beta}$

$= A_1 + A_2 + A_3$ where $A_1 = \sum_{\beta=1-m_1}^{0} \sum_{i=1-\beta}^{m_1} \rho_{i,i+\beta} = \sum_{0}^{m_1-1} (m_1 - \beta)\, T_{1\beta}$,

$A_2 = \sum_{\beta=1}^{m_2-m_1} \sum_{i=1}^{m_1} \rho_{i,i+\beta}$, $A_3 = \sum_{\beta=m_2-m_1+1}^{m_2-1} \sum_{i=1}^{m_2-\beta} \rho_{i,i+\beta}$. By (3), $T_{1\beta} \to \rho_{12}(\beta)$ so that by (2) and the Lebesgue Convergence Theorem (LCT)

(10) $A_1/n \to \lambda_1 \sum_0^\infty \rho_{12}(\beta)$, convergence being absolute.

Now $A_2 + A_3 = A_4 + A_5$ where $A_4 = n \sum_{\beta=1}^{m_2-1} T_{2\beta}$,

$T_{2\beta} = n^{-1} \sum_{i=1}^{m_1} \rho_{i,i+\beta}$, $A_5 = n \sum_{\beta=m_2-m_1+1}^{m_2-1} T_{3\beta}$, $T_{3\beta} = n^{-1} \left(\sum_{i=1}^{m_2-\beta} - \sum_{i=1}^{m_1} \right) \rho_{i,i+\beta}$.

Since $T_{2\beta} \to \lambda_1 \rho_{12}(-\beta)$ and $T_{3\beta} \to (\lambda_2 - \lambda_1)\rho_{12}(-\beta)$, (2) and the LCT imply $A_4/n \to \lambda_1 \sum_1^\infty \rho_{12}(-\beta)$, $A_5/n \to 0$, convergence being absolute. This together with (10) implies (9).

If $m_1 = m_2$ then $A_2 = 0$ and $(A_1 + A_3)/n = T_{10} + \sum_{\beta=1}^{m_1-1} T'_{1\beta}$ where $T'_{1\beta} = n^{-1} \sum_{j=1}^{m_1-\beta} (\rho_{j+\beta,j} + \rho_{j,j+\beta})$; hence by (8) and the LCT $(A_1 + A_3)/n$ converges absolutely to $\lambda_1 \Delta_{12}$ given by $(7)'$.

COROLLARY 1. Suppose (1), (4) hold and $\{\underset{\sim}{Z}_{1N}, \ldots, \underset{\sim}{Z}_{nN}\}$ are r.v.s. in R^p such that $E\, \underset{\sim}{Z}_{iN} \equiv \underset{\sim}{0}$ and

$(2)'$ $\sum_{\beta=-\infty}^{\infty} \max |E\, Z_{j+\beta,N}\, Z_{jN}| < \infty$ where the max is over j,N such that $1 \le j \le n$, $1 \le j + \beta \le n$, and $Z_{jN} = \underset{\sim}{e}'\underset{\sim}{Z}_{jN}$, $\underset{1 \times p}{e'} = (1,\ldots,1)$.

(11) For $1 \le \ell \le k$, $1 \le j \le m_\ell$ let $\underset{\sim}{Z}_{j\ell N} = \underset{\sim}{Z}_{M_\ell+j,N}$ where $M_\ell = \sum_0^{\ell-1} m_\ell$, $m_o = 0$.

$(3)'$ Suppose for $1 \le \ell \le k$ and all fixed β that as $N \to \infty$

$n^{-1} \sum_j E\, \underset{\sim}{Z}_{j+\beta,\ell N}\, \underset{\sim}{Z}'_{j\ell N} \to \lambda_\ell\, \rho_\ell(\beta)$ as $N \to \infty$ where j is summed from $\max(1,1-\beta)$ to $\min(m_\ell-\beta, m_\ell)$.

Let $Z_{i\ell N} = \underset{\sim}{e}' \underset{\sim}{Z}_{i\ell N}$. Then for S_N defined by (5), as $N \to \infty$

(6)' $\quad n^{-1}\, ES_N^2 \to T_o = \sum_{\ell=1}^{k} \lambda_\ell\, \omega_\ell^2\, \Delta_\ell$

where

(7)' $\quad \Delta_\ell = \sum_{-\infty}^{\infty} \underset{\sim}{e}' \rho_\ell(\beta) \underset{\sim}{e}.$

PROOF. Apply Theorem 1 with $k = 1$, $\omega_{\ell N} \equiv 1$ to $\{Z^*_{j\ell N} = \omega_{\ell N}\, Z_{j\ell N}\}$. It suffices to show (3)' $\Rightarrow$ (3) holds with $k = 1$ where $Z^*_{jN} \equiv Z^*_{M_\ell+j,\,\ell N}$ and $\rho_{11}(\beta) = \sum_1^k \lambda_\ell\, \omega_\ell^2\, \underset{\sim}{e}' \rho_\ell(\beta) \underset{\sim}{e}$. Consider the case $\beta > 0$. Assume without loss that $\beta < \min_1^k m_\ell$. Then

$n^{-1} \sum_1^{n-\beta} E\, Z^*_{j+\beta,N}\, Z^*_{jN} = \varepsilon_{1N} + \varepsilon_{2N}$, where $\varepsilon_{1N} = \sum_{\ell=1}^{k} \omega_{\ell N}^2\, n^{-1} \sum_{j=1}^{m_\ell-\beta}$

$E\, Z_{j\ell N}\, Z_{j+\beta,\ell N}$, $\varepsilon_{2N} = n^{-1} \sum_{\ell=1}^{k} \omega_{\ell N}\, \omega_{\ell+1,N} \sum_{j=m_\ell-\beta+1}^{m_\ell} E\, Z_{j\ell N}\, Z_{j+\beta-m_\ell,\,\ell+1,N}$,

$Z_{i,k+1,N} \equiv 0$. But $\varepsilon_{2N} = O(n^{-1})$ and by (3)' $\varepsilon_{1N} \to \rho_{11}(\beta)$. The case $\beta \leq 0$ is similar.

It is worth emphasising that (2)' ensures that the asymptotic variance given by (6)' does not depend on the nature of the dependency between subsamples. This is in contrast to the asymptotic variance given by (6): (2) allows for strong dependencies between $Z_{j\ell N}$ and $Z_{j\ell' N}$ for example, but will generally be false if $\mathcal{Y}_N$ is α-mixing, say.

Of course (2)' is implied by

(8)' $\quad \sum_{\beta=-\infty}^{\infty} \max_{N,j} |E\, Z_{j+\beta,Nh}\, Z_{jNh'} + E\, Z_{jNh}\, Z_{j+\beta,Nh'}| < \infty$ for $h,h' = 1,\ldots,p$

where $\underset{\sim}{Z}'_{jN} = (Z_{jN1},\ldots,Z_{jNp})$ and the max is over j,N as in (2)'.

EXAMPLE 1. Suppose (1), (4) hold and $\{Z_{jN}\}$ are r.v.s. such that cov $(Z_{jN}, Z_{j+\beta,N}) = \sigma_N^2 \rho_N^{|\beta|}$ where $\rho_N \to \rho$, $\sigma_N^2 \to \sigma^2$ as $N \to \infty$ and $|\rho| < 1$. Then for $\{Z_{j\ell N}\}$ given by (11),

$$n^{-1} \text{ var } \left(\sum_{\ell=1}^{k} w_{\ell N} \sum_{j=1}^{m_\ell} Z_{j\ell N}\right) \to \sigma^2 (1 + \rho) \quad (1 - \rho)^{-1} \sum_1^k \lambda_\ell w_\ell^2 \text{ as } N \to \infty .$$

Before applying the above, we give sufficient mixing conditions to ensure (8) and hence (8)'. Of course (8) is equivalent to (2) if

(12) for $N \geq 1$, for $\ell, \ell' = 1, \ldots k, EZ_{i\ell N} Z_{j\ell' N} = EZ_{j\ell N} Z_{i\ell' N}$ for $1 \leq i, j \leq \min (m_\ell, m_{\ell'})$.

For example, (12) holds if the subsamples are uncorrelated. (For definition of α-, ϕ-, and ψ-mixing for a finite sample, see Withers [2].)

DEFINITION. For Z a real r.v. define $\| Z \|_p = \begin{cases} (E\,|Z|^p)^{1/p} & \text{if } 1 \leq p < \infty \\ \text{ess sup}|Z| & \text{if } p = \infty \end{cases}$

LEMMA 1 (a). If for a given ℓ and for all N $\{Z_{1\ell N}, Z_{2\ell N}, \ldots, Z_{m_\ell \ell N}\}$ is α-mixing and

(13) for some p in $(2,\infty]$ $\sum_1^\infty \alpha(i)^{1-2/p} < \infty$, $\max_{N,i} \|Z_{i\ell N}\|_p < \infty$

then (2) and (8) hold for $\ell' = \ell$. The same is true if 'α-mixing' is replaced by 'ϕ-mixing' and (13) is replaced by

(14) for some p in $[2,\infty]$ $\sum_1^\infty \phi(i)^{1-1/p} < \infty$, $\max_{N,i} \|Z_{i\ell N}\|_p < \infty$

OR if 'α-mixing' is replaced by 'ψ-mixing' and (13) is replaced by

(15) $\sum_1^\infty \psi(i) < \infty$, $\max_{N,i} \| Z_{i\ell N} \|_2 < \infty$.

(b) For $\ell = 1, \ldots k$ if $i > m_\ell$ define $Z_{i\ell N} \equiv 0$. For $N \geq 1$ and $\ell, \ell' = 1, \ldots k$ suppose that (13) holds and

(16) $\left\{ \begin{pmatrix} Z_{1\ell N} \\ Z_{1\ell' N} \end{pmatrix}, \begin{pmatrix} Z_{2\ell N} \\ Z_{2\ell' N} \end{pmatrix}, \ldots, \begin{pmatrix} Z_{M\ell N} \\ Z_{M\ell' N} \end{pmatrix} \right\}$ is α-mixing

where $M = \max(m_\ell, m_{\ell'})$.

Then (8) holds. The same is true if 'α-mixing' is replaced by 'φ-mixing' and (13) is replaced by (14), OR if 'α-mixing' is replaced by 'ψ-mixing' and (13) is replaced by (15).

NOTE 1. In the applications in §2 - 4, $\max_{\ell,N,i} \|Z_{i\ell N}\|_\infty < \infty$, so that (2) is implied by (12) and $\sum_1^\infty \alpha(i) < \infty$.

PROOF. (a) follows from Lemma 1 of [3]. (b) follows from

LEMMA 2. Suppose $\left\{ \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \ldots, \begin{pmatrix} X_M \\ Y_M \end{pmatrix} \right\}$ are α-mixing, φ-mixing, and ψ-mixing r.v.s. in R^2 with means $\underset{\sim}{0}$. For $1 \le p \le \infty$ let $C_{Xp} = \max(\|X_k\|_p, \|X_{k+i}\|_p)$, $C_{Yp} = \max(\|Y_k\|_p, \|Y_{k+i}\|_p)$, $C_p = \max(C_{Xp}, C_{Yp})$.

Then for $1 \le k \le k + i \le M$, the following are upper bounds for $T = |E X_k Y_{k+i} + E X_{k+i} Y_i|$.

(a) $6\psi(i) C_{X1} C_{Y1}$

(b) $12\phi^{1/p}(i) C_p C_q$ for $p^{-1} + q^{-1} = 1$, $1 \le p \le \infty$

(c) $24\, \alpha(i)\, C_{X\infty} C_{Y\infty}$

(d) $60\, \alpha(i)^{1/r} C_p C_q$ for $1 \le p \le \infty$, $1 \le q \le \infty$, $r^{-1} = 1 - p^{-1} - q^{-1}$.

PROOF (d): For $t_1, t_2 \ge 0$ $\{Z_i = t_1 X_{iN} + t_2 Y_{iN}\}$ is α-mixing and hence by the proof of Lemma 1 of Withers [2] $|EZ_k Z_{k+i}| \le \alpha \|Z_k\|_p \|Z_{k+i}\|_q$ (where $\alpha = 10\, \alpha(i)^{1/r}$) $\le \alpha(t_1 C_{Xp} + t_2 C_{Yp})(t_1 C_{Xq} + t_2 C_{Yq})$. But

$T\, t_1 t_2 \leq |EZ_k\, Z_{k+i}| + t_1^{\,2}\, |EX_k\, X_{k+i}| + t_2^{\,2}\, |EY_k\, Y_{k+i}|$

$\leq \alpha\, t_1 t_2\, (C_{Xp}\, C_{Yq} + C_{Xq}\, C_{Yp}) + 2\alpha(t_1^{\,2}\, C_{Xp}\, C_{Xq} + t_2^{\,2}\, C_{Yp}\, C_{Yq})$. Minimising over t_1, t_2 we have $T \leq \alpha\, (x^2 + y^2 + 4xy) \leq 6\alpha\, C_p C_q$ where $x = (C_{Xp}\, C_{Yq})^{\frac{1}{2}}$ and $y = (C_{Xq}\, C_{Yp})^{\frac{1}{2}}$. Proof of (a) - (c) is very similar.

For simplicity we shall only apply Corollary 1 with $p = 1$. This is not really a restriction, as the case $p = 1$ contains the case of general p. In particular $\{\underset{\sim}{Z}_{1N}, \ldots, \underset{\sim}{Z}_{nN}\}$ is α-mixing for $N \geq 1$ and $\Sigma\, \alpha(i) < \infty$ $\Leftrightarrow$ the sample of size np $\{\underset{\sim}{Z}'_{1N}, \ldots, \underset{\sim}{Z}'_{nN}\}$ is α-mixing for $N \geq 1$ and $\Sigma\, \alpha(i) < \infty$. (In fact, in writing the former set of $\{\alpha(i)\}$ as $\{\alpha'(i)\}$, we have $\alpha'(ip) = \alpha(i)$.)

§3. COVARIANCE OF THE WEIGHTED EMPIRICAL PROCESS

An immediate consequence of Lemma 2 of Withers [3], Corollary 1, and Lemma 1 (a) is the following expression for the asymptotic covariance of the weighted empirical process.

COROLLARY 2. Suppose

(17) $$\left\{ \begin{pmatrix} X_{11} \\ c_{1N} \end{pmatrix}, \ldots, \begin{pmatrix} X_{nN} \\ c_{nN} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} X_{11N} \\ w_{1N} \end{pmatrix}, \ldots, \begin{pmatrix} X_{m_1 1N} \\ w_{1N} \end{pmatrix}, \begin{pmatrix} X_{12N} \\ w_{2N} \end{pmatrix}, \ldots, \begin{pmatrix} X_{m_2 2N} \\ w_{2N} \end{pmatrix}, \ldots, \begin{pmatrix} X_{1kN} \\ w_{kn} \end{pmatrix}, \ldots, \begin{pmatrix} X_{m_k kN} \\ w_{kN} \end{pmatrix} \right\},$$

and (1), (4) hold. Fix t_1, t_2 in [0,1]. Suppose for all p_1, p_2 that (2)' and (3)' hold for $\{Z_{1N}, \ldots, Z_{nN}\}$ given by (11) with $p = 1$, where

(18) $Z_{i\ell N} = \sum_{j=1}^{2} p_j\, \xi_{i\ell}\, (a_N^{-1}(t_j))$, where $\xi_{i\ell}(x) =$ $1(X_{i\ell N} \leq x) - P(X_{i\ell N} \leq x)$ and a_N is a c.d.f. on R.

Suppose also

(19) $\sum_{1}^{k} \lambda_{\ell}\, w_{\ell}^{2} > 0.$

Then (1.8) of Withers [4] holds with

(20) $K(t_1,t_2) = \sum_{\ell=1}^{k} \lambda_{\ell} w_{\ell}^{2} \Delta_{\ell}(t_1,t_2) \,/ \sum_{1}^{k} \lambda_{\ell} w_{\ell}^{2},$

where

(21) $\Delta_{\ell}(t_1,t_2) = \sum_{j=-\infty}^{\infty} \rho_{\ell}(\beta,t_1,t_2)$ and $2\rho_{\ell}(\beta,t_1,t_2)$ is the coefficient of p_1p_2 in $\rho_{\ell}(\beta)$ in (3)$'$.

The expression for $\lambda_{\ell}\Delta_{\ell}(t_1,t_2)$ is absolutely convergent. Further, (2)$'$ holds if

(22) for $N \geq 1$ $\{X_{1N},\ldots,X_{nN}\}$ is α-mixing and $\sum_{1}^{\infty} \alpha(i) < \infty$.

Also, if

(23) $\nabla_{\ell N}(\beta,x,y) = n^{-1}\sum_{j} (P(X_{j+\beta,\ell N} \leq x,\ X_{j\ell N} \leq y) - P(X_{j+\beta,\ell N} \leq x)$ $P(X_{j\ell N} \leq y))$ has a finite limit, $\lambda_{\ell}\nabla_{\ell}(\beta,x,y)$, as $N \to \infty$, where j is summed from max $(1,1-\beta)$ to min $(m_{\ell}-\beta,m)$, $\ell = 1,\ldots,k$ and if $a_N(x) \to a(x)$, a c.d.f. on R, as $N \to \infty$,

then (3)$'$ holds and

(24) $\rho_{\ell}(\beta,t_1,t_2) = \tfrac{1}{2}(\nabla(t_1,t_2) + \nabla(t_2,t_1))$, where $\nabla(t_1,t_2) = \nabla_{\ell}(\beta,a^{-1}(t_1),\ a^{-1}(t_2))$.

NOTE 2. (23) holds if for $1 \leq \ell \leq k$ either $(X_{j+\beta,\ell N},\ X_{j\ell N}) \xrightarrow{L} Y = (X_{\beta\ell},\ X_{o\ell})$ uniformly in $1 \leq j \leq m_{\ell}$, and Y has a continuous distribution,

and $\nabla_\ell(\beta,x,y) = P(X_{\beta\ell} \le x,\ X_{o\ell} \le y) - P(X_{o\ell} \le x)\ P(X_{o\ell} \le y)$,
or $\lambda_\ell = 0$.

NOTE 3. If $\nabla_\ell(\beta,x,y) \equiv \nabla(\beta,x,y)$, then (20), (21), (24) imply

(25) $K(t_1,t_2) = \sum_{\beta=-\infty}^{\infty} \nabla(\beta,\ a^{-1}(t_1),\ a^{-1}(t_2))$, convergence being absolute. (Use the fact that $\nabla(\beta,x,y) = \nabla(-\beta,y,x)$.)

NOTE 4. The only restriction on the marginal c.d.f.s is (23) for $\beta = 0$. This holds with

$$\nabla_{\ell\ell}(0,x,y) = G_\ell(\min(x,y),\infty) - G_\ell(x,y) \text{ if for all } x,y$$

$m_\ell^{-1} \sum_{j=1}^{m_\ell} F_{j\ell N}(x)\ F_{j\ell N}(y) \to G_\ell(x,y)$ as $N \to \infty$ where $F_{j\ell N}(x) = P(X_{j\ell N} \le x)$.
For example, if for all x, $\max_{j=1}^{m_\ell} |F_{j\ell N}(x) - F_\ell(x)| \to 0$ as $N \to \infty$ then
$\nabla_{\ell\ell}(0,x,y) = F_\ell(x)(1-F_\ell(y))$ for $x \le y$. If $F_{j\ell N}(x) \equiv F_\ell(x,j/m_\ell)$ then

$$\nabla_{\ell\ell}(0,x,y) = \int_0^1 F_\ell(x,\theta)\ (1-F_\ell(y,\theta))d\theta \text{ for } x \le y,$$

$$= E\ L'(x)\ L'(y) \text{ where } L'(x) = \int_0^1 W^0(F_\ell(x,\theta))dW\ (\theta)$$

and W° is the Brownian Bridge and W is an independent Wiener process; when $F_\ell(x,\theta) \equiv a(x)/a(\theta)$ for $0 \le x \le \theta \le 1$, a simpler characterisation is $L' = A\ W(a/A)$ where A is given on p.1108 of Withers [3].

§4. COVARIANCES OF SIGNED AND UNSIGNED RANK PROCESSES

Corollaries 3, 4 give the asymptotic covariance of the empirical rank process and the empirical signed rank process. They follow immediately from Lemma 2.12 of Withers [5], Lemma 1(a) and Corollary 1. For definition of H, E_N, H^+, H_-^+, S, E_N^+, σ_c, see [5]. $\dot{f}(x)$ denotes $df(x)/dx$.

COROLLARY 3. Suppose (1), (4), (17), (19) hold and that for all p_1, p_2 (2)', (11), (3)' hold for $p = 1$ and

(26) $z_{i\ell N} = \sum_{j=1}^{2} p_j \, (w_{\ell N} - c_N(t_j)) \; \xi_{i\ell} \; (H^{-1}(t_j))$ where

$\xi_{i\ell}(x) = 1(X_{i\ell N} \leq x) - P(X_{i\ell N} \leq x)$, $c_N(t) = n^{-\frac{1}{2}} \sigma_c \, E_N(t)$.

Then (1.10) of Withers [5] holds with

(27) $C(t_1,t_2) = \sum_{\ell=1}^{k} \lambda_\ell \, \Delta_\ell \, (t_1,t_2) \; / \; \sum_1^k \lambda_\ell \, w_\ell^{\,2}$, where $\Delta_\ell \, (t_1,t_2)$

is as in (21).

The expression for $\lambda_\ell \, \Delta_\ell \, (t_1,t_2)$ is absolutely convergent. Further, (2)' is implied by (22). Also, if (1), (4), (23) hold and

(28) $E_N(t) \to E(t)$ as $N \to \infty$, $0 \leq t \leq 1$ and for x in R,

$H(x) \to a(x)$ as $N \to \infty$, where $a(\cdot)$ is a c.d.f., then (3)' holds

and $\rho_\ell(\beta,t_1,t_2)$ in (21) is given by

(29) $\frac{1}{2}(\nabla(t_1,t_2) + \nabla(t_2,t_1))$ where $\nabla(t_1,t_2) = (w_\ell^{\,2} - c(t_1)w_\ell - c(t_2)w_\ell$

$+ c(t_1) \; c(t_2)) \, \nabla_\ell \, (\beta, a^{-1}(t_1), a^{-1}(t_2))$, $c(t) = (\sum_1^k \lambda_\ell \, w_\ell^2)^{\frac{1}{2}} \, E(t)$.

Also, if (1.9) of [4] holds and for $\ell = 1,\ldots,k$

(30) $n^{\frac{1}{2}} \, m_\ell^{\;-1} \sum_1^{m_\ell} (L_{i\ell N} \; (t + n^{-\frac{1}{2}}) - L_{i\ell N}(t)) \to \dot{p}_\ell(t)$, finite, as

$N \to \infty$, $0 \leq t < 1$ where $L_{i\ell N}(t) = P(X_{i\ell N} \leq H^{-1}(t))$,

then (28) holds with

(31) $E(t) = (\sum_1^k \lambda_\ell \, w_\ell^{\,2})^{-\frac{1}{2}} \sum_1^k \lambda_\ell w_\ell \, \dot{p}_\ell(t)$.

NOTE 5. If for $\ell = 1,\ldots,k$ $L_{i\ell N} \equiv L_{1\ell N}$ and $\dot{L}_{1\ell N} \to \dot{p}_\ell$ and $\{\dot{L}_{i\ell 1}, \dot{L}_{i\ell 2},\ldots\}$ is C-tight, then by Note 1.2 of [6] (1), (4) -> (1.9), (1.12) of [5] and (30). If $F_{i\ell N}(x) \equiv F_\ell(x,\theta_\ell n^{-\frac{1}{2}})$, then under mild conditions $H(x) \to a(x) = \sum_1^k \lambda_\ell F_\ell(x,0)$ as $N \to \infty$ and (30) holds with $p_\ell(t) = F_\ell(a^{-1}(t), 0)$.

COROLLARY 4. Suppose (1), (4), (17), (19) hold and that for all p_1, p_2 (2)', (3)', (11) hold for $p = 1$ and

(32) $$Z_{i\ell N} = \sum_{j=1}^{2} p_j \, (w_{\ell N} \, U_{i\ell}(t_j) - c_N^{+}(t_j) \, V_{i\ell}(t_j)),$$ where

$$U_{i\ell}(t) = \bar{U}_{i\ell}(H_-^{+}(t)), \ \bar{U}_{i\ell} = U'_{i\ell} - EU'_{i\ell}, U'_{i\ell}(x) = s(X_{i\ell N}) \, 1(|X_{i\ell N}| \le x),$$

$$V_{i\ell}(t) = \bar{V}_{i\ell}(H_-^{+}(t)), \ \bar{V}_{i\ell} = V'_{i\ell} - EV'_{i\ell}, \ V'_{i\ell}(x) = 1(|X_{i\ell N}| \le x),$$

$$c_N^{+}(t) = n^{-\frac{1}{2}} \, \sigma_c \, E_N^{+}(t).$$

Then $(1.10)^*$ of [5] holds with

(33) $$C^{+}(t_1,t_2) = \sum_{\ell=1}^{k} \lambda_\ell \Delta_\ell (t_1,t_2) \Big/ \sum_{1}^{k} \lambda_\ell w_\ell^{\,2}$$

where $\Delta_\ell \; (t_1,t_2)$ is as in (21).

The expression for $\lambda_\ell \; \Delta_\ell \; (t_1,t_2)$ is absolutely convergent. Further, (2)' is implied by (22).

Also, if (1), (4) hold and

(34) $$E_N^{+} \to E^{+} \text{ as } N \to \infty$$

and for $\underset{\sim}{Y}_{j\ell} = \begin{pmatrix} \bar{U}_{j\ell} \\ \bar{V}_{j\ell} \end{pmatrix}$ as $N \to \infty$

(35) $$n^{-1} \sum_{j} E\underset{\sim}{Y}_{j+\beta,\ell}(t_1) \, \underset{\sim}{Y}_j \, (t_2)' \to \lambda_\ell \; \Sigma(\ell,\beta,t_1,t_2)$$

where j is summed from max (1, 1-β) to min $(m_\ell-\beta, m_\ell)$, and if $H^{+}(x) \to a(x)$ as $N \to \infty$ where $a(\cdot)$ is a c.d.f., then (3)' holds and $\rho_\ell(\beta,t_1,t_2)$ in (21) is given by

(36) $\frac{1}{2}(\nabla(t_1,t_2) + \nabla \, (t_2,t_1))$, where $\nabla \, (t_1,t_2) = w_\ell^{\,2} \, \Sigma_{11} - c^{+}(t_1) w_\ell \, \Sigma_{21}$

$- c^{+}(t_2) w_\ell \, \Sigma_{12} + c^{+}(t_1) c^{+}(t_2) \, \Sigma_{22}$, $c^{+}(t) = (\sum_{1}^{k} \lambda_\ell w_\ell^{\,2})^{\frac{1}{2}} E^{+}(t)$, and

$$\underset{2\times 2}{(\Sigma_{ij})} = \Sigma(\ell,\beta,a^{-1}(t_1),a^{-1}(t_2)).$$

Also, if $(1.9)^*$ of [5] holds and for $\ell = 1,\dots,k$

(37) $n^{\frac{1}{2}} m_\ell^{-1} \sum_{1}^{m_\ell} (\mu_{i\ell N}(t + n^{-\frac{1}{2}}) - \mu_{i\ell N}(t)) \to \dot{p}_{\ell+}(t)$, finite, as

$N \to \infty$, $0 \le t < 1$ where $\mu_{i\ell N}(t) = F_{j\ell N}(H_-^+(t)) + F_{j\ell N}(-H_-^+(t)-)$

$- F_{j\ell N}(0) - F_{j\ell N}(0-)$ and $F_{j\ell N}(x) = P(X_{j\ell N} \le x)$,

then (34) holds with

(38) $E^+(t) = (\sum_{1}^{k} \lambda_\ell w_\ell^2)^{-\frac{1}{2}} \sum_{1}^{k} \lambda_\ell w_\ell \dot{p}_{\ell+}(t).$

NOTE 6. The analog of Note 5 holds; if $F_{i\ell N}(x) \equiv F_\ell(x,\theta_\ell n^{-\frac{1}{2}})$ under mild conditions $H^+(x) \to a(x) = \sum_{1}^{k} \lambda_\ell(F_\ell(x,0) - F_\ell(-x-,0))$ as $N \to \infty$ and (37) holds with $p_{\ell+}(t) = F_\ell(a^{-1}(t),0) + F_\ell(-a^{-1}(t)-,0)$.

EXAMPLE 2. If $c_{iN} \equiv 1$ and for $N \ge 1$ $\{X_{1N},\dots,X_{nN}\}$ is α-mixing and $\sum_{1}^{\infty} \alpha(i) < \infty$, $n = n_N \to \infty$ as $N \to \infty$, $H^+ \to a$ as $N \to \infty$, (34) holds and

(35)' for $i,j = 1,2$ for all x,y and $\beta \ge 0$, $n^{-1} \sum_{k=1}^{n-\beta}$ covar

$(Y_i(k+\beta,x), Y_j(k,y)) \to \Sigma_{ij}^*(\beta,x,y)$ as $N \to \infty$

where $Y_1(k,x) = S(X_{kN}) Y_2(k,x)$, $Y_2(k,x) = 1(|X_{kN}| \le x)$,

then $(1.10)^*$ of [5] holds with

(33)' $C^+(t_1,t_2) = \bar{\rho}(0,t_1,t_2) + 2 \sum_{\beta=1}^{\infty} \bar{\rho}(\beta,t_1,t_2)$ being absolutely

convergent, where $2\bar{\rho}(\beta,t_1,t_2) = \sigma_{11} - E^+(t_1)\sigma_{21} - E^+(t_2)\sigma_{12}$

$+ E^+(t_1)E^+(t_2)\sigma_{22}$,

$2\sigma_{ij} = \Sigma_{ij}(\beta,t_1,t_2) + \Sigma_{ji}(\beta,t_1,t_2) + \Sigma_{ij}(\beta,t_2,t_1) + \Sigma_{ji}(\beta,t_2,t_1)$

and $\Sigma_{ij}(\beta,t_1,t_2) = \Sigma_{ij}^*(\beta,a^{-1}(t_1),a^{-1}(t_2))$.

EXAMPLE 3. If $c_{iN} \equiv 1$ and the subsamples are independent and i.i.d. $F_{\ell N}$ and $F_{\ell N} \to F_\ell$, $F_{\ell +}(x) = F_\ell(x) - F_\ell(-x-)$, $a = \sum_1^k \lambda_\ell F_{\ell +}$ and $L_{\ell +} = F_{\ell +}(a^{-1})$, then $C^+(t_1,t_2) = \sum_{\ell=1}^k \lambda_\ell \Delta_\ell(t_1,t_2)$, where

$\Delta_\ell(t_1,t_2) = L_{\ell +}(\min(t_1,t_2)) - \mu_\ell(t_1)\mu_\ell(t_2) - \dot{\mu}_\ell(t_1)(\mu_\ell(\min(t_1,t_2)) - \mu_\ell(t_2)$
$L_{\ell +}(t_1)) - \dot{\mu}_\ell(t_2)(\mu_\ell(\min(t_1,t_2)) - \mu_\ell(t_1)L_{\ell +}(t_2)) + \dot{\mu}_\ell(t_1)\dot{\mu}_\ell(t_2)(L_{\ell +}(\min(t_1,t_2))$
$- L_{\ell +}(t_1)L_{\ell +}(t_2))$, and $\mu_\ell(t) = F_\ell(a^{-1}(t)) + F_\ell(-a^{-1}(t)-) - F_\ell(0) - F_\ell(0-)$.

In particular, if $c_{iN} \equiv 1$ and $\{X_{iN}\}$ are i.i.d. H, independent of N, and H^+ is continuous and $\theta(t) = 2H(H_-^+(t)) - t - H(0) - H(0-)$ is differentiable, then for $s \leq t$ $C^+(s,t) =$

$$s(1 + \dot{\theta}(s)\dot{\theta}(t)) - \theta(s)(\dot{\theta}(s) + \dot{\theta}(t)) - (\theta(s) - s\dot{\theta}(s))(\theta(t) - t\dot{\theta}(t));$$

hence if X_{1N} is also symmetrically distributed about 0, then T^+ in Withers [5] is the Wiener process.

§5. A C.L.T. FOR MIXING R.V.S.

The proofs of Theorems 18.5.1 - 18.5.4 of Ibragimov and Linnik [2] can readily be shown to generalise to non-stationary scquences of real r.v.s. $\underset{\sim}{X}_N' = (X_{1N}, \ldots, X_{nN})$, when $n = n_N \to \infty$ as $N \to \infty$. One thus obtains

THEOREM 2. Suppose $EX_{iN} \equiv 0$ and

(39) $\quad n^{-1}E(\sum_1^n X_{iN})^2 \to \sigma^2 < \infty$ as $N \to \infty$, and

either (a) $\underset{\sim}{X}_N$ is α-mixing and (13) holds with $Z_{i\ell N} \equiv X_{iN}$,

or (b) $\underset{\sim}{X}_N$ is ϕ-mixing and

(40) $\quad$ for some $p \geq 2$, $\max_{N,j} \| X_{jN} \|_p < \infty$, and if $p = 2$ then $\sum_1^\infty \phi(j)^{\frac{1}{2}} < \infty$.

Then $n^{-\frac{1}{2}} \sum_{1}^{n} X_{iN} \xrightarrow{L} N(0,\sigma^2)$ as $N \to \infty$.

NOTE 5. (a) with $p = \infty$ also follows from Theorem 1 of Withers [3] with p, q, k chosen as on p. 349 of [2]. M. Ghosh has pointed out to me that (6) of [3] is redundant, as

$$(41) \qquad EX_N^2 \to 0, \quad EY_N^2 \to 0 \Rightarrow EX_N Y_N \to 0.$$

In Corollary $1(g)$ of [3] the condition on r, d should be $\lambda + 5d < 1$; alternatively, one may use the version at the end of [4] in which "$d < 1/b$" should read "$d < 1/6$".

COROLLARY 5. Suppose

$$(42) \qquad 0 < \liminf_{N \to \infty} s_N{}^2/n_N < \infty, \text{ where } s_N{}^2 = \text{var}\left(\sum_{i=1}^{n} X_{iN}\right), \text{ and either}$$

(a) or (b) of Theorem 2 hold.

Then $\sum_{1}^{n} (X_{iN} - EX_{iN}) \;/\; s_N \xrightarrow{L} N(0,1)$ as $N \to \infty$.

PROOF. Apply Theorem 2 to $\{X'_{iN} = n^{\frac{1}{2}} s_N^{-1}(X_{iN} - EX_{iN})\}$.

NOTE 6. M.I. Gordin (p. 420 of [2]) has shown that for $\{X_{iN} \equiv X_i\}$ stationary and α-mixing, (a) may be weakened to

$$(43) \qquad \text{for some } p \text{ in } [2,\infty],\ \sum_{1}^{\infty} \alpha(i)^{1-1/p} < \infty,\ \max_{N,i} ||X_{iN}||_2 < \infty,$$

and that these conditions suffice for the existence of σ^2 satisfying (39).

However, his method of proof does not seem to apply to non-stationary sequences.

Applying Lemma 1 (a) and Theorem 2 to Corollary 1, we have

COROLLARY 6. Suppose (1), (4) hold and $Z_N = (\underset{\sim}{Z}_{1N},\ldots,\underset{\sim}{Z}_{nN})$ are r.v.s. in R^{pn} with $E\underset{\sim}{Z}_{iN} = \underset{\sim}{0}$ such that (3)', (11) hold and either Z_N is

α-mixing and (13) holds, or Z_N is ϕ-mixing and (14) holds, or Z_N is ψ-mixing and (15) holds, where $Z_{i\ell N} = (1,\ldots,1)\, \underset{\sim}{Z}_{i\ell N}$.

Then for S_N defined by (5), and Δ_ℓ defined by (7)',

$$n^{-\frac{1}{2}}\, S_N \xrightarrow{L} N(0, \sum_1^k \lambda_\ell\, w_\ell^2\, \Delta_\ell).$$

REFERENCES

[1] Billingsley, Patrick (1968). Convergence of Probability Measures. Wiley, New York.

[2] Ibragimov, I.A. and Linnik, Yu.V. (1971). Independent and stationary sequences of random variables. Wolters-Noordhoff, Groningen.

[3] Withers, C.S. (1975). Convergence of empirical processes of mixing rv's on [0,1]. Ann. Statist. 3, 1101-1108.

[4] ——— (1976). On the convergence of empirical processes of mixing variables. 18, No. 1. Austral. Jnl. Statist.

[5] ———. Convergence of rank processes of mixing random variables on R. Submitted to Austral. Jnl. Statist.

[6] ———. Convergence of linear rank statistics of mixing variables. Submitted.

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