

ANNALES SCIENTIFIQUES DE L'É.N.S.

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Comments on a paper of Brown and Guivarc'h : "Espaces de Poisson des groupes de Lie"

Annales scientifiques de l'É.N.S. 4^e série, tome 8, n° 3 (1975), p. 379-381

http://www.numdam.org/item?id=ASENS_1975_4_8_3_379_0

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COMMENTS ON A PAPER OF BROWN AND GUIVARC'H

BY CALVIN C. MOORE ⁽¹⁾ AND JONATHAN ROSENBERG ⁽²⁾

In a recent paper [2], Brown and Guivarc'h announce a proof of the following conjecture from [1]: Let G be a connected Lie group with radical R such that G/R has finite center; then G is type T in the sense of [1] if and only if the eigenvalues of $\text{ad}(X)$ restricted to the Lie algebra $\mathcal{L}(R)$ of R are purely imaginary for every X in the Lie algebra $\mathcal{L}(G)$ of G . The proof given, however, has a gap in it, and in particular the crucial Proposition 4 is clearly false as stated. The difficulty occurs in the next to last sentence of the proof of this Proposition for it is surely possible for $G \cap V$ to leave invariant a compact set in $\mathcal{G}_p(V)$, for instance a one point set consisting of an affine subspace containing $G \cap V$. We shall show how the difficulty can be repaired by modifying both Propositions 4 and 5; in the end, the modified version is a bit more direct than the original version. We also show that the condition in the theorem that G/R have finite center is necessary; in fact, we show that the universal covering group of $SL_2(\mathbf{R})$ fails to have property T.

Specifically, Proposition 4 should be modified to read as follows:

PROPOSITION 4'. — *Let G be a connected Lie group contained in the affine group of a vector space V . If $G \supset V$, and if G is type T, then G is type R.*

Proof. — The given proof applies directly except that the affine Grassmann manifold $\mathcal{G}_r(V)$ (use some letter other than p) must be chosen so that $0 < r < \dim V$ which is possible by the proof of Proposition 3. The next to last sentence of the proof must be changed; the point is that if a compact subset C of $\mathcal{G}_r(V)$ is invariant under a subspace V' of V , then C must consist of affine subspaces parallel to V' . In particular, if $V' = V$, we have an impossibility since $r < \dim V$. This completes the proof.

Now Proposition 5 has to be strengthened as follows:

PROPOSITION 5'. — *Let G be a connected Lie group with radical R (which is non-compact) and nil-radical N . Then there exists a homomorphism h of G onto a group $h(G)$ such that*

⁽¹⁾ Supported in part by NSF Grant MPS-74-19876.

⁽²⁾ Supported by an NSF Graduate Fellowship. MOS (1970) Subject Classification. 22A10, 60J45.

the kernel of h operates unipotently on $\mathcal{L}(\mathbf{R})$ and such that either: (i) $h(\mathbf{G}) \subset \text{GA}(\mathbf{V})$ for some vector space \mathbf{V} and $h(\mathbf{N}) \supset \mathbf{V}$, or else, (ii) $h(\mathbf{G})$ is a solvable group.

Proof. — Exactly as in the paper, one reduces to the case when \mathbf{N} is a vector group. Let \mathcal{S} be a Levi factor for $\mathcal{L}(\mathbf{G})$ so that $\mathcal{L}(\mathbf{G}) = \mathcal{L}(\mathbf{R}) + \mathcal{S}$. Now let $\mathcal{L}(\mathbf{N}_0)$ be the subspace of $\mathcal{L}(\mathbf{N})$ where \mathcal{S} acts trivially and let \mathbf{N}_0 be the corresponding vector subgroup of \mathbf{N} . If $\mathbf{N} = \mathbf{N}_0$, then as \mathcal{S} acts trivially on $\mathcal{L}(\mathbf{R})/\mathcal{L}(\mathbf{N})$ and as \mathcal{S} is semisimple, \mathcal{S} acts trivially on $\mathcal{L}(\mathbf{R})$ so that $\mathcal{L}(\mathbf{G})$ is the Lie algebra direct sum of $\mathcal{L}(\mathbf{R})$ and \mathcal{S} . The commutator subalgebra of $\mathcal{L}(\mathbf{G})$ is $[\mathcal{L}(\mathbf{R}), \mathcal{L}(\mathbf{R})] + \mathcal{S} \subset \mathcal{L}(\mathbf{N}) + \mathcal{S}$ which acts nilpotently on the radical $\mathcal{L}(\mathbf{R})$. Hence, the commutator subgroup $[\mathbf{G}, \mathbf{G}]$ of \mathbf{G} acts unipotently on $\mathcal{L}(\mathbf{R})$ and hence so does its closure \mathbf{G}_1 . In this case, we choose h to be the projection of \mathbf{G} onto \mathbf{G}/\mathbf{G}_1 and (ii) holds.

Now if $\mathbf{N}_0 \neq \mathbf{N}$, we note that \mathbf{N}_0 is a normal subgroup of \mathbf{G} since \mathbf{N} is abelian and since $\mathcal{L}(\mathbf{R}/\mathbf{N})$ is central in $\mathcal{L}(\mathbf{G}/\mathbf{N})$. Since \mathcal{S} is semisimple and acts trivially on $\mathcal{L}(\mathbf{R})/\mathcal{L}(\mathbf{N})$, we may find a subspace \mathcal{A} of $\mathcal{L}(\mathbf{R})$ complementary to $\mathcal{L}(\mathbf{N})$ which is centralized by \mathcal{S} . Since $[\mathcal{A}, \mathcal{A}]$ is also centralized by \mathcal{S} , it is contained in $\mathcal{L}(\mathbf{N}_0)$. Dividing out by \mathbf{N}_0 , let $\mathbf{G}' = \mathbf{G}/\mathbf{N}_0$, $\mathbf{R}' = \mathbf{R}/\mathbf{N}_0$, $\mathbf{N}' = \mathbf{N}/\mathbf{N}_0$, and let $\mathcal{A}' \simeq \mathcal{A}$, $\mathcal{S}' \simeq \mathcal{S}$ be the images of \mathcal{A} and \mathcal{L} in $\mathcal{L}(\mathbf{G}')$. (Note that \mathbf{R}' is the radical of \mathbf{G}' , but that \mathbf{N}' may be smaller than the nil-radical of \mathbf{G}' .) Then \mathcal{A}' is an abelian subalgebra and $\mathcal{S}' + \mathcal{A}'$ is a complement to $\mathcal{L}(\mathbf{N}')$ so that $\mathcal{L}(\mathbf{G}')$ is the semi-direct product of $\mathcal{L}(\mathbf{N}')$ and $\mathcal{S}' + \mathcal{A}'$. Now let \mathbf{H} be the connected subgroup of \mathbf{G} with Lie algebra $\mathcal{S}' + \mathcal{A}'$, and let $\overline{\mathbf{H}}$ be its closure. Then $\overline{\mathbf{H}} \cap \mathbf{N}'$ consists of elements n such that $\text{Ad}(n)$ is trivial on $\mathcal{L}(\mathbf{N}')$ and on $\mathcal{L}(\mathbf{G}')/\mathcal{L}(\mathbf{N}')$ and which stabilize $\mathcal{S}' + \mathcal{A}'$. That implies that $\text{Ad}(n)$ is the identity, or in other words, that n is in the center of \mathbf{G}' . However, by the construction of \mathbf{N}_0 , and semi-simplicity of \mathcal{S}' , this implies that $n = e$. Thus, $\overline{\mathbf{H}} \cap \mathbf{N}' = \{e\}$ so $\mathbf{H} = \overline{\mathbf{H}}$ is closed and \mathbf{G}' is the semi-direct product of \mathbf{N}' and \mathbf{H} .

We choose our vector space \mathbf{V} to be \mathbf{N}' ; for $g \in \mathbf{G}$, let g' be its image in \mathbf{G}' and write $g' = \tau(g)\rho(g)$ with $\tau(g) \in \mathbf{V}$, and $\rho(g) \in \mathbf{H}$. Now we let $h(g)v = g'vg'^{-1} + \tau(g)$ for $v \in \mathbf{V}$; then h is a homomorphism of \mathbf{G} into $\text{GA}(\mathbf{V})$. Moreover $h(\mathbf{N}) = \mathbf{V}$ and the kernel of h consists of elements $g \in \mathbf{G}$ whose projection in \mathbf{G}' lies in \mathbf{H} and which act trivially on \mathbf{V} . Thus the kernel surely acts unipotently on $\mathcal{L}(\mathbf{R})$ and (i) holds. Proposition 5' is proved.

The proof of the main theorem now proceeds as in [2] if h satisfies (i) and is trivial if h satisfies (ii).

We turn now to the second point about necessity of the condition that \mathbf{G}/\mathbf{R} have finite center. Let \mathbf{G} be semisimple with center \mathbf{Z} . By Proposition V.1 of [1] \mathbf{G} will fail to have property T if and only if there is an open semigroup \mathbf{S} in \mathbf{G} such that $\mathbf{S}\mathbf{S}^{-1} \cap \mathbf{Z}$ has infinite index in \mathbf{Z} . Now let \mathbf{G} be universal covering group of $\mathbf{G}_0 = \text{SL}_2(\mathbf{R})$, so that $\mathbf{Z} = \mathbf{Z}$, the integers, and let \mathbf{S}_0 be the open semigroup of $\text{SL}_2(\mathbf{R})$ consisting of matrices with all entries strictly positive. It is known that $\mathbf{S}_0\mathbf{S}_0^{-1}$ meets the center of \mathbf{G}_0 in only one point. Now on page 46 of [3], there is constructed a very explicit cross section $s: \mathbf{G}_0 \rightarrow \mathbf{G}$ for the group extension so that the corresponding cocycle b from $\mathbf{G}_0 \times \mathbf{G}_0$ into \mathbf{Z} defined

by $s(g)s(h) = b(g,h)s(gh)$ is explicitly computable. The cross section s is continuous and hence a homeomorphism on a dense open set D , specifically, the dense double coset of the triangular subgroup of G_0 . It is clear that $S_0 \subset D$, and a direct calculation using the formulas on page 46 of [3] shows that the cocycle is trivial on $S_0 \times S_0$ and that $s(g^{-1}) = s(g)^{-1}$ for $g \in S_0$. It follows that s is a homomorphism on S_0 and that $S = s(S_0)$ is an open semigroup in G , and that $SS^{-1} \cap Z = \{e\}$. Thus G fails to have property T.

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(Manuscrit reçu le 17 mai 1975.)

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