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Annales scientifiques de l'É.N.S. 4e série, tome 6, nº 1 (1973), p. 85-94
<http://www.numdam.org/item?id=ASENS_1973_4_6_1_85_0>

Ann. scient. Éc. Norm. Sup.,
4^e série, t. 6, 1973, p. 85 à 94.

ON A QUESTION OF SWAN IN ALGEBRAIC K-THEORY

BY PRAMOD K. SHARMA AND JAN R. STROOKER

0. In their proposal for an algebraic K-theory Karoubi and Villamayor attach functorially to every short exact sequence of rings

$$0 \to A \to B \xrightarrow{f} C \to 0$$

where f is a fibration, a long exact sequence of abelian groups

$$\begin{aligned}(1)\qquad & \ldots \to K^{-n}(A) \to K^{-n}(B) \to K^{-n}(C) \to K^{-n+1}(A) \to \ldots \\ & \to K^{-1}(C) \to K_0(A) \to K_0(B) \to K_0(C)\end{aligned}$$

in which K_0 is the usual functor of Grothendieck and K^{-n} is defined starting with the functor K^{-1} and putting $K^{-n}(R) = K^{-1}(\Omega^{n-1} R)$ for every ring R and $n \geqq 1$. For the definitions of fibration, Ω and other unexplained notions, *see* section 1.

Now the K^{-n} are homotopy functors for all $n \geqq 1$; but one knows that K_0 is not ([3], 3). It is therefore natural to enquire whether the sequence (1) remains exact if we replace K_0 by its homotopic counterpart K^0. If the new sequence H(1) were exact for all fibrations, it would mean that, rather than K^{-1}, the functor K^0 is the basic one and we may define $K^{-n}(R) = K^0(\Omega^n R)$ as there exists then a functorial isomorphism $K^{-1}(\Omega^{n-1} R) \cong K^0(\Omega^n R)$. This again is equivalent to saying that the formula

$$(2)\qquad K^{-n}(R[t, t^{-1}]) = K^{-n}(R) \oplus K^{-n+1}(R)$$

holds functorially for all rings R and $n \geqq 1$. Formula (2) in turn would imply that for a stable fibration $f : B \to C$ the sequence H(1) is exact.

In this form the question was raised by Swan in his address to the Nice congress ([10], 5) (beware of two misprints). In the paper we obtain positive results in some cases but show that in general the answer is negative.

Our investigation was begun at the Tata Institute of Fundamental Research, Bombay, where we collaborated with F. Bachmann; we also thank several authors, in particular S. M. Gersten, for making their work available to us before publication.

1. The theory of Karoubi and Villamayor has been developed by these authors in [6] and [7], by Gersten in [2], [3], [4] and [5] and by Swan in [12]. We review the facts we shall need, confining our attention to discrete rings.

For any ring R (not necessarily with 1) we write ΔR for the polynomial ring in one indeterminate (free paths on R) R[X], ER for the principal ideal (paths with base point) XR[X] and ΩR for the ideal (loops) X(1 — X) R[X]. There are two split projections p_0, $p_1 : \Delta R \rightrightarrows R$ given by $p_0(\Sigma r_i X^i) = r_0$ and $p_1(\Sigma r_i X^i) = \Sigma r_i$. Then $ER = \ker p_0$ and the restriction of p_1 to ER is called ε. Thus

$$0 \to \Omega R \to ER \xrightarrow{\varepsilon(R)} R \to 0$$

is functorially a short exact sequence of rings.

Two ring homomorphisms $f, g : A \rightrightarrows B$ are called simply homotopic if there exists a ring homomorphism $s : A \to \Delta B$ such that $f = p_0 s$ and $g = p_1 s$. The transitive closure of this relation defines homotopy between ring homomorphisms, which is compatible with composition of maps. The category of rings furnished with homotopy classes of maps as morphisms is called **Hot Rg**. A functor G from **Hot Rg** to the category of groups **Gr** (or abelian groups **Ab**) is called a homotopy functor; it may be composed with the natural functor $h : \mathbf{Rg} \to \mathbf{Hot\,Rg}$ to obtain a full embedding of functor categories

$$(\mathbf{Hot\,Rg}, \mathbf{Gr}) \xrightarrow{-h} (\mathbf{Rg}, \mathbf{Gr}).$$

This embedding has a left adjoint H, making (**Hot Rg**, **Gr**) into a reflective subcategory of (**Rg**, **Gr**); in fact, for any functor $F : \mathbf{Rg} \to \mathbf{Gr}$ the homotopy functor HF is just defined by HF(R) is the coequalizer of the maps $F p_0(R)$ and $F p_1(R) : F(\Delta R) \rightrightarrows F(R)$. The functor H is right exact. A ring R is called contractible if the identity map and the zero map of R are homotopic or, equivalently, if the map $\varepsilon(R)$ splits; ER is always a contractible ring, the splitting $ER \to E^2 R$ being given by $X \mapsto XY$. A homotopy functor F clearly vanishes on a contractible ring, but it is necessary and sufficient that $F(R) \to F(\Delta R)$ be an isomorphism for all R.

The general linear group extends to a functor $GL : \mathbf{Rg} \to \mathbf{Gr}$ if one defines GL(R) as the kernel of the split epimorphism $GL(R^+) \to GL(\mathbf{Z})$ where R^+ is the ring obtained from R by " adding a unit ". This accords with the original GL(R) in case R already possesses one. Similarly, one extends K_0 and K_1 to rings without unit. Now suppose $F : \mathbf{Rg} \to \mathbf{Gr}$ (or $\mathbf{Ab}$) is such that for every ring R the split surjection $p_0(R)$ gives rise to an exact sequence

$$F(ER) \to F(\Delta R) \to F(R). \tag{3}$$

Then $HF(R) = \operatorname{cok}(F\varepsilon(R) : F(ER) \to F(R))$ as is easily seen. The sequence (3) is exact for $F = GL$, $F = K_0$ or K_1. Indeed, GL is even left exact and the functors K_i are connected by Gersten's exact sequence ([2], 5.7) stemming from the split surjection $p_0(R)$.

Write EL(R) for the elementary group of R (extended to $\mathbf{Rg}$ as domain in the same way as GL) and consider the exact sequence of functors $1 \to EL \to GL \to K_1 \to 1$ which defines K_1. Apply the right exact functor H to obtain an exact sequence $HEL \to HGL \to HK_1 \to 1$. Now EL preserves the surjection $\varepsilon(R)$ and HEL vanishes on the contractible ring ER; hence $HEL(R) = 1$ for all rings R. Therefore $HGL = HK_1$ and it is this functor which is called K^{-1}; one knows that $K^{-1} = GL/UN$ (UN(R) is the subgroup of GL(R) generated by unipotent matrices) is a factor group of $K_1 = GL/EL$. As recalled before, one then defines $K^{-n} = K^{-1}\,\Omega^{n-1}$, $n \geqq 1$, which are all homotopy functors.

A ring homomorphism $f : B \to C$ is called a fibration provided $GL(E^n f) : GL(E^n B) \to GL(E^n C)$ is surjective for every $n \geqq 1$. A fibration is certainly surjective, but not every surjection of rings is a fibration. Put $A = \ker f$, then if f is a fibration one has the long exact sequence (1). The functors K^{-n}, $n \geqq 1$, enjoy the excision property : $K^{-n}(A)$ is defined as $\ker(K^{-n}(A^+) \to K^{-n}(\mathbf{Z}))$ and is therefore independent of the ring B in which A is an ideal. Of course, this property is also known to hold for K_0.

We now consider the process of adjoining Laurent polynomial variables. Put $\Lambda R = R[t, t^{-1}]$. A fibration $f : B \to C$ is called stable if $\Lambda^m f : \Lambda^m B \to \Lambda^m C$ is a fibration for all $m \geqq 0$. It is clear that functors from $\mathbf{Rg}$ to $\mathbf{Rg}$ which commute (up to isomorphism) with Δ and the p_i, will preserve homotopy between maps and also contractible rings. Such are the functors Δ, E, Ω and Λ which moreover commute with each other and are exact. Furthermore, the first three are known to preserve fibrations and then, by definition, all four of them stable fibrations, ([6], 2) and ([5], 2). For all rings, ε is a stable fibration.

2. A ring R is called K-semiregular if

$$K_0(R) \to K_0(R[X_1, \ldots, X_n]) = K_0(\Delta^n R)$$

is an isomorphism for any number of polynomial variables or, equivalently, if $K_0(E^n R) = 0$ for all $n \geqq 1$ ([5], 1.3).

A ring R is called K-regular if $\Lambda^m R$ is K-semiregular for all $m \geqq 0$. This notion is due to Karoubi who observed that a noetherian regular ring is certainly K-regular by a result of Bass-Heller-Swan ([7], 3.6). The functors Δ, E and Ω preserve both K-semiregular and K-regular rings ([7], 3.8), while Λ preserves K-regular rings by definition.

For any ring R with 1 there is a split exact sequence due to Bass ([1], XII, 7.4; [7], 2.6) :

$$0 \to K_1(R) \rightleftharpoons K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \rightleftharpoons K_0(R) \to 0,$$

which is functorial in R. Apply the right exact functor H, bearing in mind that $HK_1 = K^{-1}$ is a homotopy functor, to obtain a split short exact sequence

$$0 \to K^{-1}(R) \to K^{-1}(R[t, t^{-1}]) \to K^0(R) \to 0 \tag{4}$$

where we have defined K^0 as the homotopy functor HK_0. One may extend this to rings R without 1 by considering the augmentation $0 \to R \to R^+ \to \mathbf{Z} \to 0$ and observing that Λ, K^{-1} and K_0 all preserve a splitting.

Consider now the composite morphism

$$\gamma(R) : K^{-1}(R) \to K_0(\Omega R) \to K^0(\Omega R)$$

in which the first map is the connecting homomorphism $\delta(R)$ in the sequence (1) attached to the fibration $\varepsilon(R)$ and the second is passing to homotopy : $K_0(\Omega R) \to HK_0(\Omega R)$. Thus $\gamma(R)$ maps

$$\ker(K_0(\Omega R) \to K_0(ER)) \quad \text{to} \quad \operatorname{cok}(K_0(E\Omega R) \to K_0(\Omega R))$$

or, briefly, $\gamma = H\delta$. If the sequence H(1), derived from (1) by applying the functor H, were always exact, it would in particular be so for the fibration $\varepsilon(R)$ and $\gamma(R)$ should be an isomorphism for all rings R.

Suppose now that $K^{-1}(\Omega^{n-1}R) \simeq K^0(\Omega^n R)$, $n \geqq 1$, for a certain ring R, i. e. the $\gamma(\Omega^{n-1}R)$ are all isomorphisms. For this ring we obtain the formula (2) by applying (4) to the ring $\Omega^{n-1}R$ and taking into account that the functors Ω and Λ commute. This is substantially more general than ([7], 3.11).

This happy state certainly prevails for a K-semiregular ring ([3], 2.2). There we even have isomorphisms $K^{-1}(\Omega^{n-1}R) \simeq K_0(\Omega^n R) \simeq K^0(\Omega^n R)$ for all $n \geqq 1$ (the latter also holds for $n = 0$), since $K^{-1}(E\Omega^{n-1}R) = 0$ because $E\Omega^{n-1}R$ is contractible while $K_0(E\Omega^{n-1}R)$ and $K_0(E\Omega^n R)$ are both 0 because Ω preserves K-semiregularity. In this case $K^{-n}(R[t, t^{-1}])$

is expressed entirely in terms of K_0 as

$$K^{-n}(R[t, t^{-1}]) = K_0(\Omega^n R) \oplus K_0(\Omega^{n-1} R). \tag{5}$$

Remark. — From our point of view, it would be more logical to accord the " regular " terminology with that regarding fibrations. Thus Karoubi's " K-regular " would become " stably K-regular ", Gersten's " K-semiregular " would become " K-regular " and a ring with all $\gamma(\Omega^{n-1} R)$ isomorphisms would be called " K-semiregular ". Alternatively one may follow Swan and speak of a " stable fibration " as a " fibration " etc. But it is not yet clear which of these notions should be regarded as basic, so we hesitate to introduce a shift in nomenclature.

3. We now list a few situations in which we may conclude that the sequence $H(1)$ is exact.

PROPOSITION 1. — *Let* $f : B \to C$ *be a surjection of rings. In the following cases* $H(1)$ *is exact :*

(*a*) *The surjection* f *is split and* $K_0(EC) = 0$;

(*b*) f *is a stable fibration and* $\gamma(B)$ *and* $\gamma(C)$ *are isomorphisms;*

(*c*) f *is a fibration and* $K_0(EB) = K_0(EC) = 0$; *in this case the sequences* (1) *and* $H(1)$ *coincide.*

Proof :

a. First remark that any split surjection is a (stable) fibration hence the sequence (1) is exact with $K^{-1}(B) \to K^{-1}(C)$ split surjective. Therefore $K_0(A) \to K_0(B)$ is injective, while $K_0(EC) = 0$. Apply the Snake Lemma to connect kernels and cokernels of the maps $K_0\varepsilon$ plying between the K_0-sequences of Ef and f to obtain the split exact sequence

$$0 \to K^0(A) \to K^0(B) \to K^0(C) \to 0.$$

Join the exact sequences together to obtain the result.

b. Consider the splitting (4) for R and for ΩR and observe that

$$K^{-1}(\Lambda\Omega R) = K^{-1}(\Lambda\Omega R) = K^{-2}(\Lambda R).$$

The connecting homomorphisms in the sequence (1) attached to the fibration Λf are compatible with such splittings hence there is an exact sequence

$$K^0(\Omega B) \to K^0(\Omega C) \to K^0(A) \to K^0(B) \to K^0(C).$$

Together with the isomorphisms postulated in the data and the exact sequence (1) this furnishes the proof.

c. The fibration f yields a surjection $K_1(EB) \to K_1(EC)$, consequently the sequence

$$0 \to K_0(EA) \to K_0(EB) \to K_0(EC)$$

is exact. Since the last two groups are 0, so is $K_0(EA)$ and the sequences H(1) and (1) coincide.

In particular this is the case if B and C are K-semiregular. So is then A and the sequences (1) = H(1) belonging to the fibrations $\Delta^n f$ are all identical for $n \geq 0$.

4. Gersten has established some results on K-regularity in ([5], 1), based on the work of Bass-Murthy ([1], XII, 10).

First, a ring R which is (left) regular modulo some nilpotent ideal, is certainly K-regular. Further, the picture is fully clear for unital commutative noetherian rings of dimension 1 which are reduced and of finite normalization. Let A be such a ring, B its integral closure in its full ring of fractions and $\mathfrak{c}$ the conductor of A in B. The ring A is called seminormal if $\mathfrak{c}$ is square free as an ideal of B [13].

PROPOSITION 2. — *The ring* A *is* K-*semiregular if and only if it is seminormal. In that case it is even* K-*regular.*

Gersten states this for group rings $\mathbf{Z}\pi$, π a finite abelian group ([5], 1.8), but his proof really treats this as a special case of the above.

For an ideal $\mathfrak{a}$ in a Dedekind domain A he has also shown the equivalence of the following conditions (1) and (2) ([5], 1.6); (3) is equivalent by ([7], 3.7) :

(1) $\mathfrak{a}$ is square free;

(2) $\mathfrak{a}$ is K-semiregular;

(3) $A \to A/\mathfrak{a}$ is a fibration.

In this case, $\mathfrak{a}$ is even K-regular and $A \to A/\mathfrak{a}$ is a stable fibration.

Remark. — In Gersten's proof, the isomorphism should read

$$K_1(\Delta A/\mathfrak{a}, EA/\mathfrak{a}) \cong K_0(\Delta A, EA) = K_0(EA).$$

His assertion that $K_1(EA) = 0$ for a regular ring A appears to be unproven and indeed unlikely in view of the obstruction to excision for K_1 described in ([11], 4.6).

We shall now exhibit in sections :

5. A ring R for which all $\gamma(\Omega^n R)$, $n \geq 0$, are isomorphisms but $K_0(\Omega R) \neq K^0(\Omega R)$;

6. A ring R with $\gamma(R)$ not an isomorphism. For this ring, the sequence H(1) attached to the fibration $\varepsilon(R)$ is not exact.

Both rings will be ideals in Euclidean domains.

5. For a given ring k, suppose R is a contractible subring of $Ek = Xk[X]$. Contractible rings being preserved by Ω, we know that

$K^{-1}(\Omega^n R) = K^0(\Omega^n R) = 0$ for $n \geq 0$, therefore the sequence H(1) attached to the fibration $\varepsilon(R)$ consists of only zeroes. As desired, all $\gamma(\Omega^n R)$ are isomorphisms.

Put $A \subset k[X]$ for the ring R augmented by k. Then $K^{-n}(A) = K^{-n}(k)$ for $n \geq 0$, and, because Λ preserves splittings and contractible rings, we have $K^{-n}(\Lambda A) = K^{-n}(\Lambda k)$. If k for instance satisfies (2), so does A.

It is now easy to produce contractible rings with $K_0(\Omega R) \neq 0$. In order to be concrete, take k a field and R the principal ideal (X^2) in the polynomial ring $k[X]$. The mapping $X \mapsto XY$ induces a homomorphism $R \to ER$ which splits $\varepsilon(R)$, hence R is contractible. According to 4 it is not K-semiregular. Since ([1], XII, 10.5), the ring $A = k[X^2, X^3]$ is an old favourite so we shall not dwell on the computations. These show that the K_0-tail end of the sequence (1) attached to the fibration $\varepsilon(R)$ may be identified with the short exact sequence

$$0 \to \Omega k \to E k \xrightarrow{\varepsilon(k)} k \to 0,$$

hence $K_0(\Omega R) = \Omega k$ does not vanish.

Turning to A, we see that $K_0(A) = \mathbf{Z} \oplus k$ while $K^0(A) = \mathbf{Z}$; in point of fact, almost the same example illustrated Gersten's statement ([3], 3) that K_0 is not a homotopy functor.

Furthermore, $K^{-1}(A) = K^{-1}(k) = K_1(k) = U(k)$, the multiplicative group of k. But Krusemeyer and Van der Kallen have proved ([8], 12.1) that $SK_1(A)$ depends on the size of the field. For k is, for instance, the real field, one has $K_1(A) = k \oplus V$ where V is a real vector space of dimension equal to the continuum. So not only are fibrations rather special, but passing to homotopy functors by means of H looses a good deal of information.

6. Consider a prime $p \in \mathbf{Z}$ and for typographical ease also denote the ideal generated by it as p. Apply the functors Ω and $E\,\Omega$ to the short exact sequence $0 \to p^n \to p \to p/p^n \to 0$. Write down Gersten's exact sequences, connected by ε, thus

$$\begin{array}{ccccccccc} K_1(E\,\Omega\, p) & \longrightarrow & K_1(E\,\Omega\, p/p^n) & \longrightarrow & K_0(E\,\Omega\, p^n) & \longrightarrow & K_0(E\,\Omega\, p) & \longrightarrow & K_0(E\,\Omega\, p/p^n) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\varepsilon} \\ K_1(\Omega\, p) & \longrightarrow & K_1(\Omega\, p/p^n) & \longrightarrow & K_0(\Omega\, p^n) & \longrightarrow & K_0(\Omega\, p) & \longrightarrow & K_0(\Omega\, p/p^n) \end{array} \tag{6}$$

Here, $K_1(\Omega p/p^n)$ is by definition Bass' relative group $K_1((\Omega p/p^n)^+, \Omega\, p/p^n)$ Since our rings are commutative, we know the latter group splits as

$$SK_1((\Omega\, p/p^n)^+, \Omega\, p/p^n) \oplus U((\Omega\, p/p^n)^+, \Omega\, p/p^n).$$

Now notice that the ideal $\mathfrak{a} = \Omega\, p/p^n$ is nilpotent, hence a well-known Lemma ([1], IX, 1.3) asserts that the SK_1 term vanishes; the relative units are isomorphic to the ideal $\mathfrak{a}$ made into a group by the circle operation : $a \circ b = a + b + ab$. Thus $K_1(\Omega\, p/p^n) \cong \Omega\, p/p^n$ and for the same reason $K_1(E\,\Omega\, p/p^n) \cong E\,\Omega\, p/p^n$ so that $K_1\,\varepsilon(\Omega\, p/p^n)$ just yields the surjection $E\,\Omega\, p/p^n \to \Omega\, p/p^n$.

Now consider the stable fibration $\mathbf{Z} \to \mathbf{Z}/p$ and remark that the kernel p is K-regular by proposition 2. So is then $\Omega\, p$ hence $K_0(E\,\Omega\, p) = 0$. Consequently

$$K^0(\Omega\, p^n) \cong \operatorname{cok}(K_1(\Omega\, p/p^n) \to K_0(\Omega\, p^n)).$$

We claim that this last map is injective. Indeed, because of the splitting $K_1 = SK_1 \oplus U$ we need only consider the image of $U((\Omega\, p)^+, \Omega\, p)$. But this group is trivial. Thus $K^0(\Omega\, p^n) \not\cong K_0(\Omega\, p^n)$ when $n \geqq 2$.

For any regular ring R, a theorem of Bass-Heller-Swan asserts that R is K_1-regular, i. e. for the rings $\Lambda^m R$ the K_1 remains unchanged under polynomial extensions. This implies that any surjection onto a regular ring is a stable fibration ([6], 2.6; [5], 2.1).

Apply the functor Ω to the stable fibration $\mathbf{Z}/p^n \to \mathbf{Z}/p$ and take the Karoubi-Villamayor sequence (1) associated with this. Since we divide out by a nilpotent ideal, $K_0(\Omega\,\mathbf{Z}/p^n) \to K_0(\Omega\,\mathbf{Z}/p)$ is an isomorphism ([1], IX, 1.3). Now $K^{-1}(\Omega\,\mathbf{Z}/p) = K^{-2}(\mathbf{Z}/p) \cong K_2(\mathbf{Z}/p)$. The isomorphism holds because $\mathbf{Z}/p$ is a field ([12], 4.7); since $\mathbf{Z}/p$ is a finite field, $K_2(\mathbf{Z}/p) = 0$ ([9], 9.13).

We conclude that $K_0(\Omega\, p/p^n) = 0$. Going back to our diagram (6) we now know that $K^0(\Omega\, p^n) = K_0(\Omega\, p)$.

We have noticed that p is K-regular, so $K_0(\Omega\, p) = K^{-1}(p)$. To compute this group, write down a segment of the sequence (1) attached to the stable fibration $\mathbf{Z} \to \mathbf{Z}/p$:

$$\ldots \to K^{-2}(\mathbf{Z}/p) \to K^{-1}(p) \to K^{-1}(\mathbf{Z}) \to K^{-1}(\mathbf{Z}/p) \to \ldots. \tag{7}$$

The first term is trivial, and

$$K^{-1}(\mathbf{Z}) = K_1(\mathbf{Z}) = U(\mathbf{Z}) \qquad \text{while} \quad K^{-1}(\mathbf{Z}/p) = K_1(\mathbf{Z}/p) = U(\mathbf{Z}/p).$$

Since the units of $\mathbf{Z}$ go into the units of $\mathbf{Z}/p$ we find

$$K^{-1}(2) = \mathbf{Z}/2 \qquad \text{but} \quad K^{-1}(p) = 0 \quad \text{for } p > 2.$$

Thus $K^0(\Omega\, p^n) = 0$ for $p > 2$ but $\mathbf{Z}/2$ for $p = 2$.

It remains to compute the groups $K^{-1}(p^n)$. First, $K^{-1}(p^n) = HK_1(p^n)$; by definition, $K_1(p^n) = K_1((p^n)^+, p^n)$. Since $(p^n)^+$ maps surjectively to $\mathbf{Z}$, we have the excision isomorphism $K_1((p^n)^+, p^n) \cong K_1(\mathbf{Z}, p^n)$ ([9], 6.3). The latter group is known to be $U(\mathbf{Z}, p^n)$, the SK_1 term being 0 ([1], VI, 7.3). Looking at the relative units shows that $K_1(p^n) = 0$ except for $p = 2$ and $n = 1$, where $K_1(2) = \mathbf{Z}/2$. For all p^n, the K_1 and K^{-1} coincide.

The upshot is that for $R = 2^n$, $n \geqq 2$, we have

$$0 = K^{-1}(R) \ncong K^0(\Omega R) = \mathbf{Z}/2;$$

so $\gamma(R)$ is not an isomorphism. For the stable fibration $\varepsilon(R)$ the sequence H(1) fails to be exact.

Adding a unit, put $A = (2^n)^+$. There are isomorphisms

$$A \cong \mathbf{Z}[X]/(X^2 - 2^n X) \cong \mathbf{Z}[Y]/(Y^2 - 2^{2n-2})$$

described by $(m, k.2^n) \mapsto m + kX$ and $X \mapsto Y + 2^{n-1}$ respectively. One readily verifies that

$$K^{-2}(A[t, t^{-1}]) = K^{-2}(A) \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2 \quad \text{while} \quad K^{-1}(A) = \mathbf{Z}/2,$$

violating formula (2).

By keeping track of the prime factors, it is now easily seen that the ring $(m)^+$ provides a counter example to Swan's question whenever 4 divides m.

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(Manuscrit reçu le 26 juillet 1972.)

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