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WEIL-CHÂTELET GROUPS
OVER LOCAL FIELDS : ADDENDUM

BY JAMES S. MILNE

By using the structure theorems for the Néron minimal model of an
abelian variety with semi-stable reduction, as presented in [2], it is possible
to complete the proof of the following theorem. (Notations are as in [3].)

**Theorem.** — Let $\mathbf{A}$ be an abelian variety over a local field $\mathbf{K}$ (with finite
residue field) and let $\hat{\mathbf{A}}$ be the dual abelian variety. Then the pairings

$$H^r(\mathbf{K}, \mathbf{A}) \times H^{1-r}(\mathbf{K}, \hat{\mathbf{A}}) \to H^s(\mathbf{K}, \mathbf{G}_m) \approx \mathbf{Q}/\mathbf{Z},$$

as defined by Tate [4], are non-degenerate for all $r$.

After [3], we need only consider the case where $\mathbf{K}$ has characteristic
$p \neq 0$. Also we have only to prove that the map

$$\theta_p (\mathbf{A}) : H^r(\mathbf{K}, \mathbf{A})_p \to (\hat{\mathbf{A}}(\mathbf{K})^{\text{et}})^*$$

is injective, and it suffices to do this after making a finite separable field
extension. Thus we may assume that $\mathbf{A}$ and $\hat{\mathbf{A}}$ have semi-stable
reduction ([2], § 3.6) and that

$$\mathbf{A}_p(\mathbf{K}) = \mathbf{A}_p(\overline{\mathbf{K}}), \quad \hat{\mathbf{A}}_p(\mathbf{K}) = \hat{\mathbf{A}}_p(\overline{\mathbf{K}}).$$
Let $\mathfrak{C}$ be the Néron minimal model of $A$ over $R$. The Raynaud group $\mathfrak{C}^\circ$ of $\mathfrak{C}$ over $R$ is a smooth group scheme over $R$ such that: (a) there are canonical isomorphisms $\mathfrak{C} \xrightarrow{\sim} \mathfrak{C}^\circ$ and $\mathfrak{C}^\circ \xrightarrow{\sim} \mathfrak{C}^\circ_0$ (where $\mathfrak{C}^\circ$ denotes the formal completion of a scheme $\mathfrak{C}$ over $R$) and (b) there is an exact sequence $0 \to \mathfrak{O} \to \mathfrak{C}^\circ \to \mathfrak{O} \to 0$ in which $\mathfrak{O}$ is an abelian scheme and $\mathfrak{O}$ is a torus ([2], § 7.2). $\mathfrak{R} = (\mathfrak{C}^\circ)^\rho$ is identified through the isomorphism in (a) with the maximal finite flat subgroup scheme of the quasi-finite group scheme $\mathfrak{C}^\circ$. If we write $B = \mathfrak{O} \otimes_R K$, $N = \mathfrak{R} \otimes_R K$, ..., then we get a filtration $A_p = \mathfrak{O}_p \otimes_R K \supset N \supset T_p \supset 0$ of $A_p$ in which $N/T_p \approx B_p$.

Let $\mathfrak{C}'$, $\mathfrak{O}'$, $\mathfrak{O}'$, ... be the schemes corresponding, as above, to $\hat{A}$. The canonical non-degenerate pairing $A_p \times \hat{A}_p \to \mathcal{G}_m$ respects the filtrations on $A_p$ and $\hat{A}_p$, i.e. $N$ and $T'_p$ are the exact annihilators of $T'_p$ and $N'$ respectively. Indeed, the induced pairing $N \times N' \to \mathcal{G}_m$ has a canonical extension to a pairing $\mathfrak{R} \times \mathfrak{R}' \to \mathcal{G}_{m,R}$ ([2], § 1.4). This pairing is trivial on $\mathfrak{O}_p$ and $\mathfrak{O}'_p$ and the quotient pairing $\mathfrak{O}_p \times \mathfrak{O}'_p \to \mathcal{G}_{m,R}$ is the non-degenerate pairing defined by a Poincaré divisorial correspondence on $(\mathfrak{O}, \mathfrak{O}')$ ([2], § 7.4, 7.5). This shows that $T_p$ (resp. $T'_p$) is the left (resp. right) kernel in the pairing $N \times N' \to \mathcal{G}_m$. The pairing $A_p/T'_p \times N' \to \mathcal{G}_m$ is right non-degenerate. But $A_p/T'_p$ has rank $p^{2n-\mu}$ where $n = \dim (A)$ and $\mu = \dim (\mathfrak{O})$ and $N'$ has rank $p^{\mu+z}$, where $z = \dim (\mathfrak{O})$ (cf. [2], § 2.2.7). This shows that the pairing is also left non-degenerate (because $n = \mu + z$), which completes the proof of our assertion.

Consider the commutative diagram:

$$
\begin{array}{ccc}
\alpha^\circ (R)^{(p)} & \longrightarrow & H^1 (R, \alpha^\circ) \\
\downarrow & & \downarrow \\
A (K)^{(p)} & \longrightarrow & H^1 (K, A_p)
\end{array}
$$

in which the horizontal maps are boundary maps in the cohomology sequences for multiplication by $p$ on $A$ and $\alpha^\circ$. $H^1 (R, \alpha^\circ)^{(p)} \approx H^1 (R, \mathfrak{R})$ because $\alpha^\circ/\mathfrak{R}$ is smooth over $R$ with zero special fibre and so has zero cohomology groups ([1], § 11.7) (including in dimension 0). The top arrow is an isomorphism because $H^1 (R, \alpha^\circ) = 0$ (loc. cit). The cokernel of the left vertical arrow is $\Phi_\alpha (k)^{(p)}$, where $\Phi_\alpha$ is the group of connected components of $\alpha \otimes^n k$ (cf. [2], § 11.1). Using all of this, one can extract from the top diagram on p. 275 of [3] (with $m = p$) an exact commutative
It is easy to see that \( \psi_1 \) is an isomorphism if and only if
\[
[\ker \psi_1] = [\Lambda(K)^{\rho}/\alpha^\rho(R)^{\rho}],
\]
i.e. \([\ker \psi_1] = [\Phi_\rho(k)^{\rho}]\).

We shall show that
\[
[\ker \psi_1] = p^{\rho}, \quad [\Phi_\rho(k)^{\rho}] = p^{\rho} = [\Phi_\rho(k)^{\rho}],
\]
and as \([\ker \psi_2][\Phi_\rho(k)^{\rho}] = [\ker \psi_1]\), this completes the proof.

Consider first the situation : \( M \) is a finite group scheme over \( K \) and \( \mathcal{M} \) and \( \mathcal{M}' \) are finite flat group schemes over \( R \) with given embeddings \( N \to M, N' \to M \). If \( \mathcal{M} = \mathcal{O}_p \) for some abelian scheme \( \mathcal{O} \) over \( R \) and \( M = N, \mathcal{M}' = \hat{N} \), then
\[
\psi : H^1(K, M)/H^1(R, \mathcal{M}) \to H^1(R, \mathcal{M}'^\ast),
\]
the map defined by the cup-product pairing
\[
H^1(K, M) \times H^1(K, M) \to H^2(K, G_m),
\]
is an isomorphism [3]. If \( \mathcal{M} = \mu_p, M = N, \) and \( \mathcal{M}' = 0 \), then \([\ker \psi] = p\) because [3]
\[
H^1(K, \mu_p)/H^1(R, \mu_p) \cong H^1(R, \mathbb{Z}/p \mathbb{Z})^* \cong H^1(k, \mathbb{Z}/p \mathbb{Z})^*.
\]

If \( M = \mathbb{Z}/p \mathbb{Z}, \mathcal{M} = 0, \) and \( \mathcal{M}' = \mu_p, \) then \([\ker \psi] = p\) because [3]
\[
[\ker \psi_1] = p
\]
ker \( \psi = H^1(R, \mathbb{Z}/p \mathbb{Z}) \). It follows from this, and the above discussion of the structures of \( \Lambda_p \) and \( \hat{P}_p \), that \([\ker \psi_1] = p^{\rho} \).

Finally, let \( \Phi = \alpha_\hat{P}_p^\ast/\alpha_\hat{P}_p^\circ \). It is a finite étale group scheme over \( R \) such that \( \Phi \otimes_R k = \Phi_\rho, \) and there is an exact sequence
\[
0 \to \mathcal{M} \to \alpha_\hat{P}_p^\circ \to \Phi_\rho \to 0.
\]
\( \alpha_\hat{P}_p^\circ(R) = \alpha_\hat{P}_p(R), \) because \( \alpha_\hat{P}_p^\circ \) and \( \alpha_\hat{P}_p \) differ only by a scheme with empty special fibre, and \( \alpha_\hat{P}_p(R) = \alpha_\hat{P}_p(K) \). It follows that \( \Phi_\rho(K) = \Lambda_\rho(K)/N(K) \) has \( p^{\rho} \) elements. But
\[
\Phi(K) \approx \Phi(R) \approx \Phi_\rho(k) \quad \text{and so} \quad [\Phi_\rho(k)^{\rho}] = [\Phi_\rho(k)^{\rho}] = p^{\rho}.
\]
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