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F. D. VELDKAMP

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# THE CENTER OF THE UNIVERSAL ENVELOPING ALGEBRA OF A LIE ALGEBRA IN CHARACTERISTIC $p$

By F. D. VELDKAMP

INTRODUCTION. — This paper is divided into two somewhat different parts. In part I we consider the center $\mathfrak{Z}$ of the universal enveloping algebra $\mathfrak{U}$ of a Lie algebra $\mathfrak{g}$, which is the Lie algebra of a semisimple algebraic group G over a field of characteristic $p > 0$. Following H. Zassenhaus [24] we introduce a certain subalgebra $\mathcal{O}$ of $\mathfrak{Z}$ which has the structure of a polynomial algebra in $n$ variables, $n = \dim \mathfrak{g}$, and over which $\mathfrak{U}$ is a free module (*cf.* § 1). If $p > h$, the Coxeter number of G, the structure of $\mathfrak{Z}$ over $\mathcal{O}$ can be determined. Let $T_1, \ldots, T_l$ denote algebraically independent generators of the invariants in $\mathfrak{U}$ under the adjoint action of G. Then $\mathfrak{Z} = \mathcal{O}[T_1, \ldots, T_l]$, and the products $T_1^{j_1} \cdot \ldots \cdot T_l^{j_l}$ with $0 \leqq j_i < p$ form a basis of $\mathfrak{Z}$ as an $\mathcal{O}$-module [theorem (3.1)]. In the proof of this result we need certain properties of regular elements in the Lie algebra $\mathfrak{g}$, which were known for complex semisimple Lie algebras from Kostant's works [12], [13]. They form the Lie algebra counterpart of properties of regular elements in algebraic groups such as Steinberg has dealt with in his paper [21]. We recall that an element of $\mathfrak{g}$ is called regular, if its centralizer has minimal possible dimension. The main results of part II can be described as follows. The regular elements form a Zariski open set in $\mathfrak{g}$ whose complement has pure codimension 3 [theorem (4.12)]. Let $J_1, \ldots, J_l$ denote algebraically independent homogeneous generators of the algebra of AdG-invariant polynomials on $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is regular if and only if the functional matrix

$$\left(\frac{\partial J_i}{\partial \xi_j}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}$$

has rank $l$ in X [theorem (7.1)]. Furthermore, we prove some results in part II which are not needed in part I but which are of interest for their own sake, e. g., on Ad G-orbits in $\mathfrak{g}$. Though the results of part II are being applied in part I only for $p > h$, they can be proved for certain smaller $p$. Therefore, we indicate at the beginning of each section the assumptions we make on $p$. Thus, the main results are given in their greatest possible generality, whereas some less important results hold for smaller $p$ than admitted here.

In this paper " group " or " algebraic group " will always mean : affine algebraic group. An algebraic group over a field $k$ is considered as the group of points rational over the algebraic closure of $k$. In this and other respects we conform to the terminology and notations of Borel's book [1].

The author is indebted to J. E. Humphreys, M. van der Put and T. A. Springer for critical remarks and stimulating discussions.

## I. — The center of the universal enveloping algebra

1. The universal enveloping algebra. — Let G be a split semi-simple algebraic group over a perfect field $k$ of characteristic $p > 0$, $p$ not dividing the order of the Weyl group of G, $\mathfrak{g}$ its Lie algebra, and $\Sigma$ its rootsystem. By [3], théorème 2.13, and [5], G is isomorphic over $k$ to a Chevalley group, so we may assume that G is already defined and split over $\mathbf{F}_p$. Let H be a maximal torus of G, defined and split over $\mathbf{F}_p$, and $\mathfrak{h}$ the Lie algebra of H. Since the order of the fundamental group of G is invertible in $\mathbf{F}_p$ (*cf.* [22]), $\mathfrak{g}$ has a basis consisting of $H_1, \ldots, H_l$ in $\mathfrak{h}$ and rootvectors $X_\alpha$, $\alpha \in \Sigma$, derived from a Chevalley basis in characteristic 0. We consider $\mathfrak{g}$ as a Lie algebra over $k$, i. e., we only deal with points of $\mathfrak{g}$ *rational* over $k$.

Let $\mathfrak{U}$ denote the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{Z}$ the center of $\mathfrak{U}$. Following Zassenhaus [24], we introduce the subspace $\mathcal{L}$ of $\mathfrak{U}$ spanned by all $p^i$-th powers (in $\mathfrak{U}$) of elements of $\mathfrak{g}$, $i = 0, 1, 2, \ldots$, and $\mathcal{M} = \mathcal{L} \cap \mathfrak{Z}$. Let $\mathcal{O}$ be the subalgebra of $\mathfrak{Z}$ generated by 1 and $\mathcal{M}$. For $x \in \mathfrak{U}$ we define ad $x : \mathfrak{U} \to \mathfrak{U}$ by

$$\operatorname{ad} x\,(y) = xy - yx \qquad \text{for} \quad y \in \mathfrak{U}.$$

From the binomial formula it follows that $\mathrm{ad}\,(x^p) = (\mathrm{ad}\,x)^p$ (*cf.* [24], formula (1) on p. 4). $\mathfrak{g}$ has a structure of $[p]$-Lie algebra (or restricted Lie algebra in the terminology of Jacobson [10]) such that

$$H_i^{[p]} = H_i, \qquad X_\alpha^{[p]} = 0.$$

For $X \in \mathfrak{g}$, $\mathrm{ad}\,(X^p) = (\mathrm{ad}\,X)^p = \mathrm{ad}\,X^{[p]}$. It follows that the elements $H_i^p - H_i$ and $X_\alpha^p$ belong to $\mathfrak{M}$. Hence $\dim \mathfrak{L}/\mathfrak{M} \leq n$. On the other hand,

$$\mathfrak{g} \cap \mathfrak{M} = \mathfrak{g} \cap \mathfrak{Z} = 0,$$

hence $\dim \mathfrak{L}/\mathfrak{M} = n$ and $\mathfrak{M}$ has a basis over $k$ consisting of all monomials of positive total degree in the $H_i^p - H_i$ and $X_\alpha^p$. From the Poincaré- Birkhoff-Witt theorem or from [24], § 1, one deduces that $\mathcal{O} = k\,[H_i^p - H_i, X_\alpha^p \mid 1 \leq i \leq l, \alpha \in \Sigma]$, the $H_i^p - H_i$ and $X_\alpha^p$ being algebraically independent over $k$, and that $\mathfrak{U}$ is a free $\mathcal{O}$-module of rank $p^n$ having as a basis the elements

$$\prod_{\alpha \in \Sigma^+} X_{-\alpha}^{r_\alpha} \prod_{i=1}^{l} H_i^{s_i} \prod_{\alpha \in \Sigma^+} X_\alpha^{t_\alpha} \qquad \text{with} \quad 0 \leq r_\alpha,\ s_i,\ t_\alpha < p,$$

where $\Sigma^+$ denotes the set of positive roots for some ordering we have chosen in $\Sigma$.

Let D denote the quotient division ring of $\mathfrak{U}$ (*cf.* [10] or [24]), $K \subseteq D$ the quotient field of $\mathfrak{Z}$ and $Q \subseteq K$ that of $\mathcal{O}$. D being central simple over K, the dimension $[D : K]$ is a square dividing $[D : Q] = [\mathfrak{U} : \mathcal{O}] = p^n$. Hence

$$(1.1) \qquad [D : K] = p^{2m} \quad \text{for some } m, \qquad 0 \leq 2\,m \leq n.$$

The following result was shown by Zassenhaus ([24], Theorem 6).

(1.2) *The dimension of any absolutely irreducible representation of* $\mathfrak{g}$ *is at most* $p^m$.

2. Invariants. — (i) We keep the notations and conventions of the preceding section. Let $\mathfrak{S}$ denote the symmetric algebra on $\mathfrak{g}$ and $\mathfrak{T}$ the polynomial algebra on $\mathfrak{g}$, identified with $\mathfrak{S}(\mathfrak{g}^*)$, the symmetric algebra on the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. $\mathcal{H}$ will be the universal enveloping algebra of $\mathfrak{h}$, considered as a subalgebra of $\mathfrak{U}$, and identified with the symmetric algebra $\mathfrak{S}(\mathfrak{h})$. On $\mathfrak{U}$ we consider the natural filtration by the subspaces $\mathfrak{U}^{(i)}$ spanned by the products of at most $i$ elements of $\mathfrak{g}$. For $x \in \mathfrak{U}^{(i)}$, $x \notin \mathfrak{U}^{(i-1)}$, we define the filtration degree $d(x) = i$. The

corresponding graded algebra is

$$\tilde{\mathfrak{U}} = \bigoplus_{i=0}^{\infty} \tilde{\mathfrak{U}}^{(i)}, \qquad \tilde{\mathfrak{U}}^{(i)} = \mathfrak{U}^{(i)}/\mathfrak{U}^{(i-1)}.$$

The mapping $\tilde{}$ of $\mathfrak{U}$ onto $\tilde{\mathfrak{U}}$ is defined by : $\tilde{x} = x \bmod \mathfrak{U}^{(i-1)}$ if $d(x) = i$. $\tilde{\mathfrak{U}}$ is isomorphic to $\mathfrak{S}$, and we shall sometimes identify the two.

The adjoint action of G on $\mathfrak{g}$ induces actions of G on $\mathfrak{U}$, $\tilde{\mathfrak{U}}$, $\mathfrak{S}$ and $\mathfrak{P}$; the algebras of invariants will be denoted by $\mathfrak{U}^G$, etc. Notice that the isomorphism of $\tilde{\mathfrak{U}}$ onto $\mathfrak{S}$ is a G-module isomorphism. The adjoint action of $\mathfrak{g}$ on itself is extended to $\mathfrak{U}$; notice that $\mathfrak{U}^{\mathfrak{g}} = \mathfrak{Z}$. The action of the Weyl group W of G on $\mathfrak{h}$ can be extended to $\mathcal{H}$ and the polynomial algebra $\mathfrak{P}(\mathfrak{h})$, giving rise to algebras of invariants $\mathcal{H}^W$ and $\mathfrak{P}(\mathfrak{h})^W$.

(ii) This section is devoted to a description of the algebras of invariants introduced above. It is known that $\mathfrak{P}(\mathfrak{h})^W$ is generated by $l$ algebraically independent homogeneous polynomials $I_1, I_2, \ldots, I_l$ of degrees $d_1 \leq d_2 \leq \ldots \leq d_l$, respectively; these degrees do not depend on the particular choice of the $I_j$'s. In $\mathcal{H}^W$ the situation is similar; let us call the algebraically independent homogeneous generators $S_1, \ldots, S_l$ (of degrees $d_1, \ldots, d_l$). *The* $I_j$ *and* $S_j$ *are chosen once for all in this paper.*

(iii) $\mathfrak{P}^G$ is isomorphic to $\mathfrak{P}(\mathfrak{h})^W$. Indeed, restriction to $\mathfrak{h}$ of elements of $\mathfrak{P}^G$ definies an isomorphism between the two algebras. This has been proved in [18], 3.17′ on p. 33, for G an adjoint group. But the assumption that the characteristic $p$ of $k$ does not divide $|W|$ has as a consequence that $\mathfrak{g}$ is isomorphic with the Lie algebra of the adjoint group $\bar{G}$ of G, the adjoint action of G on $\mathfrak{g}$ being given by that of $\bar{G}$. We shall denote by $J_i$ the element of $\mathfrak{P}^G$ whose restriction to $\mathfrak{h}$ is $I_i$.

(iv) Similar relations exist between $\mathfrak{S}^G$ and $\mathcal{H}^W$. For consider the Killing form on $\mathfrak{g}$ :

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}(XY)) \qquad \text{for} \quad X, Y \in \mathfrak{g}.$$

This is a nondegenerate bilinear form on $\mathfrak{g}$ and so is its restriction to $\mathfrak{h}$ (*cf.* [18], 5.3 on p. 18; here the case $A_r$ for $p \nmid r+1$, which is proved quite easily, can be added). B permits us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{h}^*$ with $\mathfrak{h}$; this yields an identification of $\mathfrak{P} = \mathfrak{S}(\mathfrak{g}^*)$ with $\mathfrak{S}$ and of $\mathfrak{P}(\mathfrak{h})$ with $\mathcal{H}$. Since B is invariant under the adjoint action of G, this identification is a G-isomorphism (a W-isomorphism, respectively). Let us describe the isomorphism of $\mathfrak{S}^G$ onto $\mathcal{H}^W$ we get in this way. Restriction of an element of $\mathfrak{g}^*$ to $\mathfrak{h}$ corresponds to orthogonal projection relative to B of $\mathfrak{g}$ onto $\mathfrak{h}$. Since all rootvectors $X_\alpha$, $\alpha \in \Sigma$, are orthogonal to $\mathfrak{h}$, this orthogonal projection acts as follows :

$$H_i \mapsto H_i, \qquad X_\alpha \mapsto 0.$$

Hence restriction of an element of $\mathfrak{P}$ to $\mathfrak{h}$ is described in $\mathfrak{S}$ by the homomorphism

$$\tilde{\beta} : \ \mathfrak{S} \to \mathcal{H}$$

with

$$\tilde{\beta} : \prod_{\alpha \in \Sigma^+} X_{-\alpha}^{r_\alpha} \prod_{i=1}^{l} H_i^{s_i} \prod_{\alpha \in \Sigma^+} X_\alpha^{t_\alpha} \mapsto \prod_{i=1}^{l} H_i^{s_i} \quad \text{or } 0$$

according to whether all $r_\alpha$, $t_\alpha$ are 0 or not. $\tilde{\beta}$ definies the isomorphism of $\mathfrak{S}^G$ onto $\mathcal{H}^W$ mentioned above.

(v) The situation with $\mathfrak{U}^G$ is slightly more complicated. We consider the analog of $\tilde{\beta}$ above.

$$\beta : \ \mathfrak{U} \to \mathcal{H}$$

is the linear transformation with

$$\beta : \prod_{\alpha \in \Sigma^+} X_{-\alpha}^{r_\alpha} \prod_{i=1}^{l} H_i^{s_i} \prod_{\alpha \in \Sigma^+} X_\alpha^{t_\alpha} \mapsto \prod_{i=1} H_i^{s_i} \quad \text{or } 0$$

according to whether all $r_\alpha$, $t_\alpha$ are 0 or not. $\beta$ definies a homomorphism of $\mathfrak{U}^G$ into $\mathcal{H}$ ([16], p. 18-05). Let $\gamma$ be the automorphism of $\mathcal{H}$ defined by

$$\gamma(H) = H - \rho(H) \qquad \text{for} \quad H \in \mathfrak{h},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} d\alpha$. $\delta = \gamma \circ \beta$ is an isomorphism of $\mathfrak{U}^G$ onto $\mathcal{H}^W$. For char $(k) = 0$ this is well known (*cf.* [9] or [23]). For char $(k) = p > 0$ the same proof works provided $p > h = \max_{1 \leq i \leq l} d_i$, the Coxeter number of the group G. For convenience we shall sketch the proof.

(2.1) Proposition. — *The homomorphism $\delta$ defined above is an isomorphism of $\mathfrak{U}^G$ onto $\mathcal{H}^W$ provided $p > h$, the Coxeter number of* G.

*Proof.* — The injectivity of $\delta$ is proved as in characteristic 0. $\mathcal{H}^W$ is generated by elements of degree $\leq h$, viz. by $S_1, \ldots, S_l$, so to prove that $\delta$ is surjective it suffices to show that every $x \in \mathcal{H}^W$ with $d(x) \leq h$ can be " extended " to an element of $\mathfrak{U}^{(h)G}$ mapped on $x$ by $\delta$. This goes by induction on the degree $d(x)$. The case $d(x) = 0$ being clear, assume the proof is given for $d(x) < i$ for some $i$, $0 < i \leq h$, and let $x \in \mathcal{H}^W$ with $d(x) = i$. From (iv) it follows that there exists a $y \in \mathfrak{S}^{h(G)}$ with $\tilde{\beta}(y) = \gamma^{-1} x$. Since $p > h$, $\mathfrak{S}^{(h)}$ may be identified with the space $\mathfrak{S}\mathfrak{T}^{(h)}$ of symmetric tensors of degree $\leq h$ over $\mathfrak{g}$, by symmetrizing each element of $\mathfrak{S}^{(h)}$, and $\mathfrak{S}\mathfrak{T}^{(h)}$ can be projected onto $\mathfrak{U}^{(h)}$. This means that there exists a $y'$ in $\mathfrak{U}^{(h)}$ (corresponding to $y$) such that

$$\delta y' = x \bmod \mathcal{H}^{(i-1)W}.$$

Hence by induction hypothesis we can find a $z \in \mathfrak{U}^{(i-1)G}$ such that

$$\delta y' = x + \delta z,$$

which shows that $x$ is in the image of $\delta$.

Denote by $T_1, \ldots, T_l$ the elements of $\mathfrak{U}^G$ such that $\delta T_i = S_i$. From the above proof it is clear that $d(T_i) = d_i$, and that, if we identify $\tilde{\mathfrak{U}}$ with $\mathfrak{S}$, then the images $\tilde{T}_1, \ldots, \tilde{T}_l$ of the $T_i$'s in $\tilde{\mathfrak{U}}$ are homogeneous generators of $\mathfrak{S}^G$ mapped on the W-invariants $S_i$ by $\tilde{\beta} : \mathfrak{S}^G \to \mathcal{H}^W$.

*Remark.* — One might reasonably hope that (2.1) is valid for char $(k) = p$, $p \nmid |W|$. If that were the case, the result of the next section, theorem (3.1), would hold for the same characteristics.

3. The structure of $\mathfrak{Z}$. — We keep the notations and assumptions of the previous sections, with char $(k) = p > h$, $h$ being the Coxeter number of G. We propose to prove the following theorem.

(3.1) Theorem. — *For* char $(k) = p > h$ *we have*

$$\mathfrak{Z} = \mathcal{O}[T_1, \ldots, T_l],$$

*and* $\mathfrak{Z}$ *is a free* $\mathcal{O}$*-module with as a basis the set of elements* $T_1^{j_1} \cdot T_2^{j_2} \cdot \ldots \cdot T_l^{j_l}$, $0 \leq j_i < p$.

*Proof.* — The proof will be divided into a number of steps.

(i) We first show the elements $T_1^{j_1} \cdot T_2^{j_2} \cdot \ldots \cdot T_l^{j_l}$, $0 \leq j_i < p$, to be linearly independant over $\mathcal{O}$. We recall that $\mathcal{O} = k[H_i^p - H_i, X_\alpha^p \mid 1 \leq i \leq l, \alpha \in \Sigma]$. Assume a nontrivial relation to exist :

$$\sum_{0 \leq j_i < p} a_{j_1, \ldots, j_l} T_1^{j_1} \cdot \ldots \cdot T_l^{j_l} = 0,$$

where the $a_{j_1,\dots,j_l} \in \mathcal{O}$, not all 0. By the mapping $\tilde{}$ of $\mathfrak{U}$ onto $\tilde{\mathfrak{U}}$ we get a nontrivial relation

$$(3.2) \qquad \sum_{0 \leq j_i < p} b_{j_1,\dots,j_l} \tilde{T}_1^{j_1} \cdot \ldots \cdot \tilde{T}_l^{j_l} = 0,$$

where $b_{j_1,\dots,j_l} = \tilde{a}_{j_1,\dots,j_l}$ if $d(a_{j_1,\dots,j_l} T_1^{j_1} \cdot \ldots \cdot T_l^{j_l})$ is maximal among all degrees of the $a_{h_1,\dots,h_l} T_1^{h_1} \cdot \ldots \cdot T_l^{h_l}$, and $b_{j_1,\dots,j_l} = 0$ otherwise. Let $X_1, \ldots, X_n$ be the basis $H_1, \ldots, H_l, X_\alpha$ $(\alpha \in \Sigma)$. From the identification of $\tilde{\mathfrak{U}} = \mathfrak{S}$ with $\mathfrak{P}$ via the Killing from [see § 2, (iv)] and from Theorem (7.1) it follows that the functional matrix

$$\left(\frac{\partial \tilde{T}_i}{\partial \tilde{X}_j}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}$$

has rank $l$. This means that in $k(\tilde{X}_1, \ldots, \tilde{X}_n)$, the quotient field of $\tilde{\mathfrak{U}} = k[\tilde{X}_1, \ldots, \tilde{X}_n]$, the elements $\tilde{T}_1, \ldots, \tilde{T}_l$ are $p$-independent (*see*, e. g., [11], p. 180 ff.). But $\tilde{\mathcal{O}} = k[\tilde{X}_1^p, \ldots, \tilde{X}_n^p]$, hence relation (3.2), where the $b_{j_1,\dots,j_l} \in \tilde{\mathcal{O}}$, contradicts the $p$-independence.

(ii) Set $\mathfrak{Z}^* = \tilde{\mathcal{O}}[T_1, \ldots, T_l] \subseteq \mathfrak{Z}$. As shown in (i), $\mathfrak{Z}^*$ is a free $\mathcal{O}$-module of rank $p^l$. Let $K^*$, $Q \subseteq K^* \subseteq K$, be the quotient field of $\mathfrak{Z}^*$. The aim of this step is to show that $K^* = K$.

$\mathfrak{g}$ has an absolutely irreducible representation of degree $p^N$, $N = |\Sigma^+|$, the Steinberg representation ([19], [20]). In view of (1.2), we conclude that $m \geq N$, $m$ as in (1.1). Hence

$$[K : Q] = [D : Q]/[D : K] = p^{n-2m} \leq p^{n-2N} = p^l.$$

On the other hand,

$$[K^* : Q] = [\mathfrak{Z}^* : \mathcal{O}] = p^l,$$

hence $K^* = K$ and $m = N$.

*Remark.* — The result $m = N$ was also proved in [15] under weaker restrictions for $p$. If we use this result of [15], we do not need the existence of the Steinberg representation.

(iii) To complete the proof of the theorem it suffices to show that $\mathfrak{Z}^* = \mathfrak{Z}$. Now $\mathfrak{Z}^* \subseteq \mathfrak{Z}$ and $\mathfrak{U}^{(0)} = k$ is contained in $\mathfrak{Z}^*$, hence $\mathfrak{Z}^* = \mathfrak{Z}$ will follow by induction from the statement

(3.3) For every $z \in \mathfrak{Z}$, there exists $y \in \mathfrak{Z}^*$ such that $d(z - y) < d(z)$.

Here, we recall, $d$ is the filtration degree on $\mathfrak{U}$. $\mathfrak{Z}$ and $\mathfrak{Z}^*$ have a filtration induced by the filtration of $\mathfrak{U}$, and the corresponding graded algebras can be identified with the images $\tilde{\mathfrak{Z}}$ and $\tilde{\mathfrak{Z}}^*$ in $\tilde{\mathfrak{U}}$ under the mapping $\tilde{}$. (3.3) is equivalent with the statement

$$\tilde{\mathfrak{Z}}^* = \tilde{\mathfrak{Z}}. \tag{3.4}$$

$\tilde{\mathfrak{U}}$ is a polynomial ring, hence an integral domain, so $\tilde{\mathfrak{Z}}$ and $\tilde{\mathfrak{Z}}^*$ have a quotient field. In (ii) we have seen that $\mathfrak{Z}$ and $\mathfrak{Z}^*$ have the same quotient field, and from this one easily derives that $\tilde{\mathfrak{Z}}$ and $\tilde{\mathfrak{Z}}^*$ have the same quotient field. $\tilde{\mathfrak{U}} = k[\tilde{H}_i, \tilde{X}_\alpha]$ is obviously integral over $\tilde{\mathcal{O}} = k[\tilde{H}_i^p, \tilde{X}_\alpha^p]$, so, *a fortiori*, $\mathfrak{Z}$ is integral over $\mathfrak{Z}^*$. Therefore, to prove (3.4) it will suffise to show that $\tilde{\mathfrak{Z}}^*$ is integrally closed. This will be done in the final step of the proof.

(iv)

$$\begin{aligned}\mathfrak{Z}^* &= \mathcal{O}[T_1, \dots, T_l] \\ &= k[H_i^p - H_i, X_\alpha^p, T_i \mid 1 \leq i \leq l, \alpha \in \Sigma], \\ \tilde{\mathfrak{Z}}^* &= k[\tilde{H}_i^p, \tilde{X}_\alpha^p, \tilde{T}_i \mid 1 \leq i \leq l, \alpha \in \Sigma].\end{aligned}$$

Clearly there exists a surjective homomorphism

$$\varphi: \; k[Y_1, \dots, Y_n, Z_1, \dots, Z_l]/(Z_1^p - f_1, \dots, Z_l^p - f_l) \to \tilde{\mathfrak{Z}}^*$$

with

$$\begin{aligned}\varphi(Y_i) &= \tilde{H}_i^p \qquad (1 \leq i \leq l), \\ \varphi(Y_{l+i}) &= \tilde{X}_{\beta_i}^p \qquad (1 \leq i \leq n - l),\end{aligned}$$

where $\beta_1, \dots, \beta_{n-l}$ are the roots in $\Sigma$, in some order,

$$\varphi(Z_i) = \tilde{T}_i \qquad (1 \leq i \leq l).$$

Here $Y_1, \dots, Y_n, Z_1, \dots, Z_l$ denote algebraically independent variables over $k$, and the $f_i \in k[Y_1, \dots, Y_n]$ are such that

$$f_i(\tilde{H}_1, \dots, \tilde{H}_l, \tilde{X}_{\beta_1}, \dots, \tilde{X}_{\beta_{n-l}}) = \tilde{T}_i.$$

As we have noticed in step (i) of this proof, the elements $\tilde{T}_1, \dots, \tilde{T}_l$ are $p$-independent in the quotient field of $\tilde{\mathfrak{U}}$ over $k$. But that amounts to injectivity of $\varphi$, since $\tilde{H}_1, \dots, \tilde{H}_l, \tilde{X}_{\beta_1}, \dots, \tilde{X}_{\beta_{n-l}}$ are algebraically independent. Thus we see that

$$\mathfrak{Z}^* \cong k[Y_1, \dots, Y_n, Z_1, \dots, Z_l]/(Z_1^p - f_1, \dots, Z_l^p - f_l). \tag{3.5}$$

To show that $\tilde{\mathfrak{Z}}^*$ is integrally closed, it suffices to consider the case $k$ algebraically closed. This is quite likely to be known, but since we have no appropriate reference in the literature available, we shall prove it in the following lemma. We are indebted to M. van der Put for the proof given here.

(3.6) LEMMA. — *Let* A *be a commutative algebra without zero divisors over a field* $k$. *Let* K *denote the quotient field of* A, *and* $\bar{k}$ *the algebraic closure of* $k$. *Assume that* $K \otimes_k \bar{k}$ *is an integral domain and that* $A \otimes_k \bar{k}$ *is integrally closed. Then* A *is integrally closed.*

*Proof.* — Since the mapping $A \otimes_k \bar{k} \to K \otimes_k \bar{k}$ is injective, we may identify $A \otimes_k \bar{k}$ a with subring $A'$ of $K' = K \otimes_k \bar{k}$. As $K'$ has no zero divisors, it is contained in a quotient field of $A'$. Let $\{\omega_i \mid i \in I\}$ be a basis of $\bar{k}$ as a linear space over $k$, with $\omega_0 = 1$. Each $x \in A'$ can be written uniquely as $x = \sum_{i \in I} \omega_i x_i$, with $x_i \in A$, and similarly for $x \in K'$ with $x_i \in K$. Now assume $x \in K$ is integral over A. $x$, considered as an element of $K'$, is integral over $A'$, hence $x \in A'$. So $x = \sum_{i \in I} \omega_i x_i$, $x_i \in A$. But $x \in K$ implies $x_i = 0$ for $i \neq 0$, hence $x \in A$.

So, to complete the proof, assume $k$ algebraically closed. The injection

$$k[Y_1, \ldots, Y_n] \to k[Y_1, \ldots, Y_n, Z_1, \ldots, Z_l]/(Z_1^{p} - f_1, \ldots, Z_l^{p} - f_l)$$

defines a finite morphism $\pi$ from Spec $(\mathfrak{Z}^*)$ onto affine $n$-space $\mathbf{A}^n$. The singular points of Spec $(\mathfrak{Z}^*)$ are the points where the functional matrix

$$\left(\frac{\partial f_i}{\partial Y_j}\right)_{\substack{1 \leq i \leq \\ 1 \leq j \leq n}}$$

has rank $< l$. Since this functional matrix is the same as

$$\left(\frac{\partial J_i}{\partial \xi_j}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}},$$

if we identify $\tilde{\mathfrak{U}} = \mathfrak{S}$ with $\mathfrak{T}$ via the Killing form, it follows from proposition (7.1) that the singular points of Spec $(\tilde{\mathfrak{Z}}^*)$ are mapped by $\pi$ onto the set $\mathfrak{g} \setminus \mathfrak{r}$, where $\mathfrak{r}$ is the set of regular points of $\mathfrak{g}$ and where $\mathfrak{g}$ is identified with $\mathbf{A}^n$. By theorem (4.12), $\mathfrak{g} \setminus \mathfrak{r}$ is closed of pure codimension 3 in $\mathfrak{g}$. Since $\pi$ is a finite morphism, the set of singular points of Spec $(\tilde{\mathfrak{Z}}^*)$ has codimension 3 (*see*, e.g. the proof of Theorem 3 on p. 93-96 in [14]); thus Spec $(\tilde{\mathfrak{Z}}^*)$ has no singularities in codimension 2. By (3.5), Spec $(\tilde{\mathfrak{Z}}^*)$

is a complete intersection of hypersurfaces. By a wellknown criterion in algebraic geometry (*see*, e.g., [8], 5.8.6, p. 108), Spec $(\tilde{\mathfrak{Z}}^*)$ is a normal variety, i.e., $\tilde{\mathfrak{Z}}^*$ is integrally closed. Thus we have completed the proof of theorem (3.1).

## II. — Regular elements in Lie algebras

In this part of the paper, G will always be a connected semisimple algebraic group over an algebraically closed field $k$. About the characteristic of $k$ we shall have to make some assumptions, which will be somewhat weaker in sections 4 and 5 than in the last two sections, where we have to assume that char $(k) = 0$ or $p$ with $p$ not dividing the order of the Weyl group W of G.

H denotes a maximal torus in G, B a Borel subgroup containing H, $B^-$ the Borel subgroup opposite to B, U $(U^-)$ the unipotent radical of B $(B^-)$. The Lie algebras of these groups are denoted by the corresponding lower case german characters : $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}$, $\mathfrak{b}^-$, $\mathfrak{u}$, $\mathfrak{u}^-$. $\Sigma$ will be the root system of G, $\Sigma^+$ the set of positive roots for the ordering of $\Sigma$ corresponding to B, $\Delta = \{ \alpha_1, \ldots, \alpha_l \}$ the set of simple roots.

4. Regular elements. — *In this section we assume that the ground field $k$ has characteristic* 0 *or $p$ where $p$ is a good prime and $p \nmid r+1$* if G *has a component of type* $A_r$ (*cf.* [18], I.4.1 and 4.3, for good primes).

G acts on $\mathfrak{g}$ by the adjoint representation Ad. For $X \in \mathfrak{g}$ we define its centralizer in G by

$$Z_G(X) = Z(X) = \{ g \in G \mid \operatorname{Ad} g(X) = X \},$$

and its centralizer in $\mathfrak{g}$ by

$$\mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{z}(X) = \{ Y \in \mathfrak{g} \mid [Y, X] = 0 \}.$$

Under the assumptions on char $(k)$ we have made, $\mathfrak{z}(X)$ is the Lie algebra of Z(X) ([18], I.5.6).

If X is semisimple, $\mathfrak{z}(X)$ is reductive, that is, Z(X) is reductive. Since all results we derive for connected semisimple groups are also valid for connected reductive groups, as is easily seen, we can, without doing any harm, replace $\mathfrak{g}$ by $\mathfrak{z}(X)$ for semisimple X, as we shall sometimes do in proofs.

(4.1) LEMMA. — *For any* $X \in \mathfrak{g}$, $\dim \mathfrak{g}(X) \geqq l$.

*Proof.* — Let $X = X_s + X_n$ be the Jordan decomposition of X. $\mathfrak{g}' = \mathfrak{z}_{\mathfrak{g}}(X_s)$ is reductive of rank $l$. Now $\mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{z}_{\mathfrak{g}'}(X_n)$, and the latter has dimension $\geqq l$ by [17], Prop. (5.6).

(4.2) DEFINITION. — $X \in \mathfrak{g}$ is called *regular* if $Z_G(X)$ has dimension $l$.

Under the assumptions made on char$(k)$ this is equivalent to $\dim \mathfrak{z}_{\mathfrak{g}}(X) = l$.

(4.3) LEMMA. — *Let* $X = X_s + X_n$ *be the Jordan decomposition of* X. *Then* X *is regular in* $\mathfrak{g}$ *if and only of* $X_n$ *is regular in* $\mathfrak{z}_{\mathfrak{g}}(X_s)$.

*Proof.* — This is immediate from the fact that $\mathfrak{z}(X)$ equals the centralizer of $X_n$ in $\mathfrak{z}_{\mathfrak{g}}(X_s)$, and that the latter is reductive of rank $l$.

The following result, which is part of mathematical folklore, was proved for complex Lie groups in [13]; we give a proof valid in arbitrary characteristic. We recall that $\mathfrak{P}^G$ denotes the algebra of polynomials on $\mathfrak{g}$ which are invariant under the adjoint action of G.

(4.4) LEMMA. — *Assume* $X \in \mathfrak{h}$, $Y \in \mathfrak{u}$, $f \in \mathfrak{P}^G$. *Then* $f(X + Y) = f(X)$.

*Proof.* — Set $Y = \sum_{\alpha \in \Sigma^+} \gamma_\alpha X_\alpha$. We proceed by induction on the number of $\gamma_\alpha$'s which are $\neq 0$. Write every $\alpha \in \Sigma^+$ as a linear combination of simple roots,

$$\alpha = \sum_{i=1} n_i(\alpha)\, \alpha_i, \qquad n_i(\alpha) \in \mathbf{Z}, \qquad n_i(\alpha) \geqq 0.$$

Let $\alpha \in \Sigma^+$ and a simple root $\alpha_i$ be chosen such that $\gamma_\alpha \neq 0$ and $n_i(\alpha) \neq 0$. Let $\varphi : \mathbf{G}_m \to T$ be a one-parameter subgroup such that $\alpha_i(\varphi(x)) \neq 1$ for some $x \in \mathbf{G}_m$, but $\alpha_j(\varphi(x)) = 1$ for all $x \in \mathbf{G}_m$ if $j \neq i$. Then

$$\operatorname{Ad} \varphi(x)\, X = X,$$
$$\operatorname{Ad} \varphi(x)\, Y = \sum_{\beta \in \Sigma^+} \lambda^{n_i(\beta)} \gamma_\beta X_\beta, \qquad \text{where } \lambda = \alpha_i(\varphi(x)).$$

Since $f \in \mathfrak{P}^G$, we find

$$f(X + Y) = f\Big(X + \sum_{\beta \in \Sigma^+} \lambda^{n_i(\beta)} \gamma_\beta X_\beta\Big) \quad \text{for all } \lambda \in k^*.$$

Since $k$ is an infinite field, it follows that

$$f(X+Y) = f\Big(X + \sum_{\beta\in\Sigma^+,\ n_i(\beta)=0} \gamma_\beta X_\beta\Big),$$

hence the result follows by induction.

(4.5) Corollary. — *Assume* $X \in \mathfrak{g}$, $X = X_s + X_n$ *its Jordan decomposition. Then* $f(X) = f(X_s)$ *for* $f \in \mathfrak{T}^G$.

*Proof.* — After conjugation under G, we may assume that $X \in \mathfrak{b}$ and $X_s \in \mathfrak{h}$ (*cf.* [2]). Then $X_n \in \mathfrak{u}$, so we can apply the lemma.

From [18], p. 61-63, and [17], (5.3), we quote the following results.

(4.6) (i) *The number of nilpotent* G-*orbits in* $\mathfrak{g}$ *is finite.*

(ii) *There exist regular nilpotent elements in* $\mathfrak{g}$. *These form a single* G-*orbit.* $X = \sum_{\alpha\in\Sigma^+} \gamma_\alpha X_\alpha$ *is regular nilpotent if and only if* $\gamma_\alpha \neq 0$ *for all simple roots* $\alpha$, *and similarly for* $\sum_{\alpha\in\Sigma^+} \gamma_\alpha X_{-\alpha}$. *In particular,* $X_+ = \sum_{\alpha\in\Delta} X_\alpha$ *and* $X_- = \underset{\alpha\in\Delta}{\Delta} X_{-\alpha}$ *are regular.*

(iii) *The regular nilpotent orbit is dense and open in the set of all nilpotent elements of* $\mathfrak{g}$.

It should be observed that regular nilpotent elements are called principal nilpotents in Kostant's papers [12] and [13].

For $1 \leq i \leq l$ we introduce the following notations (*cf.* [21]); we recall that $\alpha_1, \ldots, \alpha_l$ are the simple roots.

$H_i = \operatorname{Ker}(\alpha_i)$ on H;

$U_i$ = the group generated by the $U_\alpha$'s for $\alpha \in \Sigma^+$, $\alpha \neq \alpha_i$, where $U_\alpha$ denotes the unipotent one-parameter subgroup of G normalized by H, corresponding to the root $\alpha$;

$B_i = H_i . U_i$;

$\mathfrak{h}_i = \operatorname{Ker}(d\alpha_i)$ on $\mathfrak{h}$;

$\mathfrak{u}_i = \Big\{ \sum_{\alpha\in\Sigma^+,\ \alpha\neq\alpha_i} \xi_\alpha X_\alpha \mid \xi_\alpha \in k \Big\}$;

$\mathfrak{b}_i = \mathfrak{h}_i + \mathfrak{u}_i$.

Clearly, $\mathfrak{h}_i$, $\mathfrak{u}_i$ and $\mathfrak{b}_i$ are the Lie algebras of $H_i$, $U_i$ and $B_i$, respectively. The following lemma is a Lie algebra analog of Lemma 5.1 of [21]. The proof given here is globally similar to Steinberg's proof in [21], but we shall write it out since locally there are some typical Lie algebra features.

(4.7) LEMMA. — *An element of* $\mathfrak{g}$ *is irregular if and only if it is conjugate under* Ad G *to an element of some* $\mathfrak{b}_i$.

*Proof.* — Consider $X \in \mathfrak{g}$ and let $X = X_s + X_n$ be its Jordan decomposition. After conjugation under Ad G we may assume that $X_s \in \mathfrak{h}$, $X_n \in \mathfrak{u}$ (*cf.* [2]). $\mathfrak{z}(X_s)$ is a reductive Lie algebra whose root system $\Sigma' = \Sigma'(X_s)$ consists of all roots $\alpha \in \Sigma$ such that $d\alpha(X_s) = 0$. $\Sigma'$ inherits an ordering from $\Sigma$.

Assume first $X \in \mathfrak{b}_i$. Then $d\alpha_i(X_s) = 0$, hence $\alpha_i \in \Sigma'$, and

$$X_n = \sum_{\alpha \in \Sigma^+} \gamma_\alpha X_\alpha \qquad \text{with} \quad \gamma_\alpha = 0.$$

The root $\alpha_i$ is simple in $\Sigma'$, hence $X_n$ is not regular in $\mathfrak{z}(X_s)$ by (4.6), (ii), which implies that X is not regular in $\mathfrak{g}$ by (4.3).

Conversely, assume X is irregular in $\mathfrak{g}$. $X_n$ is irregular in $\mathfrak{z}(X_s)$, and $X_n = \sum_{\alpha \in \Sigma'^+} \gamma_\alpha X_\alpha$. A good prime for G is also a good prime for $Z(X_s)$ (*cf.* [18], I.4.7), so we may apply (2.7), (ii), to conclude that $\gamma_\alpha = 0$ for some $\alpha$ which is a simple root in $\Sigma'$. By induction on ht ($\alpha$), the height of $\alpha$, we shall prove that X may be replaced by some conjugate such that $\alpha$ is simple in $\Sigma$. Then this conjugate will be in some $\mathfrak{b}_i$, which completes the proof.

So assume ht $(\alpha) > 1$. We have $(\alpha, \alpha_i) > 0$ for some $i$. Then $\alpha_i \notin \Sigma'$, for otherwise $\alpha - \alpha_i \in \Sigma'$, which would yield a contradiction to the simplicity of $\alpha$ in $\Sigma'$. Let $w_i \in N(H)$ be a representative of the reflection $\sigma_i \in W$ corresponding to $\alpha_i$, the Weyl group W being identified with $N(H)/H$. Then Ad $w_i(X_n)$ is easily seen to be in $\mathfrak{u}$. Since

$$\sigma_i \alpha = \alpha - 2\frac{(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i$$

has smaller height than $\alpha$, we may apply the inductive assumption to Ad $w_i$ (X).

The proof of the following lemma is similar to that of Lemma 2.12 of [21] and will be left to the reader.

(4.8) LEMMA. — *Let* $B' = H'U'$ *with* $B'$ *a connected solvable group,* $H'$ *a maximal torus and* $U'$ *the maximal unipotent subgroup of* $B'$, *and let* $\mathfrak{b}'$, $\mathfrak{h}'$ *and* $\mathfrak{u}'$ *be the Lie algebras of* $B'$, $H'$ *and* $U'$, *respectively, so* $\mathfrak{b}' = \mathfrak{h}' + \mathfrak{u}'$.

*If* $X \in \mathfrak{h}'$ *and* $Y \in \mathfrak{u}'$, *then there exists* $Y' \in \mathfrak{u}'$ *such that* $X + Y'$ *is conjugate to* $X + Y$ *under* Ad U', and X *and* Y' *commute.*

The following proposition is the Lie algebra analog of Corollary 5.5 of [21].

(4.9) PROPOSITION. — *In the set of irregular elements the semisimple ones are dense.*

*Proof.* — The elements of the form

$$X + Y \quad \text{with} \quad X \in \mathfrak{h}_i, Y \in \mathfrak{u}_i, \qquad d\alpha(X) \neq 0 \quad \text{for all } \alpha \in \Sigma^+, \qquad \alpha \neq \alpha_i,$$

form a dense open subset of $\mathfrak{b}_i$. These elements are semisimple; in view of the previous lemma, this assertion has only to be proved when X and Y commute. But in that case, $d\alpha(X) \neq 0$ for all roots of $\mathfrak{b}_i$ implies that $Y = 0$, hence, indeed, we have semisimplicity. The result of the present proposition now follows from lemma (4.7).

We will conclude this section by proving that the regular elements form an open subset in $\mathfrak{g}$ whose complement has pure codimension 3. The analogous result for algebraic groups has been proved by R. Steinberg [21]. First we give two ancillary results.

(4.10) LEMMA. — *For each* $i = 1, \ldots, l$, *the union of the conjugates of* $\mathfrak{b}_i$ *is closed, irreducible and of codimension* 3 *in* $\mathfrak{g}$.

*Proof* (*cf.* [21], 5.2). — Let $G_i$ denote the semisimple subgroup of G generated by $U_{\alpha_i}$ and $U_{-\alpha_i}$. Using Bruhat decomposition one easily verifies that $P_i = G_i . B_i$ is the normalizer of $\mathfrak{b}_i$ in G. Since $P_i$ is parabolic, $G/P_i$ is complete, and therefore the union of the conjugates of $\mathfrak{b}_i$ is closed, irreducible and of codimension at least

$$\dim P_i - \dim \mathfrak{b}_i = 3,$$

equality holding if and only if there exists an element of $\mathfrak{g}$ contained in only a finite number $\neq 0$ of conjugates of $\mathfrak{b}_i$. Hence the result follows from the next lemma.

(4.11) LEMMA. — *An* $X \in \mathfrak{h}_i$ *such that* $d\alpha(X) \neq 0$ *for all roots* $\alpha \neq \pm \alpha_i$ *is contained in only a finite number of conjugates of* $\mathfrak{b}_i$.

*Proof.* — This is the Lie algebra analog of the proof of (5.3, *b*) in [21]. Let Ad $y(\mathfrak{b}_i)$ be a conjugate of $\mathfrak{b}_i$ containing X. Since B normalizes $\mathfrak{b}_i$, we may assume that $y = uw$, with $u \in U^-_w$, $w \in N(H)$.

$$\mathrm{Ad}\, u^{-1}(X) = X + Y \quad \text{with} \quad Y \in \mathfrak{n}.$$

$\mathrm{Ad}\, y^{-1}(X) \in \mathfrak{b}_i$, hence

$$\mathrm{Ad}\, w^{-1}(X) + \mathrm{Ad}\, w^{-1}(Y) \in \mathfrak{b}_i.$$

Since $w^{-1} uw \in U^-$, we find for the above element

$$\begin{aligned} \mathrm{Ad}\, w^{-1}(X) + \mathrm{Ad}\, w^{-1}(Y) &= \mathrm{Ad}\,(w^{-1} u^{-1})(X) \\ &= \mathrm{Ad}\,(w^{-1} u^{-1} w)\, \mathrm{Ad}\, w^{-1}(X) \\ &= \mathrm{Ad}\, w^{-1}(X) + Z \quad \text{with} \quad Z \in \mathfrak{n}^-. \end{aligned}$$

Hence $\mathrm{Ad}\, w^{-1}(Y) \in \mathfrak{n}^-$; on the other hand, $\mathrm{Ad}\, w^{-1}(Y) \in \mathfrak{b}_i$, whence $Y = 0$. Thus we find

$$\mathrm{Ad}\, u^{-1}(X) = X.$$

Given the conditions regarding X, this is possible only for $u \in U_{\alpha_i}$. Since

$$\mathrm{Ad}\,(w^{-1} u^{-1})(X) \in \mathfrak{b}_i,$$

we get

$$\mathrm{Ad}\, w^{-1}(X) \in \mathfrak{h}_i,$$

hence $\sigma_w(\alpha_i) = \pm \alpha_i$. But then $w^{-1} uw \in G_i$, hence this element normalizes $\mathfrak{b}_i$. Therefore

$$\mathrm{Ad}\, y(\mathfrak{b}_i) = \mathrm{Ad}\, w\, \mathrm{Ad}\,(w^{-1} uw)(\mathfrak{b}_i) = \mathrm{Ad}\, w(\mathfrak{b}_i).$$

So the number of conjugates of $\mathfrak{b}_i$ containing X is finite, and in fact equal to the number of elements of the Weyl group which fix $\alpha_i$.

Combining the above two lemmas with lemma (4.7) we immediately get the following theorem.

(4.12) THEOREM. — *The set of regular elements is Zariski open in* $\mathfrak{g}$. *Its complement has pure codimension* 3 *in* $\mathfrak{g}$.

5. REGULAR NILPOTENT ELEMENTS. — We keep the notations and assumptions of the previous section. By $\mathfrak{r}$ we shall denote the set of regular elements in the Lie algebra $\mathfrak{g}$. The aim of this section is to prove some results which are needed in the subsequent sections.

In (4.6), (ii), we have seen that $X_+ = \sum_{\alpha \in \Delta} X_\alpha$ and $X_- = \sum_{\alpha \in \Delta} X_{-\alpha}$ are regular nilpotent elements.

(5.1) LEMMA. — $X_- + \mathfrak{b} \subset \mathfrak{r}$.

*Proof.* — The proof of [13], Lemma 10, p. 370, also works in the present situation.

Let $\mathfrak{g}_i$ denote the subspace of $\mathfrak{g}$ spanned by the rootvectors $X_\alpha$ with $\mathrm{ht}(\alpha) = i$, for $i \neq 0$, and $\mathfrak{g}_0 = \mathfrak{h}$. It is easily seen that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, so we have made a graded Lie algebra of $\mathfrak{g}$ in this way. We recall that $d_1, \ldots, d_l$ are the degrees of the homogeneous generators $J_1, \ldots, J_l$ of $\mathfrak{P}^G$ [*cf.* § 2, (ii) and (iii)] in characteristic 0 or $p \not| \, |W|$; we use the same numbers here in a slightly wider range of characteristics, meaning the degrees of the invariants under a group of the same type as G over a field of characteristic 0, say.

(5.2) Proposition. — (i) $\mathfrak{z}(X_+)$ *has a basis* $Z_1, \ldots, Z_l$ *with* $Z_i \in \mathfrak{g}_{d_i - 1}$.
(ii) $\mathfrak{b} = \mathfrak{z}(X_+) \oplus [X_-, \mathfrak{u}]$.

*Proof.* — It is obvious that it suffices to consider the case of almost simple G. For $\mathrm{char}(k) = 0$ the results have been proved by Kostant; *see* [12], Th. 6.7 and Cor. 8.7 for (i), [13], Lemma 12, p. 374, for (ii). The $\mathrm{char}(k) = p$ case will be derived by a reduction mod $p$ argument.

Let $\mathfrak{g}_{\mathbf{Z}}$ be an integral form of the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbf{Z}} \otimes k$ and that G is a Chevalley group corresponding to $\mathfrak{g}_{\mathbf{Z}}$. Let $H_1, \ldots, H_l$, $X_\alpha$ $(\alpha \in \Sigma)$ be a Chevalley basis of $\mathfrak{g}_{\mathbf{C}}$, contained in $\mathfrak{g}_{\mathbf{Z}}$. The $H_i \otimes 1$ and $X_\alpha \otimes 1$ form a basis of $\mathfrak{g}$ (*cf.* § 1), and will henceforth also be denoted by $H_i$, $X_\alpha$. Then $X_+ = \sum_{\alpha \in \Delta} X_\alpha$ and $X_- = \sum_{\alpha \in \Delta} X_{-\alpha}$ denote elements of $\mathfrak{g}_{\mathbf{Z}}$ as well as $\mathfrak{g}$.

Set $\mathfrak{z}(X_+)_i = \mathfrak{z}(X_+) \cap \mathfrak{g}_i$, and similarly in $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{g}_{\mathbf{Z}}$. By Kostant's results in characteristic 0,

$$\mathfrak{g}_{\mathbf{C},i} = \mathfrak{z}_{\mathbf{C}}(X_+)_i \oplus [X_-, \mathfrak{g}_{\mathbf{C},i+1}] \qquad \text{for} \quad i \geq 0,$$

$$\mathfrak{z}_{\mathbf{C}}(X_+) = \sum_{i>0} \mathfrak{z}_{\mathbf{C}}(X_+)_i.$$

Since $\mathfrak{z}_{\mathbf{C}}(X_+)_i$ is defined by linear equations with rational coefficients, it is spanned over $\mathbf{C}$ by $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$. Hence $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$ is a free $\mathbf{Z}$-submodule of $\mathfrak{g}_{\mathbf{Z},i}$ of rank $s = \dim_{\mathbf{C}} \mathfrak{z}_{\mathbf{C}}(X_+)_i$. So we can find a basis $X_1, \ldots, X_t$ of $\mathfrak{g}_{\mathbf{Z},i}$ $(i > 0)$ such that $\alpha_1 X_1, \ldots, \alpha_s X_s$ is a basis of $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$, for suitable integers $\alpha_j$. But it is clear that

$$\mathfrak{z}_{\mathbf{Z}}(X_+)_i = \mathfrak{g}_{\mathbf{Z},i} \cap \mathfrak{z}_{\mathbf{C}}(X_+)_i,$$

and therefore we may assume that all $\alpha_j = 1$. Since $[X_-, \mathfrak{g}_{\mathbf{C},i+1}]$ is spanned by $[X_-, \mathfrak{g}_{\mathbf{Z},i+1}]$, the latter is a submodule of rank $t - s$ of $\mathfrak{g}_{\mathbf{Z},i}$. Therefore, in $\mathfrak{g}_{\mathbf{Z},i}$ mod $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$ the submodule $[X_-, \mathfrak{g}_{\mathbf{Z},i+1}]$ mod $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$ is spanned by $\gamma_{s+1} X_{s+1}, \ldots, \gamma_t X_t$ mod $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$, with suitable integers $\gamma_j$. Thus we see that $\mathfrak{g}_{\mathbf{Z},i}$ $(i > 0)$ has a basis $X_1, \ldots, X_t$ with the properties

(*a*) $X_1, \ldots, X_s$ is a basis of $\mathfrak{z}_{\mathbf{Z}}(X_+)_i$, where $s = \dim \mathfrak{z}_{\mathbf{C}}(X_+)_i$;

(*b*) $[X_-, \mathfrak{g}_{\mathbf{Z},i+1}]$ has a basis consisting of vectors

$$\gamma_{s+1} X_{s+1} + Y_{s+1}, \ldots, \gamma_t X_t + Y_t$$

with suitable $\gamma_j \in \mathbf{Z}$ and $Y_j \in \mathfrak{z}_{\mathbf{Z}}(X_+)_i$.

$X_1 \otimes 1, \ldots, X_t \otimes 1$ form a basis of $\mathfrak{g}_i$ $(i > 0)$. Since $X_+$ is regular in $\mathfrak{g}$, $\dim \mathfrak{z}(X_+) = l$. Hence

$$\mathfrak{z}(X_+) = \mathfrak{z}_{\mathbf{Z}}(X_+) \otimes k,$$

since the right hand side is contained in $\mathfrak{z}(X_+)$ and has dimension $l$ by (*a*). Thus we see that $\mathfrak{z}(X_+) \subseteq \mathfrak{u}$. Similarly $\mathfrak{z}(X_-) \subseteq \mathfrak{u}^-$, hence ad $X_-$ operates injectively on $\mathfrak{b}$.

Therefore,

$$\dim_k [X_-, \mathfrak{g}_{i+1}] = \dim_{\mathbf{C}} [X_-, \mathfrak{g}_{\mathbf{C},i+1}].$$

Since clearly

$$[X_-, \mathfrak{g}_{i+1}] = [X_-, \mathfrak{g}_{\mathbf{Z},i+1}] \otimes k,$$

the vectors

$$\gamma_{s+1} X_{s+1} \otimes 1 + Y_{s+1} \otimes 1, \quad \ldots, \quad \gamma_t X_t \otimes 1 + Y_t \otimes 1$$

must be linearly independant over $k$. Consequently,

$$X_1 \otimes 1, \quad \ldots, \quad X_s \otimes 1, \quad \gamma_{s+1} X_{s+1} \otimes 1 + Y_{s+1} \otimes 1, \quad \ldots, \quad \gamma_t X_t \otimes 1 + Y_t \otimes 1$$

is a basis of $\mathfrak{g}_i$, hence

$$\mathfrak{g}_i = \mathfrak{z}(X_+)_i \oplus [X_-, \mathfrak{g}_{i+1}],$$

and therefore,

$$\mathfrak{b} = \mathfrak{z}(X_+) \oplus [X_-, \mathfrak{u}].$$

This proves (ii). But the above arguments also show that a basis for $\mathfrak{z}(X_+)$ as indicated in (i) can be obtained by reduction mod $p$. This completes the proof.

(5.3) PROPOSITION. — Ad $U(X_- + \mathfrak{z}(X_+))$ *is Zariski dense in* $X_- + \mathfrak{b}$.

*Proof.* — Since $\mathfrak{z}(X_+) \subseteq \mathfrak{b}$, it is clear that

$$\operatorname{Ad} U(X_- + \mathfrak{z}(X_+)) \subseteq X_- + \mathfrak{b}.$$

Consider the morphism

$$\alpha : \quad U \times \mathfrak{z}(X_+) \rightarrow X_- + \mathfrak{b}$$

defined by

$$\alpha(u, X) = \operatorname{Ad} u (X_- + X) \qquad \text{for} \quad u \in U, \quad X \in \mathfrak{z}(X_+).$$

Its differential in (1, 0),

$$d\alpha_{(1,0)} : \mathfrak{u} \oplus \mathfrak{z}(X_+) \to \mathfrak{b},$$

is easily seen to be

$$d\alpha_{(1,0)}(X, Y) = [X, X_-] + Y \qquad \text{for} \quad X \in \mathfrak{u}, \quad Y \in \mathfrak{z}(X_+).$$

By the previous proposition, $d\alpha_{(1,0)}$ is surjective, hence $\alpha$ is a dominant morphism (*cf.* [1], Th. (17.3), p. 75). This proves that the image of $\alpha$ is dense in $X_- + \mathfrak{b}$.

6. Regular orbits. — *In this, and the following section, we assume the groundfield $k$ to be of characteristic* 0 *or $p$ with $p$ not dividing the order of the Weyl group* W *of* G. The rest of the notations and conventions are as in the previous sections.

Let $\mathcal{O}_s$ denote the set of semisimple orbits under the adjoint action of G in $\mathfrak{g}$, and $\mathcal{O}_r$ the set of regular orbits. We recall that $J_1, \ldots, J_l$ are homogeneous generators of $\mathfrak{X}^G$ [*see* § 2, (iii)]. Define $\eta_s : \mathcal{O}_s \to k^l$ by

$$\eta_s(\operatorname{Ad} G(X)) = (J_1(X), \ldots, J_l(X)) \quad \text{for semisimple } X \in \mathfrak{g},$$

and, similarly, $\eta_r : \mathcal{O}_r \to k^l$ by

$$\eta_r(\operatorname{Ad} G(X)) = (J_1(X), \ldots, J^l(X)) \quad \text{for regular } X \in \mathfrak{g}.$$

Concerning $\eta_s$ and $\eta_r$ we have the following result.

(6.1) Proposition. — *The mappings $\eta_s : \mathcal{O}_s \to k^l$ and $\eta_r : \mathcal{O}_r \to k^l$ are bijections.*

*Proof.* — Restriction to $\mathfrak{h}$ gives an isomorphism of $\mathfrak{X}^G$ with $\mathfrak{X}(\mathfrak{h})^W$ [*see* § 2, (iii)]. It is known that $\mathfrak{X}(\mathfrak{h})^W$ separates semisimple G-orbits in $\mathfrak{g}$. This implies the result for $\eta_s$. The result for $\eta_r$ has been proved for complex Lie groups by Kostant ([13], Theorem 2, p. 360). His arguments apply to the present situation almost verbatim, so we omit them here.

(6.2) Definition. — Let $X_+$ and $X_-$ be as before. We set

$$\mathfrak{v} = X_- + \mathfrak{z}(X_+).$$

We are going to show that $\mathfrak{v}$ is a cross section for the regular orbits in $\mathfrak{g}$. We shall get this result by first showing that restriction to $\mathfrak{v}$ of polynomials on $\mathfrak{g}$ provides an isomorphism of $\mathfrak{P}^G$ onto the algebra of polynomials on $\mathfrak{v}$ : $\mathfrak{P}(\mathfrak{v})$. This key result considerably simplifies the proof that $\mathfrak{v}$ is a cross section for the regular orbits, as compared to the proof in characteristic 0 given in [13]. The proof of (6.3) given here was kindly put at the author's disposal by T. A. Springer.

(6.3) Proposition. — *The mapping* : $\mathfrak{P}^G \to \mathfrak{P}(\mathfrak{v})$ *which maps* $f \in \mathfrak{P}^G$ *on its restriction to* $\mathfrak{v}$ *is an isomorphism* (*onto*).

*Proof.* — It is trivial that $\rho$ is a homomorphism. To show that $\rho$ is injective, consider an $f \in \mathfrak{P}^G$ such that

$$\rho f = f \mid \mathfrak{v} = 0.$$

Since by (5.3), Ad U ($\mathfrak{v}$) is dense in $X_- + \mathfrak{b}$, we get

$$f \mid X_- + \mathfrak{b} = 0.$$

For every $\lambda \in k^*$, $X_- + \mathfrak{b}$ is conjugate to $\lambda X_- + \mathfrak{b}$ under Ad H, hence

$$f \mid \lambda X_- + \mathfrak{b} = 0 \qquad \text{for} \quad \lambda \in k^*.$$

This implies that

$$f \mid \mathfrak{b} = 0.$$

Since $\mathfrak{g}$ is covered by the conjugates of $\mathfrak{b}$ under Ad G ([2], Prop. 2.3), it follows that $f = 0$.

So there remains to be shown that $\rho$ is surjective. Let $Z_1, \ldots, Z_l$ be a basis of $\mathfrak{z}(X_+)$ with $Z_j \in \mathfrak{g}_{d_j - 1}$, as in proposition (5.2). Any $X \in \mathfrak{v}$ can be written as

$$X = X_- + \sum_{j=1}^{l} \xi_j Z_j \qquad \text{with} \quad \xi_j \in k.$$

Let $\varphi : \mathbf{G}_m \to H$ be a one-parameter subgroup satisfying

$$\alpha_i(\varphi(\xi)) = \xi^u \qquad \text{for} \quad 1 \leq i \leq l, \quad \xi \in k^*,$$

where $u$ is a suitably chosen fixed positive integer. For the rootvectors $X_\alpha$ we find

$$\mathrm{Ad}\, \varphi(\xi)(X_\alpha) = \xi^{u\,\mathrm{ht}(\alpha)} X_\alpha.$$

Now consider the homogeneous generators $J_1, \ldots, J_l$ of $\mathfrak{P}^G$, of degrees $d_1 \leq \ldots \leq d_l$, respectively. From

$$J_i(X) = J_i(\mathrm{Ad}\, \varphi(\xi)(X))$$

it follows that

$$J_i\left(X_- + \sum_{j=1}^{l} \xi_j Z_j\right) = J_i\left(\xi^{-u} X_- + \sum_{j=1}^{l} \xi^{u(d_j-1)} \xi_j Z_j\right) = \xi^{-ud_i} J_i\left(X_- + \sum_{j=1}^{l} \xi^{ud_j} \xi_j Z_j\right),$$

$$J_i\left(X_- + \sum_{j=1}^{l} \xi^{ud_j} \xi_j Z_j\right) = \xi^{ud_i} J_i\left(X_- + \sum_{j=1}^{l} \xi_j Z_j\right).$$

Hence $J_i \mid \mathfrak{v}$ must be of the form

$$J_i\left(X_- + \sum_{j=1}^{l} \xi_j Z_j\right) = \sum \gamma_{n_1,\ldots,n_l} \xi_1^{n_1} \ldots \xi_l^{n_l},$$

where only terms can occur with

$$\sum_{j=1}^{l} n_j d_j = d_i.$$

This condition implies that $n_j = 0$ whenever $d_j > d_i$, and that

$$(6.4) \qquad J_i\left(X_- + \sum_{j=1}^{l} \xi_j Z_j\right) = \gamma_i \xi_i + f_i(\xi_1, \ldots, \xi_{i-1}).$$

Here $f_i$ is a polynomial in those $\xi_j$ for which $d_j < d_i$, hence certainly $j < i$. We claim that $\gamma_i \neq 0$ for $1 \leq i \leq l$. For assume $\gamma_1, \ldots, \gamma_{i-1} \neq 0, \gamma_i = 0$ for some $i$, $1 \leq i \leq l$. Then $J_i \mid \mathfrak{v}$ would be a polynomial in $\xi_1, \ldots, \xi_{i-1}$, and it could be expressed as a polynomial in $J_1 \mid \mathfrak{v}, \ldots, J_{i-1} \mid \mathfrak{v}$. Since $\rho$ is already shown to be an injective homomorphism, we conclude that $J_i$, when considered as a polynomial on the entire $\mathfrak{g}$, would be a polynomial in $J_1, \ldots, J_{i-1}$, which contradicts the algebraic independence of $J_1, \ldots, J_l$. But if all $\gamma_i \neq 0$, it follows from (6.4) that $\xi_1, \ldots, \xi_l$ can be expressed as polynomials in $J_1 \mid \mathfrak{v}, \ldots, J_l \mid \mathfrak{v}$, which shows $\rho$ to be surjective.

(6.5) THEOREM. — *Every regular orbit in $\mathfrak{g}$ intersects $\mathfrak{v}$ in precisely one point, and does so transversally; every point of $\mathfrak{v}$ is regular. In order words, $\mathfrak{v}$ is a cross section for $\mathcal{O}_r$.*

*Proof.* — From (5.1), (6.1) and (6.3) it follows that every regular orbit meets $\mathfrak{v}$ in one point and that every point of $\mathfrak{v}$ is regular. From relation (6.4) and the fact that all $\gamma_i \neq 0$, it follows that the functional matrix

$$\left(\frac{\partial J_i}{\partial \xi_j}\right)_{1 \leq i,j \leq l} = \begin{pmatrix} \gamma_1 & & \Theta \\ & \ddots & \\ \star & & \gamma_l \end{pmatrix}$$

has rank $l$. Hence the tangent space to any regular orbit in its intersection with $\mathfrak{v}$ has intersection 0 with $\mathfrak{v}$, which means that these orbits intersect $\mathfrak{v}$ transversally.

From the above proof the following corollary is immediate.

(6.6) COROLLARY. — *Let $\xi_1, \ldots, \xi_n$ denote linear coordinates on $\mathfrak{g}$. The functional matrix*

$$\left(\frac{\partial J_i}{\partial \xi_j}\right)_{\substack{1 \leq i \leq \\ 1 \leq j \leq n}}$$

*has rank $l$ in the points of $\mathfrak{v}$.*

We conclude this section with some results on the relative position of the Ad G-orbits in $\mathfrak{g}$.

(6.7) PROPOSITION (Kirillov). — *Every* Ad G-*orbit in $\mathfrak{g}$ has even dimension.*

*Proof.* — The Killing form on $\mathfrak{g}$ is nondegenerate [*see* § 2, (iv)]. Hence the same proof as in characteristic 0 works (*see* [13], Prop. 15, p. 364, or [7], footnote on p. 260-03).

(6.8) THEOREM. — *Let $\eta : \mathfrak{g} \to k^l$ be the morphism given by*

$$\eta(X) = (J_1(X), \ldots, J_l(X)) \qquad \text{for} \quad X \in \mathfrak{g}.$$

*The fibers $\eta^{-1}(x)$, $x \in k^l$, have the following properties :*

(i) *$\eta^{-1}(x)$ contains only a finite number of* Ad G-*orbits.*

(ii) *$\eta^{-1}(x)$ contains a unique regular orbit, which is also the orbit whose closure is $\eta^{-1}(x)$.*

(iii) *$\eta^{-1}(x)$ contains precisely one semisimple orbit, which is the only closed orbit in $\eta^{-1}(x)$ and which is contained in the closure of every orbit in $\eta^{-1}(x)$.*

(iv) *$\eta^{-1}(x)$ consists of all* X *such that $\eta(X_s) = x$, $X_s$ being the semisimple part of $X$.*

(v) *The non-regular orbits in $\eta^{-1}(x)$ have even codimension $\geqq 2$ in $\eta^{-1}(x)$.*

*Proof.* — (i) and (ii) follow from (4.6) and (6.1), respectively, by a reasoning along similar lines as in [13], p. 366. Every G-orbit in $\mathfrak{g}$ is open in its closure, hence its closure contains a closed orbit. If X is a semisimple element in $\mathfrak{g}$, the orbit of X consists of the elements Y such that Y has the same minimum polynomial in the adjoint representation of $\mathfrak{g}$ (this polynomial has no multiple roots, hence defines a semisimple set) and $J_i(X) = J_i(Y)$ for $1 \leq i \leq l$, by (6.1), hence is closed. This proves (iii). (iv) follows from (4.5). (v) is an immediate consequence of (ii) and (6.7).

Every $\eta^{-1}(x)$ can be considered as a closed subvariety of $\mathfrak{g}$, which by (ii) of the above theorem is irreducible. With similar arguments as in [18] one can now prove the following analog of [18], III.2.7 on p. 59.

(6.9) PROPOSITION. — *For any fibrer $\eta^{-1}(x)$ as in the previous theorem we have :*

(i) *The regular elements are precisely the simple points of the variety $\eta^{-1}(x)$.*

(ii) *$\eta^{-1}(x)$ is nonsingular in codimension* 1.

(iii) *The ideal of $\eta^{-1}(x)$, in the algebra of regular functions on $\mathfrak{g}$, is generated by $J_1 - x_1, \ldots, J_l - x_l$* [where $x = (x_1, \ldots, x_l)$], *hence the latter ideal is prime and $\eta^{-1}(x)$ is a complete intersection.*

(iv) *$\eta^{-1}(x)$ is normal.*

Taking, in particular, $x = 0$, we find that the variety of nilpotent elements in $\mathfrak{g}$ is normal, which answers question III.3.10, p. 62 in [18] in the affirmative, for $p$ not dividing $|W|$.

7. A CRITERION FOR REGULARITY. — We keep the notations and conventions of the previous section. *In particular, the characteristic of the groundfield $k$ is supposed not to divide the order of the Weyl group of* G. The aim of this section is to prove the following theorem, which gives a criterion for regularity.

(7.1) THEOREM. — *$X \in \mathfrak{g}$ is regular if and only if the functional matrix*

$$M = \left(\frac{\partial J_i}{\partial \xi_j}\right)_{\substack{1 \leq j \leq l \\ 1 \leq j \leq n}}$$

*has rank $l$ in* X.

*Proof.* — (i) If X is regular, M has rank $l$ in X by (6.5) and (6.6).

(ii) Take a basis in $\mathfrak{g}$ of $H_1, \ldots, H_l$ in $\mathfrak{h}$ and rootvectors $X_\alpha$, $\alpha \in \Sigma$ (*cf.* § 1). Write any $X \in \mathfrak{g}$ as

$$X = \sum_{i=1}^{l} \xi_i H_i + \sum_{\alpha \in \Sigma} \eta_\alpha X_\alpha.$$

We consider the $\xi_i$ and $\eta_\alpha$ as coordinates on $\mathfrak{g}$. We first give a lemma on invariant polynomials, which is the Lie algebra analog of a lemma of Steinberg's ([21], lemma 8.4.6).

(7.2) LEMMA. — *Let $f \in \mathfrak{X}^{G}$ be written as an irredundant sum of monomials in the $\xi_i$, $\eta_\alpha$. Then each monomial has a total degree in the $\eta_\alpha$'s ($\alpha \in \Sigma$) which is either 0 or at least 2.*

*Proof.* — Assume there would be a monomial with total degree 1 in the $\eta_\alpha$'s, say $\prod_{i=1}^{l} \xi_i^{n_i} \eta_\alpha$ for some $\alpha$. When operated on by Ad $t$, $t \in H$, it would be multiplied by $\alpha(t)$. Because of the invariance of $f$ under Ad G, this would imply $\alpha(t) = 1$ for all $t \in H$, a contradiction.

We continue the proof of theorem (7.1).

(iii) Assume $X \in \mathfrak{h}$ is such that M has rank $l$ in X. Taking the $\xi_i$, $\eta_\alpha$ as in (ii) as coordinates on $\mathfrak{g}$, it follows from (7.2) that

$$M = \left( \left( \frac{\partial I_i}{\partial \xi_j} \right)_{1 \leq i, j \leq l} \Big| \text{-}\Theta\text{-} \right)$$

in X. It is known that

$$\det \left( \left( \frac{\partial I_i}{\partial \xi_j} \right)_{1 \leq i, j \leq l} \right) = \lambda \prod_{\alpha \in \Sigma^+} d\alpha\,(X),$$

with $\lambda \in k^*$ (*see* [6], Prop. 77.17, or [4], V, 5.5, Prop. 6). So rank $M = l$ implies that $d\alpha\,(X) \neq 0$ for all $\alpha \in \Sigma$, and therefore $\mathfrak{z}\,(X) = \mathfrak{h}$, which shows that X is regular. Since any semisimple element is conjugate to an element of $\mathfrak{h}$, we see that (7.1) holds for semisimple X.

(iv) In (4.9) it was shown that in the set of irregular elements the semi-simple ones are dense. Hence it follows from (iii) that M has rank $< l$ in all irregular elements, which completes the proof of the theorem.

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F. D. Veldkamp,
Mathematisch Instituut,
Budapestlaan 6,
Utrecht, Netherlands.