Annales scientifiques de l'É.N.S.

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Annales scientifiques de l'É.N.S. 4^e série, tome 6, nº 4 (1973), p. 457-458 http://www.numdam.org/item?id=ASENS_1973_4_6_4_457_0

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SPHERICAL FUNCTIONS ARE FOURIER TRANSFORMS OF L₁-FUNCTIONS

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In this brief note we apply a result of Kostant, (4.1) in [2], to prove the following. (All notation is as in Kostant's paper).

THEOREM 1. — Let (G, K) be an irreducible Riemannian symmetric pair of non-compact type. Fix an Iwasawa decomposition G = KAN. For each $b \in A$, $b \neq 1$, let $\mu_b \in M^1(\mathfrak{a})$ be the finite measure on \mathfrak{a} such that

$$\int_{\mathcal{K}} f(\log a(bv)) dv = \int_{\mathfrak{a}} f(x) d\mu_b(x) \qquad f \in \mathcal{K}(\mathfrak{a}).$$

Then $\mu_b \in L_1(\mathfrak{a})$ and supp μ_b is the compact set \mathfrak{a} (log b).

REMARK 2. — T. H. Koornwinder has proved this in the rank 1 case by explicitly computing μ_b (see [1]).

This result has an immediate application to the spherical functions on G. If we write $\hat{\mu}(\tau) = \int_{\mathfrak{a}} e^{-i\tau(x)} d\mu(x), \ \tau \in \mathfrak{a}^*$, for the Fourier Stieltjes transform on \mathfrak{a} , then we have

COROLLARY 3. — For $b \neq 1$, $b \in A$ and $\nu = \sigma - i \tau \in \mathfrak{a}^* + i \mathfrak{a}^*$, the spherical function $\varphi_{\nu}(b) = \int_{K} e^{\langle \nu, \log a(b\nu) \rangle} d\nu$ is, as a function of τ , the Fourier transform of the compactly supported measure $e^{\sigma} \mu_{cb} \in L_1(\mathfrak{a})$. Hence, for any tube $T = C + i \mathfrak{a}^*$ with C compact in \mathfrak{a}^* , $\varphi_{\nu}(b) \rightarrow 0$ as $\nu \rightarrow \infty$ in T.

Remark 4. — The second sentence generalizes (3.13) in [3].

Proof of Theorem 1. — The map $g_b : K \to \mathfrak{a}$ with $g_b(v) = \log a(bv)$, $v \in K$, is real analytic and, for $S \subseteq \mathfrak{a}$, $\mu_b(S) = m_{\kappa}(g_b^{-1}(S))$ where m_{κ} is Haar measure on K. We must show $\mu_b(S) = 0$ when S has Lebesgue measure zero. We claim that it suffices to show that g_b has rank equal to dim \mathfrak{a} at some point of K. For if this is so then g_b has rank equal to dim \mathfrak{a} except on a proper real analytic subvariety U of K since K is

^(*) Research supported by NSF Grant GP-32840 X.

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connected. But then dim U < dim K and hence $m_{\kappa}(U) = 0$. Now, on K - U, g_{\flat} , in appropriate coordinates, is just an orthogonal projection between Euclidean spaces. So since m_{κ} is equivalent to Lebesgue measure in any coordinate patch, Fubini's theorem shows

$$m_{\kappa}(g_{b}^{-1}(S)) = m_{\kappa}(g_{b}^{-1}(S) \cap K - U) = 0,$$

when S has Lebesgue mesure zero.

Now to see g_b has rank equal to dim \mathfrak{a} at some point, it suffices by Sard's theorem (or the theorem on functional dependence) to show that the range of g_b has interior points in \mathfrak{a} . Now Kostant shows in (4.1) of [2] that $g_b(\mathbf{K}) = \mathfrak{a}(\log b) = \operatorname{co}(\mathbf{W} \cdot \log b)$, in particular, $\mathfrak{a}(\log b)$ is a non-trivial convex W-invariant set. So by the irreducibility of the action of W on \mathfrak{a} , $0 \in \mathfrak{a}(\log b)$ and span $(\mathfrak{a}(\log b)) = \mathfrak{a}$. Thus $\mathfrak{a}(\log b)$ must have interior.

It is clear that supp $\mu_b = g_b (K) = \mathfrak{a} (\log b)$ and so is compact. \Box

REMARK 5. — The same proof holds for non-irreducible (G, K) provided $\mathfrak{a} (\log b)$ has interior in \mathfrak{a} . For instance if b is regular or more generally if b has non-zero coordinate in each irreducible factor.

Proof of Corollary 3. — The first statement follows from the definition of μ_{b} . For the second note that if $C = \{\sigma\}$, then the Riemann-Lebesgue lemma says $\varphi_{\sigma+i\tau}(b) = (e^{\sigma} \mu_{b})^{*}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. In general, $\sigma \rightarrow e^{\sigma} \mu_{b}$ is a continuous function from \mathfrak{a}^{*} to $L_{4}(\mathfrak{a})$ since μ_{b} has compact support. So it is uniformly continuous on the compact set C from which the result follows as

$$\varphi_{\sigma+i} : (b) - \varphi_{\sigma'+i} : (b) | \leq || e^{\sigma} \mu_b - e^{\sigma'} \mu_b ||_{L_1(\mathfrak{g})}. \quad \Box$$

One would like to have more precise asymptotic information on φ_{ν} as $\nu \to \infty$, but that does not seem to be obtainable by our simple methods.

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(Manuscrit reçu le 1^{er} octobre 1973.)

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