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## DUAL TOPOLOGY OF A NILPOTENT LIE GROUP

By IAN D. BROWN

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We will prove that the Kirillov correspondence between the dual of a connected, simply connected nilpotent Lie group and the space of orbits under the group in the dual of the Lie algebra is a homeomorphism. The orbit space is understood here to have the quotient topology. It should be noted that : 1<sup>o</sup> Kirillov, in his original paper [3], proved that the map from the orbit space to dual is continuous and that the bijection is a homeomorphism on an open, dense subset of the dual; 2<sup>o</sup> Olshanski, in [6], proved that the map from dual to orbit space was continuous at the identity representation; 3<sup>o</sup> Moscovici, in [5], recently showed that the bijection was a homeomorphism for groups of dimension six, extending the results of Fell and Dixmier for dimension five or less.

In what follows print capitals  $G, H$  will be Lie groups and script  $\mathfrak{g}, \mathfrak{h}$  their corresponding Lie algebras. All groups will be connected, simply connected nilpotent Lie groups. The dual of a group  $G$ , that is, the set of equivalence classes of irreducible weakly continuous unitary representations of  $G$  with the hull-kernel topology, will be denoted by  $\hat{G}$ . If  $T$  is in  $\hat{G}$ , the orbit in the dual of  $\mathfrak{g}$  corresponding to  $T$  under Kirillov's correspondence will be denoted  $\sigma(T)$ . If  $f$  is a linear form on  $\mathfrak{g}$ ,  $\mathfrak{g}(f)$  will denote

$$\{ X \in \mathfrak{g} : f([X, Y]) = 0 \text{ for all } Y \in \mathfrak{g} \}.$$

An automorphism  $a$  of  $\mathfrak{g}$  will be presumed to act on  $G$  via the exponential map, on  $\mathfrak{g}^*$  by its inverse transpose and on  $\hat{G}$  by  $a.T = T \circ a^{-1}$  for all  $T$  in  $\hat{G}$ . It can be show that  $a.\mathfrak{g}(f) = \mathfrak{g}(af)$  for any  $f$  in  $\mathfrak{g}^*$  and if the automorphisms of  $G$  have the uniform convergence on compacta topology, the map  $(a, T) \rightarrow a.T$  is continuous from  $\text{Aut}(G) \times \hat{G}$  to  $\hat{G}$ . If  $K$  is a subgroup of  $G$  and  $T$  a representation of  $K$ , let  $T \uparrow G$  be the representation of  $G$  induced by  $T$ . Then  $a.(T \uparrow G)$  is equivalent to  $(a.T) \uparrow G$ , where  $a.T$  is defined on  $a.K$ .

Recalling that the support of a representation is the set of all irreducible representations weakly contained in it, we calculate the support of an irreducible representation restricted to a normal subgroup.

LEMMA 1. — *If  $K$  is a normal subgroup of  $G$  and  $T$  is in  $\hat{G}$ , then the support of  $T|K$  is the closure in  $\hat{K}$  of  $\{t \in \hat{K} : \sigma(t) \subset \sigma(T)|\mathfrak{k}\}$ .*

*Proof.* — Choose a sequence of subalgebras  $\mathfrak{g}_i$  of  $\mathfrak{g}$  so that  $\mathfrak{g}_n = \mathfrak{g}$ ;  $\mathfrak{g}_0 = 0$  and  $\mathfrak{g}_{i+1} \supset \mathfrak{g}_i$  for all  $i$ ,  $\dim \mathfrak{g}_{i+1}/\mathfrak{g}_i = 1$  for all  $i$  and  $\mathfrak{g}_k = \mathfrak{k}$  for some  $1 \leq k \leq n$ . For  $f$  in  $\sigma(T)$ , let  $\mathfrak{h} = \sum_{i=0}^n \mathfrak{g}_i(f|\mathfrak{g}_i)$ . Then by Vergne, [8], proposition 4.1.1,  $\mathfrak{h}$  is a subalgebra of maximal dimension subordinate to  $f$  and  $\mathfrak{h} \cap \mathfrak{k}$  is a subalgebra of  $\mathfrak{k}$  of maximal dimension subordinate to  $f| \mathfrak{k}$ .

If  $\chi$  is the character on  $H$  defined by  $\chi(h) = \exp(if(\log h))$  and  $g \cdot \chi(x) = \chi(g^{-1}xg)$  for all  $x \in gHg^{-1}$ , then

$$T|K = (\chi \uparrow G)|K = \int_{G/KH} (g \cdot \chi | gHg^{-1} \cap K) \uparrow K d\mu(g)$$

from Mackey, [4], theorem 12.1, where  $\mu$  may be chosen the image, under the exponential map, of Lebesgue measure on  $\mathfrak{g}/(\mathfrak{h} + \mathfrak{k})$ .

Since  $\mathfrak{k} \cap \mathfrak{h}$  is of maximal dimension subordinate to  $f| \mathfrak{k}$ ,  $(\chi|K \cap H) \uparrow K$  is irreducible and so are all  $(g \cdot \chi | gHg^{-1} \cap K) \uparrow K$ . If  $t_0$  in  $\hat{K}$  corresponds to the  $K$  orbit of  $f| \mathfrak{k}$ , then  $t_0 = (\chi|K \cap H) \uparrow K$ . Also for  $g$  in  $G$ ,  $g \cdot t_0 = (g \cdot \chi | gHg^{-1} \cap K) \uparrow K$  and for  $h \in H$ ,  $k \in K$ ,

$$\begin{aligned} (1) \quad (kh\chi | khHh^{-1}k^{-1} \cap K) \uparrow G &= (kh\chi | kHk^{-1} \cap K) \uparrow K \\ &= k \cdot (h\chi | H \cap K) \uparrow K \\ &= (h\chi | H \cap K) \uparrow K \\ &= t_0, \end{aligned}$$

this last step because  $\mathfrak{h}$  subordinate to  $f$  implies that  $h\chi(x) = \chi(x)$  for all  $x$  in  $H$ . Since  $\mu$  is a sigma-finite standard measure on  $G/KH$  and  $g \rightarrow g \cdot t_0$  is continuous from  $G/KH$  to  $\hat{K}$ ,  $T|K$  is weakly equivalent to  $\{g \cdot t_0 : g \in G/KH\}$  from Fell, [2], theorem 3.2. To see that this is just the set mentioned in the statement of the lemma, let  $f'$  be in  $\sigma(g' \cdot t_0)$  with  $f' = g' \cdot f_0$ ,  $f_0 \in \sigma(t_0)$ . Then there is a  $g$  such that  $f' = g \cdot f|K$  and  $g \cdot t_0$  is in  $\{t \in \hat{K} : \sigma(t) \subset \sigma(T)|\mathfrak{k}\}$ . If  $t$  is in  $\hat{K}$  and  $f'$  is in  $\sigma(t) \subset \sigma(T)|\mathfrak{k}$   $f' = g \cdot f| \mathfrak{k}$  for some  $g$  in  $G$ . Then  $t = g \cdot t_0$ .  $g$  may be chosen from  $G/KH$  since the calculation (1) above shows that  $kh \cdot t_0 = t_0$  for all  $k \in K$ ,  $h \in H$ . Thus  $t$  is in  $\{g \cdot t_0 : g \in G/KH\}$  and

$$\{t \in \hat{K} : \sigma(t) \subset \sigma(T)|\mathfrak{k}\} = \{g \cdot t_0 : g \in G/KH\}.$$

Thus  $T|K$  is weakly equivalent to  $\{t \in \hat{K} : \sigma(t) \subset \sigma(T) | \mathfrak{k}\}$  and the support of  $T|K$  is the closure in  $\hat{K}$  of  $\{t \in \hat{K} : \sigma(t) \subset \sigma(T) | \mathfrak{k}\}$ .

Clearly the proof of this lemma extends without change to completely solvable real nilpotent Lie groups. With some modification it should extend to exponential solvable groups and will likely hold true wherever the Auslander-Kostant construction is used to build representation of solvable Lie groups.

Since the actual proof of the theorem will not be for all nilpotent Lie groups but only for a certain subclass from which any nilpotent group can be obtained as a quotient, we need the following lemma.

LEMMA 2. — *If the Kirillov correspondence is a homeomorphism for  $G$ , then the same is true for any quotient of  $G$  by a connected closed normal subgroup.*

*Proof.* — If  $H$  is a connected closed normal subgroup of  $G$ , then from Dixmier, [4], lemme 3,  $\widehat{G/H}$  is homeomorphic, under the obvious map, to the closed subset of  $\hat{G}$  which are the identity on  $H$ . Under the Kirillov correspondence this set is, by hypothesis, homeomorphic to the (closed) set of orbits in  $\mathfrak{g}^*$  which are zero on  $\mathfrak{h}$ . If  $\pi : G \rightarrow G/H$  and  $\mathfrak{k}$  is the Lie algebra of  $G/H$ , then  $d\pi^*$  is a homeomorphism of  $\mathfrak{k}^*$  to  $\{f \in \mathfrak{g}^* : f|_{\mathfrak{h}} = 0\}$  which commutes with the action of  $G$ -hence is a homeomorphism of the orbit spaces of the two sets. This completes the lemma.

If  $\mathfrak{g}^1 = \mathfrak{g}$  and  $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ ,  $\mathfrak{g}$  is of class  $p$  if  $\mathfrak{g}^p \neq 0$  and  $\mathfrak{g}^{p+1} = 0$ .

If the free Lie algebra on  $k$  generators is divided by the  $p + 1$ 'st member of its central descending series, the result is a nilpotent Lie algebra of class  $p$  called the model  $m(k, p)$ . Any nilpotent Lie algebra of class  $p$  such that  $\dim \mathfrak{g}/\mathfrak{g}^2 = k$  is a homomorphic image of  $m(k, p)$ . Indeed the homomorphism is uniquely defined by extending an arbitrary bijection of the  $k$  generators to  $k$  elements of  $\mathfrak{g}$  whose image in  $\mathfrak{g}/\mathfrak{g}^2$  form a basis, since these latter will generate  $\mathfrak{g}$  as a Lie algebra. Also, if  $x_1, x_2, \dots, x_k$  are a basis of  $\mathfrak{g}/\mathfrak{g}^2$  in  $m(k, p)$  then any automorphism of

$$\mathbf{R}x_1 \oplus \mathbf{R}x_2 \oplus \dots \oplus \mathbf{R}x_k$$

generates an automorphism of  $m(k, p)$ . We will use this fact in the theorem.

THEOREM. — *For any connected, simply connected nilpotent Lie group the Kirillov correspondence is a homeomorphism between the dual of the group and the orbit space with its quotient topology.*

*Proof.* — From Kirillov, [3], theorem 8.2, the map from orbit space to dual is continuous. Hence it suffices to prove that our map  $\sigma$  from dual to orbit space is continuous.

We will proceed by induction on the class of  $\mathfrak{g}$ , the abelian (class 1) case being known.

By lemma 2 and the remark after it, it will suffice to prove the result for  $m(k, p)$  for all  $k$ , assuming it for groups of class smaller than  $p$ .

If  $T_n$  converges to  $T$  in  $\hat{G}$  it suffices to show that there is a subsequence such that  $\sigma(T_n)$  converges to  $\sigma(T)$  in the orbit space.

Suppose  $T_n$  converges to  $T$  in  $\hat{G}$  and choose  $f_n$  in  $\sigma(T_n)$ ,  $f$  in  $\sigma(T)$ . If  $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}^2$ , then there is an  $s$ ,  $0 \leq s \leq k$  such that  $\dim \pi(\mathfrak{g}(f_n)) = k - s$  for an infinite number of  $n$ . Choosing a subsequence, perhaps, we can assume this for all  $n$ . Let  $a_n$  be an orthogonal transformation (choose a basis) of  $\mathfrak{g}/\mathfrak{g}^2$  taking  $\pi(\mathfrak{g}(f_n))$  to  $\pi(\mathfrak{g}(f_1))$  and extend  $a_n$  as before to an automorphism of  $\mathfrak{g}$ . Since the  $k$  dimensional orthogonal group is compact, we can choose another subsequence so that  $a_n$  converge to an automorphism  $a$ , as linear transformations of  $\mathfrak{g}/\mathfrak{g}^2$ . Then, since the coefficients of the extensions to  $\mathfrak{g}$  are polynomials in the coefficients of the automorphisms of  $\mathfrak{g}/\mathfrak{g}^2$ , the  $a_n$  converge to  $a$  as automorphisms of  $\mathfrak{g}$ . Note that  $\pi(\mathfrak{g}(a_n \cdot f_n)) = \pi(\mathfrak{g}(a_1 \cdot f_1))$  for all  $n$ . Let  $\mathfrak{k} = \mathfrak{g}(a_1 \cdot f_1) + \mathfrak{g}^2$ . If  $t_0$  is in  $\hat{K}$  and  $\sigma(t_0) \subset \sigma(a \cdot T) | \mathfrak{k}$ , then  $t_0$  is weakly contained in  $a \cdot T | \mathfrak{k}$ , by lemma 1. For any  $N$ ,  $a \cdot T | K$  is weakly contained in  $\{a_n \cdot T_n | K : n > N\}$ , hence so is  $t_0$ . Again by lemma 1, since any representation is weakly contained in its support,  $a_n \cdot T_n | K$  is weakly contained in the closure, in  $\hat{K}$ , of

$$\{t \in \hat{K} : \sigma(t) \subset \sigma(a_n \cdot T_n) | \mathfrak{k}\}.$$

Thus  $t_0$  is weakly contained (hence contained) in the closure, in  $\hat{K}$ , of  $\{t \in \hat{K} : \text{there is an } n > N \text{ with } \sigma(t) \subset \sigma(a_n \cdot T_n) | \mathfrak{k}\}$  for any  $N$ . Since  $\hat{K}$  has a countable base for its open sets (Fell, [2], lemma 2.5) we can choose a sequence  $t_i$  converging to  $t_0$  in  $\hat{K}$  with  $\sigma(t_i) \subset \sigma(a_i \cdot T_i) | \mathfrak{k}$  for all  $i$  and  $T_i$  different for all  $i$ . Hence the sequence of  $a_n$  is still infinite and  $a_n \cdot T_n$  converges to  $a \cdot T$ .

If  $\mathfrak{h} = [\mathfrak{g}(a_n \cdot f_n), \mathfrak{g}^{p-1}] = [\mathfrak{g}(a_1 \cdot f_1), \mathfrak{g}^{p-1}]$  for all  $n$ , then  $\mathfrak{h}$  is central and  $a_n \cdot T_n = \exp(i \cdot a_n \cdot f_n \circ \log)$  thereon, so  $a_n \cdot f_n | \mathfrak{h}$  converges to  $a f | \mathfrak{h}$ . Since all  $a_n \cdot f_n | \mathfrak{h}$  are zero,  $a \cdot f$  is zero on  $\mathfrak{h}$  and  $a_n \cdot T_n, a \cdot T$  are all the identity on  $H$ .

Hence the same is true of the  $t_i, t_0$  chosen above, so they belong to the closed set of elements of  $\hat{K}$  which are the identity on  $H$  and the corresponding representations on  $K/H$  converge (lemma 2). But  $K/H$  is of

class smaller than  $p$  since  $\mathfrak{k}^p \subset \mathfrak{h}$ . Thus, with the induction hypothesis, by the same argument as for lemma 2, there are  $v_i, v$  in  $\sigma(t_i), \sigma(t_0)$  such that  $v_i$  converge to  $v$ . Thus there are  $f'_i$  in  $\sigma(T_i), f'$  in  $\sigma(T)$  such that  $a_i \cdot f'_i | \mathfrak{k}$  converges to a  $f' | \mathfrak{k}$ .

By the construction of  $\mathfrak{k}$  any extension of  $a_i \cdot f'_i | \mathfrak{k}$  to  $\mathfrak{g}$  is in  $\sigma(a_i \cdot T_i)$  (Raïs, [7], lemma 4.2). If  $V$  is a vector space complementary in  $\mathfrak{g}$  to  $\mathfrak{k}$ , define  $f''_i = a_i^{-1} \cdot a \cdot f'$  on  $a_i^{-1} V$  and  $f''_i = f'_i$  on  $a_i^{-1} \mathfrak{k}$ . Then  $f''_i$  is well defined on  $\mathfrak{g}$  and  $a_i \cdot f''_i = a \cdot f'$  on  $V$  and  $a_i \cdot f''_i = a_i \cdot f'_i$  on  $\mathfrak{k}$ . Hence  $a_i \cdot f''_i$  is in  $\sigma(a_i T_i)$  and  $a_i \cdot f''_i$  converges to  $a \cdot f'$ . Thus  $a^{-1} \cdot a_i \cdot f''_i$  converges to  $f'$  and, since  $a^{-1} a_i \rightarrow 1$ ,  $f''_i$  converges to  $f'$ . Thus  $\sigma(T_i)$  converges to  $\sigma(T)$ . Hence we have a subsequence of the original sequence whose orbits converge in the quotient topology.

Q. E. D.

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*Added in Proof.* — Following the same general line of argument, Nicole Conze has recently proved the algebraic analog of this theorem : let  $\mathfrak{g}$  be a nilpotent Lie algebra over an algebraically closed field of characteristic zero,  $G$  its adjoint group,  $U(\mathfrak{g})$  its universal enveloping algebra and  $P$  the set of primitive ideals of  $U(\mathfrak{g})$ ; then the canonical bijection of  $\mathfrak{g}^*/G$  onto  $P$  is a homeomorphism, where  $P$  has the Jacobson topology and  $\mathfrak{g}^*/G$  the quotient of the Zariski topology.

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