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Stable vector bundles and the frobenius morphism

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1. Let $X$ be a curve of genus $g$, proper and smooth over an algebraically closed field, and let $E$ be a vector bundle over $X$. Mumford defines $E$ to be semi-stable if whenever $F$ is a quotient bundle of $E$, then

\[
\frac{\deg F}{\text{rank } F} \geq \frac{\deg E}{\text{rank } E},
\]

where $\deg E$ is the degree of the line bundle $\mathcal{O}_E$, $r$ the rank of $E$. If the characteristic of $X$ is $p > 0$, $E^{(p)}$ will denote Frobenius pullback of $E$.

**Theorem 1.** For each prime $p$ and integer $g > 1$, there is a curve $X$ of genus $g$ in characteristic $p$ and a semi-stable bundle $E$ of rank two on $X$ so that $E^{(p)}$ is not semi-stable.

Examples of non-ample semi-stable bundles of positive degree constructed by Serre for $p = g = 3$ and later by Tango for $p(p - 1) = 2g$ incidentally proved Theorem 1 when $p(p - 1) = 2g$.

We prove Theorem 1 by constructing a sequence of bundles $E_n$ so that $E_n^{(p)}$ is isomorphic to $E_{n-1}$, and $E_1$ is not semi-stable. In such a sequence, we must have $E_n$ semi-stable for $n \gg 0$, and then we obtain the $E$ of Theorem 1 as the first semi-stable $E_n$.

The bundles $E_n$ will be constructed in the following situation: Let $\Lambda = k[[t]]$, where $k$ is a field of characteristic $p > 0$, and let $X$ be a stable curve over $\Lambda$ with $k$-split degenerate fiber in the sense of Mumford [5]. Thus by definition $X$ is proper and flat over $\Lambda$, and its geometric fibers are reduced, connected and one dimensional. Further, all the normalizations of the components of the special fiber $X_s$ of $X$ are isomorphic to $\mathbb{P}^1$, and the singularities of $X_s$ are double points with $2k$-rational branches. Further each component $X_s$ meets at least three other
components counting itself. We also assume the generic fiber is smooth over \( K \), the quotient field of \( A \).

Let \( Y_0 \) be the universal covering scheme of \( X_0 \), i.e. there is an etale map \( p_0 \) from \( Y_0 \) to \( X_0 \) with the usual universal mapping property. \( Y_0 \) is not of finite type over \( A \). Mumford shows that the group \( G \) of covering transformations of \( Y_0 \) over \( X_0 \) is a free group on \( g \) generators, \( g \) the genus of \( X_k \). \( G \) operates freely and discontinuously in the Zariski topology of \( Y_0 \).

Section two is devoted to associating to each representation \( \rho \) of \( G \) on \( K^m \) a sequence of bundles \( E_n \) on \( X_k^{(n)} \) so that \( F^* E_n \) is isomorphic to \( E_{n-1} \), where \( X_k^{(n)} \) is the fiber product of \( X \) with the \( n \)th iterate of the Frobenius map on \( \text{Spec} \, K \), and \( F \) is relative Frobenius. The construction of \( E_n \) from \( \rho \) is analogous to the construction of a bundle \( E' \) on a smooth, compact complex variety \( X' \) from a representation \( \rho' \) of the fundamental group of \( X' \) on \( \mathbb{C}^m \). Further, the sequence \( |E_n| \) defines a stratification on \( E_1 \), which is analogous to the stratification on \( E' \) whose monodromy is \( \rho' \) [2].

Section three is devoted to the study of the bundle associated to a particular representation \( \rho \) of \( G \) on \( K^2 \) which arises in Mumford's work. We show that the \( E_1 \) associated to \( \rho \) is not semi-stable. This \( \rho \) is analogous to the following \( \rho' \) associated to a compact Riemann surface \( X' \). Let \( a_1, \ldots, a_g, b_1, \ldots, b_g \) be the usual generators of \( \pi_1(X') \), and let \( U \) be an open subset of \( \mathbb{P}^1 \), and let \( \pi \) be a covering map from \( U \) to \( X' \). Assume that the group \( G \) of covering transformations acts on \( U \) by linear fractional transformations, and that \( G \) is freely generated by the images of \( b_1, \ldots, b_g \). Such a \( \pi \) is called a Schottky uniformization. Thus we have a homomorphism from \( G \) to \( \text{PGL}(2, \mathbb{C}) \), and this lifts to a homomorphism \( \rho' \) of \( \pi_1(X) \) to \( \text{SL}(2, \mathbb{C}) \). Following Gunning, one may show the bundle \( E \) associated to \( \rho' \) is an extension

\[
0 \to L \to E \to L^{-1} \to 0
\]

where \( L^\otimes 2 \) is isomorphic to \( \Omega^1_{X, \mathbb{C}} \). In particular, \( E \) is not semi-stable. The representation \( \rho \) of \( G \) on \( K^2 \) is the rigid analytic analogue of \( \rho' \), and the bundle \( E_1 \) associated to \( \rho \) is an extension of the above type.

We conclude by noting that semi-stable bundles are not closed under symmetric product and with some examples of semi-stable bundles of positive degree which are not ample.

2. \( X \) will continue to denote a stable curve over \( A \) with smooth generic fiber and \( k \)-split degenerate fiber, \( Y_0 \) the universal covering space of \( X_0 \), and \( G \) the group of covering transformations of \( Y_0 \). There is a unique structure of a formal scheme \( Y \) with underlying space \( Y_0 \) and an etale
map $p$ of $Y$ to $\hat{X}$ which reduces to $p_o$, $\hat{X}$ being the completion of $X$ along $X_o$.

**Definition.** — Meromorphic descent data on a coherent sheaf $F$ over $Y$ is a collection of elements $h_g \in \Gamma(Y, \underline{\text{Hom}}_{\text{crys}}(F, g^* f) \otimes \Lambda K)$ for each $g \in G$ so that

$$h_g \circ g^*(h_{g'}) = h_{gg'},$$

and $h_e$ is the identity. If $\{h_g\}$ and $\{k_g\}$ are sets of meromorphic descent data on $F$ and $G$ respectively, a map from $\{h_g\}$ to $\{k_g\}$ is an element $f \in \Gamma(Y, \underline{\text{Hom}}_{\text{crys}}(F, G) \otimes \Lambda K)$ so that

$$k_g \circ f = g^*(f) \circ h_g.$$

We will show the category of coherent sheaves on $Y$ with meromorphic descent data is equivalent to the category of coherent sheaves on $X_o$.

**Lemma 1.** — Given meromorphic descent data on a coherent sheaf $F$ on $Y$, there is a coherent $F'$ with descent data $h'_g \in \underline{\text{Hom}}_{\text{crys}}(F', g^* F')$ so that $\{h_g\}$ and $\{h'_g\}$ are isomorphic. $F'$ may be taken to have no $\Lambda$ torsion.

**Proof.** — We may assume $F$ has no $\Lambda$ torsion by replacing it by its image in $F \otimes \Lambda K$. We will construct a coherent subsheaf $F'$ of $F \otimes \Lambda K$ so that the map of $F' \otimes \Lambda K$ to $F \otimes \Lambda K$ is an isomorphism and so that

$$h_g(F') = g^* F'$$

where we are regarding $h_g$ as a map of $F \otimes \Lambda K$ to $g^* (F \otimes \Lambda K)$. Suppose such an $F'$ has been constructed over a $G$ invariant open set $U$ of $Y$, and let $V$ be a quasi compact open not contained in $U$ so that $V \cap g V \subseteq U$ if $g \neq e$. $V$ exists, since $G$ acts discontinuously and has no torsion.

$F'$ may be extended to a coherent subsheaf of $F \otimes \Lambda K$ over $V \cup U$ using the following idea of Raynaud. On $V \cap U$, we may find an $N$ so that

$$F \subseteq F' \subseteq t^{-N} F,$$

where $t$ is a uniformizing parameter of $\Lambda$. Let $\bar{F}'$ be the image of $F'$ in $t^{-N} F/t^N F$. $\bar{F}'$ is a coherent sheaf on a scheme whose sheaf of local rings is $\mathcal{O}/t^{-N} \mathcal{O}$. Thus $\bar{F}'$ extends to a coherent subsheaf $\bar{F}''$ of $t^{-N} F/t^N F$ over $V \cup U$. The inverse image $\bar{F}''$ of $\bar{F}''$ in $t^{-N} F$ extends $F'$. Finally, $F'$ may be extended to the $G$ invariant open set consisting of the union of the translates of $V \cup U$ by taking the subsheaf of $F \otimes \Lambda K$ generated by $h_g^{-1} (g^* F'')$ over $U \cap g^{-1} V$. 

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Given a coherent \( F \) on \( \hat{X} \), the natural map

\[
h^F : p^* F \to g^* p^* F
\]

gives meromorphic descent data on \( p^* F \).

**Lemma 2.** — Let \( \{ h^*_x \} \) be meromorphic descent data on a coherent \( F \). There is a coherent \( H \) on \( \hat{X} \) so that \( \{ h^*_x \} \) is isomorphic to \( \{ h^*_g \} \). Further the natural map \( \varpi \),

\[
\text{Hom}_{\mathcal{E}X} (H, H') \otimes \lambda K \xrightarrow{\varpi} \text{Hom}(\{ h^*_x \}, \{ h^*_g \})
\]
is an isomorphism.

**Proof.** — By Lemma 1, we may assume \( h^*_x \) maps \( F \) to \( g^* (F) \). There is a quasi-compact open \( U \) of \( Y \) so that the translates of \( U \) by \( G \) cover \( Y \). \( \{ h^*_x \} \) gives descent data for the morphism \( U \to \hat{X} \) and so \( F \) descends to a coherent \( H \) on \( \hat{X} \), and \( \{ h^*_x \} \) is isomorphic to \( \{ h^*_g \} \).

If \( f \in \text{Hom}(\{ h^*_x \}, \{ h^*_g \}) \), then \( t^n f \) gives a morphism from \( p^* H \) to \( p^* H' \) compatible with descent data, and so a morphism from \( H \) to \( H' \). Thus \( \varpi \) is surjective. On the other hand, if \( f \in \text{Hom}_{\mathcal{E}X} (H, H') \) and if \( \varpi (f) = 0 \), then \( t^n p^* (f) = 0 \), and so \( t^n f = 0 \) for some integer \( N \). So \( f \) is zero in \( \text{Hom}_{\mathcal{E}X} (H, H') \otimes \lambda K \), and \( \varpi \) is injective.

**Proposition 1.** — There is an equivalence of categories \( \varpi \) from the category of coherent sheaves on \( X_\kappa \) to the category of coherent sheaves on \( Y \) with meromorphic descent data. If \( F \) is a coherent sheaf on \( X \), \( \varpi (F_\kappa) \) is the descent data \( \{ h^*_g \} \) on \( p^*(\hat{F}) \).

**Proof.** — Consider the category \( C \) whose objects are coherent sheaves on \( X \), with \( \text{Hom}'(F, G) = \text{Hom}_{\mathcal{E}X} (F, G) \otimes \lambda K \). \( C \) maps isomorphically to the category of coherent sheaves on \( X_\kappa \). On the other hand, Grothendieck's existence theorem and lemma 2 show that it maps isomorphically to the category of coherent sheaves with descent data on \( Y \).

Any representation \( \zeta \) of \( G \) on \( \kappa^* \) gives meromorphic descent data \( \{ h^*_x \} \) on \( \mathcal{O}_Y \), and so a bundle \( F_{p, x} = \pi^{-1} \{ h^*_x \} \). When \( \text{char } k = p > 0 \), we let \( F : X \to X^{(p)} \) be the relative Frobenius morphism. \( X^{(p)} \) is a stable curve with \( k \) split degenerate fiber, and the fundamental group of its special fiber is \( G \). Further we have

\[
F_{p, x} = F^* (F_{p, x^e}).
\]
Thus we have proven:

**Proposition 2.** — If $\varphi$ is a representation of $G$ on $K^n$ and $F_1$ is pullback of $F_\varphi$ to $X_K$, then there is a sequence of bundles $F_1, F_2, \ldots$ so that $F_{k+1}^\varphi$ is isomorphic to $F_k$.

**Remark.** — There is in fact a unique stratification on $F_1$ associated to the sequence $|F_i|$ [2]. This stratification may be defined directly, and exists even when $\text{char } K = 0$.

3. Mumford's theory [5] gives us a natural representation of $G$ on $K^2$ in the following way: Let $D$ be a positive Cartier divisor on $Y$ so that $D$ meets only one component of $Y_0$, and let $L$ be the quotient field of

$$\bigcup_{n \geq 0} \Gamma(Y, \mathcal{O}_Y(nD)).$$

$L$ does not depend on $D$, and since $G$ acts on $Y$, we get a homomorphism from $G$ to the $K$-linear automorphisms $\text{PGL} (2, K)$ of $L$. This homomorphism may be lifted to a homomorphism $\tilde{\varphi}$ of $G$ to $\text{SL} (2, K)$ since $G$ is free. We will show that $F^\varphi_{x_1}$ is not semi-stable.

**Lemma 3.** — There is a transcendance basis $\{z\}$ of $L$ over $K$ so that $z$ and $\frac{1}{z}$ are sections of $\mathcal{O}_Y \otimes \Lambda K$. Further, multiplication by $dz$ gives an isomorphism of $\mathcal{O}_Y \otimes \Lambda K$ with $\Omega^1_{Y/\Lambda} \otimes \Lambda K$.

**Proof.** — Let $\gamma \in \text{PGL} (2, K)$ be a non-identity element in the image of $G$. $\gamma$ is known to be hyperbolic, so let $P_1$ and $P_2$ be its two fixed points in $\mathbb{P}^1_K$, and let $z$ be a function on $\mathbb{P}^1_K$ having a pole at $P_1$, a zero at $P_2$, and no other poles or zeros. Identifying $L$ with the functions on $\mathbb{P}^1_K$, we get an element $z$ of $L$. Any quasi-compact open $V$ of $Y$ may be embedded via an open immersion in the formal completion of an $A$-scheme whose generic fiber is $\mathbb{P}^1_K$ so that $L$ is identified with the rational functions on $\mathbb{P}^1_K$ as above and so that the closures of $P_1$ and $P_2$ do not meet $V \cap Y_0$ ([5], Prop. 2.5, 4.20). The lemma follows using this $z$.

**Lemma 4.** — There is an exact sequence

$$0 \to L \to F^\varphi_{x_1} \to L^{-1} \to 0$$

where $F^\varphi_{x_1}$ is the bundle associated to the representation $\tilde{\varphi}$ of $G$ on $K^2$ considered above, and $L^\otimes 2 \simeq \Omega^1_{X_K/K}$.

**Proof.** — Let $\{h^x_\varphi\}$ be the meromorphic descent data on $\mathcal{O}_Y^x$ defined by $\varphi$. If

$$\varphi_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

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as a matrix, define descent data \( \{ h'_g \} \) on \( \mathcal{O}_Y \),

\[
\begin{align*}
    h'_g : \mathcal{O}_Y &\to \mathcal{O}_Y \otimes \Lambda K \\
    f &\mapsto \frac{f}{cz + d}.
\end{align*}
\]

Let \( \varphi \) be the section of \( \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y^*) \otimes \Lambda K \) defined by sending \( f \) to the vector \( (zf, f) \). \( \varphi \) is a map of the descent data \( \{ h'_g \} \) to \( \{ h_g \} \) since

\[
    g^* (z) = \frac{az + b}{cz + d}.
\]

The cokernel of \( \varphi \) is \( \mathcal{O}_Y \) with descent data \( \{ h''_g \} \),

\[
    h''_g (f) = (cz + d)f.
\]

Letting \( L \) denote the line bundle on \( X_K \) obtained from \( \{ h'_g \} \), we have an exact sequence

\[
    0 \to L \to F_{\gamma} \to L^{-1} \to 0.
\]

It remains to identify \( L^\otimes 2 \) with \( \Omega^{i}_{X_K/K} \). \( L^\otimes 2 \) is the bundle associated to the meromorphic descent data

\[
    h''_g = \frac{1}{(cz + d)^2}.
\]

Since \( g^* (dz) = \frac{dz}{(cz + d)^3} \) and since multiplication by \( dz \) gives an isomorphism of \( \Omega^{i}_{X_K/K} \) with \( \mathcal{O}_Y \otimes \Lambda K \), we see \( L^\otimes 2 \) is \( \Omega^{i}_{X_K/K} \).

Let \( F_i \) denote the pullback of \( F_{\varphi, x} \) to \( X_K \). Proposition 2 shows there is a sequence of bundles \( F_k \) on \( X_K \) so that \( F_{\varphi, k} \cong F_{k-1} \).

**Lemma 5.** — If \( g \leq p^{k-1} \), then the \( F_k \) above is semi-stable.

**Proof.** — Suppose \( F_k \) were not semi-stable. Then \( F_k \) would have a quotient bundle of negative degree. Thus \( F_i = F^j_{p^{k-1}} \) would have a quotient bundle \( L' \) of degree at most \( -p^{k-1} \). Then there is a non-zero map \( \varphi \) from either \( L \) or \( L^{-1} \) to \( L' \). The degree of \( L \) is \( g - 1 \), and so \( \varphi \) cannot exist if \( g - 1 < p^{k-1} \).

**Proof of Theorem 1.** — It suffices to show that for each \( g > 1 \) and each algebraically closed field \( k \) of characteristic \( p \), there is a stable curve of genus \( g \) over \( k[[t]] \) whose generic fiber is smooth and geometrically connected, and whose special fiber is \( k \)-split degenerate. Let \( X_0 \) be a rational curve over \( k \) with \( g \) nodes. There is a complete regular local ring \( B \) of characteristic \( p \) with residue field \( k \) and a lifting \( X \) of \( X_0 \) to Spec \( B \). 

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so that the generic fiber of $X$ is smooth and connected [1]. Pulling back by a suitably generic map of $\text{Spec } k[[t]]$ to $\text{Spec } B$ gives the desired curve.

Finally, we give two consequences of Theorem 1.

**Corollary 1.** — For each prime $p > 0$ and integer $g > 1$, there is a smooth curve $X$ of genus $g$ over an algebraically closed field $k$ of characteristic $p$ and a semi-stable bundle $E$ so that $S^p(E)$ is not semi-stable.

**Proof.** — $E^{(p)}$ is a subbundle of $S^p(E)$, and the degree of $S^p(E)$ is zero, where $E$ is the bundle of Theorem 1.

**Remark.** — Hartshorne has shown that in characteristic zero, every symmetric power of a semi-stable bundle is semi-stable [3].

**Corollary 2.** — For each prime $p > 0$ and integer $g > 1$, and each positive integer $n < \frac{g-1}{p}$, there is a semi-stable bundle of rank 2 and degree $2n$ on a curve of genus $g$ which is not ample.

**Proof.** — If $E$ is the bundle of Theorem 1, consider $E \otimes L$, where $L$ is a line bundle of degree $n$. $(E \otimes L)^{(p)}$ has a quotient of non-positive degree, so $E \otimes L$ is not ample.

It is known that if $\deg E > \frac{2g-2}{p}$, and $E$ is semi-stable of rank two, then $E$ is ample [4].

**REFERENCES**


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