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Stable vector bundles and the frobenius morphism

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1. Let $X$ be a curve of genus $g$, proper and smooth over an algebraically closed field, and let $E$ be a vector bundle over $X$. Mumford defines $E$ to be semi-stable if whenever $F$ is a quotient bundle of $E$, then

\[
\frac{\deg F}{\text{rank } F} \leq \frac{\deg E}{\text{rank } E},
\]

where $\deg E$ is the degree of the line bundle $\mathcal{O}_E$, $r$ the rank of $E$. If the characteristic of $X$ is $p > 0$, $E^{(p)}$ will denote Frobenius pullback of $E$.

**Theorem 1.** — For each prime $p$ and integer $g > 1$, there is a curve $X$ of genus $g$ in characteristic $p$ and a semi-stable bundle $E$ of rank two on $X$ so that $E^{(p)}$ is not semi-stable.

Examples of non-ample semi-stable bundles of positive degree constructed by Serre for $p = g = 3$ and later by Tango for $p(p - 1) = 2g$ incidentally proved Theorem 1 when $p(p - 1) = 2g$.

We prove Theorem 1 by constructing a sequence of bundles $E_n$ so that $E_n^{(p)}$ is isomorphic to $E_{n-1}$, and $E_1$ is not semi-stable. In such a sequence, we must have $E_n$ semi-stable for $n \geq 0$, and then we obtain the $E$ of Theorem 1 as the first semi-stable $E_n$.

The bundles $E_n$ will be constructed in the following situation: Let $\Lambda = k[[t]]$, where $k$ is a field of characteristic $p > 0$, and let $X$ be a stable curve over $\Lambda$ with $k$-split degenerate fiber in the sense of Mumford [5]. Thus by definition $X$ is proper and flat over $\Lambda$, and its geometric fibers are reduced, connected and one dimensional. Further, all the normalizations of the components of the special fiber $X_s$ of $X$ are isomorphic to $\mathbb{P}^1$, and the singularities of $X_s$ are double points with $2k$-rational branches. Further each component $X_s$ meets at least three other
components counting itself. We also assume the generic fiber is smooth over $K$, the quotient field of $A$.

Let $Y_0$ be the universal covering scheme of $X_0$, i.e. there is an etale map $p_0$ from $Y_0$ to $X_0$ with the usual universal mapping property. $Y_0$ is not of finite type over $A$. Mumford shows that the group $G$ of covering transformations of $Y_0$ over $X_0$ is a free group on $g$ generators, $g$ the genus of $X_K$. $G$ operates freely and discontinuously in the Zariski topology of $Y_0$.

Section two is devoted to associating to each representation $\gamma$ of $G$ on $K^m$ a sequence of bundles $E_n$ on $X_K^{(n)}$ so that $F^* E_n$ is isomorphic to $E_{n-1}$, where $X_K^{(n)}$ is the fiber product of $X$ with the $n^{th}$ iterate of the Frobenius map on $\text{Spec } K$, and $F$ is relative Frobenius. The construction of $E_n$ from $\gamma$ is analogous to the construction of a bundle $E'$ on a smooth, compact complex variety $X'$ from a representation $\gamma'$ of the fundamental group of $X'$ on $\mathbb{C}^n$. Further, the sequence $\{ E_n \}$ defines a stratification on $E_1$, which is analogous to the stratification on $E'$ whose monodromy is $\gamma'$ [2].

Section three is devoted to the study of the bundle associated to a particular representation $\gamma$ of $G$ on $K^2$ which arises in Mumford's work. We show that the $E_1$ associated to $\gamma$ is not semi-stable. This $\gamma$ is analogous to the following $\gamma'$ associated to a compact Riemann surface $X'$. Let $a_1, \ldots, a_g, b_1, \ldots, b_g$ be the usual generators of $\pi_1(X')$, and let $U$ be an open subset of $\mathbb{P}^1$, and let $\pi$ be a covering map from $U$ to $X'$. Assume that the group $G$ of covering transformations acts on $U$ by linear fractional transformations, and that $G$ is freely generated by the images of $b_1, \ldots, b_g$. Such a $\pi$ is called a Schottky uniformization. Thus we have a homomorphism from $G$ to $\text{PGL}(2, \mathbb{C})$, and this lifts to a homomorphism $\gamma'$ of $\pi_1(X)$ to $\text{SL}(2, \mathbb{C})$. Following Gunning, one may show the bundle $E$ associated to $\gamma'$ is an extension

$$0 \to L \to E \to L^{-1} \to 0$$

where $L^{\otimes 2}$ is isomorphic to $\Omega^1_{X,\mathbb{C}}$. In particular, $E$ is not semi-stable. The representation $\gamma$ of $G$ on $K^2$ is the rigid analytic analogue of $\gamma'$, and the bundle $E_1$ associated to $\gamma$ is an extension of the above type.

We conclude by noting that semi-stable bundles are not closed under symmetric product and with some examples of semi-stable bundles of positive degree which are not ample.

2. $X$ will continue to denote a stable curve over $A$ with smooth generic fiber and $k$-split degenerate fiber, $Y_0$ the universal covering space of $X_0$, and $G$ the group of covering transformations of $Y_0$. There is a unique structure of a formal scheme $Y$ with underlying space $Y_0$ and an etale
map $p$ of $Y$ to $\hat{X}$ which reduces to $p_o$, $\hat{X}$ being the completion of $X$ along $X_o$.

**Definition.** — Meromorphic descent data on a coherent sheaf $F$ over $Y$ is a collection of elements $h_g \in \Gamma(Y, \text{Hom}_{\mathcal{O}_Y}(F, g^* F) \otimes K)$ for each $g \in G$ so that

$$h_g \circ g^* (h_g) = h_g^g,$$

and $h_c$ is the identity. If $\{ h_c \}$ and $\{ h^g \}$ are sets of meromorphic descent data on $F$ and $G$ respectively, a map from $\{ h_c \}$ to $\{ h^g \}$ is an element $f \in \Gamma(Y, \text{Hom}_{\mathcal{O}_Y}(F, G) \otimes K)$ so that

$$k_c \circ f = g^* (f) \circ h^g.$$

We will show the category of coherent sheaves on $Y$ with meromorphic descent data is equivalent to the category of coherent sheaves on $X_o$.

**Lemma 1.** — Given meromorphic descent data on a coherent sheaf $F$ on $Y$, there is a coherent $F'$ with descent data $h' \in \text{Hom}_{\mathcal{O}_Y}(F', g^* F')$ so that $\{ h' \}$ and $\{ h^g \}$ are isomorphic. $F'$ may be taken to have no $A$ torsion.

**Proof.** — We may assume $F$ has no $A$ torsion by replacing it by its image in $F \otimes_A K$. We will construct a coherent subsheaf $F'$ of $F \otimes_A K$ so that the map of $F' \otimes_A K$ to $F \otimes_A K$ is an isomorphism and so that

$$h_g (F') = g^* F'$$

where we are regarding $h_g$ as a map of $F \otimes_A K$ to $g^* (F \otimes_A K)$. Suppose such an $F'$ has been constructed over a $G$ invariant open set $U$ of $Y$, and let $V$ be a quasi compact open not contained in $U$ so that $V \cap g V \subseteq U$ if $g \neq e$. $V$ exists, since $G$ acts discontinuously and has no torsion.

$F'$ may be extended to a coherent subsheaf of $F \otimes_A K$ over $V \cup U$ using the following idea of Raynaud. On $V \cap U$, we may find an $N$ so that

$$F \subseteq F' \subseteq t^{-N} F,$$

where $t$ is a uniformizing parameter of $A$. Let $\overline{F}'$ be the image of $F'$ in $t^{-N} F/t^N F$. $\overline{F}'$ is a coherent sheaf on a scheme whose sheaf of local rings is $\mathcal{O}_Y/t^{-N} \mathcal{O}_Y$. Thus $\overline{F}'$ extends to a coherent subsheaf $\overline{F}''$ of $t^{-N} F/t^N F$ over $V \cup U$. The inverse image $\overline{F}''$ of $\overline{F}''$ in $t^{-N} F$ extends $F'$. Finally, $F'$ may be extended to the $G$ invariant open set consisting of the union of the translates of $V \cup U$ by taking the subsheaf of $F \otimes_A K$ generated by $h^{-1}_g (g^* F'')$ over $U \cap g^{-1} V$. 

**Annales Scientifiques de l'École Normale Supérieure**
Given a coherent $F$ on $X$, the natural map

$$h^F_y: p^* F \to g^* p^* F$$

gives meromorphic descent data on $p^* F$.

**Lemma 2.** — Let $\{ h_x \}$ be meromorphic descent data on a coherent $F$. There is a coherent $H$ on $\hat{X}$ so that $\{ h_x \}$ is isomorphic to $\{ h^H_x \}$. Further the natural map $\alpha$,

$$\text{Hom}_{\hat{X}}(H, H') \otimes \Lambda K \to \text{Hom}(\{ h^H_x \}, \{ h^H'_x \})$$

is an isomorphism.

**Proof.** — By Lemma 1, we may assume $h_x$ maps $F$ to $g^* (F)$. There is a quasi-compact open $U$ of $Y$ so that the translates of $U$ by $G$ cover $Y$. $\{ h_x \}$ gives descent data for the morphism $U \to \hat{X}$ and so $F$ descends to a coherent $H$ on $\hat{X}$, and $\{ h_x \}$ is isomorphic to $\{ h^H_x \}$.

If $f \in \text{Hom}(\{ h^H_x \}, \{ h^H'_x \})$, then $t^n f$ gives a morphism from $p^* H$ to $p^* H'$ compatible with descent data, and so a morphism from $H$ to $H'$. Thus $\alpha$ is surjective. On the other hand, if $f \in \text{Hom}_{\hat{X}}(H, H')$ and if $\alpha (f) = 0$, then $t^n p^* (f) = 0$, and so $t^n f = 0$ for some integer $N$. So $f$ is zero in $\text{Hom}_{\hat{X}}(H, H') \otimes \Lambda K$, and $\alpha$ is injective.

**Proposition 1.** — There is an equivalence of categories $\alpha$ from the category of coherent sheaves on $X_k$ to the category of coherent sheaves on $Y$ with meromorphic descent data. If $F$ is a coherent sheaf on $X$, $\alpha (F_k)$ is the descent data $\{ h^F_x \}$ on $p^* (\hat{F})$.

**Proof.** — Consider the category $C$ whose objects are coherent sheaves on $X$, with $\text{Hom}'(F, G) = \text{Hom}_{\hat{X}}(F, G) \otimes \Lambda K$. $C$ maps isomorphically to the category of coherent sheaves on $X_k$. On the other hand, Grothendieck’s existence theorem and lemma 2 show that it maps isomorphically to the category of coherent sheaves with descent data on $Y$.

Any representation $\varphi$ of $G$ on $K^n$ gives meromorphic descent data $\{ h^\varphi_x \}$ on $\mathcal{O}_Y$, and so a bundle $F_{\varphi, x} = x^{-1} \{ h^\varphi_x \}$. When char $k = p > 0$, we let $F: X \to X^{(p)}$ be the relative Frobenius morphism. $X^{(p)}$ is a stable curve with $k$ split degenerate fiber, and the fundamental group of its special fiber is $G$. Further we have

$$F_{\varphi, x} = F^* (F_{\varphi, x'})$$
Thus we have proven:

**Proposition 2.** — If \( \varphi \) is a representation of \( G \) on \( K^2 \) and \( F_i \) is pullback of \( F_{\varphi} \) to \( X_K \), then there is a sequence of bundles \( F_1, F_2, \ldots \) so that \( F_{k+1}^p \) is isomorphic to \( F_k \).

**Remark.** — There is in fact a unique stratification on \( F_i \) associated to the sequence \( |F_i| \) [2]. This stratification may be defined directly, and exists even when \( \text{char } K = 0 \).

3. Mumford's theory [5] gives us a natural representation of \( G \) on \( K^2 \) in the following way: Let \( D \) be a positive Cartier divisor on \( Y \) so that \( D \) meets only one component of \( Y_0 \), and let \( L \) be the quotient field of \( \bigcup_{n=0}^{\infty} \Gamma(Y, \mathcal{O}_Y(nD)) \).

\( L \) does not depend on \( D \), and since \( G \) acts on \( Y \), we get a homomorphism from \( G \) to the \( K \)-linear automorphisms \( \text{PGL}(2, K) \) of \( L \). This homomorphism may be lifted to a homomorphism \( \gamma \) of \( G \) to \( \text{SL}(2, K) \) since \( G \) is free. We will show that \( F_{\varphi, \gamma} \) is not semi-stable.

**Lemma 3.** — There is a transcendence basis \( \{ z \} \) of \( L \) over \( K \) so that \( z \) and \( \frac{1}{z} \) are sections of \( \mathcal{O}_Y \otimes K \). Further, multiplication by \( dz \) gives an isomorphism of \( \mathcal{O}_Y \otimes K \) with \( \Omega_{Y/K}^1 \otimes K \).

**Proof.** — Let \( \gamma \in \text{PGL}(2, K) \) be a non-identity element in the image of \( G \). \( \gamma \) is known to be hyperbolic, so let \( P_1 \) and \( P_2 \) be its two fixed points in \( \mathbb{P}^1 \), and let \( z \) be a function on \( \mathbb{P}^1 \) having a pole at \( P_1 \), a zero at \( P_2 \), and no other poles or zeros. Identifying \( L \) with the functions on \( \mathbb{P}^1 \), we get an element \( z \) of \( L \). Any quasi-compact open \( V \) of \( Y \) may be embedded via an open immersion in the formal completion of an \( A \)-scheme whose generic fiber is \( \mathbb{P}^1 \) so that \( L \) is identified with the rational functions on \( \mathbb{P}^1 \) as above and so that the closures of \( P_1 \) and \( P_2 \) do not meet \( V \cap Y_0 \) ([5], Prop. 2.5, 4.20). The lemma follows using this \( z \).

**Lemma 4.** — There is an exact sequence

\[
0 \to L \to F_{\varphi, \gamma} \to L^{-1} \to 0
\]

where \( F_{\varphi, \gamma} \) is the bundle associated to the representation \( \gamma \) of \( G \) on \( K^2 \) considered above, and \( L^{\mathcal{O}_Y} \cong \Omega_{Y/K}^1 \).

**Proof.** — Let \( \{ h_{\gamma} \} \) be the meromorphic descent data on \( \mathcal{O}_Y \) defined by \( \gamma \). If

\[
\rho_{\gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
as a matrix, define descent data \( \{ h'_{\varphi} \} \) on \( \mathcal{O}_X \),

\[
h'_{\varphi} : \mathcal{O}_X \to \mathcal{O}_X \otimes_{\Lambda} K
\]

by

\[
h'_{\varphi}(f) = \frac{f}{cz + d}.
\]

Let \( \varphi \) be the section of \( \text{Hom}(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\Lambda} K \) defined by sending \( f \) to the vector \( (zf, f) \). \( \varphi \) is a map of the descent data \( \{ h'_{\varphi} \} \) to \( \{ h_{\varphi} \} \) since

\[
g^* (z) = \frac{az + b}{cz + d}.
\]

The cokernel of \( \varphi \) is \( \mathcal{O}_X \) with descent data \( \{ h''_{\varphi} \} \),

\[
h''_{\varphi}(f) = (cz + d) f.
\]

Letting \( L \) denote the line bundle on \( X_K \) obtained from \( \{ h'_{\varphi} \} \), we have an exact sequence

\[
0 \to L \to F_\varphi \to L^{-1} \to 0.
\]

It remains to identify \( L^{\otimes 2} \) with \( \Omega^1_{X_K/K} \). \( L^{\otimes 2} \) is the bundle associated to the meromorphic descent data

\[
h''_{\varphi} = \frac{1}{(cz + d)^2}.
\]

Since \( g^* (dz) = \frac{dz}{(cz + d)^2} \) and since multiplication by \( dz \) gives an isomorphism of \( \Omega^1_{X_K/K} \otimes_{\Lambda} K \) with \( \mathcal{O}_X \otimes_{\Lambda} K \), we see \( L^{\otimes 2} \) is \( \Omega^1_{X_K/K} \).

Let \( F_i \) denote the pullback of \( F_{\varphi, x} \) to \( X_K \). Proposition 2 shows there is a sequence of bundles \( F_k \) on \( X_K \) so that \( F_k^{(p^k)} \cong F_{k-1}^{(p^{k-1})} \).

**Lemma 5.** — If \( g \leq p^{k-1} \), then the \( F_k \) above is semi-stable.

**Proof.** — Suppose \( F_k \) were not semi-stable. Then \( F_k \) would have a quotient bundle of negative degree. Thus \( F_i = P_k^{(p^{k-i})} \) would have a quotient bundle \( L' \) of degree at most \( -p^{k-1} \). Then there is a non-zero map \( \varphi \) from either \( L \) or \( L^{-1} \) to \( L' \). The degree of \( L \) is \( g - 1 \), and so \( \varphi \) cannot exist if \( g - 1 < p^{k-1} \).

**Proof of Theorem 1.** — It suffices to show that for each \( g > 1 \) and each algebraically closed field \( k \) of characteristic \( p \), there is a stable curve of genus \( g \) over \( k[[t]] \) whose generic fiber is smooth and geometrically connected, and whose special fiber is \( k \)-split degenerate. Let \( X_0 \) be a rational curve over \( k \) with \( g \) nodes. There is a complete regular local ring \( B \) of characteristic \( p \) with residue field \( k \) and a lifting \( X \) of \( X_0 \) to Spec \( B \).
so that the generic fiber of \( X \) is smooth and connected \([1]\). Pulling back by a suitably generic map of \( \text{Spec } k[[t]] \) to \( \text{Spec } B \) gives the desired curve.

Finally, we give two consequences of Theorem 1.

**Corollary 1.** — For each prime \( p > 0 \) and integer \( g > 1 \), there is a smooth curve \( X \) of genus \( g \) over an algebraically closed field \( k \) of characteristic \( p \) and a semi-stable bundle \( E \) so that \( S^p(E) \) is not semi-stable.

**Proof.** — \( E^{(p)} \) is a subbundle of \( S^p(E) \), and the degree of \( S^p(E) \) is zero, where \( E \) is the bundle of Theorem 1.

**Remark.** — Hartshorne has shown that in characteristic zero, every symmetric power of a semi-stable bundle is semi-stable \([3]\).

**Corollary 2.** — For each prime \( p > 0 \) and integer \( g > 1 \), and each positive integer \( n < \frac{g-1}{p} \), there is a semi-stable bundle of rank 2 and degree \( 2n \) on a curve of genus \( g \) which is not ample.

**Proof.** — If \( E \) is the bundle of Theorem 1, consider \( E \otimes L \), where \( L \) is a line bundle of degree \( n \). \( (E \otimes L)^{(p)} \) has a quotient of non-positive degree, so \( E \otimes L \) is not ample.

It is known that if \( \deg E > \frac{2g-2}{p} \), and \( E \) is semi-stable of rank two, then \( E \) is ample \([4]\).

**REFERENCES**


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