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PROLONGATIONS
OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS. II.
INHOMOGENEOUS EQUATIONS

BY HUBERT GOLDSCHMIDT (*).

In the first part [2] of this paper, we showed that, to find local solutions of homogeneous linear partial differential equations satisfying a certain regularity condition, one need only consider those equations which are formally integrable. We show here how the methods used in proving this result can be extended to prove that the corresponding statement holds for inhomogeneous linear differential equations.

Let $D : \mathcal{E} \to \mathcal{F}$ be a differential operator. Given $f \in \mathcal{F}$, in order that there exists an element $e \in \mathcal{E}$ such that $De = f$, the germ f must satisfy the equation $D'f = 0$, where $D' : \mathcal{F} \to \mathcal{G}$ is a certain differential operator depending only on D; if f is a solution of this equation, we say that f satisfies the compatibility condition for D.

We can now summarize the main result of this paper as follows.

If the differential operator $D : \mathcal{E} \to \mathcal{F}$ *satisfies a regularity condition, then there exists a differential operator* $P : \mathcal{F} \to \mathcal{F}_1$ *such that :*

(i) *The differential operator* $P \circ D : \mathcal{E} \to \mathcal{F}_1$ *is formally integrable.*

(ii) *If* $f \in \mathcal{F}$ *satisfies the compatibility condition for* D, *then* $Pf \in \mathcal{F}_1$ *satisfies the compatibility condition for* $P \circ D$.

(iii) *For fixed* $f \in \mathcal{F}$ *satisfying the compatibility condition for* D, *the space of solutions* $e \in \mathcal{E}$ *of the inhomogeneous equation* $De = f$ *is precisely the space of solutions of the inhomogeneous equation* $(P \circ D)e = Pf$.

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We describe an explicit procedure for constructing P based on the prolongation theory of the first part of this paper which involves finitely many steps. We remark that if D is elliptic, so is $P \circ D$. These results imply the existence of solutions of analytic equations. If the differential operator $D : \mathcal{E} \rightarrow \mathcal{F}$ is analytic and satisfies our regularity assumption, then we can solve the equation $De = f$ whenever $f \in \mathcal{F}$ is an analytic germ satisfying the compatibility condition for D or, assuming moreover that D is elliptic, whenever f is a C^∞ germ satisfying the compatibility condition for D.

This paper provides the basic motivation for some of the results of [2]; we use the Spencer cohomology of a partial differential equation and the Spencer sequences as a tool in proving our results. Throughout this paper we use the notation and definitions of [2].

1. Compatibility conditions. — We begin by recalling the following result which is easily seen to follow from Theorem 3 of [2].

Theorem 1. — *Let* $\varphi : J_k(E) \rightarrow F$ *be a regular differential operator of order* k *from* E *to* F. *Then there exists a differential operator* $\varphi' : J_l(F) \rightarrow G$ *of order* l *which is an epimorphism of vector bundles such that the sequences*

$$(1)\qquad J_{k+l+r}(E) \xrightarrow{p_{l+r}(\varphi)} J_{l+r}(F) \xrightarrow{p_r(\varphi')} J_r(G)$$

are exact for all $r \geqq 0$, *and such that, for any differential operator* $\psi : J_m(F) \rightarrow H$ *of order* m *for which the sequences*

$$(2)\qquad J_{k+m+r}(E) \xrightarrow{p_{m+r}(\varphi)} J_{m+r}(F) \xrightarrow{p_r(\psi)} J_r(H)$$

are exact for all $r \geqq 0$, *we have* $m \geqq l$ *and we can write* ψ *as the composition*

$$\psi = \bar{\psi} \circ p_{m-l}(\varphi)$$

for some differential operator $\bar{\psi} : J_{m-l}(G) \rightarrow H$. *Furthermore* φ' *is essentially unique in the sense that, given any other differential operator* ψ *from* F *to a vector bundle* G_1 *satisfying these same conditions,* ψ *is a morphism from* $J_l(F)$ *to* G_1 *and there exists an isomorphism of vector bundles* $\eta : G \rightarrow G_1$ *such that* $\psi = \eta \circ \varphi'$.

Let $D = \varphi \circ j_k$ and $D' = \varphi' \circ j_l$. We say that $f \in \mathcal{F}$ satisfies the compatibility condition for the inhomogeneous differential equation $De = f$, or simply for D, if $D'f = 0$. Note that this condition is independent of the choice of D', and that $f \in \mathcal{F}$ satisfies the compatibility condition for D if and only if, for all differential operators $\psi : J_m(F) \rightarrow H$ for which the sequences (2) are exact, we have $D''f = 0$, where $D'' = \psi \circ j_m$.

We now state our main theorem from which we deduce the result stated in the introduction.

THEOREM 2. — *Let* $\varphi : J_k(E) \to F$ *be a regular differential operator from* E *to* F. *Assume that the maps* $\pi_m : R_{m+r} \to R_m$ *have constant rank, for all* $m \geq k$, $r \geq 0$. *Then there exist differential operators* $\varphi_1 : J_{k+m_0}(E) \to F_1$, $\psi : J_{m_0+l_0}(F) \to F_1$ *such that, if we set* $D = \varphi \circ j_k$, $D_1 = \varphi_1 \circ j_{k+m_0}$, $P = \psi \circ j_{m_0+l_0}$:

(i) $D_1 = P \circ D$ *and* φ_1 *is formally integrable and has the same solutions and formal solutions as* φ;

(ii) *For any differential operator* $\varphi' : J_l(F) \to G$ *of order* l, *for which the sequences* (1) *are exact for all* $r \geq 0$, *and any differential operator* $\varphi'_1 : J_q(F_1) \to G_1$ *of order* q, *for which the sequences*

$$(3) \qquad J_{k+m_0+q+s}(E) \xrightarrow{p_{q+s}(\varphi_1)} J_{q+s}(F_1) \xrightarrow{p_s(\varphi'_1)} J_s(G_1)$$

are exact for all $s \geq 0$, *there exists a differential operator* η *from* G *to* $J_m(G_1)$ *satisfying the following conditions :*

a. If $Q : \mathcal{G} \to \mathcal{J}_m(\mathcal{G}_1)$ *denotes the map induced by* η, *the diagram*

$$(4) \qquad \begin{array}{ccccc} \mathcal{E} & \xrightarrow{D} & \mathcal{F} & \xrightarrow{D'} & \mathcal{G} \\ \big\downarrow & & \big\downarrow{\scriptstyle P} & & \big\downarrow{\scriptstyle Q} \\ \mathcal{E} & \xrightarrow{D_1} & \mathcal{F}_1 & \xrightarrow{j_m \circ D'_1} & \mathcal{J}_m(\mathcal{G}_1) \end{array}$$

commutes. Hence, if $f \in \mathcal{F}$ *satisfies* $D'f = 0$, *then* $D'_1(Pf) = 0$.

b. For $f \in \mathcal{F}$ *satisfying* $D'f = 0$, *the space of solutions* $e \in \mathcal{E}$ *of the inhomogeneous differential equation* $De = f$ *coincides with the space of solutions of the equation* $D_1 e = Pf$. *More precisely,* P *induces an isomorphism from the cohomology of the complex*

$$(5) \qquad \mathcal{E} \xrightarrow{D} \mathcal{F} \xrightarrow{D'} \mathcal{G}$$

to the cohomology of the complex

$$(6) \qquad \mathcal{E} \xrightarrow{D_1} \mathcal{F}_1 \xrightarrow{D'_1} \mathcal{G}_1.$$

Hence, if $f_1 \in \mathcal{F}_1$ *satisfies* $D'_1 f_1 = 0$, *there exists* $f \in \mathcal{F}$ *with* $Pf = f_1$ *and* $D'f = 0$.

Proof. — (i) By Theorem 1 of [2], there exist integers $m_0 \geq 0$, $l_0 \geq 0$ such that the equation $R^{(l_0)}_{k+m_0} \subset J_{k+m_0}(E)$ is formally integrable and has the same formal solutions as R_k, and $R^{(l_0)}_{k+m_0+r}$ is the r-th prolongation of $R^{(l_0)}_{k+m_0}$. Let $F_1 = J_{k+m_0}(E)/R^{(l_0)}_{k+m_0}$ and let $\varphi_1 : J_{k+m_0}(E) \to F_1$ be the natural projection. Consider the commutative exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{k+m_0+l_0} & \longrightarrow & J_{k+m_0+l_0}(E) & \xrightarrow{p_{m_0+l_0}(\varphi)} & J_{m_0+l_0}(F) \\ & & \big\downarrow{\scriptstyle \pi_{k+m_0}} & & \big\downarrow{\scriptstyle \pi_{k+m_0}} & & \big\downarrow{\scriptstyle \psi} \\ 0 & \longrightarrow & R^{(l_0)}_{k+m_0} & \longrightarrow & J_{k+m_0}(E) & \xrightarrow{\varphi_1} & F_1 \end{array}$$

Since $p_{m_0+l_0}(\varphi)$ has constant rank, there exists a morphism of vector bundles $\psi : J_{m_0+l_0}(F) \to F_1$ such that $\psi \circ p_{m_0+l_0}(\varphi) = \varphi_1 \circ \pi_{k+m_0}$. Clearly we have $D_1 = P \circ D$.

(ii) Let $m = \max(m_0 + l_0 + q - l, o)$. Consider the commutative exact diagram

$$\begin{tikzcd}
J_{k+m_0+l_0+q+m}(E) \arrow[r, "p_{m_0+l_0+q+m}(\varphi)"] \arrow[d, "\pi_{k+m_0+q+m}"] & J_{m_0+l_0+q+m}(F) \arrow[r, "p_{m_0+l_0+q+m-l}(\varphi')"] \arrow[d, "p_{q+m}(\psi)"] & J_{m_0+l_0+q+m-l}(G) \arrow[d, dotted, "\gamma_1"] \\
J_{k+m_0+q+m}(E) \arrow[r, "p_{q+m}(\varphi_1)"] & J_{q+m}(F_1) \arrow[r, "p_m(\varphi'_1)"] & J_m(G_1)
\end{tikzcd}$$

Since $p_{m_0+l_0+q+m-l}(\varphi')$ has constant rank, there exists a morphism of vector bundles $\gamma_1 : J_{m_0+l_0+q+m-l}(G) \to J_m(G_1)$ such that

$$\gamma_1 \circ p_{m_0+l_0+q+m-l}(\varphi') = p_m(\varphi'_1) \circ p_{q+m}(\psi).$$

(8)

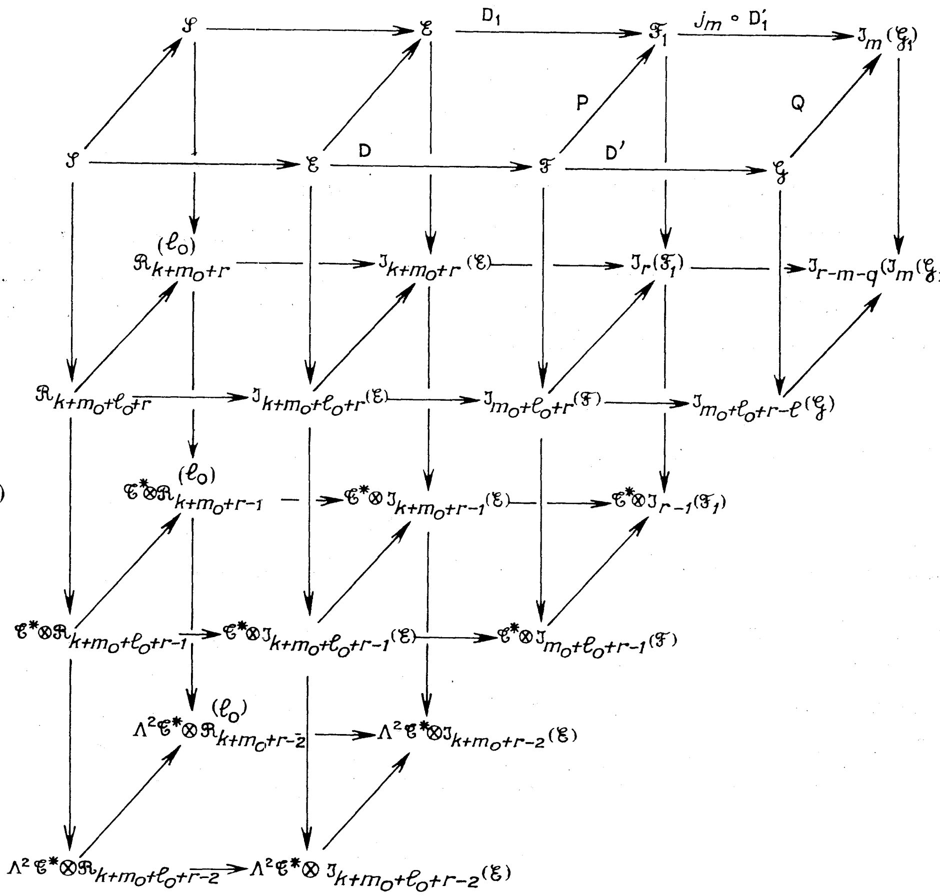

Fig. 1.

Clearly diagram (4) commutes. We note that the sequences

$$(7) \qquad 0 \longrightarrow R^{(l_0)}_{k+m_0+s} \longrightarrow J_{k+m_0+s}(E) \xrightarrow{p_s(\varphi_1)} J_s(F_1) \xrightarrow{p_{s-m-q}(p_m(\varphi'_1))} J_{s-m-q}(J_m(G_1))$$

are exact for all $s \geqq m+q$ and that the cohomology of the complex

$$\mathcal{E} \xrightarrow{D_1} \mathcal{F}_1 \xrightarrow{j_m \circ D'_1} \mathcal{J}_m(\mathcal{G}_1)$$

is the same as the cohomology of the complex (6).

If $f \in \mathcal{F}$ satisfies the equation $D'f = 0$, then $j_m(D'_1 Pf) = 0$ and so $D'_1 Pf = 0$. It follows that P determines a linear map $\tilde{P}$ from the cohomology of the complex (5) to the cohomology of the complex (6). We proceed to show that this map is an isomorphism.

We consider the commutative three-dimensional diagram (8), where r is an integer such that $r \geqq m+q+2$, and where $\mathcal{S}$ denotes the sheaf of germs of solutions $e \in \mathcal{E}$ of the differential operators φ or φ_1. By Proposition 8 of [2], we choose r such that the commutative diagram

$$\begin{array}{ccccc}
\mathcal{R}^{(l_0)}_{k+m_0+l_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+l_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+l_0+r-2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_{k+m_0+l_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}_{k+m_0+l_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{k+m_0+l_0+r-2}
\end{array}$$

establishes an isomorphism from the cohomology of the top row to the cohomology of the bottom row. Since the equation $R^{(l_0)}_{k+m_0}$ is formally integrable and the sequences (3) are exact for all $s \geqq 0$, by Theorem 4.3 of [1], we have $H^{k+m_0+q+s,2}(g^{(l_0)}_{k+m_0}) = 0$ for all $s \geqq 0$; hence because $r \geqq q+1$, by the commutativity of diagrams (33) of [2] corresponding to the equation $R^{(l_0)}_{k+m_0}$, the diagram

$$\begin{array}{ccccc}
\mathcal{R}^{(l_0)}_{k+m_0+l_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+l_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+l_0+r-2} \\
\downarrow{\scriptstyle \pi_{k+m_0+r}} & & \downarrow{\scriptstyle \pi_{k+m_0+r-1}} & & \downarrow{\scriptstyle \pi_{k+m_0+r-2}} \\
\mathcal{R}^{(l_0)}_{k+m_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+r-2}
\end{array}$$

establishes an isomorphism from the cohomology of the top row to the cohomology of the bottom row. Therefore the diagram

$$\begin{array}{ccccc}
\mathcal{R}_{k+m_0+l_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}_{k+m_0+l_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}_{k+m_0+l_0+r-2} \\
\downarrow{\scriptstyle \pi_{k+m_0+r}} & & \downarrow{\scriptstyle \pi_{k+m_0+r-1}} & & \downarrow{\scriptstyle \pi_{k+m_0+r-2}} \\
\mathcal{R}^{(l_0)}_{k+m_0+r} & \xrightarrow{D} & \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+r-1} & \xrightarrow{D} & \Lambda^2 \mathcal{T}^* \otimes \mathcal{R}^{(l_0)}_{k+m_0+r-2}
\end{array}$$

establishes a similar isomorphism, which we denote by $\tilde{\pi}$. The columns of diagram (8) are naive Spencer sequences and all the rows of this diagram are exact except possibly for the top two rows.

In order to prove that $\tilde{P}$ is bijective, we need the following two lemmas.

LEMMA 1. — *Let* $v \in J_{k+m_0+r}(E)$, $w \in J_{m_0+l_0+r}(F)$. *There exists* $u \in J_{k+m_0+l_0+r}(E)$ *satisfying*

$$p_{m_0+l_0+r}(\varphi)\, u = w$$

and

$$\pi_{k+m_0+r}\, u = v$$

if and only if

$$p_r(\varphi_1)\, v = p_r(\psi)\, w$$

and

$$p_{m_0+l_0+r-l}(\varphi')\, w = 0.$$

Proof. — The conditions on v and w for the existence of u are clearly necessary. Conversely, if they are satisfied, using the exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R_{k+m_0+l_0+r} \arrow[r] \arrow[d, "\pi_{k+m_0+r}"] & J_{k+m_0+l_0+r}(E) \arrow[r, "p_{m_0+l_0+r}(\varphi)"] \arrow[d, "\pi_{k+m_0+r}"] & J_{m_0+l_0+r}(F) \arrow[r, "p_{m_0+l_0+r-l}(\varphi')"] \arrow[d, "p_r(\psi)"] & J_{m_0+l_0+r-l}(G) \\
0 \arrow[r] & R^{(l_0)}_{k+m_0+r} \arrow[r] \arrow[d] & J_{k+m_0+r}(E) \arrow[r, "p_r(\varphi_1)"] & J_r(F_1) & \\
& 0 & & &
\end{tikzcd}$$

it is easily seen that the desired element u of $J_{k+m_0+l_0+r}(E)$ exists.

LEMMA 2. — *Let* $v \in \mathcal{C}^* \otimes \mathcal{J}_{k+m_0+l_0+r-1}(\mathcal{E})$, $w \in \mathcal{J}_{k+m_0+r}(\mathcal{E})$. *There exists* $u \in \mathcal{J}_{k+m_0+l_0+r}(\mathcal{E})$ *satisfying*

$$D u = v$$

and

$$\pi_{k+m_0+r}\, u = w$$

if and only if

$$D w = \pi_{k+m_0+r-1}\, v$$

and

$$D v = 0.$$

Proof. — The conditions on v and w for the existence of u are clearly necessary. Conversely, if they are satisfied, using the exact and commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r, "j_{k+m_0+l_0+r}"] \arrow[d] & \mathcal{J}_{k+m_0+l_0+r}(\mathcal{E}) \arrow[r, "D"] \arrow[d, "\pi_{k+m_0+r}"] & \mathcal{C}^* \otimes \mathcal{J}_{k+m_0+l_0+r-1}(\mathcal{E}) \arrow[r, "D"] \arrow[d, "\pi_{k+m_0+r-1}"] & \Lambda^2 \mathcal{C}^* \otimes \mathcal{J}_{k+m_0+l_0+r-2}(\mathcal{E}) \arrow[d, "\pi_{k+m_0+r-2}"] \\
0 \arrow[r] & \mathcal{E} \arrow[r, "j_{k+m_0+r}"] & \mathcal{J}_{k+m_0+r}(\mathcal{E}) \arrow[r, "D"] & \mathcal{C}^* \otimes \mathcal{J}_{k+m_0+r-1}(\mathcal{E}) \arrow[r, "D"] & \Lambda^2 \mathcal{C}^* \otimes \mathcal{J}_{k+m_0+r-2}(\mathcal{E})
\end{tikzcd}$$

it is easily seen that the desired element u of $\mathcal{J}_{k+m_0+l_0+r}(\mathcal{E})$ exists.

We now return to the proof of Theorem 2 and show first that $\tilde{P}$ is injective. Let f be an element of $\mathcal{F}$ with $D'f = 0$; assume that there exists $e \in \mathcal{E}$ with $D_1 e = Pf$. We shall show the existence of an element $e' \in \mathcal{E}$ satisfying the equation $De' = f$. We have

$$p_r(\varphi_1) j_{k+m_0+r}(e) = p_r(\psi) j_{m_0+l_0+r}(f).$$

Hence since $D'f = 0$, by Lemma 1 there exists $u \in \mathcal{J}_{k+m_0+l_0+r}(\mathcal{E})$ satisfying

$$p_{m_0+l_0+r}(\varphi) u = j_{m_0+l_0+r}(f)$$

and

$$\pi_{k+m_0+r} u = j_{k+m_0+r}(e).$$

Then it is easily seen that Du is an element of $\mathcal{C}^\star \otimes \mathcal{R}_{k+m_0+l_0+r-1}$. Furthermore $\pi_{k+m_0+r-1} Du = 0$. Hence by the isomorphism $\tilde{\pi}$, there exists $u' \in \mathcal{R}_{k+m_0+l_0+r}$ with $Du' = u$. Then the element $u - u'$ of $\mathcal{J}_{k+m_0+l_0+r}(\mathcal{E})$ satisfies

$$p_{m_0+l_0+r}(\varphi)\,(u - u') = j_{m_0+l_0+r}(f)$$

and

$$D(u - u') = 0.$$

This last equation implies that $u - u' = j_{k+m_0+l_0+r}(e')$, for some $e' \in \mathcal{E}$. Hence $De' = f$ and our assertion is established.

It remains to prove that $\tilde{P}$ is surjective. Let $f_1 \in \mathcal{F}_1$ satisfy $D'_1 f_1 = 0$. Then $p_{r-m-q}(\varphi'_1) j_r(f_1) = 0$ and by virtue of the exactness of sequence (7) with $s = r$, there exists $u \in \mathcal{J}_{k+m_0+r}(\mathcal{E})$ with $p_r(\varphi_1) u = j_r(f_1)$. It is easily seen that Du belongs to $\mathcal{C}^\star \otimes \mathcal{R}^{(l_0)}_{k+m_0+r-1}$. By the isomorphism $\tilde{\pi}$, there exists $v \in \mathcal{C}^\star \otimes \mathcal{R}_{k+m_0+l_0+r-1}$ satisfying

$$Dv = 0 \tag{9}$$

and

$$\pi_{k+m_0+r-1} v = D(u + u') \tag{10}$$

for some $u' \in \mathcal{R}^{(l_0)}_{k+m_0+r}$. Then $p_r(\varphi_1)(u + u') = j_r(f_1)$. By Lemma 2 and equations (9) and (10), there exists $\tilde{u} \in \mathcal{J}_{k+m_0+l_0+r}(\mathcal{E})$ with $D\tilde{u} = v$ and $\pi_{k+m_0+r}\tilde{u} = u + u'$. Then $Dp_{m_0+l_0+r}(\varphi)\tilde{u} = 0$. Hence there exists $f \in \mathcal{F}$, with $p_{m_0+l_0+r}(\varphi)\tilde{u} = j_{m_0+l_0+r}(f)$. On the other hand f satisfies

$$\begin{aligned} j_r(Pf) &= p_r(\psi) j_{m_0+l_0+r}(f) = p_r(\psi) \,.\, p_{m_0+l_0+r}(\varphi)\tilde{u} \\ &= p_r(\varphi_1) \,.\, \pi_{k+m_0+r}\tilde{u} = p_r(\varphi_1)\,(u + u') = j_r(f_1). \end{aligned}$$

Hence $Pf = f_1$. Moreover $D'f = 0$ and so $\tilde{P}$ is also surjective.

Finally, the fact that $\tilde{P}$ is injective and (i) imply that, if $f \in \mathcal{F}$ satisfies $D'f = 0$, the solutions of the inhomogeneous equation $De = f$ coincide with the solutions of the equation $D_1 e = Pf$, completing the proof of the theorem.

Theorem 1 can be generalized in a straightforward way to a statement about resolutions of arbitrary length of the differential operator $D : \mathcal{E} \to \mathcal{F}$ (*see* § 4 and 5 of [2]); the reader will find no difficulties in supplying the exact statement and adapting the proof of Theorem 2.

2. Elliptic and analytic equations. — Let $\varphi : J_k(E) \to F$ be a regular *analytic* differential operator of order k from E to F. Assume that the maps $\pi_m : R_{m+r} \to R_m$ have constant rank for all $m \geq k$, $r \geq 0$.

From Theorem 2 and Theorem 7.1 of [1], or from Corollary 4 of [2], we deduce

Theorem 3. — *For any differential operator* $\varphi' : J_l(F) \to G$ *of order* l *for which the sequences* (1) *are exact for all* $r \geq 0$, *the sequence*

$$\mathcal{E}_\omega \xrightarrow{D} \mathcal{F}_\omega \xrightarrow{D'} \mathcal{G}$$

is exact, where $D = \varphi \circ j_k$, $D' = \varphi' \circ j_l$.

We next consider analytic elliptic equations (for the definition of elliptic differential equations, *see* Definition 6.1 of [1]).

Theorem 4. — *If the equation* R_k *is elliptic, then for any differential operator* $\varphi' : J_l(F) \to G$ *of order* l *for which the sequences* (1) *are exact for all* $r \geq 0$, *the sequence*

$$\mathcal{E} \xrightarrow{D} \mathcal{F} \xrightarrow{D'} \mathcal{G}$$

is exact, where $D = \varphi \circ j_k$, $D' = \varphi' \circ j_l$.

Proof. — If R_k is elliptic, then so is R_{k+m_0} by Proposition 6.2 of [1]. Since $R^{(l_0)}_{k+m_0} \subset R_{k+m_0}$ the equation $R^{(l_0)}_{k+m_0}$ is elliptic. Hence by Theorem 2, it is sufficient to prove the theorem when R_k is formally integrable and involutive. By Corollary 3 of [2], the theorem follows from the following :

Proposition 1 (*see* D. C. Spencer [4]). — *If* $R_k \subset J_k(E)$ *is a formally integrable involutive elliptic analytic equation of order* k *on* E, *then the sophisticated Spencer sequence of* R_k

$$0 \longrightarrow \mathcal{S} \xrightarrow{j_k} \mathcal{C}^0 \xrightarrow{D^0} \mathcal{C}^1 \xrightarrow{D^1} \mathcal{C}^2 \xrightarrow{D^2} \ldots \longrightarrow \mathcal{C}^n \longrightarrow 0$$

is exact.

Proof. — We give an argument due to Spencer which follows the same lines as the original proof of H. Cartan of the $\bar{\partial}$-Poincaré lemma. Choose locally an analytic metric on X and analytic metrics on the vector bundles C^j, $0 \leq j \leq n$. Let $D^{j*} : \mathcal{C}^{j+1} \to C^j$ be the formal adjoint of D^j. Then by a theorem of Quillen (*see* Proposition 6.5 of [1]), the Laplacian $\square^j = D^{j-1} D^{j-1*} + D^{j*} D^j$ is a strongly elliptic determined operator from C^j to C^j and so $\square^j : \mathcal{C}^j \to \mathcal{C}^j$ is surjective. Hence, given a germ $u \in \mathcal{C}^j$

satisfying $D^j u = 0$, choose $\varphi \in \mathcal{C}^j$ with $\square^j \varphi = u$. Then $u - D^{j-1} D^{j-1*} \varphi$ belongs to the kernels of D^j and D^{j-1*} and therefore also to the kernel of $\square^j$. Since R_k is analytic, the differential operator $\square^j$ is strongly elliptic and analytic and so it follows, by a standard theorem on analytic elliptic equations, that $u - D^{j-1} D^{j-1*} \varphi$ is an analytic germ. By Theorem 7.2 of [1], there exists $\psi \in \mathcal{C}^{j-1}$ with $D^{j-1} \psi = u - D^{j-1*} \varphi$ and so $u = D^{j-1} (\psi + D^{j-1*} \varphi)$.

If the sophisticated Spencer sequence of an arbitary elliptic equation $R_k \subset J_k(E)$, which is formally integrable and involutive, is exact, then Theorem 4 holds when the differential operator φ is not necessarily analytic.

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