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WEIL-CHÂTELET GROUPS OVER LOCAL FIELDS : ADDENDUM

BY JAMES S. MILNE

By using the structure theorems for the Néron minimal model of an abelian variety with semi-stable reduction, as presented in [2], it is possible to complete the proof of the following theorem. (Notations are as in [3].)

THEOREM. — *Let* A *be an abelian variety over a local field* K (*with finite residue field*) *and let* $\hat{A}$ *be the dual abelian variety. Then the pairings*

$$H^r(K, A) \times H^{1-r}(K, \hat{A}) \to H^2(K, \mathbf{G}_m) \approx \mathbf{Q}/\mathbf{Z},$$

as defined by Tate [4], *are non-degenerate for all r.*

After [3], we need only consider the case where K has characteristic $p \neq 0$. Also we have only to prove that the map

$$\theta_K(A)_p : \quad H^1(K, A)_p \to (\hat{A}(K)^{(p)})^*$$

is injective, and it suffices to do this after making a finite separable field extension. Thus we may assume that A and $\hat{A}$ have semi-stable reduction ([2], § 3.6) and that

$$A_p(K) = A_p(\overline{K}), \qquad \hat{A}_p(K) = \hat{A}_p(\overline{K}).$$

Let $\mathcal{A}$ be the Néron minimal model of A over R. The Raynaud group $\mathcal{A}^{\natural}$ of $\mathcal{A}$ over R is a smooth group scheme over R such that : (a) there are canonical isomorphisms $\overline{\mathcal{A}} \xrightarrow{\approx} \overline{\mathcal{A}^{\natural}}$ and $\overline{\mathcal{A}^{0}} \xrightarrow{\approx} \overline{\mathcal{A}^{\natural 0}}$ (where $\overline{\mathcal{H}}$ denotes the formal completion of a scheme $\mathcal{H}$ over R) and (b) there is an exact sequence $0 \to \mathcal{T} \to \mathcal{A}^{\natural 0} \to \mathcal{B} \to 0$ in which $\mathcal{B}$ is an abelian scheme and $\mathcal{T}$ is a torus ([2], § 7.2). $\mathcal{N} = (\mathcal{A}^{\natural 0})_p$ is identified through the isomorphism in (a) with the maximal finite flat subgroup scheme of the quasi-finite group scheme $\mathcal{A}^{0}_{p}$. If we write $\mathrm{B} = \mathcal{B} \otimes_{\mathrm{R}} \mathrm{K}$, $\mathrm{N} = \mathcal{N} \otimes_{\mathrm{R}} \mathrm{K}$, ..., then we get a filtration $\mathrm{A}_p = \mathcal{A}^{0}_{p} \otimes_{\mathrm{R}} \mathrm{K} \supset \mathrm{N} \supset \mathrm{T}_p \supset 0$ of A_p in which $\mathrm{N}/\mathrm{T}_p \approx \mathrm{B}_p$.

Let $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{N}'$, ... be the schemes corresponding, as above, to $\hat{\mathbf{A}}$. The canonical non-degenerate pairing $\mathrm{A}_p \times \hat{\mathrm{A}}_p \to \mathbf{G}_m$ respects the filtrations on A_p and $\hat{\mathrm{A}}_p$, i. e. N and T_p are the exact annihilators of T'_p and N' respectively. Indeed, the induced pairing $\mathrm{N} \times \mathrm{N}' \to \mathbf{G}_m$ has a canonical extension to a pairing $\mathcal{N} \times \mathcal{N}' \to \mathbf{G}_{m,\mathrm{R}}$ ([2], § 1.4). This pairing is trivial on $\mathcal{T}_p$ and $\mathcal{T}'_p$ and the quotient pairing $\mathcal{B}_p \times \mathcal{B}'_p \to \mathbf{G}_{m,\mathrm{R}}$ is the non-degenerate pairing defined by a Poincaré divisorial correspondence on $(\mathcal{B}, \mathcal{B}')$ ([2], § 7.4, 7.5). This shows that T_p (resp. T'_p) is the left (resp. right) kernel in the pairing $\mathrm{N} \times \mathrm{N}' \to \mathbf{G}_m$. The pairing $\mathrm{A}_p/\mathrm{T}_p \times \mathrm{N}' \to \mathbf{G}_m$ is right non-degenerate. But $\mathrm{A}_p/\mathrm{T}_p$ has rank $p^{2n-\mu}$ where $n = \dim(\mathrm{A})$ and $\mu = \dim(\mathcal{T})$ and N' has rank $p^{\mu+2\alpha}$, where $\alpha = \dim(\mathcal{B})$ ($cf.$ [2], § 2.2.7). This shows that the pairing is also left non-degenerate (because $n = \mu + \alpha$), which completes the proof of our assertion.

Consider the commutative diagram :

$$\begin{array}{ccc} \mathcal{A}^{0}(\mathrm{R})^{(p)} & \longrightarrow & \mathrm{H}^1(\mathrm{R}, \mathcal{A}^{0}_{p}) \\ \downarrow & & \downarrow \\ \mathrm{A}(\mathrm{K})^{(p)} & \longrightarrow & \mathrm{H}^1(\mathrm{K}, \mathrm{A}_p) \end{array}$$

in which the horizontal maps are bondary maps in the cohomology sequences for multiplication by p on A and $\mathcal{A}^{0}$. $\mathrm{H}^1(\mathrm{R}, \mathcal{A}^{0}_{p}) \approx \mathrm{H}^1(\mathrm{R}, \mathcal{N})$ because $\mathcal{A}^{0}_{p}/\mathcal{N}$ is smooth over R with zero special fibre and so has zero cohomology groups ([1], § 11.7) (including in dimension 0). The top arrow is an isomorphism because $\mathrm{H}^1(\mathrm{R}, \mathcal{A}^{0}) = 0$ ($loc.\ cit.$). The cokernel of the left vertical arrow is $\Phi_0(k)^{(p)}$, where Φ_0 is the group of connected components of $\mathcal{A} \otimes_{\mathrm{R}} k$ ($cf.$ [2], § 11.1). Using all of this, one can extract from the top diagram on p. 275 of [3] (with $m = p$) an exact commutative diagram :

$$\begin{tikzcd} 0 \arrow[r] & \Phi_0(k)^{(p)} \arrow[r] \arrow[d] & H^1(K, A_p)/H^1(R, \mathcal{N}) \arrow[r] \arrow[d, "\psi_1"] & H^1(K, A)_p \arrow[r] \arrow[d, "\psi_2"] & 0 \\ & 0 \arrow[r] & H^1(R, \mathcal{N}')^* \arrow[r] \arrow[d] & \mathcal{A}'^0(R)^{(p)*} \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

It is easy to see that $\theta_K(A)_p$ is an isomorphism if and only if

$$[\ker \psi_2] = [\hat{A}(K)^{(p)}/\mathcal{A}'^0(R)^{(p)}], \qquad \text{i. e.} \quad [\ker \psi_2] = [\Phi'_0(k)^{(p)}].$$

We shall show that

$$[\ker \psi_1] = p^{2\mu}, \qquad [\Phi_0(k)^{(p)}] = p^{\mu} = [\Phi'_0(k)^{(p)}],$$

and as $[\ker \psi_2][\Phi_0(k)^{(p)}] = [\ker \psi_1]$, this completes the proof.

Consider first the situation : M is a finite group scheme over K and $\mathcal{N}$ and $\mathcal{N}'$ are finite flat group schemes over R with given embeddings $N \to M$, $N' \to \hat{M}$. If $\mathcal{N} = \mathcal{B}_p$ for some abelian scheme $\mathcal{B}$ over R and $M = N$, $\mathcal{N}' = \hat{\mathcal{N}}$, then

$$\psi: \quad H^1(K, M)/H^1(R, \mathcal{N}) \to H^1(R, \mathcal{N}')^*,$$

the map defined by the cup-product pairing

$$H^1(K, M) \times H^1(K, \hat{M}) \to H^2(K, \mathbf{G}_m),$$

is an isomorphism [3]. If $\mathcal{N} = \boldsymbol{\mu}_p$, $M = N$, and $\mathcal{N}' = 0$, then $[\ker \psi] = p$ because [3]

$$H^1(K, \boldsymbol{\mu}_p)/H^1(R, \boldsymbol{\mu}_p) \approx H^1(R, \mathbf{Z}/p\,\mathbf{Z})^* \approx H^1(k, \mathbf{Z}/p\,\mathbf{Z})^*.$$

If $M = \mathbf{Z}/p\,\mathbf{Z}$, $\mathcal{N} = 0$, and $\mathcal{N}' = \boldsymbol{\mu}_p$, then $[\ker \psi] = p$ because [3] $\ker \psi = H^1(R, \mathbf{Z}/p\,\mathbf{Z})$. It follows from this, and the above discussion of the structures of A_p and $\hat{A}_p$, that $[\ker \psi_1] = p^{2\mu}$.

Finally, let $\Phi = \mathcal{A}^{\natural}/\mathcal{A}^{\natural 0}$. It is a finite étale group scheme over R such that $\Phi \otimes_R k = \Phi_0$, and there is an exact sequence

$$0 \to \mathcal{N} \to \mathcal{A}_p^{\natural} \to \Phi_p \to 0.$$

$\mathcal{A}_p^{\natural}(R) \approx \mathcal{A}_p(R)$, because $\mathcal{A}_p^{\natural}$ and $\mathcal{A}_p$ differ only by a scheme with empty special fibre, and $\mathcal{A}_p(R) \approx \mathcal{A}_p(K)$. It follows that $\Phi_p(K) = A_p(K)/N(K)$ has p^{μ} elements. But

$$\Phi(K) \approx \Phi(R) \approx \Phi_0(k) \qquad \text{and so} \qquad [\Phi_0(k)^{(p)}] = [\Phi_0(k)_p] = p^{\mu}.$$

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